Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 83 (2008)

Artikel: Jenkins-Strebel differentials with poles
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DOI: https://doi.org/10.5169/seals-99028

Comment. Math. Helv. 83 (2008), 211–240 **Commentarii Mathematici Helvetici**

Jenkins–Strebel differentials with poles

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Abstract. Given any compact Riemann surface with finitely many punctures, we show that there exists a unique *Jenkins–Strebel differential* on the Riemann surface with prescribed heights. In addition, the differential has second order poles at the distinguished punctures with prescribed leading coefficients. As a corollary, we obtain the solution of the moduli problem.

Mathematics Subject Classification (2000). 30F30, 30F60, 32G15.

Keywords. Jenkins–Strebel differentials, moduli, heights.

Introduction

The theory of quadratic differentials has long played a central role in the study of Teichmüller spaces. One aspect of quadratic differentials is their geometric properties.

At present, Jenkins–Strebel differentials (quadratic differentials with closed trajectories) turn out to be useful. For example, the solutions of a large variety of function theoretic extremal problems on Riemann surfaces can be described by these differentials. See e.g. [12], [30], [31].

In particular, Jenkins–Strebel differentials with second order poles are also of interest. They show up in the Penner–Strebel triangulation of the moduli space, which is important in computing its homology. String theorists care about these cases too.

With respect to Jenkins–Strebel differentials with second order poles, characteristic punctured disks take the place of annuli around the distinguished punctures. There are several types of existence theorems for these differentials. For example, one can prescribe the lengths of the circumferences of the annuli, their heights or the moduli of the annuli. For punctured disks, one can prescribe reduced moduli or circumferences.

K. Strebel [28], [31] obtained the existence and uniqueness theorems for Jenkins–Strebel differentials with characteristic punctured disks. Later B. Zwiebach [35]

*Partially supported by the Post-doctor fellowship of China.

extended the existence and uniqueness theorem to Jenkins–Strebel differentials with punctured disks and annuli.

In [17] we discussed Jenkins–Strebel differentials on compact Riemann surfaces by geometrical methods. By extending Strebel's methods, and using the geometrical methods, in this paper we will describe the most general situation for closed Riemann surfaces with possible punctures. It shows that the geometrical methods is also useful when constructing Jenkins–Strebel differentials with second order poles.

Suppose S is a compact Riemann surface of genus g with n punctures. Also we suppose that S is hyperbolic, that is, $3g + n - 3 > 0$. Denote by $\{Q_1, Q_2, \dots, Q_q\}$ $(q \le n)$ the distinguished punctures on the Riemann surface S.

A system of simple closed curves $\{\gamma_k\}_{1\le k\le p}$ on S is called admissible if none of the γ_k is homotopic to zero, and if any two distinct curves neither intersect nor are freely homotopic. For the definitions, please see [17], [31].

The following theorem claims that the height problem on S is always solvable if one prescribes the heights of annuli and the negative leading coefficients.

Theorem 3.1. *For arbitrary $h_k > 0$, $1 \le k \le p$, and $a_j > 0$, $1 \le j \le q$, there is a Jenkins–Strebel differential φ on S with the following properties:*

(i) *The differential φ has p characteristic annuli $\{R_k\}$ with type $\{\gamma_k\}$. In the φ-metric these annuli have heights $\{h_k\}$.*

(ii) *φ has q punctured disks $\{D_j\}$ which are swept out by closed trajectories around the marked punctures $\{Q_j\}$. The closed horizontal trajectories in D_j have the same φ-length a_j. Equivalently, φ has a second order pole at Q_j with leading coefficient $-\left(\frac{a_j}{2\pi}\right)^2$, $j = 1, 2, \dots, q$.*

Moreover the quadratic differential φ is uniquely determined.

As a special case, Theorem 3.1 implies the following result due to Strebel [31].

Theorem 3.4. *There is a unique Jenkins–Strebel differential φ on S whose characteristic domains are q punctured disks with specified circumferences a_j, $1 \le j \le q$.*

Theorem 3.1 can be applied to prove the following result, which claims the moduli problem is solvable if and only if the given array of moduli is admissible. For the definitions, please see Section 4.

Theorem 4.6. *If $M = (m_1, m_2, \dots, m_p)$ is admissible on S, then for any given $A = (a_1, a_2, \dots, a_q) \in \mathbb{R}_+^q$ there is a Jenkins–Strebel differential φ which has p characteristic annuli with homotopic type $\{\gamma_k\}$ and with conformal moduli $\{m_k\}$. At the puncture Q_j the differential φ has a second order pole with leading coefficient $-\left(\frac{a_j}{2\pi}\right)^2$, $j = 1, 2, \dots, q$.*

Moreover the differential φ is uniquely determined.

Furthermore, Strebel [31] solved an extremal problem associated with punctured disks on Riemann surfaces. He dealt with the case when there are finitely many punctured disks (no additional annuli). Also he showed that the solutions can be described by the Jenkins–Strebel differentials with second order poles. We will deal with the general cases. That is, there are not only punctured disks but also annuli.

Let ζ_j be a fixed local parameter on S near the distinguished puncture Q_j, $1 \leq j \leq q$. Then we have

Theorem 4.7. *Suppose that $M = (m_1, m_2, \dots, m_p)$ is admissible. Then, for any real number $\tilde{m}_j$, $1 \leq j \leq q$, there is a Jenkins–Strebel differential φ which has characteristic annuli $\{R_k\}$ with type $\{\gamma_k\}$. The characteristic annulus R_k has modulus m_k, $1 \leq k \leq p$.*

Around the puncture Q_j, the differential φ has a characteristic punctured disk D_j with reduced modulus (with respect to the given local parameter ζ_j)

$$M_j = \tilde{m}_j + c, \quad 1 \leq j \leq q,$$

for some constant c independent of j.

In addition, φ is uniquely determined up to a positive constant factor. In particular, the annuli $\{R_k\}$ and the punctured disks $\{D_j\}$ are uniquely determined.

The paper is organized as follows.

In Section 1 we introduce some terminologies and develop various background necessary for our proofs. In Section 2 we describe Jenkins–Strebel differentials with poles on Riemann surfaces. The object of Section 3 is to give the proof of the existence and uniqueness of the height theorem. The solution of moduli problems is left to Section 4. In the last section we give the proofs of some basic results.

Notational conventions. Throughout the paper we follow the conventions in [31]. That is, annuli are denoted by the letter k and punctured disks by the letter j.

Denote by Q_j the puncture on Riemann surfaces and by a_j the circumference of a punctured disk. Moreover, θ_k denotes the twisting angle and l_k denotes the circumference of a characteristic annuli.

Denote by Δ the unit disk $\{|z| < 1\}$, and denote the unit punctured disk by

$$\Delta^* \equiv \{0 < |z| < 1\}.$$

For any annulus R, we denote by $M(R)$ its conformal modulus.

If $f : U \to V$ is a quasiconformal homeomorphism, then we let $K[f]$ be the maximal dilatation of f.

Acknowledgements. I would like to express my indebtedness to Professor Zhong Li for his encouragement and help during this work. I also thank Professor Guizhen Cui, Yunping Jiang, Yuefei Wang and Shengjian Wu for their help during the preparation of this paper.

We thank the referees of this paper for their many thoughtful suggestions and valuable comments.

1. Preliminaries

A punctured disk on the complex plane $\mathbb{C}$ always has infinite conformal modulus. It is, however, possible to assign it a finite number which is called the *reduced modulus*, introduced by Teichmüller.

Suppose that D is a punctured disk in the z-plane with puncture z_0. For any sufficiently small $r > 0$, we denote by $D(r)$ the 2-connected region

$$D \backslash \{z : |z - z_0| \leq r\}.$$

Let $m(r)$ denote the conformal modulus of $D(r)$. Then for any $0 < r' < r$ we have

$$m(r) + \frac{1}{2\pi} \log \frac{r}{r'} \leq m(r'),$$

or equivalently

$$m(r) + \frac{1}{2\pi} \log r \leq m(r') + \frac{1}{2\pi} \log r'.$$

Hence the function $m(r) + \frac{1}{2\pi} \log r$ is increasing as $r \to 0^+$ and thus the limit

$$\lim_{r \to 0^+} \left(m(r) + \frac{1}{2\pi} \log r \right)$$

exists. This limit is called the reduced modulus of the punctured disk $D \subset \mathbb{C}$ with respect to the local parameter z near z_0, see e.g. [13], [31], [33].

The definition of reduced modulus can be extended to general Riemann surfaces. Suppose that Q is a puncture of a Riemann surface Ω and suppose

$$z : U \to \mathbb{C}, \quad z(Q) = 0,$$

is a local patch near Q. For any punctured disk $D \subset S$ around Q, as above, for any sufficiently small $r > 0$ we denote

$$D(r) \equiv D \backslash \{|z| \leq r\}.$$

Denote by $m(r) \equiv M(D(r))$ the modulus of the domain $D(r)$.

Definition 1.1. The limit

$$\lim_{r\to 0^+}\left(m(r)+\frac{1}{2\pi}\log r\right)$$

is called the reduced modulus of the punctured disk D (with respect to the local uniformizer z).

Suppose that h is a holomorphic analytic homeomorphism between the punctured disk D and the punctured disk $\{0<|z|<\rho\}$ with $h(Q)=0$ and $\frac{dh}{dz}(0)=1$. Then the number ρ is called the *mapping radius* of the punctured disk D with respect to the local parameter z.

Lemma 1.2 ([31]). *The reduced modulus of a punctured disk D with respect to the local parameter z is equal to* $\frac{1}{2\pi}\log\rho$.

For our purpose, in the remainder of this paper we need the notions of *canonical modulus* and *canonical local parameter*.

Recall that Δ^* is the unit punctured disk. For any hyperbolic Riemann surface S, let

$$\pi\colon\Delta^*\to S,\quad \pi(0)=Q$$

be an annular (unbranched) covering map induced by a simple loop around the puncture Q. Then π is unique up to rotations of Δ^*. The local parameter $\eta\equiv\pi^{-1}|_D$ can be regarded as a local uniformizer at the neighborhood of Q.

Then we have the following definition.

Definition 1.3. For any punctured disk $D\subset S$ around the puncture Q, the local parameter $\eta\equiv\pi^{-1}|_D$ is called the canonical local parameter of S at the neighborhood of Q.

The canonical modulus $M_{D,S}$ is defined to be the reduced modulus of D with respect to the canonical local parameter $\eta\equiv\pi^{-1}|_D$.

Evidently we have $M_{D,S}\le 0$. In comparison with the reduced modulus, the canonical modulus of a punctured disk is independent of the choice of local parameters near the puncture. The number

$$r_{D,S}\equiv e^{2\pi M_{D,S}}$$

is called the *canonical mapping radius* of the punctured disk $D\subset S$.

Note that an orientation preserving homeomorphism $f\colon U\to V$ between two regions in $\mathbb{C}$ is a K-quasiconformal homeomorphism if and only if

$$\limsup_{r\to 0^+}\frac{\max_{|z-\zeta|=r}|f(z)-f(\zeta)|}{\min_{|z-\zeta|=r}|f(z)-f(\zeta)|}\le K,\quad \text{a.e. } \zeta\in U.$$

Since Riemann surfaces have conformal structures, it makes sense to speak of quasiconformal homeomorphisms between two Riemann surfaces. With respect to quasiconformal maps we have the following result.

Lemma 1.4. *Let $Q \subset S$ (resp. $Q' \subset S'$) be a marked puncture on S (resp. S'). Let $f: S \to S'$ be a K-quasiconformal homeomorphism with $f(Q) = Q'$.*

If $D \subset S$ is any punctured disk around Q, then $f(D) \subset S'$ is a punctured disk around the puncture $f(Q)$. In addition, the canonical moduli $M_{D,S}$ and $M_{f(D),S'}$ of the punctured disks D and $f(D)$ satisfy that

$$K \cdot M_{D,S} - \frac{2K-1}{\pi} \log 2 \leq M_{f(D),S'} \leq \frac{M_{D,S}}{K} + \frac{2}{\pi} \log 2 - \frac{\log 2}{\pi K}.$$

In particular, $M_{f(D),S'} \to -\infty$ if and only if $M_{D,S} \to -\infty$.

Proof. Recall that $\pi : \Delta^* \to S$ is the annular covering map and $\eta \equiv \pi^{-1}|_D$ is the canonical parameter of S near Q.

Similarly, there is an annular covering map $\Delta^* \to S'$ induced by a simple loop around the puncture $f(Q)$.

Denote by $\tilde{D}$ (resp. $\tilde{D}'$) the lifting image of D (resp. D') which encloses the center $0 \in \Delta$. Let

$$z: \tilde{D} \to \{z : 0 < |z| < e^{2\pi M}\}$$

be the holomorphic homeomorphism, where $\frac{dz}{d\eta}(0) = 1$ and $M \equiv M_{D,S}$ is the canonical modulus of the punctured disk D.

By applying the Koebe-$\frac{1}{4}$ Theorem, we obtain that

$$\left\{\eta : 0 < |\eta| < \frac{e^{2\pi M}}{4}\right\} \subset \tilde{D} \subset \Delta^*.$$

Lift $f: S \to S'$ to a K-quasiconformal homeomorphism $F: \Delta^* \to \Delta^*$ with $F(0) = 0$. By applying Mori's Theorem (see the Appendix), we have

$$|F(\eta)| \geq 4^{1-K} |\eta|^K, \quad \eta \in \Delta^*.$$

From the fact $F(\tilde{D}) = \tilde{D}'$, it follows that

$$\left\{\eta : 0 < |\eta| < 4^{1-K} \left(\frac{e^{2\pi M}}{4}\right)^K\right\} \subset \tilde{D}' \subset \Delta^*.$$

Hence the canonical modulus $M' \equiv M_{f(D),S'}$ satisfies

$$M' \geq \frac{\log 4^{1-K} \left(\frac{e^{2\pi M}}{4}\right)^K}{2\pi} = K \cdot M - \frac{2K-1}{\pi} \log 2.$$

Interchanging M and M', we obtain the desired result. $\square$

Lemma 1.5. *Let $\gamma_0 \subset S$ be a simple loop around the puncture Q which encloses a punctured disk D_0. Then there exists a positive constant $m \equiv m(S, \gamma_0)$ with the following property:*

Any annulus $R \subset S$ of homotopic type γ_0 with modulus $M(R) \geq m$ has at least one of its boundary components lying inside the punctured disk D_0.

Proof. For any $0 < r < 1$, we denote by Λ_r the conformal modulus of the 2-connected region $\Delta \backslash [0, r]$.

Using the annular covering $\pi : \Delta^* \to S$, we can lift γ_0 to a simple closed curve $\tilde{\gamma}_0 \subset \Delta^*$ which surrounds 0. If we set

$$c \equiv \inf\{|\zeta| : \zeta \in \tilde{\gamma}_0\},$$

then the positive constant $m \equiv \Lambda_c$ has the desired property. □

Note that a *quadratic differential* $\varphi = \varphi(z)dz^2$ on the Riemann surface S is a holomorphic section of the square $(T^*)^{\otimes 2}$ of the tangent bundle T^*.

Obviously, φ induces a 'singular' metric $ds = \sqrt{|\varphi(z)|}|dz|$ on S. For any piecewise smooth curve $\gamma \subset S$, the infimum

$$h = \inf_{\tilde{\gamma} \sim \gamma} \int_{\tilde{\gamma}} |\Im \sqrt{\varphi}|,$$

where $\tilde{\gamma}$ varies over all piecewise smooth curves in the homotopy class of γ, is called the φ-height of γ.

If a quadratic differential φ has a double pole at z_0, then φ has the form

$$\varphi \equiv \varphi(z)\, dz^2 = \left(\frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + \cdots\right) dz^2 = \frac{a_{-2}}{\xi^2}\, d\xi^2.$$

The local parameter ξ is uniquely determined up to a complex constant factor. It is called the *normalized uniformizer* near z_0. The leading coefficient a_{-2} is an invariant datum, i.e. it is independent of the choice of coordinate patches near the puncture z_0.

Figure 1 shows the local trajectory structures near a double pole. Depending on the leading coefficient $a_{-2,}$, we have three cases to distinguish.

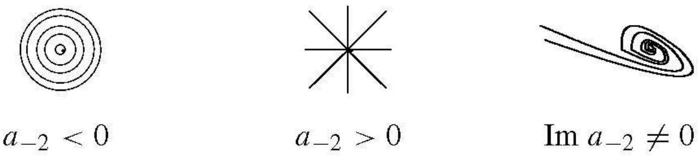

Figure 1

If z_0 is a double pole of a quadratic differential with leading coefficient $-a^2$ $(a > 0)$, then horizontal trajectories near z_0 are closed and they have the same φ-length $2\pi a$.

Note that a non-zero quadratic differential is called a Jenkins–Strebel differential if its non-closed trajectories cover a set of measure zero. Hence a Jenkins–Strebel differential decomposes the Riemann surface into characteristic regions, which are swept out by closed trajectories. These characteristic regions consist of annuli or punctured disks.

If a quadratic differential has a pole of order ≥ 3, then there is a neighborhood of the pole such that each trajectory ray entering it tends to it. Therefore a Jenkins–Strebel differential has only simple poles or double poles, and the leading coefficient of the double pole must be negative. See e.g. [31].

In [17] we considered Jenkins–Strebel differentials with only characteristic annuli. In this paper we allow that Jenkins–Strebel differentials have characteristic punctured disks near the distinguished punctures.

The next two lemmas on the inequalities of weighted sum of moduli and weighted sum of the reciprocals of moduli are well known. For their complete proofs we refer to [31].

Lemma 1.6 ([31]). *Let $\varphi \neq 0$ be a non-zero Jenkins–Strebel differential on S with type $\{\gamma_k\}_{1\leq k\leq p}$, and its characteristic annuli $\{R_k\}$ have heights $\{h_k\}$. If $\{\tilde{R}_k\}$ is a system of non-overlapping annuli on S with the homotopy type $\{\gamma_k\}$, then their conformal moduli $\tilde{M}_k \equiv M(R_k)$ satisfy*

$$\sum_k \frac{h_k^2}{\tilde{M}_k} \geq \sum_k \frac{h_k^2}{M_k},$$

with equality holds if and only if $\tilde{R}_k = R_k$ for each k.

Lemma 1.7 ([31]). *Let $\varphi \neq 0$ be a Jenkins–Strebel differential on S. Suppose that its characteristic regions consist of a finite number of annuli $\{R_k\}$ with finite conformal moduli $\{M_k\}$ and a finite number of punctured disks $\{D_j\}$ around the distinguished punctures $\{Q_j\}$. Furthermore, suppose that $\{D_j\}$ have reduced moduli $\{M_j\}$ (with respect to a fixed system of local coordinates near the punctures $\{Q_j\}$).*

Suppose that φ has finite reduced norm

$$\sum_k a_k^2 M_k + \sum_j a_j^2 M_j,$$

where a_k, $1 \leq k \leq p$, is the φ-length of closed horizontal trajectories homotopic to γ_k, and a_j, $1 \leq j \leq q$, is the φ-length of closed horizontal trajectories around Q_j.

Let $\{\tilde{R}_k\}$ and $\{\tilde{D}_j\}$ be non-overlapping regions homotopic to the annuli $\{R_k\}$ and the punctured disks $\{D_j\}$, respectively. Moreover we suppose that $\{\tilde{R}_k\}$ have conformal moduli $\{\tilde{M}_k\}$ and $\{\tilde{D}_j\}$ have reduced moduli $\{\tilde{M}_j\}$ (with respect to the *same system of coordinates near $\{Q_j\}$). Then we have*

$$\sum_k a_k^2 M_k + \sum_j a_j^2 M_j \geq \sum_k a_k^2 \tilde{M}_k + \sum_j a_j^2 \tilde{M}_j.$$

The equality holds if and only if $\tilde{R}_k = R_k$, $\tilde{D}_j = D_j$ for all k and j. In addition, as an easy consequence, we obtain that

$$\inf_{j,k}\{\tilde{M}_k - M_k,\ \tilde{M}_j - M_j\} \leq 0.$$

The equality holds if and only if $\tilde{R}_k = R_k$, $\tilde{D}_j = D_j$ for all k, j.

Remark 1.8. Lemmas 1.6 and 1.7 can be extended, in a rather straightforward manner, to the case when S is a Riemann surface with boundaries and φ is a Jenkins–Strebel differential which is real along the boundary curves.

In fact, we can construct the double surface of S, denoted by $\tilde{S}$. Since φ is real along the boundary curves, by reflection it can be continued to a Jenkins–Strebel differential on $\tilde{S}$. Therefore we have the corresponding results on the bordered Riemann surface.

2. Differentials with poles

Suppose that $\mathcal{P}$ is a pair of 'pants', namely it is a bordered surface by cutting off the interiors of 3 disjoint closed disks from the Riemann sphere. Denote by $\{\gamma_1, \gamma_2, \gamma_3\}$ its boundary components.

Then we have the following result.

Lemma 2.1. *Let $(h_1, h_2, h_3) \neq 0$ be a fixed non-zero triple of numbers, where $h_i \geq 0$. If (l_1, l_2, l_3) is a non-negative triple such that $l_i \neq 0$ if and only if $h_i \neq 0$, then there is a conformal structure P on $\mathcal{P}$ with the following properties:*

(i) *The Riemann surface P admits a Jenkins–Strebel differential φ with type $\{\gamma_k\}$ (k corresponds to those $h_k \neq 0$). The boundary components $\{\gamma_k\}$ are closed horizontal trajectories of φ. And the characteristic annuli $\{R_k\}$ have φ-circumferences $\{l_k\}$ and φ-heights $\{h_k\}$.*

(ii) *If $h_i = 0$, then the corresponding boundary component γ_i is a puncture of P and φ has at most a simple pole at this puncture.*

Proof. If each component of (h_1, h_2, h_3) is not zero, then this lemma is just Lemma 2.1 in [17].

Supposing that $l_i = 0$ for at least one i, then we will deal with two cases:

1. There is only one component of (l_1, l_2, l_3) is 0. Without loss of generality we assume that $l_3 = 0$. We can divide the case 1 into two subcases:

(i) $l_1 > l_2 > 0$ (the subcase $0 < l_1 < l_2$ can be treated by the same way).

Let $A_1A_1'A_2A_2'A_3A_3' = \bigcup_{i=1,2} \tilde{D}_i$ be a 'hexagon' in the z-plane, where $\tilde{D}_1, \tilde{D}_2$ are two rectangles and

$$A_1 = \left(\frac{l_1}{2}, \frac{h_1}{2}\right), \quad A_1' = \left(0, \frac{h_1}{2}\right), \quad A_2 = \left(0, -\frac{h_2}{2}\right),$$

$$A_2' = \left(\frac{l_2}{2}, -\frac{h_2}{2}\right), \quad A_3 = \left(\frac{l_2}{2}, 0\right), \quad A_3' = \left(\frac{l_1}{2}, 0\right).$$

See Figure 2(a). Denote by $\varphi \equiv dz^2$ the quadratic differential in the 'pentagon' $A_1A_1'A_2A_2'A_3$.

(ii) $l_1 = l_2 \neq 0$.

Let $A_1A_1'A_2A_2'A_3 = \bigcup_{i=1,2} \tilde{D}_i$ be a 'pentagon' in the z-plane, where $\tilde{D}_1, \tilde{D}_2$ are two rectangles and

$$A_1 = \left(\frac{l_1}{2}, \frac{h_1}{2}\right), \quad A_1' = \left(0, \frac{h_1}{2}\right), \quad A_2 = \left(0, -\frac{h_2}{2}\right),$$

$$A_2' = \left(\frac{l_1}{2}, -\frac{h_2}{2}\right), \quad A_3 = \left(\frac{l_1}{2}, 0\right).$$

See Figure 2 (b). Denote by $\varphi \equiv dz^2$ the quadratic differential in the 'pentagon' $A_1A_1'A_2A_2'A_3$.

2. Two components of (l_1, l_2, l_3) are zero. Without loss of generality we assume that $l_2 = l_3 = 0$ and $l_1 \neq 0$.

Let $A_1A_1'A_2A_2' = \tilde{D}_i$ be a 'rectangle' in the z-plane, where

$$A_1 = \left(\frac{l_1}{2}, \frac{h_1}{2}\right), \quad A_1' = \left(0, \frac{h_1}{2}\right), \quad A_2 = (0, 0), \quad A_2' = \left(\frac{l_1}{2}, 0\right).$$

See Figure 2 (c). Denote by $\varphi \equiv dz^2$ the quadratic differential in the 'pentagon' $A_1A_1'A_2A_2'A_3$.

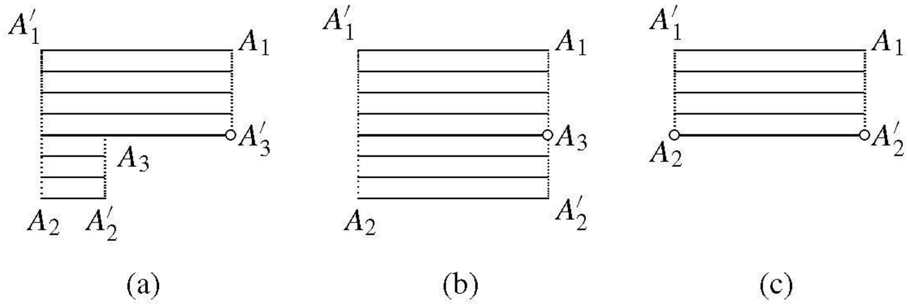

Figure 2

In the case 1 (i), the surface $\tilde{P} = A_1A_1'A_2A_2'A_3A_3'$ has a mirror image $\tilde{P}^* = A_1A_1'A_2A_2'A_3A_3'^*$. See e.g. [31]. The surfaces $\tilde{P}^*$ and $\tilde{P}$ can be glued along the boundary components

$$\{A_1'A_2,\ A_2'A_3,\ A_3A_3',\ A_3'A_1\}$$

to form a new Riemann surface, denoted by P.

Similarly, in the case 1 (ii) the surface $\tilde{P}$ and its mirror image $\tilde{P}^*$ can be glued along the boundary components

$$\{A_1'A_2,\ A_2'A_3,\ A_3A_1\}$$

to form a new surface P.

In case 2, the Riemann surface $\tilde{P} = A_1A_1'A_2A_2'$ and its mirror image $\tilde{P}^* = A_1A_1'A_2A_2'^*$ can be glued along the boundary components $\{A_1'A_2,\ A_2A_2',\ A_2'A_1\}$ to form a new surface P.

In Figure 2 (a), (b) and (c) the symbol '$\circ$' denotes the punctures of P.

By analytic continuation, the quadratic differential φ on $\tilde{P}$ and the reflection differential φ^* on $\tilde{P}^*$ can be joined together to form a new quadratic differential on P, denoted by the same notation φ.

In case 1 (i) the surface P has a puncture at A_3' and φ has a simple pole at A_3'. In case 1 (ii) the surface P has a puncture at A_3 and A_3 is a removable pole of φ. In case 2 the differential φ has two simple poles at the punctures A_2 and A_2'. This construction shows that the surface P has the desired properties.

When $l_i \neq 0$, we call A_i the marked point of the boundary curve γ_i on P. □

From Lemma 2.1, and repeating the similar methods in [17], we obtain

Corollary 2.2 ([31]). *Suppose that $\{\gamma_1, \gamma_2, \dots, \gamma_p\}$ is an admissible curves system on the punctured Riemann surface S. Then, for arbitrary $h_k > 0$, $1 \le k \le p$, there exists a Jenkins–Strebel differential φ with type $\{\gamma_k\}$ and φ-heights $\{h_k\}$.*

Moreover the differential φ is uniquely determined.

Analogous to Lemma 2.1, we have the following result.

Lemma 2.3. *Let (h_1, h_2, h_3) be a fixed non-zero triple of numbers with $0 \le h_i \le +\infty$. Also we suppose that $0 < h_i < +\infty$ for at least one i.*

For any non-negative triple (l_1, l_2, l_3) satisfying that $l_i = 0$ if and only if $h_i = 0$, we can put a conformal structure on $\mathcal{P}$ such that the resulting bordered Riemann surface P has the following properties:

(i) *P admits a Jenkins–Strebel differential φ of type $\{\gamma_k\}$, where $k \in \{i : 0 < h_i < +\infty\}$. The boundary components $\{\gamma_k\}$ are closed horizontal trajectories of φ. In the φ-metric the characteristic annulus R_k with homotopic type γ_k has circumferences l_k and heights h_k, where $k \in \{i : 0 < h_i < +\infty\}$.*

(ii) *If j satisfies that $h_j = +\infty$, then the boundary component of P corresponding to γ_j is a puncture, denoted by Q_j. In the φ-metric the characteristic punctured disk D_j around Q_j has circumference l_j. Equivalently φ has a second order pole at Q_j with the leading coefficient $-\big(\frac{l_j}{2\pi}\big)^2$.*

(iii) *All the left boundary components of P are punctures and φ has at most a simple pole at these punctures.*

The proof of Lemma 2.3 will be postponed to Section 5.

Recall that the Riemann surface S is of type (g, n). If

$$\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_N\}$$

is a maximal finite admissible curves system on S, then $N = 3g + n - 3$. And Γ divides S into $2g + n - 2$ pairs of 'pants' $\{\mathscr{P}_i\}_{1 \le i \le 2g+n-2}$. Let $\{\gamma_{i\mu}\}_{\mu=1,2,3}$ denote the boundary components of $\mathscr{P}_i$.

Let $H \equiv (h_1, h_2, \dots, h_{3g+n-3}) \in \mathbb{R}_+^{3g+n-3}$ be the heights associated with the maximal admissible system Γ and let

$$A \equiv (a_1, a_2, \dots, a_q) \in \mathbb{R}_+^q$$

be the leading coefficients at the marked punctures $\{Q_j\}$.

Suppose that $v = (L_v, \Theta_v) \in \mathbb{R}_+^{3g+n-3} \times \mathbb{R}^{3g+n-3}$, where

$$L_v = (l_1, l_2, \dots, l_{3g+n-3}) \in \mathbb{R}_+^{3g+n-3},$$

and

$$\Theta_v = (\theta_1, \theta_2, \dots, \theta_{3g+n-3}) \in \mathbb{R}^{3g+n-3}.$$

Lemma 2.3 immediately implies that there exists a conformal structure on $\mathscr{P}_i$ such that the corresponding Riemann surface P_i has the following properties:

(1) P_i admits a Jenkins–Strebel differential φ_i. If the boundary component $\gamma_{i\mu}$ is not a puncture, then $\gamma_{i\mu}$ is a closed horizontal trajectory of φ_i. Also in the φ_i-metric, the characteristic annuli $\{R_{i\mu}\}$ have lengths $\{l_{i\mu}\}$ and heights $\{h_{i\mu}/2\}$.

(2) If Q_{ij} is a marked puncture of P_i, then φ_i has a double pole at Q_{ij} with leading coefficient $-\big(\frac{a_{ij}}{2\pi}\big)^2$.

(3) All the left boundary components of P_i are punctures and φ_i has at most simple poles at these punctures.

Let L_v be the lengths of boundary trajectories and let Θ_v be the twisting angles between two pairs of adjacent 'pants'. With the help of 3-graphs [4], as in [17] we can construct a Riemann surface $h_H^A(v)$ with a Jenkins–Strebel differential φ_v on $h_H^A(v)$. The differential φ_v is of type Γ and its characteristic annulus R_k has φ_v-heights h_k, where $1 \leq k \leq 3g + n - 3$. Moreover, φ has the leading coefficient $-\left(\frac{a_j}{2\pi}\right)^2$ at the puncture Q_j, $1 \leq j \leq q$.

Note that the Teichmüller space of the Riemann surface S is defined to be the space of Teichmüller deformations of the complex structure on S, denoted by $T_{g,n}$. See e.g. [1], [15]. For any Riemann surface $\tilde{S}$ of the same type (g, n), we denote by $[\tilde{S}] \in T_{g,n}$ the equivalent class of conformal structures which includes $\tilde{S}$.

By sending $h_H^A(v) = [h_H^A(v)] \in T_{g,n}$, we obtain a map

$$h_H^A : \mathbb{R}_+^{3g+n-3} \times \mathbb{R}^{3g+n-3} \to T_{g,n}. \tag{1}$$

In addition, we have

Theorem 2.4. *Given the maximal admissible system Γ, and positive arrays $A = (a_1, a_2, \dots, a_q)$ and $H = (h_1, h_2, \dots, h_{3g+n-3})$, the map*

$$h_H^A : \mathbb{R}_+^{3g+n-3} \times \mathbb{R}^{3g+n-3} \to T_{g,n}$$

defined in (1)*, is a homeomorphism.*

The proof of Theorem 2.4 will also be postponed to Section 5.

3. The main theorem

With the help of Theorem 2.4 we are ready to prove the following generalized height theorem on punctured Riemann surfaces.

Recall that S is a compact Riemann surface with distinguished punctures $\{Q_j\}_{1 \leq j \leq q}$.

Theorem 3.1. *For arbitrary $h_k > 0$, $1 \leq k \leq p$, and $a_j > 0$, $1 \leq j \leq q$, there is a Jenkins–Strebel differential φ on S with the following properties:*

(i) *The differential φ has p characteristic annuli $\{R_k\}$ with type $\{\gamma_k\}$. In the φ-metric these annuli have heights $\{h_k\}$.*

(ii) *φ has q punctured disks $\{D_j\}$ which are swept out by closed trajectories around the marked punctures $\{Q_j\}$. The closed horizontal trajectories in D_j have the same φ-length a_j. Equivalently, φ has a second order pole at Q_j with leading coefficient $-\left(\frac{a_j}{2\pi}\right)^2$, $j = 1, 2, \dots, q$.*

Moreover the quadratic differential φ is uniquely determined.

Proof. At first we prove the uniqueness part.

We assume, by contradiction, that there are two quadratic differentials φ_i, $i = 1, 2$, on S such that the characteristic annuli R_{1k} and R_{2k} have the same height h_k, $1 \le k \le p$. Moreover φ_1 and φ_2 have the same leading coefficient $-\left(\frac{a_j}{2\pi}\right)^2$ at the marked puncture Q_j, where $1 \le j \le q$.

Let η_j be the canonical local parameter of S near the puncture Q_j. Denote by D_{ij} the characteristic punctured disks of φ_i around Q_j, where $i = 1, 2$ and $1 \le j \le q$.

Using the local normalized uniformizer ζ_{ij}, the punctured disk D_{ij} has the form

$$D_{ij} = \{\zeta_{ij} : 0 < |\zeta_{ij}| < r_{ij}\},$$

where $\frac{d\zeta_{ij}}{d\eta_j}(0) = 1$ and r_{ij} is nothing else than the mapping radius of φ_i with respect to ζ_j. Then $M_{ij} \equiv \frac{1}{2\pi}\log r_{ij}$ is the canonical modulus of D_{ij}, where $1 \le i \le 2$ and $1 \le j \le q$.

Picking a sufficient small ρ with mapping radius $r_{ij} \ge \rho$ for $1 \le i \le 2, 1 \le j \le q$, we denote

$$S_1(\rho) \equiv S \setminus \cup \{|\zeta_{1j}| < \rho\}.$$

Let $M_j(\rho)$ resp. $\tilde{M}_j(\rho)$) denote the conformal modulus of the annulus $D_{1j} \setminus \bigcup\{|\zeta_{1j}| < \rho\}$ (resp. $D_{2j} \setminus \bigcup\{|\zeta_{1j}| < \rho\}$), $1 \le j \le q$. Denote by M_{ik} the conformal modulus of the annulus R_{ik} , where $1 \le i \le 2$, $1 \le k \le p$.

In the Riemann surface $S_1(\rho)$, by applying Lemma 1.6 to the differential φ_1, we have

$$\sum_k \frac{h_k^2}{M_{1k}} + \sum_j \frac{\left(\frac{a_j}{2\pi}\log\frac{r_{1j}}{\rho}\right)^2}{M_j(\rho)} \le \sum_k \frac{h_k^2}{M_{2k}} + \sum_j \frac{\left(\frac{a_j}{2\pi}\log\frac{r_{1j}}{\rho}\right)^2}{\tilde{M}_j(\rho)}. \tag{2}$$

Evidently as $\rho \to 0^+$, we have

$$M_j(\rho) = \frac{1}{2\pi}\log\frac{r_{1j}}{\rho} \quad \text{and} \quad \tilde{M}_j(\rho) + \frac{1}{2\pi}\log\rho \to M_{2j} = \frac{1}{2\pi}\log r_{2j}.$$

By adding the term $\sum_j \frac{a_j^2}{2\pi}\log\rho$ to both sides of the inequality (2) and letting $\rho \to 0$, we deduce that

$$\sum_k a_{1k}h_k + \sum_j a_j^2 M_{1j} \le \sum_k a_{2k}h_k + \sum_j a_j^2(2M_{1j} - M_{2j}),$$

or

$$\sum_k a_{1k}h_k - \sum_j a_j^2 M_{1j} \le \sum_k a_{2k}h_k - \sum_j a_j^2 M_{2j}. \tag{3}$$

Interchanging φ_1 and φ_2, the opposite inequality holds too. Therefore

$$\sum_k a_{1k}h_k - \sum_j a_j^2 M_{1j} = \sum_k a_{2k}h_k - \sum_j a_j^2 M_{2j}. \tag{4}$$

On the other hand, from Lemma 1.7 it follows that

$$\sum_k a_{1k}^2 M_{1k} + \sum_j a_j^2 M_{1j} \geq \sum_k a_{1k}^2 M_{2k} + \sum_j a_j^2 M_{2j}. \tag{5}$$

Combining (4) and (5) one obtains

$$\sum_k 2a_{1k}h_k \geq \sum_k \left(a_{2k} + \frac{a_{1k}^2}{a_{2k}}\right) h_k.$$

The elementary inequality implies that

$$2a_{1k} \leq a_{2k} + \frac{a_{1k}^2}{a_{2k}}, \quad 1 \leq k \leq p,$$

with equality if and only if $a_{1k} = a_{2k}$. Hence

$$a_{1k} = a_{2k}, \quad 1 \leq k \leq p.$$

From Lemma 1.7 it follows that the characteristic annuli and the characteristic punctured disks of φ_1 and φ_2 are identical, which proves the uniqueness part.

We now show the existence part.

Since $\{\gamma_1, \gamma_2, \ldots, \gamma_p\}$ is an admissible system on S, we conclude $p \leq 3g+n-3$.

If $p = 3g + n - 3$, then $\{\gamma_1, \gamma_2, \ldots, \gamma_p\}$ is a maximal admissible curves system on S. Denote

$$H \equiv (h_1, h_2, \ldots, h_{3g+n-3}) \in \mathbb{R}_+^{3g+n-3}.$$

Let $A \equiv (a_1, a_2, \ldots, a_q) \in \mathbb{R}_+^q$ be the leading coefficients at the punctures $\{Q_j\}$. From Theorem 3.2, it follows that

$$(h_H^A)^{-1} \colon T_{g,n} \to \mathbb{R}_+^{3g+n-3} \times \mathbb{R}^{3g+n-3}$$

is a homeomorphism. By considering the point $[S] \in T_{g,n}$, we conclude that there is a Jenkins–Strebel differential φ_H^A on S with type $\{\gamma_k\}$. Its annuli $\{R_k\}$ have φ_H^A-heights $\{h_k\}$ and its characteristic punctured disks D_j have φ_H^A-circumferences $\{a_j\}$.

Now we assume $p < 3g+n-3$. By adding $3g+n-3-p$ simple closed curves $\{\gamma_{p+1}, \ldots, \gamma_{3g+n-3}\}$ to $\{\gamma_1, \gamma_2, \ldots, \gamma_p\}$, we obtain a maximal admissible system

$$\{\gamma_1, \gamma_2, \ldots, \gamma_p, \gamma_{p+1}, \ldots, \gamma_{3g+n-3}\}$$

on S, denoted by Γ.

For any positive vector $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{3g+n-3-p})$, by applying the same argument as above, we obtain a differential φ_ε on S with type Γ. Its characteristic annuli have φ_ε-heights

$$(h_1, \ldots, h_p, \varepsilon_1, \ldots, \varepsilon_{3g+n-3-p}).$$

and its characteristic punctured disks have φ_ε-circumferences $\{a_j\}$.

Now we have the following result. Its proof will be postponed to Section 5.

Lemma 3.2. *If $\varepsilon_i \leq \varepsilon_i(0)$ for some $\varepsilon_i(0) > 0$, where $1 \leq i \leq 3g + n - 3 - p$, then the quadratic differentials $\{\varphi_\varepsilon\}$ are locally uniformly bounded on S.*

Let us proceed with the proof of Theorem 3.1.

Letting ε tend to 0, Lemma 3.2 implies that the quadratic differentials $\{\varphi_\varepsilon\}$ are locally uniformly bounded on S. Hence φ_ε converges locally uniformly to some quadratic differential φ with type $\{\gamma_k\}_{1\leq k\leq p}$. Also the characteristic annuli of φ have φ-heights $(h_1, h_2, \ldots, h_p)$ and its characteristic punctured disks have φ-circumferences $\{a_j\}_{1\leq j\leq q}$, as desired. $\square$

Denote by $\mathcal{J}_S$ the set consisting of all Jenkins–Strebel differentials on S of homotopic type $\{\gamma_k\}_{1\leq k\leq p}$ and with second order poles at the marked punctures $\{Q_j\}$. It is permitted that for some k there is no annulus and for some j there is no punctured disk. Moreover we assume that $0 \in \mathcal{J}_S$.

Applying Theorems 23.3 and 24.7 of [31], with the assistance of Lemma 3.2 we have

Lemma 3.3. *The space $\mathcal{J}_S$ is closed under locally uniform convergence on S. In other words, if $\varphi_n \in \mathcal{J}_S$ and $\varphi_n \to \varphi$ locally uniformly on S, then $\varphi \in \mathcal{J}_S$. Moreover the lengths $a_{nj} \to a_j$, the heights $h_{nk} \to h_k$ and the moduli $M_{nk} \to M_k$ for each j and k.*

Conversely, if $\{a_{nj}\}$ and $\{h_{nk}\}$ are bounded, then the sequence $\{\varphi_n\} \subset \mathcal{J}_S$ contains a subsequence which converges locally uniformly on S.

By taking the admissible curves system to be $\emptyset$, we obtain a new proof of the following result due to Strebel.

Theorem 3.4. *For any distinct q marked punctures on S, there is a unique Jenkins–Strebel differential φ whose characteristic domains are q-punctured disks with specified circumferences a_j, $1 \leq j \leq q$.*

4. Solution of the moduli problems

Recall that $\{\gamma_k\}$ is a system of admissible curves on S. If $\{R_k\} \subset S$ are disjoint 2-connected regions with type $\{\gamma_k\}$, then their conformal moduli $\{M(R_k)\}$ are bounded from above. It leads to the following definition.

Definition 4.1. A moduli array $M = (m_1, m_2, \ldots, m_p)$ $(m_k > 0)$ is called admissible on S, if there is a system of non-overlapping ring regions $\{R_k\} \subset S$ homotopic to $\{\gamma_k\}$ and their conformal moduli $\{M(R_k)\}$ satisfy

$$m_k < M(R_k), \quad k = 1, 2, \ldots, p.$$

Denote by $\mathcal{M}$ the set consisting of all admissible vectors M on S. Clearly $\mathcal{M} \subset \mathbb{R}_+^p$ is open.

Remark 4.2. For any $M = (m_1, m_2, \dots, m_p) \in \mathbb{R}_+^p$, the Strebel Moduli Theorem shows that there exists a Jenkins–Strebel differential of type $\{\gamma_k\}$. Its characteristic annuli have moduli $\{\lambda\, m_k\}$ for some constant $\lambda > 0$ independent of k. In addition, the differential φ is unique up to a positive constant factor.

Obviously $M \in \mathcal{M}$ if and only if $\lambda > 1$.

For any

$$C = (c_1, c_2, \dots, c_p) \in \mathbb{R}_+^p,$$

there is a unique Jenkins–Strebel differential φ_c on S such that the moduli of the characteristic annuli $\{R_k^c\}$ maximize the sum $\sum_k c_k^2 \tilde{m}_k$, among all possible choices of disjoint ring domains $\{\tilde{R}_k\} \subset S$ homotopic to $\{\gamma_k\}$. See e.g. Theorem 21.11 in [31]. Again, for each annulus R_k^c with conformal modulus $M(R_k^c) > 0$, the number c_k is just the φ_c-length of closed trajectories in R_k^c.

Let $\{m_k^c\}$ denote the conformal moduli of the characteristic annuli $\{R_k^c\}$ (If some annulus R_k^c disappears, then we set $m_k^c = 0$).

To obtain some properties of the space $\mathcal{M}$, we will give another criterion in determining whether $M \in \mathcal{M}$ or not.

Lemma 4.3. *If $M = (m_1, m_2, \dots, m_p) \in \mathbb{R}_+^p$, then $M \in \mathcal{M}$ if and only if for each $C = \{c_k\} \in \mathbb{R}_+^p$,*

$$\sum_k c_k^2 m_k^c > \sum_k c_k^2 m_k.$$

Proof. If $M \in \mathcal{M}$, then it is immediate that, $\sum_k c_k^2 m_k^c > \sum_k c_k^2 m_k$ for each $C \in \mathbb{R}_+^p$.

Conversely, we assume that $M \notin \mathcal{M}$. Thus there exists a differential φ_0 with type $\{\gamma_k\}$ and its annuli have moduli $\{\lambda\, m_k\}$ for some $0 < \lambda \le 1$. Letting $(c_{01}, c_{02}, \dots, c_{0p})$ be the φ_0-lengths of its characteristic annuli, we have

$$\sum_k c_{0k}^2 \lambda\, m_k \le \sum_k c_{0k}^2 m_k,$$

which contradicts our assumption. □

Lemma 4.3 immediately implies the following result.

Theorem 4.4. *$\mathcal{M}$ is strictly convex in $\mathbb{R}_+^p$. That is, $M, M' \in \mathcal{M}$ implies that*

$$t\,M + (1-t)\,M' \in \mathcal{M} \quad \textit{for all } t \in [0,1].$$

Recall that $\{\gamma_k\}_{1\leq k\leq p}$ is an admissible curves system on S.

Given $A=(a_1,a_2,\dots,a_q)\in\mathbb{R}^q_+$, for any

$$V=(v_1,v_2,\dots,v_p)\in\mathbb{R}^p_+,$$

from Theorem 3.3 it follows that there is a unique Jenkins–Strebel differential φ_V with homotopic type $\{\gamma_k\}$. Its characteristic annuli have φ_V-heights $\{v_k\}$ and φ_V has the leading coefficient $-\left(\frac{a_j}{2\pi}\right)^2$ at the puncture Q_j, $1\leq j\leq q$.

Denote by $\{m^v_k\}_{1\leq k\leq p}$ the moduli of its characteristic annuli. It is clear that $M_V\equiv(m^v_1,m^v_2,\dots,m^v_p)\in\mathcal{M}$. By setting $F_A(V)=M_V$ we immediately obtain a map

$$F_A\colon\mathbb{R}^p_+\to\mathcal{M}. \tag{6}$$

Furthermore we have:

Theorem 4.5. *For any $A=(a_1,a_2,\dots,a_q)\in\mathbb{R}^q_+$, the map $F_A\colon\mathbb{R}^p_+\to\mathcal{M}$ defined in* (6) *is a homeomorphism.*

The proof of this theorem is postponed to Section 5. The following is an equivalent statement of Theorem 4.5.

Theorem 4.6. *If $M=(m_1,m_2,\dots,m_p)$ is admissible on S, then for any given $A=(a_1,a_2,\dots,a_q)\in\mathbb{R}^q_+$ there is a Jenkins–Strebel differential φ which has p characteristic annuli with homotopic type $\{\gamma_k\}$ and with conformal moduli $\{m_k\}$. At the puncture Q_j the differential φ has a second order pole with leading coefficient $-\left(\frac{a_j}{2\pi}\right)^2$, $j=1,2,\dots,q$.*

Moreover the differential φ is uniquely determined.

Theorem 4.5 and 4.6 can be applied to solve the following moduli problem.

Moduli problem. Given arrays $M=(m_1,\dots,m_p)\in\mathbb{R}^p_+$ and $A=(a_1,\dots,a_q)\in\mathbb{R}^q_+$, can one find a quadratic differential φ with the following properties?

The characteristic annuli of φ are homotopic to $\{\gamma_k\}$ and have conformal moduli $\{m_k\}$. Also φ has second order poles at $\{Q_j\}$ with prescribed leading coefficient $\{-\left(\frac{a_j}{2\pi}\right)^2\}$.

As we have shown in Theorem 4.6, the moduli problem is solvable if and only if $M\in\mathcal{M}$. In particular the moduli problem is always solvable for sufficiently small $M>0$.

Recall that ζ_j is a fixed local parameter near the distinguished puncture Q_j, $1\leq j\leq q$. The remainder of this section is to prove the following result.

Theorem 4.7. *Suppose that $M=(m_1,m_2,\dots,m_p)\in\mathcal{M}$. That is, M is admissible on S.*

Then, for any real numbers $\tilde{m}_j$, $1 \le j \le q$, *there is a Jenkins–Strebel differential* φ *which has characteristic annuli* $\{R_k\}$ *homotopic to* $\{\gamma_k\}$. *The characteristic annulus* R_k *has modulus* m_k, $1 \le k \le p$.

Around the puncture Q_j, *the differential* φ *has a characteristic punctured disk* D_j *with reduced modulus (with respect to the given local parameter* ζ_j*)*

$$M_j = \tilde{m}_j + c, \quad 1 \le j \le q,$$

for some constant c *independent of* j.

In addition, φ *is uniquely determined up to a positive constant factor. In particular, the annuli* $\{R_k\}$ *and the punctured disks* $\{D_j\}$ *are uniquely determined.*

Proof. To prove the uniqueness, let φ and $\hat{\varphi}$ be two solutions whose annuli R_k and $\hat{R}_k$ have the same conformal moduli m_k, $1 \le k \le p$, and whose punctured disks D_j and $\hat{D}_j$ have reduced moduli (with respect to the given local parameter ζ_j)

$$M_j = \tilde{m}_j + c, \quad \hat{M}_j = \tilde{m}_j + \hat{c}, \quad 1 \le j \le q,$$

respectively. By applying Lemma 1.7 to the quadratic differential φ, we conclude that

$$\inf_{1 \le j \le q} (\hat{M}_j - M_j) = \hat{c} - c \le 0.$$

Similarly, by starting with the quadratic differential $\hat{\varphi}$, we conclude that $c - \hat{c} \le 0$. Hence $c = \hat{c}$.

From Lemma 1.7, it follows that

$$R_k \equiv \hat{R}_k, \quad D_j \equiv \hat{D}_j, \quad 1 \le k \le p,\ 1 \le j \le q.$$

Hence φ and $\hat{\varphi}$ have the same trajectory structures, which implies that $\hat{\varphi} = \lambda\varphi$ for some $\lambda \in \mathbb{R}^+$.

To prove the existence, we denote by $\mathcal{C}$ the set consisting of all real numbers $\{c\}$ with the following properties:

(1) There exists a system of disjoint ring domains $\{\tilde{R}_k\}$ and punctured disks $\{\tilde{D}_j\}$ on S such that $\{\tilde{R}_k\}$ is homotopic to $\{\gamma_k\}$ and $\{\tilde{D}_j\}$ is around the distinguished punctures $\{Q_j\}$.

(2) The 2-connected domain $\tilde{R}_k$ has conformal modulus $\ge m_k$, $1 \le k \le p$. With respect to the given local parameter ζ_j, the punctured disk $\tilde{D}_j$ has reduced modulus $\ge \tilde{m}_j + c$, $1 \le j \le q$.

From $M = (m_1, m_2, \dots, m_p) \in \mathcal{M}$, it immediately follows that $\mathcal{C} \ne \emptyset$ (see e.g. Theorem 4.6). Furthermore it is obvious that

$$c_0 \equiv \sup \mathcal{C} < \infty.$$

By using the normal family argument, we conclude that there exists a system of disjoint ring domains $\{R_k\}$ and punctured disks $\{D_j\}$ on S such that their conformal moduli and reduced moduli (with respect to the given local parameter $\{\zeta_j\}$) realize the number c_0.

Now we can prove that the regions $\{R_k\}$ and $\{D_j\}$ are associated with some Jenkins–Strebel differential on S.

In terms of the normalized local parameter z_j,

$$D_j = \{z_j : 0 < |z_j| < r_j\}, \quad 1 \leq j \leq q,$$

where $\frac{dz_j}{d\zeta_j}(0) = 1$ and r_j is the reduced mapping radius with respect to ζ_j. Choose a sufficiently small number ρ with $0 < \rho < r_j$, where $1 \leq j \leq q$. Denote $m_j(\rho)$ to be the modulus of the ring domain

$$R_j(\rho) \equiv \{z_j : \rho < |z_j| < r_j\}, \quad 1 \leq j \leq q.$$

We claim that R_k, $1 \leq k \leq p$, and $R_j(\rho)$, $1 \leq j \leq q$, are characteristic ring domains of some quadratic differential φ_ρ on the truncated Riemann surface $S(\rho) \equiv S \backslash \cup_j \{z_j : |z_j| < \rho\}$.

Otherwise, we would have a system of ring domains $\{\tilde{R}_k\}$ and $\{\tilde{R}_j(\rho)\}$ on $S(\rho)$ with conformal moduli $M(\tilde{R}_k) = (1+\varepsilon)m_k$ and

$$M(\tilde{R}_j(\rho)) = (1+\varepsilon)\frac{1}{2\pi}\log\frac{r_j}{\rho},$$

for some $\varepsilon > 0$.

By adding the punctured disks $\{z_j : |z_j| < \rho\}$ to the truncated Riemann surface $S(\rho)$, we obtain a system of ring domains with conformal moduli $\{(1+\varepsilon)m_k\}$ and punctured disks with reduced moduli (with respect to the given local parameter ζ_j)

$$\tilde{M}_j \geq (1+\varepsilon)\frac{1}{2\pi}\log\frac{r_j}{\rho} + \frac{1}{2\pi}\log\rho > \frac{1}{2\pi}\log r_j = M_j, \quad 1 \leq j \leq q.$$

It contradicts the original assumption that $c_0 = \sup \mathcal{C}$.

Thus the ring domains $\{R_k\}_{1\leq k\leq p}$ and $\{R_j(\rho)\}_{1\leq j\leq q}$ are associated with a quadratic differential φ_ρ on $S(\rho)$. And the boundary components $\{|z_j| = \rho\}$ are the closed trajectories of φ_ρ. Hence

$$\varphi_\rho = \varphi|_{S(\rho)}$$

for some Jenkins–Strebel differential φ on S.

In conclusion, the characteristic domains of φ consist of ring domains with moduli $\{m_k\}$ and punctured disks with reduced moduli $\{\tilde{m}_j + c_0\}$ (with respect to the given local parameters $\{\zeta_j\}$), as desired. □

5. Proof of some basic results

Now we can begin the proofs of some basic results.

Proof of Lemma 2.3. If $h_i \neq +\infty$ for each $i \in \{1, 2, 3\}$, Lemma 2.3 follows from Lemma 2.1.

Now we assume that there is at least one j such that $h_j = +\infty$. We form a new triple $(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3)$ by setting $\tilde{h}_i = h_i$ if $h_i \neq +\infty$, otherwise setting $\tilde{h}_i = 1$.

From the new triple $(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3)$, by applying Lemma 2.1 we can construct a Riemann surface $\tilde{P}$ with the following properties:

◇ The Riemann surface $\tilde{P}$ admits a Jenkins–Strebel differential $\tilde{\varphi}$ of type $\{\gamma_k\}$, where $k \in \{0 < \tilde{h}_k < +\infty\}$. The boundary components $\{\gamma_k\}$ are closed horizontal trajectories of $\tilde{\varphi}$. In the $\tilde{\varphi}$-metric the characteristic annulus $\tilde{R}_k \subset \tilde{P}$ has circumference l_k and height $\tilde{h}_k$.

◇◇ When $\tilde{h}_i = 0$, each boundary component γ_i is a puncture of $\tilde{P}$ and $\tilde{\varphi}$ has at most a simple pole at this puncture.

Recall that Δ^* is the punctured unit disk. Denote by

$$\varphi_* \equiv -\left(\frac{l_j}{2\pi}\right)^2 \frac{d\zeta^2}{\zeta^2}$$

the quadratic differential in Δ^*. We can easily check that all concentric circles $\{\zeta : |\zeta| = r\}$ $(0 < r < 1)$ are horizontal trajectories of φ_* with the same φ_*-length l_j.

As in Figure 2, for each j satisfying that $h_j = +\infty$, we denote by A_j the marked point on boundaries γ_j.

By identifying the marked points $A_j \in \gamma_j$ and $1 \in \{|\zeta| = 1\}$, and by isometrically welding the boundary components $\{|\zeta| = 1\}$ and γ_j (in the φ_*- and $\tilde{\varphi}$-metric, respectively), we can join together the Riemann surfaces Δ^* and $\tilde{P}$. The weld process preserves their induced orientations. Since the curves $\{|\zeta| = 1\})$ and γ_j are both horizontal trajectories and have the same length l_j, this welding is possible. Denote by P the resulting Riemann surface.

The Jenkins–Strebel differentials $\tilde{\varphi}$ and φ_* are joined to form a new Jenkins–Strebel differential on P, denoted by φ. Hence the Riemann surface P and the differential φ on P have the desired properties, which establishes Lemma 2.3. □

Proof of Theorem 2.4. Note that the spaces $\mathbb{R}_+^{3g+n-3} \times \mathbb{R}^{3g+n-3}$ and $T_{g,n}$ are both homeomorphic to the $6g + 2n - 6$ dimensional Euclidean space $\mathbb{R}^{6g+n-6}$. To prove that the map $h_H^A : \mathbb{R}_+^{3g+n-3} \times \mathbb{R}^{3g+n-3} \to T_{g,n}$ is a homeomorphism, it is sufficient to check that h_H^A is continuous, injective and proper.

The proof of Theorem 2.4 is divided into several steps.

Step 1. Prove that $h_H^A : \mathbb{R}_+^{3g+n-3} \times \mathbb{R}^{3g+n-3} \to T_{g,n}$ is continuous.

We assume that, as $n \to +\infty$,

$$v_n \equiv (L_n, \Theta_n) \to v_0 \equiv (L_0, \Theta_0), \tag{7}$$

in the space $\mathbb{R}_+^{3g+n-3} \times \mathbb{R}^{3g+n-3}$. For simplicity of notations we set $S_n \equiv h_H^A(v_n)$ and $S_0 \equiv h_H^A(v_0)$.

From the assumption (7) we deduce that $\{S_n\}$ lie in a compact set of $T_{g,n}$, see e.g. [8]. Therefore there is a subsequence of $\{S_n\}$ which tends to some $S_* \in T_{g,n}$. For convenience of notation we call the subsequence $\{S_n\}$ again. Therefore there are Teichmüller extremal homeomorphisms

$$F_n \colon S_n \to S_*, \quad n = 1, 2, \dots \tag{8}$$

with maximal dilatations $K_n \equiv K[F_n] \to 1$.

Let η_{nj} be the canonical local parameter of S_n at the neighborhood of Q_j. For any characteristic punctured domain $D_{nj} \subset S_n$, in terms of the normalized local parameters ζ_{nj} we have

$$D_{nj} = \{\zeta_{nj} : 0 < |\zeta_{nj}| < r_{nj}\}, \quad n = 0, 1, 2, \dots,$$

where $\frac{d\zeta_{nj}}{d\eta_{nj}}(0) = 1$ and r_{nj} is the canonical mapping radii of D_{nj}. If we denote

$$S_n^k \equiv S_n \Big\backslash \left\{ 0 < |\zeta_{nj}| < \frac{r_{nj}}{2^k} \right\}, \quad k = 1, 2, \dots,$$

then the Riemann surfaces sequence $\{S_n^k\}$ is an exhaustion of the Riemann surface S_n. For each k, Lemma 3.5 in [8] implies that $[S_n^k] \to [S_0^k]$ in the reduced Teichmüller space. Hence there are Teichmüller deformations

$$F_{nk} \colon S_n^k \to S_0^k, \tag{9}$$

with maximal dilatations $K_n^k \equiv K[F_{nk}] \to 1$ as $n \to \infty$.

Combining (8) with (9), we obtain a quasiconformal map

$$F_n \circ F_{nk}^{(-1)} \colon S_0^k \to S_*.$$

Furthermore, for each fixed k the maximal dilatations satisfy

$$K[F_n \circ F_{nk}^{(-1)}] \le K[F_n] \cdot K[F_{nk}] \to 1, \quad n \to +\infty.$$

We can therefore pass to a subsequence (denoted again by $F_n \circ F_{nk}^{(-1)}$) such that, as $n \to +\infty$, the quasiconformal homeomorphism $F_n \circ F_{nk}^{(-1)}$ locally uniformly converges to a quasiconformal homeomorphism

$$F^k \colon S_0^k \to S_*.$$

Since $F_n \circ F_{nk}^{(-1)}$ induces the same isomorphism between the fundamental groups of S_0^k and S_*, we conclude that F^k is univalent.

By using the standard argument we know that there is a conformal homeomorphism $F\colon S_0 \to S_*$. This implies that

$$[S_*] = [S_0] \in T_{g,n},$$

which proves the continuity of the map $h_H^A\colon \mathbb{R}_+^{3g+n-3} \times \mathbb{R}^{3g+n-3} \to T_{g,n}$.

Step 2. Prove that $h_H^A\colon \mathbb{R}_+^{3g+n-3} \times \mathbb{R}^{3g+n-3} \to T_{g,n}$ is injective.

Since the proof is similar to the uniqueness part of the proof of Theorem 3.1, we omit it here.

Step 3. Show that $h_H^A\colon \mathbb{R}_+^{3g+n-3} \times \mathbb{R}^{3g+n-3} \to T_{g,n}$ is a proper map.

To prove the properness of the map h_H^A, we must show that if any sequence $\{v_n'\} \subset \mathbb{R}_+^{3g+n-3} \times \mathbb{R}^{3g+n-3}$ approaches the boundary of $\mathbb{R}_+^{3g+n-3} \times \mathbb{R}^{3g+n-3}$, then the surfaces $\{h_H^A(v_n')\}$ approach the boundary of $T_{g,n}$.

Let

$$v_n' = (l_{n1}, \dots, l_{n(3g-3)}, \theta_{n1}, \dots, \theta_{n(3g-3)}).$$

The assumption that $\{v_n'\}$ approaches the boundary of $\mathbb{R}_+^{3g+n-3} \times \mathbb{R}^{3g+n-3}$ implies that at least one of the following holds:

(i) $l_{nk_0} \to +\infty$ for some fixed $1 \le k_0 \le 3g+n-3$ as $n \to +\infty$.

(ii) For some fixed $1 \le k_0 \le 3g+n-3$, $l_{nk_0} \to +0$ as $n \to +\infty$.

(iii) As $n \to +\infty$, then $c < l_{nk} < C$, $1 \le k \le 3g+n-3$, and

$$\sum_{k=1}^{3g+n-3} |\theta_{nk}| \to +\infty,$$

for two positive constants $c, C > 0$ independent of n.

Letting $v_0' = (1, 1, \dots, 1; 0, 0, \dots, 0) \in \mathbb{R}_+^{3g+n-3} \times \mathbb{R}^{3g+n-3}$, as before we set

$$S_0' \equiv h_H^A(v_0') \quad \text{and} \quad S_n' \equiv h_H^A(v_n').$$

Also we let φ_n, $n = 0, 1, \dots$, be the corresponding quadratic differential on the Riemann surface S_n'.

Let $f_n\colon S_0' \to S_n'$ be the extremal quasiconformal homeomorphism which is homotopic to the identity. And let $K_n \equiv K[f_n]$ be the maximal dilatation. If $\{S_n'\}$ do not go to the boundary of $T_{g,n}$, then we may assume (selecting a subsequence if necessary) that

$$K_n \le K, \tag{10}$$

for some $K \ge 1$ independent of n.

When $n = 0, 1, 2, \ldots$, let

$$M_{nk} \equiv M(R_{nk}), \quad 1 \le k \le 3g + n - 3,$$

be the conformal modulus of the characteristic annulus of φ_n. Also we denote by M_{nj} the canonical moduli of the punctured disks D_{nj}, $1 \le j \le q$.

Now, for notational simplicity, we use the same notations as in Step 1. That is, η_{nj} is the canonical local parameter of S'_n at the neighborhood of Q_j. For any characteristic punctured domain $D_{nj} \subset S'_n$ of φ_n, in terms of the normalized local parameters ζ_{nj} we have

$$D_{nj} = \{\zeta_{nj} : 0 < |\zeta_{nj}| < r_{nj}\}, \quad n = 0, 1, 2, \ldots, \tag{11}$$

where $\frac{d\zeta_{nj}}{d\eta_{nj}}(0) = 1$ and r_{nj} is the canonical mapping radii of D_{nj}.

We denote by

$$\delta_j \equiv \left\{\zeta_{0j} : |\zeta_{0j}| = \frac{r_{0j}}{2}\right\}, \quad 1 \le j \le q$$

the closed curves on the Riemann surface S'_0, where ζ_{0j} is defined in (11). Lemma 1.4 and the Koebe Distortion Theorem (see the Appendix) show that the curves $\{f_n(\delta_j)\}$ lie outside the punctured disk

$$\left\{\zeta_{nj} : 0 < |\zeta_{nj}| < \frac{r_{nj}}{C_j}\right\} \subset S'_n,$$

where $C_j \equiv C_j(S_0, K, r_{0j}) > 1$ is independent of n. Hence we obtain a map

$$f_n : S'_0 \backslash \bigcup_j \left\{0 < |\zeta_{0j}| < \frac{r_{0j}}{2}\right\} \to S'_n \backslash \bigcup_j \left\{0 < |\zeta_{nj}| < \frac{r_{nj}}{C_j}\right\}.$$

Let $\tilde{M}_{nk}$ be the modulus of the region $f(R_{0k})$, where $1 \le k \le 3g + n - 3$. We have

$$\frac{1}{\tilde{M}_{nk}} \le \frac{K}{M_{0k}}, \quad 1 \le k \le p, \ n = 1, 2, \ldots. \tag{12}$$

Since $\mathcal{R}_{0j} \equiv \left\{\zeta_{0j} : \frac{r_{0j}}{2} < |\zeta_{0j}| < r_{0j}\right\} \subset S'_0$ has conformal modulus $\frac{\log 2}{2\pi}$, the modulus $\tilde{M}_{nj} \equiv M(f_n(\mathcal{R}_{0j}))$ satisfies

$$\tilde{M}_{nj} \ge \frac{\log 2}{2K\pi}, \quad 1 \le j \le q. \tag{13}$$

Summing the inequalities (12) and (13) over all annuli on the Riemann surface $S'_0 \backslash \cup \left\{0 < |\zeta_{0j}| < \frac{r_{0j}}{2}\right\}$, we have

$$\sum_k \frac{h_k^2}{\tilde{M}_{nk}} + \sum_j \frac{\left(\frac{a_j \log C_j}{2\pi}\right)^2}{\tilde{M}_{nj}} \le K \cdot \left(\sum_k \frac{h_k^2}{M_{0k}} + \sum_j \frac{\left(\frac{a_j \log C_j}{2\pi}\right)^2}{\frac{\log 2}{2\pi}}\right). \tag{14}$$

Together with the inequality (14), from Lemma 1.6 it immediately follows that

$$\begin{aligned}\sum_k \frac{h_k^2}{M_{nk}} + \sum_j \frac{\left(\frac{a_j \log C_j}{2\pi}\right)^2}{\frac{\log C_j}{2\pi}} &\leq \sum_k \frac{h_k^2}{\tilde{M}_{nk}} + \sum_j \frac{\left(\frac{a_j \log C_j}{2\pi}\right)^2}{\tilde{M}_{nj}}, \\ &\leq K \cdot \left(\sum_k \frac{h_k^2}{M_{0k}} + \sum_j \frac{\left(\frac{a_j \log C_j}{2\pi}\right)^2}{\frac{\log 2}{2\pi}}\right),\end{aligned}$$

that is,

$$\sum_k h_k \cdot l_{nk} + \sum_j \frac{a_j^2 \log C_j}{2\pi} \leq K \cdot \left(\sum_k h_k \cdot l_{0k} + \sum_j \frac{(a_j \log C_j)^2}{2\pi \log 2}\right). \tag{15}$$

If $l_{nk_0} \to +\infty$ for some fixed k_0 as $n \to +\infty$, then the left-hand side of inequality (15) approaches $+\infty$ but the right-hand side remains bounded. This contradicts the assumption (10). Hence

$$d_T(S_0', S_n') = \log K_n \to +\infty,$$

which proves the case (i).

Combining Ahlfors' Lemma and Wolpert's Lemma (see the Appendix), we can prove the case (ii), see e.g. [1].

The case (iii) follows from the discreteness of Teichmüller modular group acting on $T_{g,n}$; see [17] for details.

In conclusion, we proved Theorem 2.4. □

Proof of Lemma 3.2. Denote $\varepsilon_0 \equiv (\varepsilon_1(0), \ldots, \varepsilon_{3g+n-3-p}(0))$. Theorem 2.4 implies that there exists a Jenkins–Strebel differential φ_0 on S with type Γ. Its characteristic annuli have φ_0-heights

$$(h_1, \ldots, h_p, \varepsilon_1(0), \ldots, \varepsilon_{3g+n-3-p}(0)),$$

and its characteristic punctured disks $\{D_j^0\}$ around $\{Q_j\}$ have φ_0-circumferences $\{a_j\}$.

Let η_j be the canonical local parameter of S near the puncture Q_j, where $1 \leq j \leq q$. Then by using the normalized local parameter ζ_j^0, we have

$$D_j^0 = \{p : 0 < |\zeta_j^0(p)| < r_j\}, \quad 1 \leq j \leq q,$$

where $\frac{d\zeta_j^0}{d\eta_j}(0) = 1$ and r_j is the canonical mapping radius of D_j^0.

For each ε, we denote by $\{D_j^\varepsilon\}$ the characteristic punctured disks of the differential φ_ε. Then

$$D_j^\varepsilon = \{p : 0 < |\zeta_j^\varepsilon(p)| < r_j^\varepsilon\}, \quad 1 \leq j \leq q,$$

where $\frac{d\zeta_j^\varepsilon}{d\eta_j}(0) = 1$ and r_j^ε is the canonical mapping radius of D_j^ε.

From Lemma 1.5 it follows that, for each l there exists $m_l > 0$ such that any ring domain $\tilde{R} \subset S$ around Q_j with modulus $M(\tilde{R}) > m_l$ has at least one of its boundary components lying inside the punctured disk $\{0 < |\zeta_j^0(p)| < \frac{r_j}{2^l}\}$.

If we denote

$$S^l \equiv S \backslash \left\{ 0 < |\zeta_j^0| < \frac{r_j}{2^l} \right\}, \quad l = 1, 2, \ldots,$$

then for any $\rho_j^\varepsilon > 0$ which satisfies $m_l < \frac{1}{2\pi} \log \frac{r_j^\varepsilon}{\rho_j^\varepsilon} < 2m_l$, we have

$$S^l \subset S \backslash \cup_j \{0 < |\zeta_j^\varepsilon(p)| < \rho_j^\varepsilon\}.$$

Let M_k resp. M_k^ε) be the moduli of the characteristic annuli of the Jenkins–Strebel differentials φ_0 resp. φ_ε), where $1 \le k \le 3g+n-3$. Denote $\|\varphi_\varepsilon\|_{S^l} \equiv \iint_{S^l} |\varphi_\varepsilon|\, dxdy$. Then Lemma 1.6 implies that

$$\begin{aligned}
\|\varphi_\varepsilon\|_{S^l} &\le \|\varphi_\varepsilon\|_{S \backslash \cup_j \{0 < |\zeta_j^\varepsilon| < \rho_j^\varepsilon\}} \\
&= \sum_{k=1}^{p} \frac{h_k^2}{M_k^\varepsilon} + \sum_{i=1}^{3g+n-3-p} \frac{\varepsilon_i^2}{M_{p+i}^\varepsilon} + \sum_{j=1}^{q} \frac{\left(\frac{a_j}{2\pi} \log \frac{r_j^\varepsilon}{\rho_j^\varepsilon}\right)^2}{\frac{1}{2\pi} \log \frac{r_j^\varepsilon}{\rho_j^\varepsilon}} \\
&\le \sum_{k=1}^{p} \frac{h_k^2}{M_k} + \sum_{i=1}^{3g+n-3-p} \frac{\varepsilon_i^2}{M_{p+i}} + \sum_{j=1}^{q} \frac{\left(\frac{a_j}{2\pi} \log \frac{r_j^\varepsilon}{\rho_j^\varepsilon}\right)^2}{M_j^l} \\
&\le \sum_{k=1}^{p} \frac{h_k^2}{M_k} + \sum_{i=1}^{3g+n-3-p} \frac{\varepsilon_i(0)^2}{M_{p+i}} + \sum_{j=1}^{q} \frac{(2m_l a_j)^2}{M_j^l} \\
&\le \|\varphi_{\varepsilon_0}\|_{S^l} + \sum_{j=1}^{q} \frac{(2m_l a_j)^2}{\frac{l \log 2}{2\pi}},
\end{aligned}$$

where $M_j^l = \frac{l \log 2}{2\pi}$ is the conformal modulus of $\{\frac{r_j}{2^l} < |\zeta_j^0| < r_j\}$.

Thus the norm $\|\varphi_\varepsilon\|_{S^l}$ is bounded from above independent of ε, from which we deduce that the quadratic differentials $\{\varphi_\varepsilon\}$ are locally uniformly bounded on S. It completes the proof of Lemma 3.2. □

Proof of Theorem 4.5. To prove Theorem 4.5 it suffices to show $F_A\colon \mathbb{R}_+^p \to \mathcal{M}$ is continuous, injective and proper.

The continuity of F_V follows from Lemma 3.3.

To prove F_A is injective, we assume that there are $V_1, V_2 \in \mathbb{R}_+^p$ such that $F_A(V_1) = F_A(V_2)$.

Let M_{ij}, $i = 1, 2$, be the reduced moduli of the punctured disk around Q_j with respect to the same fixed local parameter ζ_j, $1 \le j \le q$. When proving Theorem 3.1, with respect to the differential φ_1 we have actually established the following inequality:

$$\sum_k \frac{v_{1k}^2}{M_{1k}} - \sum_j a_j^2 M_{1j} \le \sum_k \frac{v_{1k}^2}{M_{2k}} - \sum_j a_j^2 M_{2j}$$

(see the claim (3)). The fact $M_{1k} = M_{2k}$, $1 \le k \le p$, implies that

$$\sum_j a_j^2 M_{1j} \ge \sum_j a_j^2 M_{2j}.$$

Interchanging φ_1 and φ_2, the opposite inequality holds too. We therefore have

$$\sum_j a_j^2 M_{1j} = \sum_j a_j^2 M_{2j}.$$

Lemma 1.7 then implies the injectivity of the map F_A.

Brouwer's theorem on invariance of domain shows that $F_A(\mathbb{R}_+^p)$ is an open domain in $\mathcal{M}$. If $F_A(\mathbb{R}_+^p) \subsetneqq \mathcal{M}$, then there is a point $M_0 \in \mathcal{M}$ but $M_0 \in \partial F_A(\mathbb{R}_+^p)$. That is, there is a sequence

$$\{V_n\}_{n=1,2\dots} \subset \mathbb{R}_+^p$$

approaching the boundary of $\mathbb{R}_+^p$, but $\{F_A(V_n)\}$ approaches an interior point M_0 of $\mathcal{M}$.

The assertion that $V_n = (v_{n1}, v_{n2}, \dots, v_{np})$ approaches the boundary of $\mathbb{R}_+^p$ is equivalent to one of the following:

$\diamond$ For all n, the sequence $\{V_n\}$ remains bounded but $v_{nk_0} \to +0$ for some fixed $1 \le k_0 \le p$.

$\diamond\diamond$ When $n \to +\infty$, the Euclidean norm $\|V_n\| \to \infty$.

Denote by φ_n the unique Jenkins–Strebel differential which realizes the data V_n and A. That is, φ_n has a second order pole at Q_j with leading coefficient $-\left(\frac{a_j}{2\pi}\right)^2$, $1 \le j \le q$, and its characteristic annuli have φ_n-heights $V_n \in \mathbb{R}_+^p$.

In the case ($\diamond$), from Lemma 3.2 and 3.3 it follows that the k_0-th component of $F_A(V_n)$ approaches 0^+, which contradicts the assumption that $M_0 \in \mathcal{M}$.

In the case ($\diamond\diamond$), the sequence $\{V_n/\|V_n\|\}$ remains bounded but $V_n/\|V_n\| \nrightarrow 0^+$. Lemma 3.3 shows that the quadratic differential $\varphi_n/\|V_n\|$ locally uniformly converges to a non-zero Jenkins–Strebel differential φ_0, with homotopic type $\{\gamma_k\}$. From $a_j/\|V_n\| \to 0$, it follows that the quadratic differential φ_0 has no second order pole at Q_j, $1 \le j \le q$.

Since φ_n and $\varphi_n/\|V_n\|$ have the same trajectory structures, from Lemma 3.3 we obtain that the annuli of φ_0 have moduli M_0. This contradicts our previous assumption that $M_0 \in \mathcal{M}$.

Therefore $F_A(\mathbb{R}_+^p) = \mathcal{M}$, as desired. $\square$

Appendix. Some known results

In previous proofs we needed several well-known results on conformal maps or quasiconformal homeomorphisms. To make this paper self-contained, we add it here. For their complete proofs, please see [1], [2], [24].

Koebe Distortion Theorem. *If $f : \Delta \to \mathbb{C}$ be a univalent function with $f(0) = 0$ and $f'(0) = 1$, then*

$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2}.$$

Mori's Theorem. *If $f : \Delta \to \Delta$ is a K-quasiconformal map with $f(0) = 0$, then*

$$|f(z)| \le 4^{1-\frac{1}{K}} |z|^{\frac{1}{K}}, \quad z \in \Delta.$$

Suppose that X is a hyperbolic Riemann surface. Then X has a canonical metric of constant curvature -1. This metric is unique and denoted by $d\rho_X$.

Let $d_X(x, y)$ denote the hyperbolic distance between two points $x, y \in X$. Then we have the following two results.

Ahlfors's Lemma. *Let $f : X \to Y$ be a holomorphic map between hyperbolic Riemann surfaces. Then either*

(i) *f is a locally covering map, or*

(ii) *$f^*(d\rho_Y) < d\rho_X$, and hence $d_Y(f(x), f(y)) < d_X(x, y)$ for any pair of distinct points $x, y \in X$.*

In particular, if hyperbolic Riemann surfaces $X \subset \tilde{X}$, then

$$d\rho_{\tilde{X}} < d\rho_X.$$

Wolpert's Lemma. *Let $h: S_1 \to S_2$ be a K-quasiconformal homeomorphism between hyperbolic Riemann surfaces S_1, S_2.*

If $\alpha_1 \subset S_1$ is a closed hyperbolic geodesic, then the hyperbolic geodesic α_2 in the homotopy class of $f(\alpha_1)$ satisfies

$$\frac{L(\alpha_1)}{K} \le L(\alpha_2) \le K \cdot L(\alpha_1),$$

where $L(\alpha_i)$, $i = 1, 2$, denotes the hyperbolic length.

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Received March 7, 2005; revised December 27, 2005

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