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# Logarithmic plurigenera of smooth affine surfaces with finite Picard groups

Hideo Kojima

*Dedicated to Professor Hisao Yoshihara  
on the occasion of his sixtieth birthday*

**Abstract.** Let  $S$  be a smooth complex affine surface with finite Picard group. We prove that if  $\bar{\kappa}(S) = 1$  (resp.  $\bar{\kappa}(S) = 2$ ), then  $\bar{P}_2(S) > 0$  (resp.  $\bar{P}_6(S) > 0$ ) and determine the surface  $S$  when  $\bar{\kappa}(S) \geq 0$  and  $\bar{P}_6(S) = 0$ . Moreover, we prove that if  $\text{Pic}(S) = (0)$ ,  $\Gamma(S, \mathcal{O}_S)^* = \mathbb{C}^*$  and  $\bar{P}_2(S) = 0$ , then  $S \cong \mathbb{C}^2$ .

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**Keywords.** Affine surfaces with finite Picard groups, logarithmic Kodaira dimension, logarithmic plurigenera.

## 1. Introduction

Throughout the present article, we work over the field of complex numbers  $\mathbb{C}$ .

In [19] and [30], Kuramoto and Tsunoda considered the problem finding the smallest positive integer  $m$  such that  $\bar{P}_m(X) > 0$  for a smooth open algebraic surface  $X$  with  $\bar{\kappa}(X) \geq 0$ , where  $\bar{P}_m(X)$  and  $\bar{\kappa}(X)$  denote respectively the logarithmic  $m$ -genus and the logarithmic Kodaira dimension of  $X$ . It follows from [30, Theorem 3.3] that a smooth affine surface  $S$  has non-negative logarithmic Kodaira dimension if and only if  $\bar{P}_{12}(S) > 0$ . Recently, in [17] and [18], the author studied the problem for  $\mathbb{Q}$ -homology planes minus non-empty reduced algebraic curves, homology planes and complements of projective plane curves. In particular, we have the following results.

**Theorem A** (cf. [18, Theorem 1.1]). *Let  $X$  be a  $\mathbb{Q}$ -homology plane (for the definition, see Definition 2.6) and  $C$  a non-empty reduced algebraic curve on  $X$ . Then  $\bar{\kappa}(X - C) = -\infty$  if and only if  $\bar{P}_2(X - C) = 0$ .*

**Theorem B** (cf. [18, Theorem 1.3]). *Let  $S$  be a homology plane (for the definition, see Definition 2.6). Then  $S \cong \mathbb{C}^2$  if and only if  $\bar{P}_2(S) = 0$ .*

**Theorem C** (cf. [17, Theorem 1.1] and [18, Theorem 1.2]). *Let  $B \subset \mathbb{P}^2$  be a (not necessarily irreducible) plane curve. Then  $\bar{\kappa}(\mathbb{P}^2 - B) = -\infty$  if and only if  $\bar{P}_6(\mathbb{P}^2 - B) = 0$ . Moreover, if  $\bar{\kappa}(\mathbb{P}^2 - B) \geq 0$  and  $\bar{P}_3(\mathbb{P}^2 - B) = 0$ , then  $B$  can be constructed as either Orevkov's curve  $C_4$  or Orevkov's curve  $C_4^*$  (for the definitions, see [26], [29]).*

In the present article, we shall study logarithmic plurigenera of smooth affine surfaces with finite Picard groups. In Section 2, we recall some results on open algebraic surfaces which will be used later. Moreover, we prove that every smooth affine surface with  $\bar{\kappa} \geq 0$  and  $\bar{P}_2 = 0$  is rational (cf. Lemma 2.9). In Sections 3 and 4, we shall prove the following result.

**Theorem 1.1.** *Let  $S$  be a smooth affine surface with finite Picard group. Then the following assertions hold true.*

- (1) *If  $\bar{\kappa}(S) = 1$ , then  $\bar{P}_2(S) > 0$ .*
- (2) *If  $\bar{\kappa}(S) = 2$ , then  $\bar{P}_6(S) > 0$ .*
- (3) *The surface  $S$  is isomorphic to the surface  $Y\{2, 4, 4\}$  (see [2, (8.53), (8.54)]) if and only if  $\bar{\kappa}(S) \geq 0$  and  $\bar{P}_6(S) = 0$ .*

Here we recall the surface  $Y\{2, 4, 4\}$ . Let  $V_0 = \mathbb{P}^1 \times \mathbb{P}^1$ . Let  $\ell_1, \ell_2$  and  $\ell_3$  be three distinct irreducible curves with  $\ell_i \sim \ell$ , where  $\ell$  is a fiber of a fixed ruling on  $V_0$ , and let  $\bar{\ell}_1, \bar{\ell}_2$ , and  $\bar{\ell}_3$  be three distinct curves with  $\bar{\ell}_i \sim M_0$ , where  $M_0$  is a minimal section of  $V_0$ . Set  $P_1 := \ell_1 \cap \bar{\ell}_1$ ,  $P_2 := \ell_2 \cap \bar{\ell}_1$ ,  $P_3 := \ell_2 \cap \bar{\ell}_3$  and  $P_4 := \ell_3 \cap \bar{\ell}_3$ . Let  $\mu_0: V_1 \rightarrow V_0$  be the blowing-up with centers  $P_1, \dots, P_4$ . Set  $E_1 := \mu_0^{-1}(P_1)$  and  $E_4 := \mu_0^{-1}(P_4)$ . Let  $\mu_1: V_2 \rightarrow V_1$  be the blowing-up with centers  $Q_1 := E_1 \cap \mu'_0(\ell_1)$  and  $Q_2 := E_4 \cap \mu'_0(\ell_3)$ . Set  $V := V_2$  and  $D := \mu'_1(E_1 + E_4 + \mu'_0(\sum_{i=1}^3 (\ell_i + \bar{\ell}_i)))$ . Then the surface  $Y\{2, 4, 4\}$  is the surface  $V - D$ .

In Section 5, we shall prove the following results.

**Theorem 1.2.** *Let  $X$  be a smooth affine surface with finite Picard group and  $C$  a non-empty reduced algebraic curve on  $X$ . Then  $\bar{\kappa}(X - C) = -\infty$  if and only if  $\bar{P}_2(X - C) = 0$ .*

**Theorem 1.3.** *Let  $S = \text{Spec } A$  be a smooth affine surface with  $\text{Pic}(S) = (0)$ . Then the following assertions hold true.*

- (1)  *$\bar{\kappa}(S) = -\infty$  if and only if  $\bar{P}_2(S) = 0$ .*
- (2) *Assume further that  $A^* = \mathbb{C}^*$ , where  $A^*$  denotes the multiplicative group consisting of invertible elements of  $A$  and  $\mathbb{C}^* = \mathbb{C} - \{0\}$ . Then  $S \cong \mathbb{C}^2$  if and only if  $\bar{P}_2(S) = 0$ .*

By [23, Lemma 1.1 (1)], the Picard group of every  $\mathbb{Q}$ -homology plane is finite. In particular, the Picard group of every homology plane is trivial. So, Theorem 1.2 (resp. Theorem 1.3) includes Theorem A (resp. Theorem B).

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## 2. Preliminary results

Given a connected smooth quasi-projective variety  $S$ , we denote by  $\bar{p}_g(S)$  (resp.  $\bar{P}_n(S)$  ( $n \geq 2$ ),  $\bar{\kappa}(S)$ ) the logarithmic geometric genus of  $S$  (resp. the logarithmic  $n$ -genus of  $S$ , the logarithmic Kodaira dimension of  $S$ ). For the definitions, see [9], [21]. By a  $(-n)$ -curve, we mean a smooth complete rational curve with self-intersection number  $(-n)$ . A reduced effective divisor  $D$  is called an SNC-divisor (resp. NC-divisor) if  $D$  has only simple normal crossings (resp. normal crossings). For a  $\mathbb{Q}$ -divisor  $G = \sum \alpha_i C_i$ , where the  $C_i$  are irreducible components of  $G$  and  $\alpha_i \in \mathbb{Q}$ , we write as  $\lfloor G \rfloor = \sum \lfloor \alpha_i \rfloor C_i$ , where  $\lfloor \alpha_i \rfloor$  is the greatest integer  $\leq \alpha_i$ . For an effective divisor  $F$ , we denote by  $\#F$  the number of all irreducible components in  $\text{Supp } F$ .

We recall some basic notions in the theory of peeling (cf. [21, Chapter 2]). Let  $(V, D)$  be a pair of a smooth projective surface  $V$  and an SNC-divisor  $D$  on  $V$ . We call such a pair  $(V, D)$  an *SNC-pair*. A connected curve  $T$  consisting of irreducible components of  $D$  (a connected curve in  $D$ , for short) is a twig if the dual graph of  $T$  is a linear chain and  $T$  meets  $D - T$  in a single point at one of the end components of  $T$ . A connected curve  $R$  (resp.  $F$ ) in  $D$  is a rod (resp. fork) if  $R$  (resp.  $F$ ) is a connected component of  $D$  and the dual graph of  $R$  (resp.  $F$ ) is a linear chain (resp. the dual graph of the exceptional curves of the minimal resolution of a non-cyclic quotient singularity). A connected curve  $E$  in  $D$  is rational (resp. admissible) if each irreducible component of  $E$  is rational (resp. if there are no  $(-1)$ -curves in  $\text{Supp } E$  and the intersection matrix of  $E$  is negative definite). An admissible rational twig  $T$  in  $D$  is maximal if  $T$  is not extended to an admissible rational twig with more irreducible components of  $D$ .

Let  $\{T_\lambda\}$  (resp.  $\{R_\mu\}$ ,  $\{F_\nu\}$ ) be the set of all admissible rational maximal twigs (resp. all admissible rational rods, all admissible rational forks), where no irreducible components of  $T_\lambda$ 's belong to  $R_\mu$ 's or  $F_\nu$ 's. Then there exists a unique decomposition of  $D$  as a sum of effective  $\mathbb{Q}$ -divisors  $D = D^\# + \text{Bk}(D)$  such that the following two conditions (1) and (2) are satisfied:

$$(1) \quad \text{Supp}(\text{Bk}(D)) = \left( \bigcup_\lambda T_\lambda \right) \cup \left( \bigcup_\mu R_\mu \right) \cup \left( \bigcup_\nu F_\nu \right).$$



(2)  $(D^\# + K_V \cdot Z) = 0$  for every irreducible component  $Z$  of  $\text{Supp}(\text{Bk}(D))$ .

We call the  $\mathbb{Q}$ -divisor  $\text{Bk}(D)$  the bark of  $D$  and say that  $D^\#$  is produced by the peeling of  $D$ .

**Lemma 2.1.** *Let  $(V, D)$  be an SNC-pair. Then we have*

$$h^0(V, n(D + K_V)) = h^0(V, \lfloor n(D^\# + K_V) \rfloor)$$

for every integer  $n \geq 0$ .

*Proof.* See [21, Lemma 3.10.1 (p. 106)]. □

**Definition 2.2.** A morphism  $\phi$  from a smooth algebraic surface to a smooth algebraic curve is called a  $\mathbb{P}^1$ -fibration if a general fiber of  $\phi$  is isomorphic to  $\mathbb{P}^1$ . Similarly, an  $\mathbb{A}^1$ -fibration and a  $\mathbb{C}^*$ -fibration are defined, where  $\mathbb{C}^* = \mathbb{A}^1 - \{0\}$ . A  $\mathbb{C}^*$ -fibration is said to be *untwisted* if it is a Zariski-locally trivial fibration on a non-empty Zariski open subset of the base. Otherwise, it is said to be *twisted*.

**Lemma 2.3.** *Let  $S$  be a smooth affine rational surface with  $\bar{\kappa}(S) = 1$ . Then there exists a  $\mathbb{C}^*$ -fibration  $\phi: S \rightarrow T$  onto a smooth rational curve  $T$ .*

*Proof.* See [10, Theorem (2.3)]. (See also [21, Theorem 1.7.1 (p. 201)].) □

For a topological space  $T$ ,  $e(T)$  denotes the topological Euler characteristic of  $T$ . We recall some well-known results on the topological Euler characteristics of some affine surfaces (cf. Lemmas 2.4 and 2.5).

**Lemma 2.4.** *Let  $S$  be a smooth affine surface. Then the following assertions hold true.*

- (1) *If  $\bar{\kappa}(S) = 2$ , then  $e(S) > 0$ .*
- (2) *If  $\bar{\kappa}(S) = 0$  or  $1$ , then  $e(S) \geq 0$ .*

*Proof.* (1) See [24, Theorem 1.4]. (See also [5].)

(2) It follows from [5, Section 5] that if  $e(S) < 0$ , then  $\bar{\kappa}(S) = -\infty$ . So the assertion follows. □

The following result is usually called the Suzuki–Zaidenberg formula (cf. [32, Lemma 3.2] and [3]).

**Lemma 2.5.** *Let  $S$  be a smooth affine surface and  $\phi: S \rightarrow T$  a morphism onto a smooth curve  $T$ . Then*

$$e(S) = e(T)e(f) + \sum_i (e(f_i) - e(f)),$$

where  $f$  is a general fiber of  $\phi$  and the summation is over all the singular fibers of  $\phi$ . Further,  $e(f_i) \geq e(f)$  for all  $i$  and the equality holds if and only if either  $f \cong \mathbb{A}^1$  or  $f \cong \mathbb{C}^*$  and  $(f_i)_{\text{red}} \cong f$ .

We recall some results on  $\mathbb{Q}$ -homology planes.

**Definition 2.6.** A smooth algebraic surface  $S$  is called a  $\mathbb{Q}$ -homology plane (resp. a homology plane) if  $H_i(S; \mathbb{Q}) = (0)$  (resp.  $H_i(S; \mathbb{Z}) = (0)$ ) for any positive integer  $i$ .

It is well-known that every  $\mathbb{Q}$ -homology plane is affine and rational (see [2], [27], [7], [6]).

**Lemma 2.7.** Let  $S$  be a  $\mathbb{Q}$ -homology plane and  $\phi: S \rightarrow T$  a  $\mathbb{C}^*$ -fibration onto a smooth curve  $T$ . Then  $T$  is isomorphic to  $\mathbb{P}^1$  or  $\mathbb{A}^1$ . Moreover, the following assertions hold true.

- (1) If  $T \cong \mathbb{P}^1$ , then  $\phi$  is untwisted, all fibers of  $\phi$  are irreducible and there exists exactly one fiber  $f$  with  $f_{\text{red}} \cong \mathbb{A}^1$  (all the other fibers are isomorphic to  $\mathbb{C}^*$ , if taken with reduced structure).
- (2) If  $T \cong \mathbb{A}^1$  and  $\phi$  is untwisted, then all fibers of  $\phi$  are irreducible except for one singular fiber which consists of two irreducible components. If  $T \cong \mathbb{A}^1$  and  $\phi$  is twisted, then all fibers are irreducible and there exists exactly one fiber  $f$  with  $f_{\text{red}} \cong \mathbb{A}^1$  (all the other fibers are isomorphic to  $\mathbb{C}^*$ , if taken with reduced structure).
- (3) If  $S$  is a homology plane, then  $T \cong \mathbb{P}^1$ .

*Proof.* The assertions (1) and (2) follow from [23, Lemma 1.4]. The assertion (3) follows from [4, Theorems 3 and 4].  $\square$

We shall prove some results on smooth affine surfaces with  $\bar{p}_g = 0$  or  $\bar{P}_2 = 0$  (cf. Lemmas 2.8 and 2.9).

**Lemma 2.8.** Let  $S$  be a smooth affine rational surface with  $\bar{p}_g(S) = 0$  and with finite Picard group. Then the following assertions hold true.

- (1) If  $\bar{\kappa}(S) \geq 0$ , then  $e(S) = 0$  or  $1$ . Moreover,  $e(S) = 1$  if and only if  $S$  is a  $\mathbb{Q}$ -homology plane.
- (2) If  $\bar{\kappa}(S) = 2$ , then  $e(S) = 1$ . In particular,  $S$  is a  $\mathbb{Q}$ -homology plane.

*Proof.* Let  $(V, D)$  be an SNC-pair such that  $V - D \cong S$  and let  $D = \sum_{i=1}^r D_i$  be the decomposition of  $D$  into irreducible components, where  $r = \#D$ . Since  $\text{Pic}(S) = \text{Pic}(V - D)$  is finite, we have  $r \geq \rho(V)$ , where  $\rho(V)$  denotes the Picard number of  $V$ . Since  $\bar{p}_g(S) = h^0(V, D + K_V) = 0$  and  $V$  is a rational surface, we

infer from [20, Lemma I.2.1.3] that  $D_i \cong \mathbb{P}^1$  for  $1 \leq i \leq r$  and the dual graph of  $D$  is a tree. So,  $D$  is simply connected and  $e(D) = 1 + r$ . Thus,

$$e(S) = e(V) - e(D) = \rho(V) - r + 1 \leq 1.$$

Lemma 2.4 implies that  $e(S) = 0$  or  $1$  (resp.  $e(S) = 1$ ) provided  $\bar{\kappa}(S) = 0$  or  $1$  (resp.  $\bar{\kappa}(S) = 2$ ). Suppose that  $e(S) = 1$ . Then  $r = \rho(V)$  and so the natural homomorphism  $H^2(V; \mathbb{Q}) \rightarrow H^2(D; \mathbb{Q})$  is an isomorphism. By [22, Lemma 2.1 (3)],  $S$  is a  $\mathbb{Q}$ -homology plane.  $\square$

**Lemma 2.9.** *Let  $S$  be a smooth affine surface with  $\bar{\kappa}(S) \geq 0$  and  $\bar{P}_2(S) = 0$ . Then  $S$  is a rational surface.*

*Proof.* Let  $S$  be a smooth affine surface with  $\bar{\kappa}(S) \geq 0$ . It suffices to show that  $\bar{P}_2(S) > 0$  if  $S$  is not a rational surface. Let  $(V, D)$  be an SNC-pair such that  $V - D \cong S$ . We treat the following four cases separately.

*Case 1.*  $\kappa(V) = 2$ . It then follows that  $P_2(V) > 0$  (see [1, Theorem 9.1]). Hence,  $\bar{P}_2(S) \geq P_2(V) > 0$ .

*Case 2.*  $V$  is an irrational ruled surface. In this case, there exists an  $\mathbb{P}^1$ -fibration  $p: V \rightarrow B$  onto a smooth projective curve  $B$  with  $g(B) = h^1(V, \mathcal{O}_V) (> 0)$ , where  $g(B)$  denotes the genus of  $B$ . Let  $D' = \sum_{i=1}^{\ell} D_i$  ( $\ell \geq 0$ ) be the sum of all components of  $D$  that are not fiber components of  $p$ . Since  $\bar{\kappa}(S) = \bar{\kappa}(V - D) \geq 0$ , we have  $(F \cdot D) = (F \cdot D') \geq 2$  for a fiber  $F$  of  $p$ . It then follows from [20, Lemmas I.2.3.1 and I.2.3.2] that  $\bar{\kappa}(V - D') \geq 0$ . By [28, Proposition 2.2], we have  $\bar{P}_2(V - D') > 0$ , here we note that the divisor  $D'$  is semi-stable in the sense of [28] because it is an SNC-divisor and contains no rational curves. Hence,  $\bar{P}_2(S) = \bar{P}_2(V - D) \geq \bar{P}_2(V - D') > 0$ .

*Case 3.*  $\kappa(V) = 0$ . If  $P_2(V) > 0$ , then  $\bar{P}_2(S) \geq P_2(V) > 0$ . So we may assume that  $P_2(V) = 0$ . Then  $V$  is a hyperelliptic surface and so there exists an elliptic fibration  $f: V \rightarrow E$  onto a smooth projective elliptic curve  $E$ . Since  $V - D = S$  is affine,  $D$  contains an irreducible curve  $D_1$  that is not a fiber component of  $f$ . Then  $g(D_1) > 0$ , i.e.,  $D_1$  is semi-stable in the sense of [28]. Since  $\bar{\kappa}(V - D_1) \geq \kappa(V) = 0$ , we have  $\bar{P}_2(V - D_1) > 0$  by [28, Proposition 2.2]. Hence,  $\bar{P}_2(S) \geq \bar{P}_2(V - D_1) > 0$ .

*Case 4.*  $\kappa(V) = 1$ . Then, there exists an elliptic fibration  $f: V \rightarrow B$  onto a smooth projective curve  $B$ . We may assume that  $P_2(V) = 0$  (cf. Case 3). Then  $p_g(V) = 0$  and  $g(B) \leq 1$ . We note that  $D$  contains an irreducible component  $D_1$  that is not a fiber component of  $f$  because  $S = V - D$  is affine. Assume that  $g(B) = 1$ . Then,  $g(D_1) \geq 1$ . Hence, by using the same argument as in Case 3, we know that  $\bar{P}_2(S) \geq \bar{P}_2(V - D_1) > 0$ .

Assume that  $g(B) = 0$ . By the canonical bundle formula for elliptic fibrations, we know that  $P_2(V) > 0$  if  $\chi(\mathcal{O}_V) \geq 1$ . So, we may assume further that  $\chi(\mathcal{O}_V) = 0$ . Then  $h^1(V, \mathcal{O}_V) = 1$ . Let  $\alpha: V \rightarrow E$  be the Albanese mapping of  $V$ , where  $E$  is a smooth projective elliptic curve. Then, by using the same argument as in the case  $g(B) = 1$ , we know that  $\bar{P}_2(S) > 0$ .

The proof of Lemma 2.9 is thus completed.  $\square$

Now, we recall some results on log del Pezzo surfaces of rank one. Let  $\bar{V}$  be a normal projective surface with only quotient singular points, let  $\pi: V \rightarrow \bar{V}$  be the minimal resolution of the singularities on  $\bar{V}$  and let  $D$  be the reduced exceptional divisor with respect to  $\pi$ . We often denote  $(V, D)$  and  $\bar{V}$  interchangeably. Since  $\bar{V}$  has only quotient singular points,  $D$  is an SNC-divisor and  $D^\# + K_V \equiv \pi^*(K_{\bar{V}})$  (for the definition of  $D^\#$ , see before Lemma 2.1).

**Definition 2.10.** The above surface  $\bar{V}$  (or the above pair  $(V, D)$ ) is called a log del Pezzo surface if the anticanonical divisor  $-K_{\bar{V}}$  is ample. A log del Pezzo surface is said to have rank one if its Picard number equals one. In the present article, we call a log del Pezzo surface of rank one an *LDP1-surface*.

Hereafter in the present section, we assume that  $\bar{V}$  is an LDP1-surface and we use the same notation as above.

**Lemma 2.11.** *With the same notation and assumptions as above, the following assertions hold true.*

- (1)  $-(D^\# + K_V)$  is a nef and big  $\mathbb{Q}$ -Cartier divisor. Moreover, for any irreducible curve  $F$ ,  $-(D^\# + K_V \cdot F) = 0$  if and only if  $F$  is a component of  $D$ .
- (2) Any  $(-n)$ -curve with  $n \geq 2$  on  $V$  is a component of  $D$ .
- (3)  $V$  is a rational surface.

*Proof.* See [34, Lemma 1.1].  $\square$

**Lemma 2.12.** *There is no  $(-1)$ -curve  $E$  on  $V$  such that the divisor  $E + D$  has negative definite intersection matrix.*

*Proof.* See [33, Lemma 1.4].  $\square$

By Lemma 2.11 (1), if  $C$  is an irreducible curve not contained in  $\text{Supp } D$ , then  $-(C \cdot D^\# + K_V)$  takes value in  $\{n/p \mid n \in \mathbb{N}\}$ , where  $p$  is the smallest positive integer such that  $pD^\#$  is an integral divisor. So we can find an irreducible curve  $C$  such that  $-(C \cdot D^\# + K_V)$  attains the smallest positive value. We denote the set of such

irreducible curves by  $MV(V, D)$ . The pair  $(V, D)$  is said to be of the first kind if there exists an irreducible curve  $C \in MV(V, D)$  such that  $|C + D + K_V| \neq \emptyset$ . Otherwise, the pair  $(V, D)$  is said to be of the second kind.

**Lemma 2.13.** *Assume that  $(V, D)$  is of the first kind and that  $\bar{V}$  has a singular point  $P$  that is not a rational double point. Then the following assertions hold true.*

- (1) *Let  $C \in MV(V, D)$  be an irreducible curve such that  $|C + D + K_V| \neq \emptyset$ . Then there exists a unique decomposition of  $D$  as a sum of effective integral divisors  $D = D' + D''$  such that the following two conditions are satisfied:*

- (i)  $(C \cdot D_i) = (D'' \cdot D_i) = (K_V \cdot D_i) = 0$  for any component  $D_i$  of  $D'$ .
- (ii)  $C + D'' + K_V \sim 0$ .

- (2) *The singular point  $P$  is a cyclic quotient singular point and the other singular points on  $\bar{V}$  are rational double points.*

*Proof.* The assertion (1) follows from [33, Lemma 2.1]. We prove the assertion (2). With the same notation as in the assertion (1), we know that  $\text{Supp } D' \cap \text{Supp } D'' = \emptyset$  and each connected component of  $D'$  can be contracted to a rational double point. By the hypothesis that  $P$  is not a rational double point, we have  $D'' \neq 0$ . Since  $C + D'' + K_V \sim 0$ ,  $|C + K_V| = |-D''| = \emptyset$ . So  $C$  is a smooth rational curve and  $(C \cdot D'') = (C \cdot -C - K_V) = 2$ . Further, for every irreducible component  $D_i$  of  $D''$ , we have  $(D_i \cdot C + D'' - D_i) = (D_i \cdot -K_V - D_i) = 2$ . Hence we know that  $D'' = \pi^{-1}(P)$  and  $D''$  is a linear chain of smooth rational curves.  $\square$

**Lemma 2.14.** *Assume that  $(V, D)$  is of the second kind and  $\rho(V) \geq 3$ . Then every irreducible curve  $C \in MV(V, D)$  is a  $(-1)$ -curve.*

*Proof.* See [33, Lemma 2.2] and [8, Proposition 3.6].  $\square$

**Lemma 2.15.** *Let  $\Phi: V \rightarrow \mathbb{P}^1$  be a  $\mathbb{P}^1$ -fibration. Assume that there exists a singular fiber  $F$  whose configuration is given as one of (i) and (ii) in Figure 1 and that  $C \in MV(V, D)$ , where  $C$  is the unique  $(-1)$ -curve in  $\text{Supp } F$ . Then each singular fiber of  $\Phi$  consists of  $(-2)$ -curves and  $(-1)$ -curves, say  $E_1$  and  $E_2$  (possibly  $E_1 = E_2$ ), and  $E_i \in MV(V, D)$  for  $i = 1, 2$ .*

*Proof.* See [33, Lemma 1.6 (3)].  $\square$

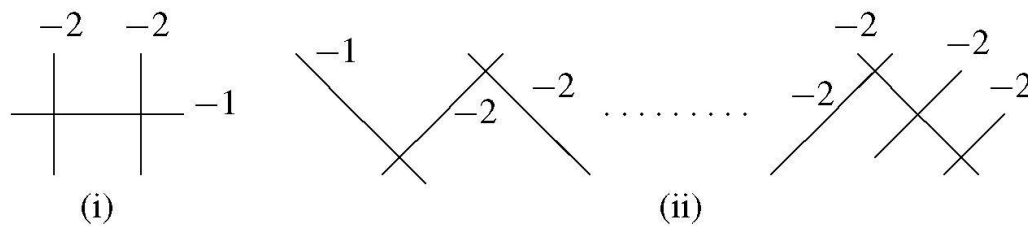


Figure 1

### 3. Proof of the assertion (1) of Theorem 1.1

In this section, we shall prove the assertion (1) of Theorem 1.1.

Let  $S$  be a smooth affine surface with  $\bar{\kappa}(S) = 1$  and with finite Picard group. If  $S$  is not a rational surface, then  $\bar{P}_2(S) \geq 1$  by Lemma 2.9. So we may assume that  $S$  is a rational surface. Further, we may assume that  $\bar{p}_g(S) = 0$ . Lemma 2.8 (1) then implies that  $e(S) = 0$  or 1 and  $e(S) = 1$  if and only if  $S$  is a  $\mathbb{Q}$ -homology plane. By Lemma 2.3, there exists a  $\mathbb{C}^*$ -fibration  $\phi: S \rightarrow T$  onto a smooth rational curve  $T$ . The  $\mathbb{C}^*$ -fibration  $\phi$  is extended to a  $\mathbb{P}^1$ -fibration  $\Phi: V \rightarrow \mathbb{P}^1$ , where  $V$  is a smooth projective rational surface such that  $D := V - S$  is an SNC-divisor. Since  $\bar{p}_g(S) = 0$  and  $S$  is a rational surface,  $D$  is a tree of smooth rational curves by [20, Lemma I.2.1.3].

The following lemma can be proved by using the same argument as in [18, Section 3]. For the sake of completeness, we shall reproduce the proof.

**Lemma 3.1.** *With the same notation and assumptions as above, assume further that  $e(S) = 0$ . Then  $\bar{P}_2(S) > 0$ .*

*Proof.* Since  $e(S) = 0$ , it follows from Lemma 2.5 that every fiber of  $\phi$  is isomorphic to  $\mathbb{C}^*$  if taken with reduced structure. We shall consider the following two cases separately.

*Case 1.*  $\phi$  is twisted. In this case,  $D$  contains exactly one irreducible component  $H$  that is not a fiber component of  $\Phi$ . The curve  $H$  is then a 2-section of  $\Phi$  and hence  $\Phi|_H: H \rightarrow \mathbb{P}^1$  is a double covering. Since  $H \cong \mathbb{P}^1$ , there exist two branch points  $Q_1, Q_2 (\in \mathbb{P}^1)$  of  $\Phi|_H$ . Set  $F_i := \Phi^{-1}(Q_i)$  for  $i = 1, 2$ .

Suppose that  $\text{Supp}(F_i) \cap S \neq \emptyset$  for  $i = 1$  or 2. Since  $D$  is connected and  $\#(\text{Supp}(F_i) \cap H) = 1$ ,  $(F_i|_S)_{\text{red}}$  contains an affine line. This is a contradiction. So,  $\text{Supp}(F_i) \subset \text{Supp} D$  for  $i = 1, 2$  and hence  $T$  is contained in  $\mathbb{C}^*$  as a Zariski open subset. Since  $\bar{\kappa}(T) \geq \bar{\kappa}(\mathbb{C}^*) = 0$ , it follows from [19, Proposition 1] that  $\bar{P}_2(S) > 0$ .

*Case 2.*  $\phi$  is untwisted. In this case,  $D$  contains exactly two sections  $H_1$  and  $H_2$  of  $\Phi$  and each component of  $D - (H_1 + H_2)$  is a fiber component of  $\Phi$ . Since  $D$  is a tree

of smooth rational curves, we may assume that  $(E \cdot D - E) \geq 3$  for any  $(-1)$ -curve  $E \subset \text{Supp}(D - (H_1 + H_2))$  (i.e.,  $(V, D)$  is minimal along fibers (cf. [4, p. 87])).

**Claim 1.**  $(H_1 \cdot H_2) = 0$ .

*Proof.* Suppose that  $(H_1 \cdot H_2) > 0$ . Let  $P$  be a point of  $H_1 \cap H_2$  and set  $F_P := \Phi^{-1}(\Phi(P))$ . Since  $D$  is a connected SNC-divisor and  $\text{Supp } F_P \cap H_1 \cap H_2 = \{P\}$ ,  $\text{Supp}(F_P)$  contains no components of  $D$ . Then,  $\text{Supp } F_P \cap S \cong \mathbb{A}^1$ , a contradiction. Hence,  $(H_1 \cdot H_2) = 0$ .  $\square$

**Claim 2.** Let  $F$  be a fiber of  $\Phi$ . Then  $F$  is reducible if and only if  $F|_S (\neq \emptyset)$  is a singular fiber of  $\phi$ .

*Proof.* The “if” part is clear because every singular fiber of  $\phi$  is multiple. We shall prove the “only if” part. Suppose that  $F$  is a reducible fiber of  $\Phi$ . Then  $F$  contains a  $(-1)$ -curve  $E$ . We note that, if the coefficient of  $E$  in  $F$  equals one, then [2, Lemma (7.3)] implies that  $(E \cdot F_{\text{red}} - E) = 1$  and  $F$  contains another  $(-1)$ -curve. Suppose that  $\text{Supp } F \subset \text{Supp } D$ . Then  $E$  meets both  $H_1$  and  $H_2$  because  $H_1$  and  $H_2$  are sections of  $\Phi$  and  $(E \cdot D - E) \geq 3$ . Then the coefficient of  $E$  in  $F$  equals one and hence  $F$  has another  $(-1)$ -curve  $E'$ . Then  $(E' \cdot D - E') = (E' \cdot F_{\text{red}} - E') \leq 2$ , a contradiction. Suppose that  $F|_S (\neq \emptyset)$  is not a singular fiber of  $\phi$ . Let  $F_0$  be the component of  $F$  with  $F_0 \cap S \neq \emptyset$ . Since the coefficient of  $F_0$  in  $F$  equals one,  $F$  has a  $(-1)$ -curve other than  $F_0$ . So we may assume that  $E \neq F_0$ . Then  $E$  is a component of  $D$ . Since  $H_1$  and  $H_2$  are sections of  $\Phi$  and  $(E \cdot D - E) \geq 3$ , we know that  $E$  meets both  $H_1$  and  $H_2$ . So  $(F_0 \cdot H_1) = (F_0 \cdot H_2) = 0$ . Since  $F_0|_S \cong \mathbb{C}^*$  and  $\text{Supp}(F_{\text{red}} - F_0) \subset \text{Supp } D$ ,  $\text{Supp } D$  is not connected, a contradiction. Therefore, we know that  $F|_S (\neq \emptyset)$  is a singular fiber of  $\phi$ .  $\square$

**Claim 3.** Let  $f$  be a singular fiber of  $\phi$  and let  $F$  be the fiber of  $\Phi$  containing  $f$ . Then the weighted dual graph of  $F_{\text{red}}$  is linear. Moreover,  $F$  has exactly one  $(-1)$ -curve, say  $E$ ,  $\text{Supp}(F_{\text{red}} - E)$  consists of two connected components and  $(F|_S)_{\text{red}} = E|_S$ .

*Proof.* The fiber  $F$  has exactly one irreducible component, say  $F_0$ , with  $F_0 \cap S \neq \emptyset$ . If  $F$  contains no components meeting both  $H_1$  and  $H_2$ , then the assertions follow from [2, Lemma (7.6)]. Suppose that  $F$  contains a component  $F_1$  meeting both  $H_1$  and  $H_2$ . Then  $F_0 \neq F_1$ . Since  $F_0|_S \cong \mathbb{C}^*$ ,  $F_{\text{red}} - F_0$  is not connected. This is a contradiction because  $D$  is connected.  $\square$

Let  $m_1 f_1, \dots, m_r f_r$  be all the singular fibers of  $\phi$  with respective multiplicities  $m_1, \dots, m_r$ , where  $f_i \cong \mathbb{C}^*$  ( $1 \leq i \leq r$ ), and let  $F_i$  ( $1 \leq i \leq r$ ) be the fiber of  $\Phi$  containing  $m_i f_i$ . Let  $E_i = \bar{f}_i$  ( $1 \leq i \leq r$ ) be the closure of  $f_i = (m_i f_i)_{\text{red}}$  in  $V$ .

**Claim 4.** (1)  $T \cong \mathbb{A}^1$ .

(2)  $r \geq 2$ .



*Proof.* (1) If  $T \cong \mathbb{P}^1$ , then  $D$  cannot be connected by Claims 1 and 3. This is a contradiction. If  $T \cong \mathbb{P}^1 \setminus \{s \text{ points}\}$  ( $s \geq 2$ ), then  $D$  contains a loop of (smooth rational) curves. So,  $\bar{p}_g(S) = h^0(V, D + K_V) > 0$ , which contradicts our assumption.

(2) Suppose that  $r \leq 1$ . Since  $\phi$  is untwisted,  $S$  contains  $\mathbb{C}^* \times \mathbb{C}^*$  as a Zariski open subset. Then

$$0 = \bar{\kappa}(\mathbb{C}^* \times \mathbb{C}^*) \geq \bar{\kappa}(S) = 1,$$

which is a contradiction. Hence,  $r \geq 2$ .  $\square$

Set  $P_0 := \mathbb{P}^1 \setminus T$  and  $F_0 := \Phi^{-1}(P_0)$ . By Claim 2,  $F_0$  is irreducible. Let  $(F_i)_{\text{red}} - E_i = \sum_{j=1}^{r_i} C_{ij}$  ( $1 \leq i \leq r$ ) be the decomposition of  $(F_i)_{\text{red}} - E_i$  into irreducible components. Then,

$$D = F_0 + H_1 + H_2 + \sum_{i=1}^r \left( \sum_{j=1}^{r_i} C_{ij} \right).$$

Since  $(H_1 \cdot H_2) = 0$  by Claim 1, we obtain a birational morphism  $\rho: V \rightarrow \mathbb{F}_a$  onto a Hirzebruch surface  $\mathbb{F}_a$  of degree  $a$  such that  $\bar{H}_1 := \rho(H_1)$  and  $\bar{H}_2 := \rho(H_2)$  are sections of a  $\mathbb{P}^1$ -fibration  $\Phi \circ \rho^{-1}: \mathbb{F}_a \rightarrow \mathbb{P}^1$  and  $(\bar{H}_1 \cdot \bar{H}_2) = 0$ . Since  $(\bar{H}_1 \cdot \bar{H}_2) = 0$ , we may assume that  $(\bar{H}_1^2) \leq 0$ . Then,  $\bar{H}_1$  is a minimal section and  $\bar{H}_2 \sim \bar{H}_1 + a\ell$ , where  $\ell$  is a fiber of the  $\mathbb{P}^1$ -fibration  $\Phi \circ \rho^{-1}$ . Moreover, we may assume that  $(H_1^2) = (\bar{H}_1^2) = -a$ , namely,  $V$  is obtained from  $\mathbb{F}_a$  by starting the blowing-ups with centers at points on  $\bar{H}_2$  or fibers of  $\Phi \circ \rho^{-1}$ , while no points on  $\bar{H}_1$  are blown up.

Since  $K_{\mathbb{F}_a} \sim -2\bar{H}_1 - (a+2)\ell \sim -\bar{H}_1 - \bar{H}_2 - 2\ell$ , we have

$$K_V \sim -H_1 - \rho^*(\bar{H}_2) - 2F_0 + \sum_{i=1}^r \left( \sum_{j=1}^{r_i} \lambda_{ij} C_{ij} \right) + \sum_{i=1}^r \lambda_i E_i,$$

where  $\lambda_i \geq 0$  and  $\lambda_{ij} \geq 0$  for  $1 \leq i \leq r$  and  $1 \leq j \leq r_i$ . We set

$$\rho^*(\bar{H}_2) = H_2 + \sum_{i=1}^r \left( \sum_{j=1}^{r_i} \mu_{ij} C_{ij} \right) + \sum_{i=1}^r \mu_i E_i$$

and

$$F_i = \sum_{j=1}^{r_i} \alpha_{ij} C_{ij} + m_i E_i$$

for  $1 \leq i \leq r$ . Since  $D = F_0 + H_1 + H_2 + \sum_{i=1}^r (\sum_{j=1}^{r_i} C_{ij})$ , we have

$$D + K_V \sim \sum_{i=1}^r \left( \sum_{j=1}^{r_i} (\lambda_{ij} - \mu_{ij} + 1) C_{ij} \right) + \sum_{i=1}^r (\lambda_i - \mu_i) E_i - F_0.$$



By Claim 2, the weighted dual graph of  $F_i$  ( $1 \leq i \leq r$ ) looks like the one given in [20, p. 79]. As seen from [20, I.4.9.3 (p. 80)], we know that

$$F_i = \sum_{j=1}^{r_i} (\lambda_{ij} - \mu_{ij} + 1) C_{ij} + (\lambda_i - \mu_i + 1) E_i$$

for  $1 \leq i \leq r$ . Then,

$$D + K_V \sim \sum_{i=1}^r F_i - \sum_{i=1}^r E_i - F_0 \sim (r-1)F_0 - \sum_{i=1}^r E_i.$$

Since  $r \geq 2$  by Claim 4 (2) and  $2E_i \leq m_i E_i \leq F_i$  for  $i = 1, \dots, r$ , we have

$$2(D + K_V) \sim 2(r-1)F_0 - 2 \sum_{i=1}^r E_i \sim (r-2)F_0 + \sum_{i=1}^r (F_i - 2E_i) \geq 0.$$

Therefore,  $\bar{P}_2(S) > 0$ . □

**Remark 3.2.** As seen from the proof of Lemma 3.1, we know that a smooth affine surface with  $\bar{\kappa} = 1$  and  $e = 0$  has positive logarithmic bigenus.

In the following lemma, we shall treat the case  $e(S) = 1$ .

**Lemma 3.3.** *With the same notation and assumptions as above, assume further that  $e(S) = 1$ . Then  $\bar{P}_2(S) > 0$ .*

*Proof.* We shall consider the following two cases separately.

*Case 1.*  $\phi$  is untwisted. In this case,  $D$  contains exactly two sections  $H_1$  and  $H_2$  of  $\Phi$  and each component of  $D - (H_1 + H_2)$  is a fiber component of  $\Phi$ . As seen from [4, Section 3], we may assume that:

- (i)  $(H_1 \cdot H_2) = 0$ .
- (ii) If  $G$  is a  $(-1)$ -curve contained in  $\text{Supp}(D - (H_1 + H_2))$ , then  $(G \cdot D - G) \geq 2$ .  
Moreover, if  $(G \cdot D - G) = 2$ , then  $(G \cdot H_1) = (G \cdot H_2) = 1$ .

By Lemma 2.7,  $\phi$  has a singular fiber  $f$  satisfying one of the following two conditions (a) and (b):

- (a)  $f$  is irreducible and  $f_{\text{red}} \cong \mathbb{A}^1$ .
- (b)  $f_{\text{red}}$  consists of two irreducible components  $f_1$  and  $f_2$ , where  $f_1 \cong \mathbb{A}^1$  and  $f_2 \cong \mathbb{A}^1$  or  $\mathbb{C}^*$ , and  $\#(f_1 \cap f_2) \leq 1$ .

Let  $E$  (resp.  $E_i$  ( $i = 1, 2$ )) be the closure of  $f$  (resp.  $f_i$  ( $i = 1, 2$ )) in  $V$  if the condition (a) (resp. the condition (b)) holds.

If the condition (a) holds, then  $E$  is a smooth rational curve. Moreover, by the assumptions (i) and (ii),  $E$  must be a  $(-1)$ -curve and  $(E \cdot D) = 1$ . Hence,  $\bar{P}_n(S) = \bar{P}_n(V - (E + D))$  for any integer  $n > 0$ . If the condition (b) holds, then  $E_1$  is a smooth rational curve. By the assumptions (i) and (ii), we may assume that  $E_1$  is a  $(-1)$ -curve and  $(E_1 \cdot D) = 1$ . Then,  $\bar{P}_n(S) = \bar{P}_n(V - (E_1 + D))$  for any integer  $n > 0$ .

Set  $S' := S \setminus f$  (resp.  $S' := S \setminus f_1$ ) if the condition (a) (resp. the condition (b)) holds. Then  $e(S') = 0$ ,  $\text{Pic}(S')$  is finite and  $\bar{\kappa}(S') = \bar{\kappa}(S) = 1$ . We infer from Lemma 3.1 that  $\bar{P}_2(S') > 0$ . Hence,  $\bar{P}_2(S) = \bar{P}_2(S') > 0$ .

*Case 2.*  $\phi$  is twisted. In this case,  $D$  contains exactly one component  $H$  that is not a fiber component of  $\Phi$ . The curve  $H$  is a 2-section of  $\Phi$  and so  $\Phi|_H: H \rightarrow \mathbb{P}^1$  is a double covering. Since  $H \cong \mathbb{P}^1$ , there exist two branch points  $Q_0, Q_\infty (\in \mathbb{P}^1)$  of  $\Phi|_H$ . Set  $F_0 := \Phi^{-1}(Q_0)$  and  $F_\infty := \Phi^{-1}(Q_\infty)$ . By Lemma 2.7, we may assume that  $\text{Supp}(F_\infty) \subset \text{Supp} D$  and  $\text{Supp}(F_0) \not\subset \text{Supp} D$ . Then  $F_0 \cap S$  is written as  $m_0 f_0$  with  $f_0 \cong \mathbb{A}^1$ . Let  $m_i f_i$  ( $1 \leq i \leq r$ ) exhaust all singular fibers with  $f_i \cong \mathbb{C}^*$  and  $m_i \geq 2$ . Let  $E_i$  be the closure of  $f_i$  in  $V$  and set  $F_i := \Phi^{-1}(\Phi(E_i))$ ,  $1 \leq i \leq r$ . As seen from [21, p. 241], we may assume that the following conditions are satisfied:

- (1) The dual graph of  $F_i$  ( $1 \leq i \leq r$ ) is a linear chain and  $E_i$  is a unique  $(-1)$ -curve in  $\text{Supp}(F_i)$ . The fiber  $F_i$  meets the 2-section  $H$  at the terminal components.
- (2) The dual graph of  $F_\infty + H$  is given as in Figure 2.
- (3) The dual graph of the fiber  $F_0$  is that for  $F_\infty$  with the corresponding components  $A_0, B_0, C_0$  and with  $A_0$  either contained in  $\text{Supp} D$  or not.

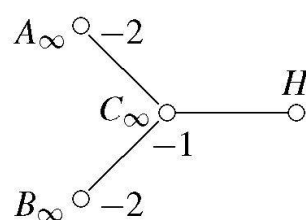


Figure 2

We need some explanation about the condition (3) as above (cf. [21, p. 241]). If  $A_0$ , which is the closure of  $f_0$  in  $V$ , is a  $(-1)$ -curve, then  $\bar{P}_n(S) = \bar{P}_n(S - f_0)$  for any integer  $n > 0$ . Since  $e(S - f_0) = 0$  and  $\bar{\kappa}(S - f_0) = 1$ , it follows from Lemma 3.1 that  $\bar{P}_2(S) > 0$ . In this case, Lemma 3.3 follows. So we may assume that  $(A_0^2) \leq -2$  and that  $S$  is NC-minimal (for the definition, see [21, Definition 4.4.1 (p. 232)]). Then the contractions of  $\text{Supp}(F_0)$  except for  $A_0$  makes its dual graph look like that for  $F_\infty$  with the component  $A_0$  not contained in  $\text{Supp} D$ .

By virtue of the proof of [21, Theorem 4.6.5 (p. 243)], we know that

$$D + K_V \sim_{\mathbb{Q}} -F + (B_0 + C_0) + \frac{1}{2}F + \frac{1}{2}(A_{\infty} + B_{\infty}) + \sum_{i=1}^r (F_i - E_i),$$

where  $F$  is a general fiber of  $\Phi$ . So,

$$2(D + K_V) \sim -F + 2(B_0 + C_0) + (A_{\infty} + B_{\infty}) + \sum_{i=1}^r (2F_i - 2E_i),$$

here we note that  $V$  is a rational surface. Since  $2F_i - 2E_i = F_i + (F_i - 2E_i) \geq F_i$  for  $1 \leq i \leq r$ , we have  $2(D + K_V) \geq 0$  if  $r \geq 1$ . If  $r = 0$ , then

$$N := r - \frac{1}{2} - \sum_{i=1}^r \frac{1}{m_i} = -\frac{1}{2} < 0.$$

Then [21, Theorem 4.6.5 (1) (p. 243)] implies that  $\bar{\kappa}(S) = -\infty$ , which is a contradiction. Therefore, we know that  $\bar{P}_2(S) > 0$ .  $\square$

The proof of the assertion (1) of Theorem 1.1 is thus completed.

#### 4. Proof of the assertions (2) and (3) of Theorem 1.1

The assertion (3) of Theorem 1.1 easily follows from [2, Theorem (8.70) 1)] and the assertions (1) and (2) of Theorem 1.1. Indeed, if  $S$  is a smooth affine surface with  $\bar{\kappa}(S) \geq 0$ , with  $\bar{P}_6(S) = 0$  and with finite Picard group, then  $\bar{\kappa}(S) = 0$  by the assertions (1) and (2) of Theorem 1.1. By [2, Theorem (8.70) 1)],  $S$  is then isomorphic to the surface  $Y\{2, 4, 4\}$ , here we note that the surface  $Y\{3, 3, 3\}$  and the surface  $Y\{2, 3, 6\}$  have positive logarithmic 6-genera. From now on, we shall prove the assertion (2) of Theorem 1.1.

Let  $S$  be a smooth affine surface with  $\bar{\kappa}(S) = 2$  and with finite Picard group. By virtue of Lemma 2.9, we may assume that  $S$  is a rational surface. Moreover, by virtue of Lemma 2.8 (2), we may assume further that  $S$  is a  $\mathbb{Q}$ -homology plane.

Let  $(V, D)$  be a pair of a smooth projective rational surface and an NC-divisor  $D$  on  $V$  such that  $V - D \cong S$  and  $(E \cdot D - E) \geq 3$  for any  $(-1)$ -curve  $E \subset \text{Supp } D$ . Since  $S$  is a  $\mathbb{Q}$ -homology plane, the divisor  $D$  is a tree of smooth rational curves by [23, Lemma 1.1 (1)]. In particular,  $D$  is an SNC-divisor.

**Lemma 4.1.** *The pair  $(V, D)$  is almost minimal (for the definition, see [21, Chapter 2, Section 3]).*

*Proof.* The assertion can be verified by using the same argument as in the proof of [17, Lemma 4.3].  $\square$

Let  $D + K_V \equiv (D + K_V)^+ + (D + K_V)^-$  be the Zariski–Fujita decomposition of  $D + K_V$  (see [21, p. 69]), where  $(D + K_V)^+$  is the nef part of  $D + K_V$ . By Lemma 4.1,  $\text{Supp}((D + K_V)^-) \subset \text{Supp } D$  and  $D - (D + K_V)^-$  is an effective  $\mathbb{Q}$ -Cartier divisor (for more details, see [21, Chapter 2, Section 3]). Moreover,  $D^\# = D - (D + K_V)^-$  (for the definition of  $D^\#$ , see Section 2). Since  $\bar{\kappa}(S) = 2$ , it follows from Lemma 4.1 that  $D^\# + K_V$  is a nef and big  $\mathbb{Q}$ -Cartier divisor.

**Lemma 4.2.** *For an integer  $n \geq 2$ , set  $K_n := (n - 1)K_V - \lfloor -(n - 1)D^\# \rfloor + \lfloor D^\# \rfloor$ . Then we have*

$$\bar{P}_n(S) \geq \frac{1}{2}(K_V + K_n \cdot K_n) + 1. \quad (4.1)$$

*Proof.* Since  $V$  is a rational surface and the SNC-pair  $(V, D)$  is almost minimal, the assertion follows from [30, Proposition 3.1], where we note that the  $\mathbb{Q}$ -divisor  $D_m$  in [30, Proposition 3.1] is  $D^\#$ .  $\square$

Since  $\text{Supp } D$  is connected, so is  $\text{Supp}(\lfloor D^\# \rfloor)$ . Set  $\ell := (\lfloor D^\# \rfloor \cdot D - \lfloor D^\# \rfloor)$  and let  $C_1, C_2, \dots, C_\ell$  exhaust all irreducible components of  $D - \lfloor D^\# \rfloor$  meeting  $\lfloor D^\# \rfloor$ . Since  $D$  is a tree of smooth rational curves and  $\bar{\kappa}(V - D) = 2$ , it follows from [21, Corollary 2.11.1 (p. 82)] that  $\ell$  equals the number of the maximal admissible rational twigs in  $D$ . For each  $i$ ,  $1 \leq i \leq \ell$ , the coefficient of the curve  $C_i$  in  $D^\#$  equals  $(a_i - 1)/a_i$ , where  $a_i$  is an integer  $\geq 2$ . Let  $D^{(i)}$  ( $1 \leq i \leq \ell$ ) be the maximal admissible rational twig in  $D$  with  $C_i \subset \text{Supp}(D^{(i)})$ .

Since  $D^\# + K_V$  is a nef and big  $\mathbb{Q}$ -Cartier divisor, it follows from the Kawamata–Viehweg vanishing theorem (see [11] and [31]) and the Riemann–Roch theorem that

$$h^0(V, K_V + K_n - \lfloor D^\# \rfloor) = \frac{1}{2}(K_V + K_n - \lfloor D^\# \rfloor \cdot K_n - \lfloor D^\# \rfloor) + 1 \quad (4.2)$$

for any integer  $n \geq 2$ . Since

$$n(D + K_V) \geq nK_V - \lfloor -(n - 1)D^\# \rfloor + \lfloor D^\# \rfloor = K_V + K_n,$$

we have  $\bar{P}_n(S) = h^0(V, n(D + K_V)) \geq h^0(V, K_V + K_n - \lfloor D^\# \rfloor)$  for any integer  $n \geq 2$ . By (4.1) and (4.2), we know that if  $\bar{P}_n(S) = 0$ , then

$$(\lfloor D^\# \rfloor \cdot (2n - 1)K_V - 2\lfloor -(n - 1)D^\# \rfloor + \lfloor D^\# \rfloor) \leq 0. \quad (4.3)$$

**Lemma 4.3.** *With the same notation and assumptions as above, assume further that  $\bar{P}_2(S) = 0$ . Then the following assertions hold true.*

- (1)  $\ell = 3$ .

(2)  $\lfloor D^\# \rfloor$  is irreducible.

*Proof.* If  $\ell \leq 2$ , then the dual graph of  $D$  is linear. Since  $\bar{\kappa}(S) = 2$ , it follows from [21, Corollary 2.11.1 (p. 82)] that the intersection matrix of  $D$  is negative definite, which is a contradiction. So,  $\ell \geq 3$ . Since  $\lfloor D^\# \rfloor$  is a tree of smooth rational curves, we know that  $(\lfloor D^\# \rfloor \cdot \lfloor D^\# \rfloor + K_V) = -2$ . Then, by (4.3) for  $n = 2$ , we have

$$0 \geq (\lfloor D^\# \rfloor \cdot 3K_V + 2D + \lfloor D^\# \rfloor) = 2\ell - 6.$$

Hence,  $\ell = 3$ . This proves the assertion (1). The assertion (2) easily follows from the assertion (1).  $\square$

From now on, we assume that  $\bar{P}_6(S) = 0$ . Then  $\bar{P}_2(S) = \bar{P}_3(S) = 0$ . Set  $D_0 := \lfloor D^\# \rfloor$ , which is a smooth irreducible rational curve by Lemma 4.3 (2). Then, for any integer  $n \geq 2$ , we have

$$\begin{aligned} & (D_0 \cdot (2n - 1)K_V - 2\lfloor -(n - 1)D^\# \rfloor + D_0) \\ &= 2(1 - 2n) - 2 \sum_{i=1}^3 \left[ -(n - 1) \left( 1 - \frac{1}{a_i} \right) \right]. \end{aligned} \quad (4.4)$$

**Lemma 4.4.** *With the same notation and assumptions as above, the weighted dual graph of  $D$  is given as one of (1)–(18) in Figure 3, where the weights of the vertices corresponding to  $D_0$  and  $(-2)$ -curves of  $D$  are omitted.*

*Proof.* We assume that  $a_1 \leq a_2 \leq a_3$ . By (4.3) and (4.4) for  $n = 3$ , we have

$$5 \geq - \sum_{i=1}^3 \left[ -2 \left( 1 - \frac{1}{a_i} \right) \right].$$

Since  $2 \leq a_1 \leq a_2 \leq a_3$ , it follows that  $a_1 = 2$ .

Since  $(D \cdot D^\# + K_V) = (D_0 \cdot D^\# + K_V)$  and  $D$  and  $D^\# + K_V$  are big  $\mathbb{Q}$ -Cartier divisors, we infer from the Hodge index theorem that

$$0 < (D_0 \cdot D^\# + K_V).$$

Since  $(D_0 \cdot D^\# + K_V) = (D_0 \cdot D_0 + K_V) + 3 - \sum_{i=1}^3 1/a_i$  and  $a_1 = 2$ , we have

$$\frac{1}{a_2} + \frac{1}{a_3} < \frac{1}{2}.$$

In particular, we know that  $3 \leq a_2 \leq a_3$  and that if  $a_2 = 3$  (resp.  $a_2 = 4$ ), then  $a_3 \geq 7$  (resp.  $a_3 \geq 5$ ).



By (4.3) and (4.4) for  $n = 6$ , we know that

$$11 \geq - \sum_{i=1}^3 \left\lfloor -5 \left( 1 - \frac{1}{a_i} \right) \right\rfloor.$$

Since  $a_1 = 2$  and  $3 \leq a_2 \leq a_3$ , we have

$$8 \geq - \sum_{i=2}^3 \left\lfloor -5 \left( 1 - \frac{1}{a_i} \right) \right\rfloor$$

and

$$\left\lfloor -5 \left( 1 - \frac{1}{a_i} \right) \right\rfloor \leq -4$$

for  $i = 2, 3$ . Then  $\lfloor -5(1 - (1/a_i)) \rfloor = -4$  for  $i = 1, 2$  and hence  $a_3 \leq 5$ . Therefore,  $(a_2, a_3) = (4, 5)$  or  $(5, 5)$ .

By using [21, Lemma 3.4.2 (p. 92)], we can determine the possible weighted dual graphs of  $D - D_0$ . This proves Lemma 4.4.  $\square$

Set  $D' := D - D_0$ . Then  $D'$  can be contracted to three cyclic quotient singular points.

**Lemma 4.5.** *With the same notation and assumptions as above, the pair  $(V, D')$  is an LDP1-surface.*

*Proof.* Since  $S = V - D$  is a  $\mathbb{Q}$ -homology plane, we know that  $\rho(V) = 1 + \#D'$ . If  $\bar{\kappa}(V - D') = -\infty$ , then  $(V, D')$  is an LDP1-surface by [33, Remark 1.2 (2)]. So we shall prove that  $\bar{\kappa}(V - D') = -\infty$ .

Set  $D_1 := D^{(1)} (= C_1)$ . Since  $D_1$  is a  $(-2)$ -curve and  $(D_1 \cdot D' - D_1) = 0$ , the coefficient of  $D_1$  in  $(D')^\#$  equals zero. So,  $\bar{\kappa}(V - D') = \bar{\kappa}(V - (D' - D_1))$  by Lemma 2.1. We treat three cases (4), (5) and (18) as in Figure 3.

*Case (4).* In this case,  $\rho(V) = 7$ ,  $(K_V^2) = 3$  and

$$D^\# = D_0 + \frac{1}{2}D_1 + \frac{3}{4}D_2 + \frac{4}{5}D_3 + \frac{3}{5}D_4 + \frac{2}{5}D_5 + \frac{1}{5}D_6.$$

So  $(D_0 \cdot D^\# + K_V) = 1/20$  and  $(D^\# + K_V)^2 = (K_V \cdot D_0) + 91/20$ . By the log Miyaoka–Yau inequality (see [13], [12]), we know that

$$(0 <) (D^\# + K_V)^2 = (K_V \cdot D_0) + \frac{91}{20} \leq 3e(V - D) = 3.$$

Hence,  $(D_0^2) \geq 0$ . By virtue of [21, Corollary 2.11.1 (p. 82)], we know that  $\bar{\kappa}(V - (D - D_1)) = -\infty$ . Therefore,

$$\bar{\kappa}(V - D') = \bar{\kappa}(V - (D' - D_1)) = \bar{\kappa}(V - (D - D_1)) = -\infty.$$

*Case (5).* Since  $S = V - D$  is affine, the intersection matrix of  $D$  is not negative definite. Then  $(D_0^2) \geq -1$  and so  $\bar{\kappa}(V - (D - D_1)) = -\infty$  by [21, Corollary 2.11.1 (p. 82)]. Therefore,

$$\bar{\kappa}(V - D') = \bar{\kappa}(V - (D' - D_1)) = \bar{\kappa}(V - (D - D_1)) = -\infty.$$

*Case (18).* In this case,  $(D')^\# = 0$ . So,  $\bar{\kappa}(V - D') = \kappa(V) = -\infty$  by Lemma 2.1.

The other cases can be treated similarly.  $\square$

Now, we shall consider the cases (1)–(18) as in Figure 3 separately. We shall use the same notation as in Figure 3.

*Cases (1) and (9).* We consider Case (1); Case (9) can be treated similarly. In this case,  $\rho(V) = 4$  and  $(D')^\# = (1/2)D_2 + (3/5)D_3$ . Let  $C$  be an irreducible curve in  $MV(V, D')$  (for the definition, see Section 2). By virtue of Lemmas 2.13 and 2.14, we know that  $C$  is a  $(-1)$ -curve and  $|C + D' + K_V| = \emptyset$ . So,  $(C \cdot D_i) = 0$  or  $1$  for  $i = 1, 2, 3$ . Since  $-(C \cdot (D')^\# + K_V) = 1 - (1/2)(D_2 \cdot C) - (3/5)(C \cdot D_3) > 0$ , it follows that  $(C \cdot D_2) = 0$  or  $(C \cdot D_3) = 0$ . Then the intersection matrix of  $C + D'$  is negative definite, which contradicts Lemma 2.12. Therefore, Case (1) does not take place.

*Cases (2), (3), (10) and (11).* We consider Case (2); Cases (3), (10) and (11) can be treated similarly. In this case,  $\rho(V) = 5$  and  $(D')^\# = (1/2)D_2 + (2/5)D_3 + (1/5)D_4$ . Let  $C$  be an irreducible curve in  $MV(V, D')$ . By virtue of Lemmas 2.13 and 2.14, we know that  $C$  is a  $(-1)$ -curve and  $|C + D' + K_V| = \emptyset$ . So,  $(C \cdot D_i) = 0$  or  $1$  for  $i = 1, 2, 3, 4$  and  $(C \cdot D_3 + D_4) = 0$  or  $1$ . If  $(C \cdot D_1) = 0$ , then we know that the intersection matrix of  $C + D'$  is negative definite, which contradicts Lemma 2.12. So,  $(C \cdot D_1) = 1$ .

Suppose that  $(C \cdot D_4) = 0$ . Since the intersection matrix of  $C + D'$  is not negative definite, we know that  $(C \cdot D_2) = (C \cdot D_3) = 1$ . Let  $\mu: V \rightarrow V'$  be the contraction of  $C, D_1, D_3, D_4$ . Then  $V' \cong \mathbb{P}^2$  and  $(\mu_*(D_2)^2) = -4 + 2 + 4 + 4 = 6$ . This is a contradiction. Hence,  $(C \cdot D_4) = 1$  and so  $(C \cdot D_3) = 0$ . Then a divisor  $F := 2C + D_1 + D_4$  defines a  $\mathbb{P}^1$ -fibration  $\Phi := \Phi|_F: V \rightarrow \mathbb{P}^1$  and  $D_3$  is a section of  $\Phi$ . By virtue of Lemma 2.15, we know that  $(C \cdot D_2) = 1$  (i.e.,  $D_2$  is not a fiber component of  $\Phi$ ). In particular,  $D_2$  is a 2-section of  $\Phi$ . Since  $\rho(V) = 5$ ,  $\Phi$  has a singular fiber  $G$  other than  $F$ . Lemma 2.11 (2) implies that  $G = E_1 + E_2$ , where  $E_1$  and  $E_2$  are  $(-1)$ -curves and  $(E_1 \cdot E_2) = 1$ . Then we can easily see that either  $E_1 + D'$  or  $E_2 + D'$  has negative definite intersection matrix. This contradicts Lemma 2.12. Therefore, Case (2) does not take place.

*Cases (4), (5) and (12).* We consider Case (4); Cases (5) and (12) can be treated similarly. In this case,  $\rho(V) = 7$  and  $(D')^\# = (1/2)D_2$ . Let  $C$  be an irreducible curve in  $MV(V, D')$ .



Assume that  $|C + D' + K_V| \neq \emptyset$ . Then Lemma 2.13 implies that  $(C \cdot D_2) = 2$ ,  $C + D_2 + K_V \sim 0$  and  $C$  is a smooth rational curve. Then

$$(0 <) - (C \cdot (D')^\# + K_V) = -(C \cdot K_V) - 1$$

and so  $(C^2) \geq 0$  and  $-(C \cdot (D')^\# + K_V) \geq 1$ . Since  $V$  contains a  $(-1)$ -curve  $E$  and  $-(E \cdot (D')^\# + K_V) \leq 1$ , we have  $(C^2) = 0$ . So,  $C$  defines a  $\mathbb{P}^1$ -fibration  $\Phi_{|C|}: V \rightarrow \mathbb{P}^1$ . Let  $G_1$  and  $G_2$  be the fibers of  $\Phi_{|C|}$  containing  $D_1$  and  $D_3 + D_4 + D_5 + D_6$ , respectively. Then  $G_1 \neq G_2$ ,  $\#G_1 \geq 3$  and  $\#G_2 \geq 6$ . Then we have

$$7 = \rho(V) \geq 2 + (\#G_1 - 1) + (\#G_2 - 1) \geq 9,$$

which is a contradiction. Therefore,  $|C + D' + K_V| = \emptyset$ . In particular,  $(C \cdot D_i) = 0$  or 1 for  $i = 1, 2, \dots, 6$ . By Lemma 2.14,  $C$  is a  $(-1)$ -curve.

If  $(C \cdot D_2) = 0$ , then there exists an effective divisor  $\Delta$  with  $\text{Supp } \Delta \subset \text{Supp } (D_1 + D_3 + D_4 + D_5 + D_6)$  such that  $2C + \Delta$  defines a  $\mathbb{P}^1$ -fibration  $\Phi_{|2C+\Delta|}: V \rightarrow \mathbb{P}^1$  and  $D_2$  is a fiber component of  $\Phi_{|2C+\Delta|}$ . This contradicts Lemma 2.15. So,  $(C \cdot D_2) = 1$ .

Suppose that  $(C \cdot D_1) = 1$ . Since the intersection matrix of  $C + D'$  is not negative definite, we know that  $(C \cdot D_3 + D_4 + D_5 + D_6) = (C \cdot D_j) = 1$  for some  $j$ ,  $3 \leq j \leq 6$ . Then  $F := 2C + D_1 + D_j$  defines a  $\mathbb{P}^1$ -fibration  $\Phi := \Phi_{|F|}: V \rightarrow \mathbb{P}^1$  and  $D_2$  is a 2-section of  $\Phi$ . We may assume that  $j = 3$  or 4. Consider the case  $j = 3$ . Then  $D_4$  is a section of  $\Phi$ . Let  $F'$  be the fiber of  $\Phi$  containing  $D_5$  and  $D_6$ . By considering Lemma 2.11 (2), we know that  $F' = E + D_5 + D_6 + E'$ , where  $E$  and  $E'$  are  $(-1)$ -curves and  $(E \cdot D_5) = (E' \cdot D_6) = 1$ . Since  $-(E \cdot (D')^\# + K_V)$ ,  $-(E' \cdot (D')^\# + K_V) > 0$  and  $(D')^\# = (1/2)D_2$ , we have  $(E \cdot D_2) = (E' \cdot D_2) = 1$ . Then  $G := 4E' + 3D_6 + 2D_5 + D_2 + D_4$  defines a  $\mathbb{P}^1$ -fibration  $\Phi' := \Phi_{|G|}: V \rightarrow \mathbb{P}^1$  and  $D_3$  is a section of  $\Phi'$ . Let  $G'$  be the fiber of  $\Phi'$  containing  $D_1$ . Then  $\#G' \geq 3$ . However, this contradicts  $\rho(V) = 7$  because  $\rho(V) \geq \#G + \#G' \geq 8$ . Consider the case  $j = 4$ . Then  $D_3$  and  $D_5$  are sections of  $\Phi$ . Let  $F'$  be a fiber of  $\Phi$  containing  $D_6$ . By considering Lemma 2.11 (2), we know that  $F' = E + D_6 + E'$ , where  $E$  and  $E'$  are  $(-1)$ -curves. By using the same argument as in the case  $j = 3$ , we can derive a contradiction. Thus, we see that the case  $(C \cdot D_1) = 1$  does not take place.

Suppose that  $(C \cdot D_1) = 0$ . Then,  $(C \cdot D_3 + D_4 + D_5 + D_6) = (C \cdot D_j) = 1$  for some  $j$ ,  $3 \leq j \leq 6$ . If  $j = 3$  or 6, then we can derive a contradiction by using the same argument as in the previous paragraph. So, we may assume that  $j = 4$ . Then  $F := 2(C + D_4) + D_3 + D_5$  defines a  $\mathbb{P}^1$ -fibration  $\Phi := \Phi_{|F|}: V \rightarrow \mathbb{P}^1$ ,  $D_6$  is a section of  $\Phi$  and  $D_2$  is a 2-section of  $\Phi$ . Let  $F'$  be the fiber of  $\Phi$  containing  $D_1$ . By considering Lemma 2.11 (2), we know that  $F' = E_2 + D_1 + E'_2$ , where  $E_2$  and  $E'_2$  are  $(-1)$ -curves. Since  $D_2$  is a 2-section of  $\Phi$  and  $-(E_2 \cdot (D')^\# + K_V)$ ,  $-(E'_2 \cdot (D')^\# + K_V) > 0$ , we know that  $(E_2 \cdot D_2) = (E'_2 \cdot D_2) = 1$ . We may assume further that  $(E'_2 \cdot D_6) = 1$  since  $D_6$  is a section of  $\Phi$ . Then  $(E_2 \cdot D') = (E_2 \cdot D_1 + D_2) = 2$  and so  $E_2 + D'$  has negative definite intersection matrix. This contradicts Lemma 2.12.

Therefore, Case (4) does not take place.

*Cases (6), (7), (15) and (17).* Let  $\pi: V \rightarrow \bar{V}$  be the contraction of  $D'$ . Then  $\bar{V}$  has a unique rational triple point and two rational double points. Namely,  $\bar{V}$  is a dP3-surface in the sense of [34]. However, by [34, Main Theorem], these cases do not take place.

*Cases (8) and (18).* Let  $\pi: V \rightarrow \bar{V}$  be the contraction of  $D'$ . The  $\bar{V}$  is a Gorenstein log del Pezzo surface. However, by [25, Lemma 3], these cases do not take place.

*Cases (13), (14) and (16).* We treat Case (13); Cases (14) and (16) can be treated similarly. In this case,  $\rho(V) = 6$  and

$$(D')^\# = \frac{2}{5}(D_2 + D_4) + \frac{1}{5}(D_3 + D_5).$$

Let  $C$  be an irreducible curve in  $MV(V, D')$ . By virtue of Lemmas 2.13 and 2.14, we know that  $C$  is a  $(-1)$ -curve and  $|C + D' + K_V| = \emptyset$ . So,  $(C \cdot D_i) = 0$  or  $1$  ( $i = 1, \dots, 5$ ),  $(C \cdot D_2 + D_3) = 0$  or  $1$  and  $(C \cdot D_4 + D_5) = 0$  or  $1$ . Since the intersection matrix of  $C + D'$  is not negative definite, we may assume that one of the following six cases (i)–(vi) takes place.

- (i)  $(C \cdot D_1) = (C \cdot D_3) = 1$  and  $(C \cdot D_j) = 0$  if  $j \neq 1, 3$ .
- (ii)  $(C \cdot D_1) = (C \cdot D_3) = (C \cdot D_4) = 1$  and  $(C \cdot D_2) = (C \cdot D_5) = 0$ .
- (iii)  $(C \cdot D_1) = (C \cdot D_3) = (C \cdot D_5) = 1$  and  $(C \cdot D_2) = (C \cdot D_4) = 0$ .
- (iv)  $(C \cdot D_1) = (C \cdot D_2) = (C \cdot D_4) = 1$ .
- (v)  $(C \cdot D_3) = (C \cdot D_4) = 1$  and  $(C \cdot D_1) = 0$ .
- (vi)  $(C \cdot D_3) = (C \cdot D_5) = 1$  and  $(C \cdot D_1) = 0$ .

*Case (i).* Then  $F := 2C + D_1 + D_3$  defines a  $\mathbb{P}^1$ -fibration  $\Phi := \Phi|_F: V \rightarrow \mathbb{P}^1$  and  $D_4 + D_5$  is contained in a fiber of  $\Phi$ . This contradicts Lemma 2.15.

*Case (ii).* Then  $F := 2C + D_1 + D_3$  defines a  $\mathbb{P}^1$ -fibration  $\Phi := \Phi|_F: V \rightarrow \mathbb{P}^1$  and  $D_5$  is contained in a fiber  $G$  of  $\Phi$ . By Lemma 2.11 (2), we know that  $G = E + D_5 + E'$ , where  $E$  and  $E'$  are  $(-1)$ -curves with  $(E \cdot D_5) = (E' \cdot D_5) = 1$ . Since  $D_4$  is a 2-section of  $\Phi$ , we may assume that  $(E \cdot D_4) = 1$ . Then  $-(C \cdot (D')^\# + K_V) = -(E \cdot (D')^\# + K_V)$  and so  $E \in MV(V, D')$ . On the other hand, since  $(E \cdot D_4) = (E \cdot D_5) = 1$ , we have  $|E + D' + K_V| \neq \emptyset$ . This is a contradiction.

*Case (iii).* By using the same argument as in Case (i), we know that this case does not take place.

*Case (iv).* Let  $\mu: V \rightarrow W$  be the contraction of  $C, D_1, D_4$  and  $D_5$ . Then  $W = \mathbb{F}_2$ , a Hirzebruch surface of degree two, and  $M_2 := \mu_*(D_3)$  is the minimal section of  $\mathbb{F}_2$ . Moreover,  $(\mu_*(D_2)^2) = 7$ . On the other hand, since  $(\mu_*(D_2) \cdot M_2) = 1$ , we have  $\mu_*(D_2) \sim \alpha M_2 + (2\alpha + 1)\ell$ , where  $\ell$  is a fiber of the ruling on  $\mathbb{F}_2$ . Hence,  $(\mu_*(D_2)^2) = 2\alpha^2 + 2\alpha$  is even. This is a contradiction.

*Case (v).* Then  $F := 5C + 3D_3 + 2D_4 + D_2 + D_5$  defines a  $\mathbb{P}^1$ -fibration  $\Phi := \Phi|_F: V \rightarrow \mathbb{P}^1$  and  $D_1$  is contained in a fiber  $G$  of  $\Phi$ . Since  $\#G \geq 3$ , we have

$$\rho(V) = 6 \geq 2 + (\#F - 1) + (\#G - 1) \geq 8.$$

This is a contradiction.

*Case (vi).* Then  $F := 2C + D_3 + D_5$  defines a  $\mathbb{P}^1$ -fibration  $\Phi := \Phi|_F: V \rightarrow \mathbb{P}^1$ ,  $D_2$  and  $D_4$  are sections of  $\Phi$  and  $D_1$  is contained in a fiber  $G$  of  $\Phi$ . By Lemma 2.11 (2), we know that  $G = E + D_1 + E'$ , where  $E$  and  $E'$  are  $(-1)$ -curves and  $(E \cdot D_1) = (E' \cdot D_1) = 1$ . Since  $D_2$  and  $D_4$  are sections of  $\Phi$ , we can easily see that one of  $E + D'$  and  $E' + D'$  has negative definite intersection matrix. This contradicts Lemma 2.12.

Therefore, Case (13) does not take place.

Thus, we know that Cases (1)–(18) do not take place. This proves the assertion (2) of Theorem 1.1.

The proof of Theorem 1.1 is thus completed.

## 5. Proofs of Theorems 1.2 and 1.3

In this section, we shall prove Theorems 1.2 and 1.3 by using results in previous sections.

*Proof of Theorem 1.2.* Let  $X$  be a smooth affine surface with finite Picard group and  $C$  a non-empty reduced algebraic curve on  $X$ . Set  $S := X - C$ . Then  $\text{Pic}(S)$  is finite. It suffices to show that  $\bar{\kappa}(S) = -\infty$  provided  $\bar{P}_2(S) = 0$ . Let  $C = \bigcup_{i=1}^r C_i$  be the decomposition of  $C$  into irreducible components.

Suppose to the contrary that  $\bar{\kappa}(S) \geq 0$ . Lemma 2.9 then implies that  $S$  is a rational surface. Let  $(V, D)$  be an SNC-pair with  $V - D \cong S$  and  $D = \sum_{j=1}^s D_j$  the decomposition of  $D$  into irreducible components. Since  $V$  is a rational surface and  $\bar{p}_g(V - D) = \bar{p}_g(S) = 0$ ,  $D$  is a tree of smooth rational curves by [20, Lemma I.2.1.3]. Hence, we know that  $C$  is a disjoint union of topologically contractible curves. In particular,  $e(S) = e(X) - r$ . By virtue of Lemma 2.8, we know that  $e(S) = 0$  or 1 and  $e(S) = 1$  if  $\bar{\kappa}(S) = 2$ .

Now, let  $(V', D')$  be an SNC-pair with  $V' - D' \cong X$ . Since  $\bar{p}_g(X) = \bar{p}_g(S) = 0$ ,  $D'$  is a tree of smooth rational curves. Let  $D' = \sum_{t=1}^k D'_t$  be the decomposition of  $D'$  into irreducible components. Then

$$e(X) = e(V') - e(D') = \rho(V') - k + 1.$$

Since  $\text{Pic}(X)$  is finite, we have  $k \geq \rho(V')$ . So,

$$e(X) = \rho(V') - k + 1 \leq 1.$$

Hence,  $e(S) = e(X) - e(C) \leq 1 - r \leq 0$  because  $r \geq 1$ . Lemma 2.8 (1) implies that  $\bar{\kappa}(S) = 0$  or 1. By Theorem 1.1 (1),  $\bar{\kappa}(S) \neq 1$ . Since every smooth affine surface with  $\bar{\kappa} = 0$  and  $\bar{P}_2 = 0$  has positive topological Euler characteristic by [14, Theorem 0.1] (see also [2, Section 8]), we know that  $\bar{\kappa}(S) \neq 0$ . The proof of Theorem 1.2 is thus completed.  $\square$

*Proof of Theorem 1.3.* The assertion (2) is a consequence of the assertion (1) and the algebraic characterization of  $\mathbb{C}^2$  due to Fujita, Miyanishi and Sugie (see [20], [21]). We shall prove the assertion (1). Let  $S = \text{Spec } A$  be a smooth affine surface with  $\text{Pic}(S) = (0)$ . It suffices to show that  $\bar{P}_2(S) > 0$  provided  $\bar{\kappa}(S) \geq 0$ .

If  $\bar{\kappa}(S) = 0$ , then, by virtue of [2, Theorem (8.70) 3)], we know that  $\bar{p}_g(S) > 0$ . Hence,  $\bar{P}_2(S) > 0$ . If  $\bar{\kappa}(S) = 1$ , then  $\bar{P}_2(S) > 0$  by Theorem 1.1 (1). Suppose that  $\bar{\kappa}(S) = 2$  and  $\bar{P}_2(S) = 0$ . By virtue of Lemmas 2.8 and 2.9, we know that  $S$  is a  $\mathbb{Q}$ -homology plane. In particular,  $S$  is a homology plane because  $\text{Pic}(S) = (0)$ . However, by Theorem B (cf. [18, Theorem 1.3]), we know that  $\bar{P}_2(S) > 0$ . This is a contradiction. The proof of Theorem 1.3 is thus completed.  $\square$

## References

- [1] L. Bădescu, *Algebraic surfaces*. Universitext, Springer-Verlag, New York 2001. [Zbl 0965.14001](#) [MR 1805816](#)
- [2] T. Fujita, On the topology of non-complete algebraic surfaces. *J. Fac. Sci. Univ. Tokyo* **29** (1982), 503–566. [Zbl 0513.14018](#) [MR 0687591](#)
- [3] R. V. Gurjar, A new proof of Suzuki’s formula. *Proc. Indian Acad. Sci.* **107** (1997), 237–242. [Zbl 0920.14007](#) [MR 1467428](#)
- [4] R. V. Gurjar and M. Miyanishi, Affine surfaces with  $\bar{\kappa} \leq 1$ . In *Algebraic Geometry and Commutative Algebra in honor of M. Nagata*, Vol. I, Kinokuniya, Tokyo 1987, 99–124. [Zbl 0707.14015](#) [MR 0977756](#)
- [5] R. V. Gurjar and A. J. Parameswaran, Open surfaces with non-positive Euler characteristic. *Compositio Math.* **99** (1995), 213–229. [Zbl 0849.14013](#) [MR 1361739](#)
- [6] R. V. Gurjar and C. R. Pradeep,  $\mathbb{Q}$ -homology planes are rational, III. *Osaka J. Math.* **36** (1999), 259–335. [Zbl 0954.14013](#) [MR 1736480](#)
- [7] R. V. Gurjar, C. R. Pradeep and A. R. Shastri, On rationality of logarithmic  $\mathbb{Q}$ -homology planes II. *Osaka J. Math.* **34** (1997), 725–743. [Zbl 0890.14022](#) [MR 1613057](#)
- [8] R. V. Gurjar and D.-Q. Zhang,  $\pi_1$  of smooth points of a log del Pezzo surface is finite: I. *J. Math. Sci. Tokyo* **1** (1994), 137–180. [Zbl 0841.14017](#) [MR 1298542](#)
- [9] S. Iitaka, *Algebraic Geometry*. Grad. Texts in Math. 76, Springer-Verlag, Berlin, New York 1981. [Zbl 0491.14006](#) [MR 0637060](#)
- [10] Y. Kawamata, On the classification of non-complete algebraic surfaces. In *Algebraic geometry* (Proceedings Summer Meeting, Copenhagen 1978), Lecture Notes in Math. 732, Springer-Verlag, Berlin 1978, 215–232. [Zbl 0407.14012](#) [MR 055700](#)



- [11] Y. Kawamata, A generalization of Kodaira–Ramanujam’s vanishing theorem. *Math. Ann.* **261** (1982), 43–46. [Zbl 0476.14007](#) [MR 0675204](#)
- [12] R. Kobayashi, Uniformization of complex surfaces. *Adv. Stud. Pure Math.* **18** (1990), 313–394. [Zbl 0755.32024](#) [MR 1145252](#)
- [13] R. Kobayashi, S. Nakamura and F. Sakai, A numerical characterization of ball quotients for normal surfaces with branch loci. *Proc. Japan Acad. Ser. A Math. Sci.* **65** (1989), 238–241. [Zbl 0704.53053](#) [MR 1030189](#)
- [14] H. Kojima, Open rational surfaces with logarithmic Kodaira dimension zero. *Internat. J. Math.* **10** (1999), 619–642. [Zbl 1066.14506](#) [MR 1708080](#)
- [15] H. Kojima, Open surfaces of logarithmic Kodaira dimension zero in arbitrary characteristic. *J. Math. Soc. Japan* **53** (2001), 933–955. [Zbl 1074.14534](#) [MR 1852890](#)
- [16] H. Kojima, Rank one log del Pezzo surfaces of index two. *J. Math. Kyoto Univ.* **43** (2003), 100–122. [Zbl 1066.14042](#) [MR 2028702](#)
- [17] H. Kojima, On the logarithmic plurigenera of complements of plane curves. *Math. Ann.* **332** (2005), 1–15. [Zbl 1098.14046](#) [MR 2139247](#)
- [18] H. Kojima, On the logarithmic bigenera of some affine surfaces. *Affine Algebraic Geometry* (Takayuki Hibi, ed.), Osaka University Press, Osaka 2007, 257–273. [Zbl 1132.14049](#) [MR 2327242](#)
- [19] Y. Kuramoto, On the logarithmic plurigenera of algebraic surfaces. *Compositio Math.* **43** (1981), 343–364. [Zbl 0464.14008](#) [MR 0632434](#)
- [20] M. Miyanishi, *Non-complete algebraic surfaces*. Lecture Notes in Math. 857, Springer-Verlag, Berlin 1981. [Zbl 0456.14018](#) [MR 0635930](#)
- [21] M. Miyanishi, *Open algebraic surfaces*. CRM Monogr. Ser. 12, Amer. Math. Soc., Providence, RI, 2000. [Zbl 0964.14030](#) [MR 1800276](#)
- [22] M. Miyanishi and T. Sugie,  $\mathbb{Q}$ -homology planes with  $\mathbb{C}^{**}$ -fibrations. *Osaka J. Math.* **28** (1991), 1–26. [Zbl 0749.14029](#) [MR 1096924](#)
- [23] M. Miyanishi and T. Sugie, Homology planes with quotient singularities. *J. Math. Kyoto Univ.* **31** (1991), 755–788. [Zbl 0790.14034](#) [MR 1127098](#)
- [24] M. Miyanishi and S. Tsunoda, Absence of affine lines on the homology planes of general type. *J. Math. Kyoto Univ.* **32** (1992), 443–450. [Zbl 0794.14017](#) [MR 1183360](#)
- [25] M. Miyanishi and D.-Q. Zhang, Gorenstein log del Pezzo surfaces of rank one. *J. Algebra* **118** (1988), 63–84. [Zbl 0664.14019](#) [MR 0961326](#)
- [26] S. Yu Orevkov, On rational cuspidal curves I. Sharp estimate for degree via multiplicities. *Math. Ann.* **324** (2002), 657–673. [Zbl 1014.14010](#) [MR 1942244](#)
- [27] C. R. Pradeep and A. R. Shastri, On rationality of logarithmic  $\mathbb{Q}$ -homology planes I. *Osaka J. Math.* **34** (1997), 429–456. [Zbl 0890.14021](#) [MR 1483859](#)
- [28] F. Sakai, Semi-stable curves on algebraic surfaces and logarithmic pluricanonical maps, *Math. Ann.* **254** (1980), 89–102. [Zbl 0431.14011](#) [MR 0597076](#)
- [29] K. Tono, On Orevkov’s rational cuspidal plane curves. Preprint.
- [30] S. Tsunoda, Structure of open algebraic surfaces, I. *J. Math. Kyoto Univ.* **23** (1983), 95–125. [Zbl 0605.14034](#) [MR 0692732](#)

- [31] E. Viehweg, Vanishing theorems. *J. Reine. Angew. Math.* **335** (1982), 1–8. [Zbl 0485.32019](#) [MR 0667459](#)
- [32] M. Zaidenberg, Isotrivial families of curves on affine surfaces and the characterization of the affine plane. *Math. USSR-Izv.* **30** (1988), 503–532. [Zbl 0666.14018](#) [MR 0903623](#)
- [33] D.-Q. Zhang, Logarithmic del Pezzo surfaces of rank one with contractible boundaries. *Osaka J. Math.* **25** (1988), 461–497. [Zbl 0736.14016](#) [MR 0957874](#)
- [34] D.-Q. Zhang, Logarithmic del Pezzo surfaces with rational double and triple singular points. *Tohoku Math. J.* **41** (1989), 399–452. [Zbl 0666.14017](#) [MR 1007096](#)

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