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## On Frobenius-destabilized rank-2 vector bundles over curves

Herbert Lange and Christian Pauly

**Abstract.** Let  $X$  be a smooth projective curve of genus  $g \geq 2$  over an algebraically closed field  $k$  of characteristic  $p > 0$ . Let  $\mathcal{M}_X$  be the moduli space of semistable rank-2 vector bundles over  $X$  with trivial determinant. The relative Frobenius map  $F: X \rightarrow X_1$  induces by pull-back a rational map  $V: \mathcal{M}_{X_1} \dashrightarrow \mathcal{M}_X$ . In this paper we show the following results.

- (1) For any line bundle  $L$  over  $X$ , the rank- $p$  vector bundle  $F_*L$  is stable.
- (2) The rational map  $V$  has base points, i.e., there exist stable bundles  $E$  over  $X_1$  such that  $F^*E$  is not semistable.
- (3) Let  $\mathcal{B} \subset \mathcal{M}_{X_1}$  denote the scheme-theoretical base locus of  $V$ . If  $g = 2$ ,  $p > 2$  and  $X$  ordinary, then  $\mathcal{B}$  is a 0-dimensional local complete intersection of length  $\frac{2}{3}p(p^2 - 1)$  and the degree of  $V$  equals  $\frac{1}{3}p(p^2 + 2)$ .

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### Introduction

Let  $X$  be a smooth projective curve of genus  $g \geq 2$  over an algebraically closed field  $k$  of characteristic  $p > 0$ . Denote by  $F: X \rightarrow X_1$  the relative  $k$ -linear Frobenius map. Here  $X_1 = X \times_{k, \sigma} k$ , where  $\sigma: \text{Spec}(k) \rightarrow \text{Spec}(k)$  is the Frobenius of  $k$  (see e.g. [R], Section 4.1). We denote by  $\mathcal{M}_X$ , respectively  $\mathcal{M}_{X_1}$ , the moduli space of semistable rank-2 vector bundles on  $X$ , respectively  $X_1$ , with trivial determinant. The Frobenius  $F$  induces by pull-back a rational map (the Verschiebung)

$$V: \mathcal{M}_{X_1} \dashrightarrow \mathcal{M}_X, \quad [E] \mapsto [F^*E].$$

Here  $[E]$  denotes the S-equivalence class of the semistable bundle  $E$ . It is shown [MS] that  $V$  is generically étale, hence separable and dominant, if  $X$  or equivalently  $X_1$  is an ordinary curve. Our first result is

**Theorem 1.** *Over any smooth projective curve  $X_1$  of genus  $g \geq 2$  there exist stable rank-2 vector bundles  $E$  with trivial determinant, such that  $F^*E$  is not semistable. In other words,  $V$  has base points.*

Note that this is a statement for an arbitrary curve of genus  $g \geq 2$  over  $k$ , since associating  $X_1$  to  $X$  induces an automorphism of the moduli space of curves of genus  $g$  over  $k$ . The existence of Frobenius-destabilized bundles was already proved in [LP2], Theorem A.4, by specializing the so-called Gunning bundle on a Mumford-Tate curve. The proof given in this paper is much simpler than the previous one. Given a line bundle  $L$  over  $X$ , the generalized Nagata–Segre theorem asserts the existence of rank-2 subbundles  $E$  of the rank- $p$  bundle  $F_*L$  of a certain (maximal) degree. Quite surprisingly, these subbundles  $E$  of maximal degree turn out to be stable and Frobenius-destabilized.

In the case  $g = 2$  the moduli space  $\mathcal{M}_X$  is canonically isomorphic to the projective space  $\mathbb{P}_k^3$  and the set of strictly semistable bundles can be identified with the Kummer surface  $\text{Kum}_X \subset \mathbb{P}_k^3$  associated to  $X$ . According to [LP2], Proposition A.2, the rational map

$$V: \mathbb{P}_k^3 \dashrightarrow \mathbb{P}_k^3$$

is given by polynomials of degree  $p$ , which are explicitly known in the cases  $p = 2$  [LP1] and  $p = 3$  [LP2]. Let  $\mathcal{B}$  be the scheme-theoretical base locus of  $V$ , i.e., the subscheme of  $\mathbb{P}_k^3$  determined by the ideal generated by the 4 polynomials of degree  $p$  defining  $V$ . Clearly its underlying set equals (see [O1], Theorem A.6)

$$\text{supp } \mathcal{B} = \{E \in \mathcal{M}_{X_1} \cong \mathbb{P}_k^3 \mid F^*E \text{ is not semistable}\}$$

and  $\text{supp } \mathcal{B} \subset \mathbb{P}_k^3 \setminus \text{Kum}_{X_1}$ . Since  $V$  has no base points on the ample divisor  $\text{Kum}_{X_1}$ , we deduce that  $\dim \mathcal{B} = 0$ . Then we show

**Theorem 2.** *Assume  $p > 2$ . Let  $X_1$  be an ordinary curve of genus  $g = 2$ . Then the 0-dimensional scheme  $\mathcal{B}$  is a local complete intersection of length*

$$\frac{2}{3}p(p^2 - 1).$$

Since  $\mathcal{B}$  is a local complete intersection, the degree of  $V$  equals  $\deg V = p^3 - l(\mathcal{B})$  where  $l(\mathcal{B})$  denotes the length of  $\mathcal{B}$  (see e.g. [O1], Proposition 2.2). Hence we obtain the

**Corollary.** *Under the assumption of Theorem 2*

$$\deg V = \frac{1}{3}p(p^2 + 2).$$

The underlying idea of the proof of Theorem 2 is rather simple: we observe that a vector bundle  $E \in \text{supp } \mathcal{B}$  corresponds via adjunction to a subbundle of the rank- $p$  vector bundle  $F_*(\theta^{-1})$  for some theta characteristic  $\theta$  on  $X$  (Proposition 3.1). This is the motivation to introduce Grothendieck's Quot-scheme  $\mathcal{Q}$  parametrizing rank-2

subbundles of degree 0 of the vector bundle  $F_*(\theta^{-1})$ . We prove that the two 0-dimensional schemes  $\mathcal{B}$  and  $\mathcal{Q}$  decompose as disjoint unions  $\coprod \mathcal{B}_\theta$  and  $\coprod \mathcal{Q}_\eta$  where  $\theta$  and  $\eta$  vary over theta characteristics on  $X$  and  $p$ -torsion points of  $JX_1$  respectively and that  $\mathcal{B}_\theta$  and  $\mathcal{Q}_0$  are isomorphic, if  $X$  is ordinary (Proposition 4.6). In particular since  $\mathcal{Q}$  is a local complete intersection,  $\mathcal{B}$  also is.

In order to compute the length of  $\mathcal{B}$  we show that  $\mathcal{Q}$  is isomorphic to a determinantal scheme  $\mathcal{D}$  defined intrinsically by the 4-th Fitting ideal of some sheaf. The non-existence of a universal family over the moduli space of rank-2 vector bundles of degree 0 forces us to work over a different parameter space constructed via the Hecke correspondence and carry out the Chern class computations on this parameter space.

The underlying set of points of  $\mathcal{B}$  has already been studied in the literature. In fact, using the notion of  $p$ -curvature, S. Mochizuki [Mo] describes points of  $\mathcal{B}$  as “dormant atoms” and obtains, by degenerating the genus-2 curve  $X$  to a singular curve, the above mentioned formula for their number ([Mo], Corollary 3.7, p. 267). Moreover he shows that for a general curve  $X$  the scheme  $\mathcal{B}$  is reduced. In this context we also mention the recent work of B. Osserman [O1], [O2], which explains the relationship of  $\text{supp } \mathcal{B}$  with Mochizuki’s theory.

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## 1. Stability of the direct image $F_*L$

Let  $X$  be a smooth projective curve of genus  $g \geq 2$  over an algebraically closed field of characteristic  $p > 0$  and let  $F: X \rightarrow X_1$  denote the relative Frobenius map. Let  $L$  be a line bundle over  $X$  with

$$\deg L = g - 1 + d,$$

for some integer  $d$ . Applying the Grothendieck–Riemann–Roch theorem to the morphism  $F$ , we obtain

**Lemma 1.1.** *The slope of the rank- $p$  vector bundle  $F_*L$  equals*

$$\mu(F_*L) = g - 1 + \frac{d}{p}.$$

The following result will be used in Section 3.

**Proposition 1.2.** *If  $g \geq 2$ , then the vector bundle  $F_*L$  is stable for any line bundle  $L$  on  $X$ .*

*Proof.* Suppose that the contrary holds, i.e.,  $F_*L$  is not stable. Consider its Harder–Narasimhan filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \dots \subset E_l = F_*L,$$

such that the quotients  $E_i/E_{i-1}$  are semistable with  $\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$  for all  $i \in \{1, \dots, l-1\}$ . If  $F_*L$  is not semistable, we denote  $E := E_1$ . If  $F_*L$  is semistable, we denote by  $E$  any proper semistable subbundle of the same slope. Then clearly

$$\mu(E) \geq \mu(F_*L). \tag{1}$$

In case  $r = \text{rk } E > \frac{p-1}{2}$ , we observe that the quotient bundle

$$Q = \begin{cases} F_*L/E_{l-1} & \text{if } F_*L \text{ is not semistable,} \\ F_*L/E & \text{if } F_*L \text{ is semistable,} \end{cases}$$

is also semistable and that its dual  $Q^*$  is a subbundle of  $(F_*L)^*$ . Moreover, by relative duality  $(F_*L)^* = F_*(L^{-1} \otimes \omega_X^{\otimes 1-p})$  and by assumption  $\text{rk } Q^* \leq p - r \leq \frac{p-1}{2}$ . Hence, replacing if necessary  $E$  and  $L$  by  $Q^*$  and  $L^{-1} \otimes \omega_X^{\otimes 1-p}$ , we may assume that  $E$  is semistable and  $r \leq \frac{p-1}{2}$ .

Now, by [SB], Corollary 2, we have the inequality

$$\mu_{\max}(F^*E) - \mu_{\min}(F^*E) \leq (r-1)(2g-2), \tag{2}$$

where  $\mu_{\max}(F^*E)$  (resp.  $\mu_{\min}(F^*E)$ ) denotes the slope of the first (resp. last) graded piece of the Harder–Narasimhan filtration of  $F^*E$ . The inclusion  $E \subset F_*L$  gives, by adjunction, a nonzero map  $F^*E \rightarrow L$ . Hence

$$\text{deg } L \geq \mu_{\min}(F^*E) \geq \mu_{\max}(F^*E) - (r-1)(2g-2) \geq p\mu(E) - (r-1)(2g-2).$$

Combining this inequality with (1) and using Lemma 1.1, we obtain

$$g-1 + \frac{d}{p} = \mu(F_*L) \leq \mu(E) \leq \frac{g-1+d}{p} + \frac{(r-1)(2g-2)}{p},$$

which simplifies to

$$(g-1) \leq (g-1) \left( \frac{2r-1}{p} \right).$$

This is a contradiction, since we have assumed  $r \leq \frac{p-1}{2}$  and therefore  $\frac{2r-1}{p} < 1$ .  $\square$

**Remark 1.3.** We observe that the vector bundles  $F_*L$  are destabilized by Frobenius, because of the nonzero canonical map  $F^*F_*L \rightarrow L$  and clearly  $\mu(F^*F_*L) > \deg L$ . For further properties of the bundles  $F_*L$ , see [JRXY], Section 5.

**Remark 1.4.** In the context of Proposition 1.2 we mention the following open question: given a finite separable morphism between smooth curves  $f: Y \rightarrow X$  and a line bundle  $L \in \text{Pic}(Y)$ , is the direct image  $f_*L$  stable? For a discussion, see [B2].

## 2. Existence of Frobenius-destabilized bundles

Let the notation be as in the previous section. We recall the generalized Nagata–Segre theorem, proved by Hirschowitz, which says

**Theorem 2.1.** *For any vector bundle  $G$  of rank  $r$  and degree  $\delta$  over any smooth curve  $X$  and for any integer  $n$ ,  $1 \leq n \leq r - 1$ , there exists a rank- $n$  subbundle  $E \subset G$ , satisfying*

$$\mu(E) \geq \mu(G) - \left(\frac{r-n}{r}\right)(g-1) - \frac{\varepsilon}{rn}, \quad (3)$$

where  $\varepsilon$  is the unique integer with  $0 \leq \varepsilon \leq r-1$  and  $\varepsilon + n(r-n)(g-1) \equiv n\delta \pmod{r}$ .

**Remark 2.2.** The previous theorem can be deduced (see [L], Remark 3.14) from the main theorem of [Hir] (for its proof, see <http://math.unice.fr/~ah/math/Brill/>).

*Proof of Theorem 1.* We apply Theorem 2.1 to the rank- $p$  vector bundle  $F_*L$  on  $X_1$  and  $n = 2$ , where  $L$  is a line bundle of degree  $g-1+d$  on  $X$ , with  $d \equiv -2g+2 \pmod{p}$ : There exists a rank-2 vector bundle  $E \subset F_*L$  such that

$$\mu(E) \geq \mu(F_*L) - \frac{p-2}{p}(g-1). \quad (4)$$

Note that our assumption on  $d$  was made to have  $\varepsilon = 0$ .

Now we will check that any  $E$  satisfying inequality (4) is stable with  $F^*E$  not semistable.

(i)  $E$  is stable: Let  $N$  be a line subbundle of  $E$ . The inclusion  $N \subset F_*L$  gives, by adjunction, a nonzero map  $F^*N \rightarrow L$ , which implies (see also [JRXY], Proposition 3.2 (i))

$$\deg N \leq \mu(F_*L) - \frac{p-1}{p}(g-1).$$

Comparing with (4) we see that  $\deg N < \mu(E)$ .

(ii)  $F^*E$  is not semistable. In fact, we claim that  $L$  destabilizes  $F^*E$ . For the proof note that Lemma 1.1 implies

$$\mu(F_*L) - \frac{p-2}{p}(g-1) = \frac{2g-2+d}{p} > \frac{g-1+d}{p} = \frac{\deg L}{p} \quad (5)$$

since  $g \geq 2$ . Together with (4) this gives  $\mu(E) > \frac{\deg L}{p}$  and hence

$$\mu(F^*E) > \deg L.$$

This implies the assertion, since by adjunction we obtain a nonzero map  $F^*E \rightarrow L$ .

Replacing  $E$  by a subsheaf of suitable degree, we may assume that inequality (4) is an equality. In that case, because of our assumption on  $d$ ,  $\mu(E)$  is an integer, hence  $\deg E$  is even. In order to get trivial determinant, we may tensorize  $E$  with a suitable line bundle.

This shows the existence of a stable rank-2 vector bundle  $E$  with  $F^*E$  not semistable, which is equivalent to the existence of base points of  $V$  (see e.g. [O1], Theorem A.6).  $\square$

### 3. Frobenius-destabilized bundles in genus 2.

From now on we assume that  $X$  is an ordinary curve of genus  $g = 2$  and the characteristic of  $k$  is  $p > 2$ . Recall that  $\mathcal{M}_X$  denotes the moduli space of semistable rank-2 vector bundles with trivial determinant over  $X$  and  $\mathcal{B}$  the scheme-theoretical base locus of the rational map

$$V: \mathcal{M}_{X_1} \cong \mathbb{P}_k^3 \dashrightarrow \mathbb{P}_k^3 \cong \mathcal{M}_X,$$

which is given by polynomials of degree  $p$ .

First of all we will show that the 0-dimensional scheme  $\mathcal{B}$  is the disjoint union of subschemes  $\mathcal{B}_\theta$  indexed by theta characteristics of  $X$ .

**Proposition 3.1.** (a) *Let  $E$  be a vector bundle on  $X_1$  such that  $E \in \text{supp } \mathcal{B}$ . Then we have*

(i) *There exists a unique theta characteristic  $\theta$  on  $X$  such that  $\text{Hom}(E, F_*(\theta^{-1})) \neq 0$ .*

(ii) *Any rank-2 vector bundle  $E$  of degree 0 satisfying  $\text{Hom}(E, F_*(\theta^{-1})) \neq 0$  is a subbundle of  $F_*(\theta^{-1})$ , i.e. the quotient  $F_*(\theta^{-1})/E$  is torsion free.*

(b) *Let  $\theta$  be a theta characteristic on  $X$ . Any rank-2 subbundle  $E \subset F_*(\theta^{-1})$  of degree 0 has the following properties*

(i)  *$E$  is stable and  $F^*E$  is not semistable,*

- (ii)  $F^*(\det E) = \mathcal{O}_X$ ,
- (iii)  $\dim \text{Hom}(E, F_*(\theta^{-1})) = 1$  and  $\dim H^1(E^* \otimes F_*(\theta^{-1})) = 5$ ,
- (iv)  $E$  is a rank-2 subbundle of maximal degree.

*Proof.* (a) By [LS], Corollary 2.6, we know that, for every  $E \in \text{supp } \mathcal{B}$  the bundle  $F^*E$  is the nonsplit extension of  $\theta^{-1}$  by  $\theta$ , for some theta characteristic  $\theta$  on  $X$  (note that  $\text{Ext}^1(\theta^{-1}, \theta) \cong k$ ). By adjunction we get a nonzero homomorphism  $\psi: E \rightarrow F_*(\theta^{-1})$ , which shows (i). Uniqueness of  $\theta$  will be proved below.

As for (ii), we have to show that  $\psi$  is of maximal rank. Suppose it is not, then there is a line bundle  $N$  on the curve  $X_1$  such that  $\psi$  factorizes as  $E \rightarrow N \rightarrow F_*(\theta^{-1})$ . By stability of  $E$  we have  $\deg N > 0$ . On the other hand, by adjunction, we get a nonzero homomorphism  $F^*N \rightarrow \theta^{-1}$  implying  $p \cdot \deg N \leq -1$ , a contradiction. Hence  $\psi: E \rightarrow F_*(\theta^{-1})$  is injective. Moreover  $E$  is even a subbundle of  $F_*(\theta^{-1})$ , since otherwise there exists a subbundle  $E' \subset F_*(\theta^{-1})$  with  $\deg E' > 0$  and which fits into the exact sequence

$$0 \longrightarrow E \longrightarrow E' \xrightarrow{\pi} T \longrightarrow 0,$$

where  $T$  is a torsion sheaf supported on an effective divisor. Varying  $\pi$ , we obtain a family of bundles  $\ker \pi \subset E'$  of dimension  $> 0$  and  $\det \ker \pi = \mathcal{O}_{X_1}$ . This would imply (see proof of Theorem 1)  $\dim \mathcal{B} > 0$ , a contradiction.

Finally, since  $\theta$  is the maximal destabilizing line subbundle of  $F^*E$ , it is unique.

(b) We observe that inequality (4) holds for the pair  $E \subset F_*(\theta^{-1})$ . Hence, by the proof of Theorem 1,  $E$  is stable and  $F^*E$  is not semistable.

Let  $\varphi: F^*E \rightarrow \theta^{-1}$  denote the homomorphism adjoint to the inclusion  $E \subset F_*(\theta^{-1})$ . The homomorphism  $\varphi$  is surjective, since otherwise  $F^*E$  would contain a line subbundle of degree  $> 1$ , contradicting [LS], Satz 2.4. Hence we get an exact sequence

$$0 \rightarrow \ker \varphi \rightarrow F^*E \rightarrow \theta^{-1} \rightarrow 0. \tag{6}$$

On the other hand, let  $N$  denote a line bundle on  $X_1$  such that  $E \otimes N$  has trivial determinant, i.e.  $N^{-2} = \det E$ . Applying Corollary 2.6 in [LS] to the bundle  $F^*(E \otimes N)$  we get an exact sequence

$$0 \rightarrow \tilde{\theta} \otimes F^*N^{-1} \rightarrow F^*E \rightarrow \tilde{\theta}^{-1} \otimes F^*N^{-1} \rightarrow 0,$$

for some theta characteristic  $\tilde{\theta}$ . By uniqueness of the destabilizing subbundle of maximal degree of  $F^*E$ , this exact sequence must coincide with (6) up to a nonzero constant. This implies that  $F^*N \otimes \tilde{\theta} = \theta$ , hence  $(F^*N)^2 = \mathcal{O}_X$ . So we obtain that  $\mathcal{O}_X = \det(F^*E) = F^*(\det E)$  proving (ii).

By adjunction the equality  $\dim \text{Hom}(E, F_*(\theta^{-1})) = \dim \text{Hom}(F^*E, \theta^{-1}) = 1$  holds. Moreover by Riemann–Roch we obtain  $\dim H^1(E^* \otimes F_*(\theta^{-1})) = 5$ . This proves (iii).

Finally, suppose that there exists a rank-2 subbundle  $E' \subset F_*(\theta^{-1})$  with  $\deg E' \geq 1$ . Then we can consider the kernel  $E = \ker \pi$  of a surjective morphism  $\pi : E' \rightarrow T$  onto a torsion sheaf with length equal to  $\deg E'$ . By varying  $\pi$  and after tensoring  $\ker \pi$  with a suitable line bundle of degree 0, we construct a family of dimension  $> 0$  of stable rank-2 vector bundles with trivial determinant which are Frobenius-destabilized, contradicting  $\dim \mathcal{B} = 0$ . This proves (iv).  $\square$

It follows from Proposition 3.1 (a) that the scheme  $\mathcal{B}$  decomposes as a disjoint union

$$\mathcal{B} = \coprod_{\theta} \mathcal{B}_{\theta},$$

where  $\theta$  varies over the set of all theta characteristics of  $X$  and

$$\text{supp } \mathcal{B}_{\theta} = \{E \in \text{supp } \mathcal{B} \mid E \subset F_*(\theta^{-1})\}.$$

Tensor product with a 2-torsion point  $\alpha \in JX_1[2] \cong JX[2]$  induces an isomorphism of  $\mathcal{B}_{\theta}$  with  $\mathcal{B}_{\theta \otimes \alpha}$  for every theta characteristic  $\theta$ . We denote by  $l(\mathcal{B})$  and  $l(\mathcal{B}_{\theta})$  the length of the schemes  $\mathcal{B}$  and  $\mathcal{B}_{\theta}$ . From the preceding we deduce the relations

$$l(\mathcal{B}) = 16 \cdot l(\mathcal{B}_{\theta}) \quad \text{for every theta characteristic } \theta. \tag{7}$$

### 4. Grothendieck’s Quot-scheme

Let  $\theta$  be a theta characteristic on  $X$ . We consider the functor  $\underline{\mathcal{Q}}$  from the opposite category of  $k$ -schemes of finite type to the category of sets defined by

$$\begin{aligned} \underline{\mathcal{Q}}(S) = \{ & \sigma : \pi_{X_1}^*(F_*(\theta^{-1})) \rightarrow \mathcal{G} \rightarrow 0 \mid \mathcal{G} \text{ coherent over } X_1 \times S, \text{ flat over } S, \\ & \deg \mathcal{G}|_{X_1 \times \{s\}} = \text{rk } \mathcal{G}|_{X_1 \times \{s\}} = p - 2 \text{ for all } s \in S\} / \cong \end{aligned}$$

where  $\pi_{X_1} : X_1 \times S \rightarrow X_1$  denotes the natural projection and  $\sigma \cong \sigma'$  for quotients  $\sigma$  and  $\sigma'$  if and only if there exists an isomorphism  $\lambda : \mathcal{G} \rightarrow \mathcal{G}'$  such that  $\sigma' = \lambda \circ \sigma$ .

Grothendieck showed in [G] (see also [HL], Section 2.2) that the functor  $\underline{\mathcal{Q}}$  is representable by a  $k$ -scheme, which we denote by  $\mathcal{Q}$ . A  $k$ -point of  $\mathcal{Q}$  corresponds to a quotient  $\sigma : F_*(\theta^{-1}) \rightarrow G$ , or equivalently to a rank-2 subsheaf  $E = \ker \sigma \subset F_*(\theta^{-1})$  of degree 0 on  $X_1$ . By Proposition 3.1 (a) (ii) any subsheaf  $E$  of degree 0 is a subbundle of  $F_*(\theta^{-1})$ , which implies that any sheaf  $\mathcal{G} \in \underline{\mathcal{Q}}(S)$  is locally free (see also [MuSa] or [L], Lemma 3.8). Moreover we note that by Proposition 3.1 (b) (iv) the bundle  $E$  has maximal degree as a subbundle of  $F_*(\theta^{-1})$ .

Hence taking the kernel of  $\sigma$  induces a bijection of  $\underline{\mathcal{Q}}(S)$  with the following set, which we also denote by  $\underline{\mathcal{Q}}(S)$

$$\begin{aligned} \underline{\mathcal{Q}}(S) = \{ & \mathcal{E} \hookrightarrow \pi_{X_1}^*(F_*(\theta^{-1})) \mid \mathcal{E} \text{ locally free sheaf over } X_1 \times S \text{ of rank } 2, \\ & \pi_{X_1}^*(F_*(\theta^{-1}))/\mathcal{E} \text{ locally free, } \deg \mathcal{E}|_{X_1 \times \{s\}} = 0 \text{ for all } s \in S\} / \cong \end{aligned}$$

By Proposition 3.1 (b) the scheme  $\mathcal{Q}$  decomposes as a disjoint union

$$\mathcal{Q} = \coprod_{\eta} \mathcal{Q}_{\eta},$$

where  $\eta$  varies over the  $p$ -torsion points  $\eta \in JX_1[p]_{\text{red}} = \ker(V : JX_1 \rightarrow JX)$ . We also denote by  $V$  the Verschiebung of  $JX_1$ , i.e.  $V(L) = F^*L$ , for  $L \in JX_1$ . The set-theoretical support of  $\mathcal{Q}_{\eta}$  equals

$$\text{supp } \mathcal{Q}_{\eta} = \{E \in \text{supp } \mathcal{Q} \mid \det E = \eta\}.$$

Because of the projection formula, the tensor product with a  $p$ -torsion point  $\beta \in JX_1[p]_{\text{red}}$  induces an isomorphism of  $\mathcal{Q}_{\eta}$  with  $\mathcal{Q}_{\eta \otimes \beta}$ . This implies the relation

$$l(\mathcal{Q}) = p^2 \cdot l(\mathcal{Q}_0), \tag{8}$$

since  $X_1$  is assumed to be ordinary. Moreover, by Proposition 3.1 we have the set-theoretical equality

$$\text{supp } \mathcal{Q}_0 = \text{supp } \mathcal{B}_{\theta}.$$

**Proposition 4.1.** (a)  $\dim \mathcal{Q} = 0$ .

(b) *The scheme  $\mathcal{Q}$  is a local complete intersection at any  $k$ -point  $e = (E \subset F_*(\theta^{-1})) \in \mathcal{Q}$ .*

*Proof.* Assertion (a) follows from the preceding remarks and  $\dim \mathcal{B} = 0$ . By [HL], Proposition 2.2.8, assertion (b) follows from the equality  $\dim_{[E]} \mathcal{Q} = 0 = \chi(\underline{\text{Hom}}(E, G))$ , where  $E = \ker(\sigma : F_*(\theta^{-1}) \rightarrow G)$  and  $\underline{\text{Hom}}$  denotes the sheaf of homomorphisms. □

Let  $\mathcal{N}_{X_1}$  denote the moduli space of semistable rank-2 vector bundles of degree 0 over  $X_1$ . We denote by  $\mathcal{N}_{X_1}^s$  and  $\mathcal{M}_{X_1}^s$  the open subschemes of  $\mathcal{N}_{X_1}$  and  $\mathcal{M}_{X_1}$  corresponding to stable vector bundles. Recall (see [La1], Theorem 4.1) that  $\mathcal{N}_{X_1}^s$  and  $\mathcal{M}_{X_1}^s$  universally corepresent the functors (see e.g. [HL], Definition 2.2.1) from the opposite category of  $k$ -schemes of finite type to the category of sets defined by

$$\begin{aligned} \underline{\mathcal{N}}_{X_1}^s(S) &= \{\mathcal{E} \text{ locally free sheaf over } X_1 \times S \text{ of rank } 2 \mid \mathcal{E}|_{X_1 \times \{s\}} \text{ stable,} \\ &\quad \deg \mathcal{E}|_{X_1 \times \{s\}} = 0 \text{ for all } s \in S\} / \sim, \end{aligned}$$

$$\begin{aligned} \underline{\mathcal{M}}_{X_1}^s(S) &= \{\mathcal{E} \text{ locally free sheaf over } X_1 \times S \text{ of rank } 2 \mid \mathcal{E}|_{X_1 \times \{s\}} \text{ stable} \\ &\quad \text{for all } s \in S, \det \mathcal{E} = \pi_S^* M \text{ for some line bundle } M \text{ on } S\} / \sim, \end{aligned}$$

where  $\pi_S : X_1 \times S \rightarrow S$  denotes the natural projection and  $\mathcal{E}' \sim \mathcal{E}$  if and only if there exists a line bundle  $L$  on  $S$  such that  $\mathcal{E}' \cong \mathcal{E} \otimes \pi_S^* L$ . We denote by  $\langle \mathcal{E} \rangle$  the equivalence class of the vector bundle  $\mathcal{E}$  for the relation  $\sim$ .

Consider the determinant morphism

$$\det: \mathcal{N}_{X_1} \rightarrow JX_1, \quad [E] \mapsto \det E,$$

and denote by  $\det^{-1}(0)$  the scheme-theoretical fibre over the trivial line bundle on  $X_1$ . Since  $\mathcal{N}_{X_1}^s$  universally corepresents the functor  $\underline{\mathcal{N}}_{X_1}^s$ , we have an isomorphism

$$\mathcal{M}_{X_1}^s \cong \mathcal{N}_{X_1}^s \cap \det^{-1}(0).$$

**Remark 4.2.** If  $p > 0$ , it is not known whether the canonical morphism  $\mathcal{M}_{X_1} \rightarrow \det^{-1}(0)$  is an isomorphism (see e.g. [La2], Section 3).

In the sequel we need the following relative version of Proposition 3.1 (b) (ii). By a  $k$ -scheme we always mean a  $k$ -scheme of finite type.

**Lemma 4.3.** *Let  $S$  be a connected  $k$ -scheme and let  $\mathcal{E}$  be a locally free sheaf of rank-2 over  $X_1 \times S$  such that  $\deg \mathcal{E}|_{X_1 \times \{s\}} = 0$  for all points  $s$  of  $S$ . Suppose that  $\mathrm{Hom}(\mathcal{E}, \pi_{X_1}^*(F_*(\theta^{-1}))) \neq 0$ . Then we have the exact sequence*

$$0 \longrightarrow \pi_X^*(\theta) \longrightarrow (F \times \mathrm{id}_S)^*\mathcal{E} \longrightarrow \pi_X^*(\theta^{-1}) \longrightarrow 0.$$

In particular

$$(F \times \mathrm{id}_S)^*(\det \mathcal{E}) = \mathcal{O}_{X_1 \times S}.$$

*Proof.* First we note that by flat base change for  $\pi_{X_1}: X_1 \times S \rightarrow X_1$ , we have an isomorphism  $\pi_{X_1}^*(F_*(\theta^{-1})) \cong (F \times \mathrm{id}_S)_*(\pi_X^*(\theta^{-1}))$ . Hence the nonzero morphism  $\mathcal{E} \rightarrow \pi_{X_1}^*(F_*(\theta^{-1}))$  gives via adjunction a nonzero morphism

$$\varphi: (F \times \mathrm{id}_S)^*\mathcal{E} \longrightarrow \pi_X^*(\theta^{-1}).$$

We know by the proof of Proposition 3.1 (b) that the fibre  $\varphi_{(x,s)}$  over any closed point  $(x, s) \in X \times S$  is a surjective  $k$ -linear map. Hence  $\varphi$  is surjective by Nakayama and we have an exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow (F \times \mathrm{id}_S)^*\mathcal{E} \longrightarrow \pi_X^*(\theta^{-1}) \longrightarrow 0,$$

with  $\mathcal{L}$  locally free sheaf of rank 1. By [K], Section 5, the rank-2 vector bundle  $(F \times \mathrm{id}_S)^*\mathcal{E}$  is equipped with a canonical connection

$$\nabla: (F \times \mathrm{id}_S)^*\mathcal{E} \longrightarrow (F \times \mathrm{id}_S)^*\mathcal{E} \otimes \Omega_{X \times S/S}^1.$$

We note that  $\Omega_{X \times S/S}^1 = \pi_X^*(\omega_X)$ , where  $\omega_X$  denotes the canonical line bundle of  $X$ . The first fundamental form of the connection  $\nabla$  is an  $\mathcal{O}_{X \times S}$ -linear homomorphism

$$\psi_\nabla: \mathcal{L} \longrightarrow \pi_X^*(\theta^{-1}) \otimes \pi_X^*(\omega_X) = \pi_X^*(\theta).$$

The restriction of  $\psi_{\nabla}$  to the curve  $X \times \{s\} \subset X \times S$  for any closed point  $s \in S$  is an isomorphism (see e.g. proof of [LS], Corollary 2.6). Hence the fibre of  $\psi_{\nabla}$  is a  $k$ -linear isomorphism over any closed point  $(x, s) \in X \times S$ . We conclude that  $\psi_{\nabla}$  is an isomorphism, by Nakayama’s lemma and because  $\mathcal{L}$  is a locally free sheaf of rank 1.

We obtain the second assertion of the lemma, since

$$(F \times \text{id}_S)^*(\det \mathcal{E}) = \det(F \times \text{id}_S)^*\mathcal{E} = \mathcal{L} \otimes \pi_X^*(\theta^{-1}) = \mathcal{O}_{X_1 \times S}. \quad \square$$

**Proposition 4.4.** *We assume  $X$  ordinary.*

(a) *The forgetful morphism*

$$i: \mathcal{Q} \hookrightarrow \mathcal{N}_{X_1}^s, \quad e = (E \subset F_*(\theta^{-1})) \mapsto E$$

*is a closed embedding.*

(b) *The restriction  $i_0$  of  $i$  to the subscheme  $\mathcal{Q}_0 \subset \mathcal{Q}$  factors through  $\mathcal{M}_{X_1}^s$ , i.e. there is a closed embedding*

$$i_0: \mathcal{Q}_0 \hookrightarrow \mathcal{M}_{X_1}^s.$$

*Proof.* (a) Let  $e = (E \subset F_*(\theta^{-1}))$  be a  $k$ -point of  $\mathcal{Q}$ . To show that  $i$  is a closed embedding at  $e \in \mathcal{Q}$ , it is enough to show that the differential  $(di)_e: T_e\mathcal{Q} \rightarrow T_{[E]}\mathcal{N}_{X_1}$  is injective – note that  $\mathcal{Q}$  is proper. Since the bundle  $E$  is stable, the Zariski tangent spaces identify with  $\text{Hom}(E, G)$  and  $\text{Ext}^1(E, E)$  respectively (see e.g. [HL], Proposition 2.2.7 and Corollary 4.5.2). Moreover, if we apply the functor  $\text{Hom}(E, \cdot)$  to the exact sequence associated with  $e \in \mathcal{Q}$

$$0 \longrightarrow E \longrightarrow F_*(\theta^{-1}) \longrightarrow G \longrightarrow 0,$$

the coboundary map  $\delta$  of the long exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Hom}(E, E) &\longrightarrow \text{Hom}(E, F_*(\theta^{-1})) \\ &\longrightarrow \text{Hom}(E, G) \xrightarrow{\delta} \text{Ext}^1(E, E) \longrightarrow \dots \end{aligned}$$

identifies with the differential  $(di)_e$ . By Proposition 3.1 (b) we obtain that the map  $\text{Hom}(E, E) \rightarrow \text{Hom}(E, F_*(\theta^{-1}))$  is an isomorphism. Thus  $(di)_e$  is injective.

(b) We consider the composite map

$$\alpha: \mathcal{Q} \xrightarrow{i} \mathcal{N}_{X_1}^s \xrightarrow{\det} JX_1 \xrightarrow{V} JX,$$

where the last map is the isogeny given by the Verschiebung on  $JX_1$ , i.e.  $V(L) = F^*L$  for  $L \in JX_1$ . The morphism  $\alpha$  is induced by the natural transformation of functors  $\underline{\alpha}: \underline{\mathcal{Q}} \Rightarrow \underline{JX}$ , defined by

$$\underline{\mathcal{Q}}(S) \longrightarrow \underline{JX}(S), \quad (\mathcal{E} \hookrightarrow \pi_X^*(F_*(\theta^{-1}))) \mapsto (F \times \text{id}_S)^*(\det \mathcal{E}).$$

Using Lemma 4.3 this immediately implies that  $\alpha$  factors through the inclusion of the reduced point  $\{\mathcal{O}_X\} \hookrightarrow JX$ . Hence the image of  $\mathcal{Q}$  under the composite morphism  $\det \circ i$  is contained in the kernel of the isogeny  $V$ , which is the reduced scheme  $JX_1[p]_{\text{red}}$ , since we have assumed  $X$  ordinary. Taking connected components we see that the image of  $\mathcal{Q}_0$  under  $\det \circ i$  is the reduced point  $\{\mathcal{O}_{X_1}\} \hookrightarrow JX_1$ , which implies that  $i_0(\mathcal{Q}_0)$  is contained in  $\mathcal{N}_{X_1}^s \cap \det^{-1}(0) \cong \mathcal{M}_{X_1}^s$ .  $\square$

In order to compare the two schemes  $\mathcal{B}_\theta$  and  $\mathcal{Q}_0$  we need the following lemma.

**Lemma 4.5.** (1) *The closed subscheme  $\mathcal{B} \subset \mathcal{M}_{X_1}^s$  corepresents the functor  $\underline{\mathcal{B}}$  which associates to a  $k$ -scheme  $S$  the set*

$$\begin{aligned} \underline{\mathcal{B}}(S) = \{ & \mathcal{E} \text{ locally free sheaf over } X_1 \times S \text{ of rank } 2 \mid \mathcal{E}|_{X_1 \times \{s\}} \text{ stable for all } s \in S, \\ & 0 \rightarrow \mathcal{L} \rightarrow (F \times \text{id}_S)^* \mathcal{E} \rightarrow \mathcal{M} \rightarrow 0 \text{ for some locally free sheaves } \mathcal{L}, \mathcal{M} \\ & \text{over } X \times S \text{ of rank } 1, \deg \mathcal{L}|_{X \times \{s\}} = -\deg \mathcal{M}|_{X \times \{s\}} = 1 \text{ for all } s \in S, \\ & \det \mathcal{E} = \pi_S^* M \text{ for some line bundle } M \text{ on } S\} / \sim . \end{aligned}$$

(2) *The closed subscheme  $\mathcal{B}_\theta \subset \mathcal{M}_{X_1}^s$  corepresents the subfunctor  $\underline{\mathcal{B}}_\theta$  of  $\underline{\mathcal{B}}$  defined by  $\langle \mathcal{E} \rangle \in \underline{\mathcal{B}}_\theta(S)$  if and only if the set-theoretical image of the classifying morphism of  $\mathcal{L}$*

$$\Phi_{\mathcal{L}} : S \longrightarrow \text{Pic}^1(X), \quad s \longmapsto \mathcal{L}|_{X \times \{s\}},$$

*is the point  $\theta \in \text{Pic}^1(X)$ .*

*Proof.* We denote by  $\mathfrak{M}_X$  the algebraic stack parametrizing rank-2 vector bundles with trivial determinant over  $X$ . Let  $\mathfrak{M}_X^{ss}$  and  $\mathfrak{M}_X^s$  denote the open substacks of  $\mathfrak{M}_X$  parametrizing semistable and stable bundles. We similarly denote the corresponding stacks of bundles over  $X_1$ . The Shatz stratification [Sh] of  $\mathfrak{M}_X$  induced by the degree of the first piece of the Harder–Narasimhan filtration reduces in the case of rank-2 vector bundles to a filtration of the stack  $\mathfrak{M}_X$

$$\mathfrak{M}_X^{ss} \subset \mathfrak{M}_X^{\leq 1} \subset \mathfrak{M}_X^{\leq 2} \subset \dots \subset \mathfrak{M}_X^{\leq n} \subset \dots \subset \mathfrak{M}_X$$

by open substacks  $\mathfrak{M}_X^{\leq n}$ . It follows from the semicontinuity of the Harder–Narasimhan filtration ([Sh], Section 5) that, for every integer  $n$ , there is a closed reduced substack  $\mathfrak{M}_X^n$  of  $\mathfrak{M}_X^{\leq n}$  parametrizing vector bundles having a maximal destabilizing line subbundle of degree  $n$ . Note that  $\mathfrak{M}_X^n$  is the complement of  $\mathfrak{M}_X^{\leq n-1}$  in  $\mathfrak{M}_X^{\leq n}$ . It can be shown (see e.g. [He], Folgerung 2.1.10) that the stacks  $\mathfrak{M}_X^n$  and  $\mathfrak{M}_X$  are smooth. Let  $\mathfrak{V} : \mathfrak{M}_{X_1} \rightarrow \mathfrak{M}_X$  denote the morphism of stacks induced by pull-back under the Frobenius map  $F : X \rightarrow X_1$ . It follows from [LS], Corollary 2.6, that the restriction of  $\mathfrak{V}$  to the open substack  $\mathfrak{M}_{X_1}^{ss}$  determines a morphism of stacks

$$\mathfrak{V}^{ss} : \mathfrak{M}_{X_1}^{ss} \longrightarrow \mathfrak{M}_X^{\leq 1}.$$

We will use the following facts about the stack  $\mathfrak{M}_X$ .

- The pull-back of  $\mathcal{O}_{\mathbb{P}^3}(1)$  by the natural map  $\mathfrak{M}_X^{ss} \rightarrow \mathcal{M}_X \cong \mathbb{P}^3$  extends to a line bundle, which we denote by  $\mathcal{O}(1)$ , over the moduli stack  $\mathfrak{M}_X^{\leq 1}$  and  $\text{Pic}(\mathfrak{M}_X^{\leq 1}) = \mathbb{Z} \cdot \mathcal{O}(1)$ . Moreover, for any positive integer  $l$ , there is a natural isomorphism  $H^0(\mathfrak{M}_X^{\leq 1}, \mathcal{O}(l)) \cong H^0(\mathcal{M}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}(l))$  (see [BL], Propositions 8.3 and 8.4).
- The closed substack  $\mathfrak{M}_X^1$  is the base locus of the linear system  $|\mathcal{O}(1)|$  over the stack  $\mathfrak{M}_X^{\leq 1}$  (see Proposition A).

In order to prove part (1) it will be enough to show that the functor  $\underline{\mathcal{B}}$  defined in the lemma coincides with the fibre product functor  $\mathcal{B} \times_{\mathcal{M}_{X_1}^s} \underline{\mathcal{M}}_{X_1}^s$  – we recall that  $\mathcal{M}_{X_1}^s$  universally corepresents the functor  $\underline{\mathcal{M}}_{X_1}^s$ .

We now compute the fibre product functor  $\mathcal{B} \times_{\mathcal{M}_{X_1}^s} \underline{\mathcal{M}}_{X_1}^s$ . Let  $S$  be a  $k$ -scheme and consider a vector bundle  $\mathcal{E} \in \mathfrak{M}_{X_1}^s(S)$ . Since the subscheme  $\mathcal{B}$  is defined as the base locus of the linear system  $V^*|\mathcal{O}_{\mathbb{P}^3}(1)|$ , we obtain that  $\langle \mathcal{E} \rangle \in [\mathcal{B} \times_{\mathcal{M}_{X_1}^s} \underline{\mathcal{M}}_{X_1}^s](S)$  if and only if  $\mathcal{E}$  lies in the base locus of  $\mathfrak{V}^{ss*}|\mathcal{O}(1)|$  – here we use the isomorphism  $|\mathcal{O}_{\mathbb{P}^3}(1)| \cong |\mathcal{O}(1)|$ , or equivalently  $\mathfrak{V}^{ss}(\mathcal{E}) := (F \times \text{id}_S)^*\mathcal{E} \in \mathfrak{M}_X^{\leq 1}(S)$  lies in the base locus of  $|\mathcal{O}(1)|$ , which is the closed substack  $\mathfrak{M}_X^1$ .

We now consider the universal exact sequence defined by the Harder–Narasimhan filtration over  $\mathfrak{M}_X^1$ :

$$0 \rightarrow \mathcal{L} \rightarrow (F \times \text{id}_S)^*\mathcal{E} \rightarrow \mathcal{M} \rightarrow 0,$$

with  $\mathcal{L}, \mathcal{M}$  locally free sheaves over  $X \times S$  such that  $\text{deg } \mathcal{L}|_{X \times \{s\}} = -\text{deg } \mathcal{M}|_{X \times \{s\}} = 1$  for any  $s \in S$ . This shows that the two sets  $[\mathcal{B} \times_{\mathcal{M}_{X_1}^s} \underline{\mathcal{M}}_{X_1}^s](S)$  and  $\underline{\mathcal{B}}(S)$  coincide. This proves (1).

As for (2), we add the condition that the family  $\mathcal{E}$  is Frobenius-destabilized by the theta-characteristic  $\theta$ . □

**Remark 4.6.** Note that in Lemma 4.5 we do not need to assume  $X$  ordinary.

**Proposition 4.7.** *We assume  $X$  ordinary. There is a scheme-theoretical equality*

$$\mathcal{B}_\theta = \mathcal{Q}_0$$

as closed subschemes of  $\mathcal{M}_{X_1}$ .

*Proof.* Since  $\mathcal{B}_\theta$  and  $\mathcal{Q}_0$  corepresent the two functors  $\underline{\mathcal{B}}_\theta$  and  $\underline{\mathcal{Q}}_0$  it will be enough to show that there is a canonical bijection between the set  $\underline{\mathcal{B}}_\theta(S)$  and  $\underline{\mathcal{Q}}_0(S)$  for any  $k$ -scheme  $S$ . We recall that

$$\begin{aligned} \underline{\mathcal{Q}}_0(S) = \{ \mathcal{E} \hookrightarrow \pi_{X_1}^*(F_*(\theta^{-1})) \mid \mathcal{E} \text{ locally free sheaf over } X_1 \times S \text{ of rank 2,} \\ \pi_{X_1}^*(F_*(\theta^{-1}))/\mathcal{E} \text{ locally free, } \det \mathcal{E} \cong \mathcal{O}_{X_1 \times S} \} / \cong \end{aligned}$$

Note that the property  $\det \mathcal{E} \cong \mathcal{O}_{X_1 \times S}$  is implied as follows: by Proposition 4.4 (b) we have  $\det \mathcal{E} \cong \pi_S^* L$  for some line bundle  $L$  over  $S$  and by Lemma 4.3 we conclude that  $L = \mathcal{O}_S$ .

First we show that the natural map  $\mathcal{Q}_0(S) \rightarrow \mathcal{M}_{X_1}^s(S)$  is injective. Suppose that there exist  $\mathcal{E}, \mathcal{E}' \in \mathcal{Q}_0(S)$  such that  $\langle \mathcal{E} \rangle = \langle \mathcal{E}' \rangle$ , i.e.  $\mathcal{E}' \cong \mathcal{E} \otimes \pi_S^*(L)$  for some line bundle  $L$  on  $S$ . Then by Lemma 4.3 we have two inclusions

$$i: \pi_X^*(\theta) \rightarrow (F \times \text{id}_S)^* \mathcal{E}, \quad i': \pi_X^*(\theta) \otimes \pi_S^*(L^{-1}) \rightarrow (F \times \text{id}_S)^* \mathcal{E}.$$

Composing with the projection  $\sigma: (F \times \text{id}_S)^* \mathcal{E} \rightarrow \pi_X^*(\theta^{-1})$  we see that the composite map  $\sigma \circ i'$  is identically zero. Hence the two subbundles  $\pi_X^*(\theta)$  and  $\pi_X^*(\theta) \otimes \pi_S^*(L^{-1})$  coincide, which implies  $\pi_S^*(L) = \mathcal{O}_{X_1 \times S}$ .

Therefore the two sets  $\mathcal{Q}_0(S)$  and  $\mathcal{B}_\theta(S)$  are naturally subsets of  $\mathcal{M}_{X_1}^s(S)$ .

We now show that  $\mathcal{Q}_0(S) \subset \mathcal{B}_\theta(S)$ . Consider  $\mathcal{E} \in \mathcal{Q}_0(S)$ . By Proposition 3.1 (b) the bundle  $\mathcal{E}|_{X_1 \times \{s\}}$  is stable for all  $s \in S$ . By Lemma 4.3 we can take  $\mathcal{L} = \pi_X^*(\theta)$  and  $\mathcal{M} = \pi_X^*(\theta^{-1})$ , so that  $\langle \mathcal{E} \rangle \in \mathcal{B}_\theta(S)$ .

Hence it remains to show that  $\mathcal{B}_\theta(S) \subset \mathcal{Q}_0(S)$ . Consider a sheaf  $\mathcal{E}$  with  $\langle \mathcal{E} \rangle \in \mathcal{B}_\theta(S)$  – see Lemma 4.5 (2). As in the proof of Lemma 4.3 we consider the canonical connection  $\nabla$  on  $(F \times \text{id}_S)^* \mathcal{E}$ . Its first fundamental form is an  $\mathcal{O}_{X \times S}$ -linear homomorphism

$$\psi_\nabla: \mathcal{L} \rightarrow \mathcal{M} \otimes \pi_X^*(\omega_X),$$

which is surjective on closed points  $(x, s) \in X \times S$ . Hence we can conclude that  $\psi_\nabla$  is an isomorphism. Moreover taking the determinant, we obtain

$$\mathcal{L} \otimes \mathcal{M} = \det(F \times \text{id}_S)^* \mathcal{E} = \pi_S^* M,$$

for some line bundle  $M$  on  $S$ . Combining both isomorphisms we deduce that

$$\mathcal{L} \otimes \mathcal{L} = \pi_X^*(\omega_X) \otimes \pi_S^* M.$$

Hence its classifying morphism  $\Phi_{\mathcal{L} \otimes \mathcal{L}}: S \rightarrow \text{Pic}^2(X)$  factorizes through the inclusion of the reduced point  $\{\omega_X\} \hookrightarrow \text{Pic}^2(X)$ . Moreover the composite map of  $\Phi_{\mathcal{L}}$  with the duplication map [2]

$$\Phi_{\mathcal{L} \otimes \mathcal{L}}: S \xrightarrow{\Phi_{\mathcal{L}}} \text{Pic}^1(X) \xrightarrow{[2]} \text{Pic}^2(X)$$

coincides with  $\Phi_{\mathcal{L} \otimes \mathcal{L}}$ . We deduce that  $\Phi_{\mathcal{L}}$  factorizes through the inclusion of the reduced point  $\{\theta\} \hookrightarrow \text{Pic}^1(X)$ . Note that the fibre  $[2]^{-1}(\omega_X)$  is reduced, since  $p > 2$ . Since  $\text{Pic}^1(X)$  is a fine moduli space, there exists a line bundle  $N$  over  $S$  such that

$$\mathcal{L} = \pi_X^*(\theta) \otimes \pi_S^*(N).$$

We introduce the vector bundle  $\mathcal{E}_0 = \mathcal{E} \otimes \pi_S^*(N^{-1})$ . Then  $\langle \mathcal{E}_0 \rangle = \langle \mathcal{E} \rangle$  and we have an exact sequence

$$0 \longrightarrow \pi_X^*(\theta) \longrightarrow (F \times \text{id}_S)^* \mathcal{E}_0 \xrightarrow{\sigma} \pi_X^*(\theta^{-1}) \longrightarrow 0,$$

since  $\pi_S^* M = \pi_S^* N^2$ . By adjunction the morphism  $\sigma$  gives a nonzero morphism

$$j: \mathcal{E}_0 \longrightarrow (F \times \text{id}_S)_*(\pi_X^*(\theta^{-1})) \cong \pi_{X_1}^*(F_*(\theta^{-1})).$$

We now show that  $j$  is injective. Suppose it is not. Then there exists a subsheaf  $\tilde{\mathcal{E}}_0 \subset \pi_{X_1}^*(F_*(\theta^{-1}))$  and a surjective map  $\tau: \mathcal{E}_0 \rightarrow \tilde{\mathcal{E}}_0$ . Let  $\mathcal{K}$  denote the kernel of  $\tau$ . Again by adjunction we obtain a map  $\alpha: (F \times \text{id}_S)^* \tilde{\mathcal{E}}_0 \rightarrow \pi_X^*(\theta^{-1})$  such that the composite map

$$(F \times \text{id}_S)^* \mathcal{E}_0 \xrightarrow{\tau^*} (F \times \text{id}_S)^* \tilde{\mathcal{E}}_0 \xrightarrow{\alpha} \pi_X^*(\theta^{-1})$$

coincides with  $\sigma$ . Here  $\tau^*$  denotes the map  $(F \times \text{id}_S)^* \tau$ . Since  $\sigma$  is surjective,  $\alpha$  is also surjective. We denote by  $\mathcal{M}$  the kernel of  $\alpha$ . The induced map  $\bar{\tau}: \pi_X^*(\theta) = \ker \sigma \rightarrow \mathcal{M}$  is surjective, because  $\tau^*$  is surjective. Moreover the first fundamental form of the canonical connection  $\tilde{\nabla}$  on  $(F \times \text{id}_S)^* \tilde{\mathcal{E}}_0$  induces an  $\mathcal{O}_{X \times S}$ -linear homomorphism  $\psi_{\tilde{\nabla}}: \mathcal{M} \rightarrow \pi_X^*(\theta)$  and the composite map

$$\psi_{\tilde{\nabla}}: \pi_X^*(\theta) \xrightarrow{\bar{\tau}} \mathcal{M} \xrightarrow{\psi_{\tilde{\nabla}}} \pi_X^*(\theta)$$

coincides with the first fundamental form of  $\nabla$  of  $(F \times \text{id}_S)^* \mathcal{E}_0$ , which is an isomorphism. Therefore  $\bar{\tau}$  is an isomorphism too. So  $\tau^*$  is an isomorphism and  $(F \times \text{id}_S)^* \mathcal{K} = 0$ . We deduce that  $\mathcal{K} = 0$ .

In order to show that  $\mathcal{E}_0 \in \underline{\mathcal{Q}}_0(S)$ , it remains to verify that the quotient sheaf  $\pi_{X_1}(F_*(\theta^{-1}))/\mathcal{E}_0$  is flat over  $S$ . We recall that flatness implies locally freeness because of maximality of degree. But flatness follows from [HL], Lemma 2.1.4, since the restriction of  $j$  to  $X_1 \times \{s\}$  is injective for any closed  $s \in S$  by Proposition 3.1 (a).  $\square$

Since  $\mathcal{Q}_0$  represents the functor  $\underline{\mathcal{Q}}_0$ , we obtain the following

**Corollary 4.8.** *The scheme  $\mathcal{B}_\theta$  represents the functor  $\underline{\mathcal{B}}_\theta$  defined in Lemma 4.5.*

Combining Proposition 4.7 with relations (7) and (8), we obtain

**Corollary 4.9.** *We have*

$$l(\mathcal{B}) = \frac{16}{p^2} \cdot l(\mathcal{Q}).$$

### 5. Determinantal subschemes

In this section we introduce a determinantal subscheme  $\mathcal{D} \subset \mathcal{N}_{X_1}$ , whose length will be computed in the next section. We also show that  $\mathcal{D}$  is isomorphic to Grothendieck’s Quot-scheme  $\mathcal{Q}$ . We first define a determinantal subscheme  $\tilde{\mathcal{D}}$  of a variety  $JX_1 \times Z$  covering  $\mathcal{N}_{X_1}$  and then we show that  $\tilde{\mathcal{D}}$  is a  $\mathbb{P}^1$ -fibration over an étale cover of  $\mathcal{D} \subset \mathcal{N}_{X_1}$ .

Since there does not exist a universal bundle over  $X_1 \times \mathcal{M}_{X_1}$ , following an idea of Mukai [Mu], we consider the moduli space  $\mathcal{M}_{X_1}(x)$  of stable rank-2 vector bundles on  $X_1$  with determinant  $\mathcal{O}_{X_1}(x)$  for a fixed point  $x \in X_1$ . According to [N1] the variety  $\mathcal{M}_{X_1}(x)$  is a smooth intersection of two quadrics in  $\mathbb{P}^5$ . Let  $\mathcal{U}$  denote a universal bundle on  $X_1 \times \mathcal{M}_{X_1}(x)$  and denote

$$\mathcal{U}_x := \mathcal{U}|_{\{x\} \times \mathcal{M}_{X_1}(x)}$$

considered as a rank-2 vector bundle on  $\mathcal{M}_{X_1}(x)$ . Then the projectivized bundle

$$Z := \mathbb{P}(\mathcal{U}_x)$$

is a  $\mathbb{P}^1$ -bundle over  $\mathcal{M}_{X_1}(x)$ . The variety  $Z$  parametrizes pairs  $(F_z, l_z)$  consisting of a stable vector bundle  $F_z \in \mathcal{M}_{X_1}(x)$  and a non-trivial linear form  $l_z: F_z(x) \rightarrow k_x$  on the fibre of  $F_z$  over  $x$  defined up to a non-zero constant. Thus to any  $z \in Z$  one can associate an exact sequence

$$0 \rightarrow E_z \rightarrow F_z \rightarrow k_x \rightarrow 0$$

uniquely determined up to a multiplicative constant. Clearly  $E_z$  is semistable, since  $F_z$  is stable, and  $\det E_z = \mathcal{O}_{X_1}$ . Hence we get a diagram (the so-called Hecke correspondence)

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & \mathcal{M}_{X_1} \cong \mathbb{P}^3 \\ \pi \downarrow & & \\ \mathcal{M}_{X_1}(x) & & \end{array}$$

with  $\varphi(z) = [E_z]$  and  $\pi(z) = F_z$ . We note that there is an isomorphism  $\varphi^{-1}(E) \cong \mathbb{P}^1$  (see e.g. [Mu], (3.7)) and that  $\pi(\varphi^{-1}(E)) \subset \mathcal{M}_{X_1}(x) \subset \mathbb{P}^5$  is a conic for any stable  $E \in \mathcal{M}_{X_1}^s$  (see e.g. [NR2]). On  $X_1 \times Z$  there exists a “universal” bundle, which we denote by  $\mathcal{V}$  (see [Mu], (3.8)). It has the property

$$\mathcal{V}|_{X_1 \times \{z\}} \cong E_z, \quad \text{for all } z \in Z.$$

Let  $\mathcal{L}$  denote a Poincaré bundle on  $X_1 \times JX_1$ . By abuse of notation we also denote by  $\mathcal{V}$  and  $\mathcal{L}$  their pull-backs to  $X_1 \times JX_1 \times Z$ . We denote by  $\pi_{X_1}$  and  $q$  the

canonical projections

$$X_1 \xleftarrow{\pi_{X_1}} X_1 \times JX_1 \times Z \xrightarrow{q} JX_1 \times Z.$$

We consider the map  $m$  given by tensor product

$$m: JX_1 \times \mathcal{M}_{X_1} \longrightarrow \mathcal{N}_{X_1}, \quad (L, E) \longmapsto L \otimes E.$$

Note that the restriction of  $m$  to the stable locus  $m^s: JX_1 \times \mathcal{M}_{X_1}^s \longrightarrow \mathcal{N}_{X_1}^s$  is an étale map of degree 16. We denote by  $\psi$  the composite map

$$\psi: JX_1 \times Z \xrightarrow{\text{id}_{JX_1} \times \varphi} JX_1 \times \mathcal{M}_{X_1} \xrightarrow{m} \mathcal{N}_{X_1}, \quad \psi(L, z) = L \otimes E_z.$$

Let  $D \in |\omega_{X_1}|$  be a smooth canonical divisor on  $X_1$ . We introduce the following sheaves over  $JX_1 \times Z$

$$\mathcal{F}_1 = q_*(\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^*(F_*(\theta^{-1}) \otimes \omega_{X_1}))$$

and

$$\mathcal{F}_0 = \bigoplus_{y \in D} (\mathcal{L}^* \otimes \mathcal{V}^*|_{\{y\} \times JX_1 \times Z}) \otimes k^{\oplus p}.$$

The next proposition is an even degree analogue of [LN], Theorem 3.1.

**Proposition 5.1.** (a) *The sheaves  $\mathcal{F}_0$  and  $\mathcal{F}_1$  are locally free of rank  $4p$  and  $4p - 4$  respectively and there is an exact sequence*

$$0 \longrightarrow \mathcal{F}_1 \xrightarrow{\gamma} \mathcal{F}_0 \longrightarrow R^1 q_*(\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^*(F_*(\theta^{-1}))) \longrightarrow 0.$$

Let  $\tilde{\mathcal{D}} \subset JX_1 \times Z$  denote the subscheme defined by the 4-th Fitting ideal of the sheaf  $R^1 q_*(\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^*(F_*(\theta^{-1})))$ . We have set-theoretically

$$\text{supp } \tilde{\mathcal{D}} = \{(L, z) \in JX_1 \times Z \mid \dim \text{Hom}(L \otimes E_z, F_*(\theta^{-1})) = 1\},$$

and  $\dim \tilde{\mathcal{D}} = 1$ .

(b) *Let  $\delta$  denote the  $l$ -adic ( $l \neq p$ ) cohomology class of  $\tilde{\mathcal{D}}$  in  $JX_1 \times Z$ . Then*

$$\delta = c_5(\mathcal{F}_0 - \mathcal{F}_1) \in H^{10}(JX_1 \times Z, \mathbb{Z}_l).$$

*Proof.* We consider the canonical exact sequence over  $X_1 \times JX_1 \times Z$  associated to the effective divisor  $\pi_{X_1}^* D$

$$\begin{aligned} 0 \rightarrow \mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^* F_*(\theta^{-1}) &\xrightarrow{\otimes D} \mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^*(F_*(\theta^{-1}) \otimes \omega_{X_1}) \\ &\rightarrow \mathcal{L}^* \otimes \mathcal{V}^*|_{\pi_{X_1}^* D} \otimes k^{\oplus p} \rightarrow 0. \end{aligned}$$

By Proposition 1.2 the rank- $p$  vector bundle  $F_*(\theta^{-1})$  is stable and since

$$1 - \frac{2}{p} = \mu(F_*(\theta^{-1})) > \mu(L \otimes E) = 0 \quad \text{for all } (L, E) \in JX_1 \times \mathcal{M}_{X_1},$$

we obtain

$$\dim H^1(L^* \otimes E^* \otimes F_*(\theta^{-1}) \otimes \omega_{X_1}) = \dim \text{Hom}(F_*(\theta^{-1}), L \otimes E) = 0.$$

This implies

$$R^1 q_*(\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^*(F_*(\theta^{-1}) \otimes \omega_{X_1})) = 0.$$

By the base change theorems the sheaf  $\mathcal{F}_1$  is locally free. Taking direct images by  $q$  (note that  $q_*(\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^* F_*(\theta^{-1})) = 0$  because it is a torsion sheaf), we obtain the exact sequence

$$0 \longrightarrow \mathcal{F}_1 \xrightarrow{\gamma} \mathcal{F}_0 \longrightarrow R^1 q_*(\mathcal{L}^* \otimes \mathcal{V}^* \otimes \pi_{X_1}^*(F_*(\theta^{-1}))) \longrightarrow 0.$$

with  $\mathcal{F}_1$  and  $\mathcal{F}_0$  as in the statement of the proposition. Note that by Riemann–Roch we have

$$\text{rk } \mathcal{F}_1 = 4p - 4 \quad \text{and} \quad \text{rk } \mathcal{F}_0 = 4p.$$

It follows from the proof of Proposition 3.1 (a) that any nonzero homomorphism  $L \otimes E \longrightarrow F_*(\theta^{-1})$  is injective. Moreover by Proposition 3.1 (b) (iii) for any subbundle  $L \otimes E \subset F_*(\theta^{-1})$  we have  $\dim \text{Hom}(L \otimes E, F_*(\theta^{-1})) = 1$ , or equivalently  $\dim H^1(L^* \otimes E^* \otimes F_*(\theta^{-1})) = 5$ . Using the base change theorems we obtain the following series of equivalences

$$\begin{aligned} (L, z) \in \text{supp } \tilde{\mathcal{D}} &\iff \text{rk } \gamma_{(L,z)} < 4p - 4 = \text{rk } \mathcal{F}_1 \\ &\iff \dim H^1(L^* \otimes E^* \otimes F_*(\theta^{-1})) \geq 5 \\ &\iff \dim \text{Hom}(L \otimes E, F_*(\theta^{-1})) \geq 1 \\ &\iff \dim \text{Hom}(L \otimes E, F_*(\theta^{-1})) = 1. \end{aligned}$$

Finally we clearly have the equality  $\text{supp } \psi(\tilde{\mathcal{D}}) = \text{supp } \mathcal{Q}$ . Since  $\dim \mathcal{Q} = 0$  and since  $\varphi^{-1}(E) \cong \mathbb{P}^1$  for  $E$  stable, we deduce that  $\dim \tilde{\mathcal{D}} = 1$ . This proves part (a).

Part (b) follows from Porteous’ formula, which says that the fundamental class  $\delta \in H^{10}(JX_1 \times Z, \mathbb{Z}_l)$  of the determinantal subscheme  $\tilde{\mathcal{D}}$  is given (with the notation of [ACGH], p. 86) by

$$\begin{aligned} \delta &= \Delta_{4p-(4p-5), 4p-4-(4p-5)}(c_t(\mathcal{F}_0 - \mathcal{F}_1)) \\ &= \Delta_{5,1}(c_t(\mathcal{F}_0 - \mathcal{F}_1)) \\ &= c_5(\mathcal{F}_0 - \mathcal{F}_1). \end{aligned}$$

□

Let  $M$  be a sheaf over a  $k$ -scheme  $S$ . We denote by

$$\text{Fitt}_n[M] \subset \mathcal{O}_S$$

the  $n$ -th Fitting ideal sheaf of  $M$ .

We now define the 0-dimensional subscheme  $\mathcal{D} \subset \mathcal{N}_{X_1}^s$ , which is supported on  $\text{supp } \mathcal{Q}$ , by defining a scheme structure  $\mathcal{D}_E$  for every  $E \in \text{supp } \mathcal{Q}$ . Note that

$$\mathcal{D} = \coprod_{E \in \text{supp } \mathcal{Q}} \mathcal{D}_E.$$

Consider a bundle  $E \in \mathcal{N}_{X_1}^s$  with  $E \in \text{supp } \mathcal{Q}$ , i.e.

$$\dim \text{Hom}(E, F_*(\theta^{-1})) \geq 1 \iff \dim H^1(E^* \otimes F_*(\theta^{-1})) \geq 5.$$

The GIT-construction of the moduli space  $\mathcal{N}_{X_1}^s$  realizes  $\mathcal{N}_{X_1}^s$  as a quotient of an open subset  $\mathcal{U}$  of a Quot-scheme by the group  $\mathbb{P}\text{GL}(N)$  for some  $N$ . It can be shown (see e.g. [La2], Section 3) that  $\mathcal{U}$  is a principal  $\mathbb{P}\text{GL}(N)$ -bundle for the étale topology over  $\mathcal{N}_{X_1}^s$ . Hence there exists an étale neighbourhood  $\tau: \bar{U} \rightarrow U$  of  $E$  over which the  $\mathbb{P}\text{GL}(N)$ -bundle is trivial, i.e., admits a section. The universal bundle over the Quot-scheme restricts to a bundle  $\mathcal{E}$  over  $X_1 \times \bar{U}$ . Choose a point  $\bar{E} \in \bar{U}$  over  $E$ . We denote by  $\mathcal{D}_{\bar{E}}$  the connected component supported at  $\bar{E}$  of the scheme defined by the Fitting ideal sheaf

$$\text{Fitt}_4[R^1\pi_{\bar{U}*}(\mathcal{E}^* \otimes \pi_{X_1}^* F_*(\theta^{-1}))].$$

**Lemma 5.2.** *Let  $\tau: \bar{U} \rightarrow U$  be an étale map and  $y \in \bar{U}$ ,  $x \in U$  such that  $\tau(y) = x$ . Let  $\bar{\Lambda} \subset \bar{U}$  be a 0-dimensional scheme supported at  $y$ . Then the restriction of  $\tau$  to  $\bar{\Lambda}$  induces an isomorphism of  $\bar{\Lambda}$  with its scheme-theoretical image in  $\Delta = \tau(\bar{\Lambda}) \subset U$ , i.e.*

$$\tau|_{\bar{\Lambda}}: \bar{\Lambda} \xrightarrow{\sim} \Delta \subset U.$$

*Proof.* We denote by  $A = \mathcal{O}_{\bar{U},y}$ ,  $B = \mathcal{O}_{U,x}$  the local rings at the points  $y, x$  and by  $\mathfrak{m}_A, \mathfrak{m}_B$  their maximal ideals. Let  $I \subset \mathfrak{m}_A$  denote the ideal defining the scheme  $\bar{\Lambda}$ . Since  $\dim \bar{\Lambda} = 0$  there exists an integer  $n$  such that  $\mathfrak{m}_A^n \subset I$ . The natural map  $B \hookrightarrow A \rightarrow A/I$  factorizes as follows

$$\beta: B \twoheadrightarrow B/\mathfrak{m}_B^n \xrightarrow{\alpha} A/\mathfrak{m}_A^n \twoheadrightarrow A/I.$$

Note that  $\alpha$  is an isomorphism, since  $\tau$  is étale (see e.g. [Mum], Corollary 1 of Theorem III.5.3). This shows that  $\beta$  is surjective, hence  $\tau|_{\bar{\Lambda}}$  is an isomorphism.  $\square$

**Proposition–Definition 5.3.** *For  $E \in \text{supp } \mathcal{Q}$  we define  $\mathcal{D}_E$  as the scheme-theoretical image  $\tau(\mathcal{D}_{\bar{E}}) \subset \mathcal{N}_{X_1}^s$  under the étale map  $\tau$ . Then the scheme  $\mathcal{D}_E$  does not depend on the étale neighbourhood  $\tau : \bar{U} \rightarrow U$  of  $E$  and the point  $\bar{E}$ .*

*Proof.* Consider for  $i = 1, 2$  étale neighbourhoods  $\tau_i : \bar{U}_i \rightarrow U$  such that universal bundles  $\mathcal{E}_i$  exist over  $X_1 \times \bar{U}_i$ , and points  $\bar{E}_i \in \bar{U}_i$  lying over  $E \in U$ . Because of Lemma 5.2 it will be enough to show that the schemes  $\mathcal{D}_{\bar{E}_1}$  and  $\mathcal{D}_{\bar{E}_2}$  are isomorphic.

Consider the fibre product  $\bar{U} = \bar{U}_1 \times_U \bar{U}_2$  and the point  $\bar{E} = (\bar{E}_1, \bar{E}_2) \in \bar{U}$ . The two projections  $\pi_i : \bar{U} \rightarrow \bar{U}_i$  for  $i = 1, 2$  are étale. Moreover  $(\text{id}_{X_1} \times \pi_i)^* \mathcal{E}_i \sim \mathcal{E}$ , where  $\mathcal{E}$  denotes the universal bundle over  $X_1 \times \bar{U}$ . Since the formation of the Fitting ideal and taking the higher direct image  $R^1 \pi_{\bar{U}*}$  commutes with the flat base changes  $\pi_1$  and  $\pi_2$  (see [E], Corollary 20.5), we obtain for  $i = 1, 2$

$$\pi_i^{-1} [\text{Fitt}_4(R^1 \pi_{\bar{U}_i*} (\mathcal{E}_i^* \otimes \pi_{X_1}^* F_*(\theta^{-1})))] \cdot \mathcal{O}_{\bar{U}} = \text{Fitt}_4(R^1 \pi_{\bar{U}*} (\mathcal{E}^* \otimes \pi_{X_1}^* F_*(\theta^{-1}))).$$

This shows that the connected components supported at  $\bar{E}$  of the fibres  $\pi_i^{-1}(\mathcal{D}_{\bar{E}_i})$  equal  $\mathcal{D}_{\bar{E}}$ . Applying Lemma 5.2 to  $\pi_i$  and  $\mathcal{D}_{\bar{E}_i}$  we obtain isomorphisms  $\pi_i : \mathcal{D}_{\bar{E}_i} \rightarrow \mathcal{D}_{\bar{E}}$  and we are done.  $\square$

**Lemma 5.4.** (a) *Let  $S$  be a  $k$ -scheme and  $\mathcal{E}$  a sheaf over  $X_1 \times S$  with  $\langle \mathcal{E} \rangle \in \mathcal{N}_{X_1}^s(S)$ . We suppose that the set-theoretical image of the classifying morphism of  $\mathcal{E}$*

$$\Phi_{\mathcal{E}} : S \longrightarrow \mathcal{N}_{X_1}^s, \quad s \longmapsto \mathcal{E}|_{X_1 \times \{s\}}$$

*is a point. Then there exists an Artinian ring  $A$ , a morphism  $\varphi : S \rightarrow \Delta := \text{Spec}(A)$  and a locally free sheaf  $\mathcal{E}_0$  over  $X_1 \times \Delta$  such that*

- (1)  $\mathcal{E} \sim (\text{id}_{X_1} \times \varphi)^* \mathcal{E}_0$
- (2) *the natural map  $\mathcal{O}_{\Delta} \rightarrow \varphi_* \mathcal{O}_S$  is injective.*
- (b) *There exists a universal family  $\mathcal{E}_0$  over  $X_1 \times \mathcal{D}$ .*

*Proof.* (a) Since the set-theoretical support of  $\text{im } \Phi_{\mathcal{E}}$  is a point  $x \in \mathcal{N}_{X_1}^s$ , there exists an Artinian ring  $A$  such that  $\Phi_{\mathcal{E}}$  factorizes through the inclusion  $\Delta := \text{Spec}(A) \hookrightarrow \mathcal{N}_{X_1}^s$ . As explained above there exists an étale neighbourhood  $\tau : \bar{U} \rightarrow U$  of  $x$  such that there is a universal bundle  $\mathcal{E}^{\text{univ}}$  over  $X_1 \times \bar{U}$ . Choose  $y \in \bar{U}$  such that  $\tau(y) = x$  and denote by  $\bar{\Lambda} \subset \bar{U}$  the connected component supported at  $y$  of the fibre  $\tau^{-1}(\Delta)$ . By Lemma 5.2 there is an isomorphism  $\tau : \bar{\Lambda} \xrightarrow{\sim} \Delta$ . Denote by  $\mathcal{E}_0$  the restriction of  $\mathcal{E}^{\text{univ}}$  to  $X_1 \times \bar{\Lambda} \cong X_1 \times \Delta$ . This shows property (1). As for (2), we consider the ideal  $I \subset A$  defined by  $\tilde{I} = \ker(\mathcal{O}_{\text{Spec}(A)} \rightarrow \varphi_* \mathcal{O}_S)$ , where  $\tilde{I}$  denotes the associated  $\mathcal{O}_{\text{Spec}(A)}$ -module. If  $I \neq 0$ , we replace  $A$  by  $A/I$  and we are done.

(b) We take  $\Delta = \mathcal{D}_E$  and  $\bar{\Lambda} = \mathcal{D}_{\bar{E}}$  and proceed as in (a).  $\square$

**Proposition 5.5.** *The subscheme  $\mathcal{D} \subset \mathcal{N}_{X_1}^s$  represents the functor  $\underline{\mathcal{D}}$  which associates to any  $k$ -scheme  $S$  the set*

$$\underline{\mathcal{D}}(S) = \{\mathcal{E} \text{ locally free sheaf over } X_1 \times S \text{ of rank } 2 \mid \deg \mathcal{E}|_{X_1 \times \{s\}} = 0 \\ \text{for all } s \in S, \text{Fitt}_4[R^1\pi_{S*}(\mathcal{E}^* \otimes \pi_{X_1}^*(F_*(\theta^{-1})))] = 0\} / \sim .$$

*Proof.* Consider a sheaf  $\mathcal{E}$  over  $X_1 \times S$  with  $\langle \mathcal{E} \rangle \in \underline{\mathcal{N}}_{X_1}^s(S)$ . Then  $\langle \mathcal{E} \rangle$  is an element of  $[\mathcal{D} \times_{\mathcal{N}_{X_1}^s} \underline{\mathcal{N}}_{X_1}^s](S)$  if and only if the classifying map  $\Phi_{\mathcal{E}} : S \rightarrow \mathcal{N}_{X_1}^s$  factorizes as  $S \xrightarrow{\varphi} \mathcal{D} \subset \mathcal{N}_{X_1}^s$ . By Lemma 5.4 (b) there exists a universal family  $\mathcal{E}_0$  over  $X_1 \times \mathcal{D}$  and we have  $\mathcal{E} \sim (\text{id}_{X_1} \times \varphi)^* \mathcal{E}_0$ . Since  $\mathcal{D}$  is defined (over an étale cover) by a Fitting ideal and since the formation of the Fitting ideal commutes with any base change, we deduce that  $[\mathcal{D} \times_{\mathcal{N}_{X_1}^s} \underline{\mathcal{N}}_{X_1}^s](S) = \underline{\mathcal{D}}(S)$ . Since  $\mathcal{N}_{X_1}^s$  universally corepresents the functor  $\underline{\mathcal{N}}_{X_1}^s$ , this shows that  $\mathcal{D}$  corepresents the functor  $\underline{\mathcal{D}}$ . The existence of a universal family  $\mathcal{E}_0$  over  $X \times \mathcal{D}$  implies that  $\mathcal{D}$  represents the functor  $\underline{\mathcal{D}}$ .  $\square$

**Proposition 5.6.** *There is a scheme-theoretical equality*

$$\tilde{\mathcal{D}} = \psi^{-1} \mathcal{D}.$$

*Proof.* In order to show that the subschemes  $\tilde{\mathcal{D}}$  and  $\psi^{-1} \mathcal{D}$  of  $JX_1 \times Z$  coincide, it is enough to show that the two subsets  $\text{Mor}(S, \tilde{\mathcal{D}})$  and  $\text{Mor}(S, \psi^{-1} \mathcal{D})$  of  $\text{Mor}(S, JX_1 \times Z)$  coincide for any  $k$ -scheme  $S$ . Consider  $\Phi \in \text{Mor}(S, JX_1 \times Z)$  and denote  $\mathcal{E}_{\Phi} := (\text{id}_{X_1} \times \Phi)^*(\mathcal{L} \otimes \mathcal{V})$ . By definition of  $\tilde{\mathcal{D}}$  we have  $\Phi \in \text{Mor}(S, \tilde{\mathcal{D}})$  if and only if  $\text{Fitt}_4[R^1\pi_{S*}(\mathcal{E}_{\Phi}^* \otimes \pi_{X_1}^*(F_*(\theta^{-1})))] = 0$ . On the other hand  $\Phi \in \text{Mor}(S, \psi^{-1}(\mathcal{D}))$  if and only if  $\psi \circ \Phi \in \text{Mor}(S, \mathcal{D})$ . The latter set equals  $\underline{\mathcal{D}}(S)$  by Proposition 5.5. Since  $(\psi \circ \Phi)^* \mathcal{E}_0 \sim \mathcal{E}_{\Phi}$ , we are done.  $\square$

**Proposition 5.7.** *There is a scheme-theoretical equality*

$$\mathcal{D} = \mathcal{Q}.$$

*Proof.* We note that  $\underline{\mathcal{D}}(S)$  and  $\underline{\mathcal{Q}}(S)$  are subsets of  $\underline{\mathcal{N}}_{X_1}^s(S)$  (the injectivity of the map  $\underline{\mathcal{Q}}(S) \rightarrow \underline{\mathcal{N}}_{X_1}^s(S)$  is proved similarly as in the proof of Proposition 4.7). Since  $\mathcal{D}$  and  $\mathcal{Q}$  corepresent the two functors  $\underline{\mathcal{D}}$  and  $\underline{\mathcal{Q}}$ , it will be enough to show that the set  $\underline{\mathcal{D}}(S)$  coincides with  $\underline{\mathcal{Q}}(S)$  for any  $k$ -scheme  $S$ .

We first show that  $\underline{\mathcal{D}}(S) \subset \underline{\mathcal{Q}}(S)$ . Consider a sheaf  $\mathcal{E}$  with  $\langle \mathcal{E} \rangle \in \underline{\mathcal{D}}(S)$ . For simplicity we denote the sheaf  $\mathcal{E}^* \otimes \pi_{X_1}^*(F_*(\theta^{-1}))$  by  $\mathcal{H}$ . By [Ha], Theorem 12.11, there is an isomorphism

$$R^1\pi_{S*} \mathcal{H} \otimes k(s) \cong H^1(X_1 \times \{s\}, \mathcal{H}|_{X_1 \times \{s\}}) \quad \text{for all } s \in S.$$

From our assumption  $\text{Fitt}_4[R^1\pi_{S*} \mathcal{H}] = 0$ , we obtain  $\dim H^1(X_1 \times \{s\}, \mathcal{H}|_{X_1 \times \{s\}}) \geq 5$ , or equivalently  $\dim H^0(X_1 \times \{s\}, \mathcal{H}|_{X_1 \times \{s\}}) \geq 1$ , i.e., the vector bundle  $\mathcal{E}|_{X_1 \times \{s\}}$

is a subsheaf, hence by Proposition 3.1 (a) (ii) a subbundle, of  $F_*(\theta^{-1})$ . This implies that the set-theoretical image of the classifying map  $\Phi_{\mathcal{E}}$  is contained in  $\text{supp}\mathcal{Q}$ . Taking connected components of  $S$ , we can assume that the image of  $\Phi_{\mathcal{E}}$  is a point. Therefore we can apply Lemma 5.4: there exists a locally free sheaf  $\mathcal{E}_0$  over  $X_1 \times \Delta$  such that  $\mathcal{E} \sim (\text{id}_{X_1} \times \varphi)^* \mathcal{E}_0$ . For simplicity we write  $\mathcal{H}_0 = \mathcal{E}_0^* \otimes \pi_{X_1}^*(F_*(\theta^{-1}))$ . In particular  $\mathcal{H} = (\text{id}_{X_1} \times \varphi)^* \mathcal{H}_0$ . Since the projection  $\pi_{\Delta}: X_1 \times \Delta \rightarrow \Delta$  is of relative dimension 1, taking the higher direct image  $R^1\pi_{\Delta*}$  commutes with the (not necessarily flat) base change  $\varphi: S \rightarrow \Delta$  ([Ha], Proposition 12.5), i.e., there is an isomorphism

$$\varphi^* R^1\pi_{\Delta*} \mathcal{H}_0 \cong R^1\pi_{S*} \mathcal{H}.$$

Since the formation of Fitting ideals also commutes with any base change (see [E], Corollary 20.5), we obtain

$$\text{Fitt}_4[R^1\pi_{S*} \mathcal{H}] = \text{Fitt}_4[R^1\pi_{\Delta*} \mathcal{H}_0] \cdot \mathcal{O}_S.$$

Since  $\text{Fitt}_4[R^1\pi_{S*} \mathcal{H}]$  is equal to 0 and  $\mathcal{O}_{\Delta} \rightarrow \varphi_* \mathcal{O}_S$  is injective, we deduce that  $\text{Fitt}_4[R^1\pi_{\Delta*} \mathcal{H}_0] = 0$ . Since by Proposition 3.1 (b) (iii)  $\dim R^1\pi_{\Delta*} \mathcal{H}_0 \otimes k(s_0) = 5$  for the closed point  $s_0 \in \Delta$ , we have  $\text{Fitt}_5[R^1\pi_{\Delta*} \mathcal{H}_0] = \mathcal{O}_{\Delta}$ . We deduce by [E], Proposition 20.8, that the sheaf  $R^1\pi_{\Delta*} \mathcal{H}_0$  is a free  $A$ -module of rank 5. By [Ha], Theorem 12.11 (b), we deduce that there is an isomorphism

$$\pi_{\Delta*} \mathcal{H}_0 \otimes k(s_0) \cong H^0(X_1 \times s_0, \mathcal{H}|_{X_1 \times \{s_0\}})$$

Again by Proposition 3.1 (b) (iii) we obtain  $\dim \pi_{\Delta*} \mathcal{H}_0 \otimes k(s_0) = 1$ . In particular the  $\mathcal{O}_{\Delta}$ -module  $\pi_{\Delta*} \mathcal{H}_0$  is not zero and therefore there exists a nonzero global section  $i \in H^0(\Delta, \pi_{\Delta*} \mathcal{H}_0) = H^0(X_1 \times \Delta, \mathcal{E}_0^* \otimes \pi_{X_1}^* F_*(\theta^{-1}))$ . We pull-back  $i$  under the map  $\text{id}_{X_1} \times \varphi$  and we obtain a nonzero section

$$j = (\text{id}_{X_1} \times \varphi)^* i \in H^0(X_1 \times S, \mathcal{E}^* \otimes \pi_{X_1}^* F_*(\theta^{-1})).$$

Now we apply Lemma 4.3 and we continue as in the proof of Proposition 4.7. This shows that  $\langle \mathcal{E} \rangle \in \underline{\mathcal{Q}}(S)$ .

We now show that  $\underline{\mathcal{Q}}(S) \subset \underline{\mathcal{D}}(S)$ . Consider a sheaf  $\mathcal{E} \in \underline{\mathcal{Q}}(S)$ . The nonzero global section  $j \in H^0(X_1 \times S, \mathcal{H}) = H^0(S, \pi_{S*} \mathcal{H})$  determines by evaluation at a point  $s \in S$  an element  $\alpha \in \pi_{S*} \mathcal{H} \otimes k(s)$ . The image of  $\alpha$  under the natural map

$$\varphi^0(s): \pi_{S*} \mathcal{H} \otimes k(s) \longrightarrow H^0(X_1 \times \{s\}, \mathcal{H}|_{X_1 \times \{s\}})$$

coincides with  $j|_{X_1 \times \{s\}}$  which is nonzero. Also, as  $\dim H^0(X_1 \times \{s\}, \mathcal{H}|_{X_1 \times \{s\}}) = 1$ , we obtain that  $\varphi^0(s)$  is surjective. Hence by [Ha], Theorem 12.11, the sheaf  $R^1\pi_{S*} \mathcal{H}$  is locally free of rank 5. Again by [E], Proposition 20.8, this is equivalent to  $\text{Fitt}_4[R^1\pi_{S*} \mathcal{H}] = 0$  and  $\text{Fitt}_5[R^1\pi_{S*} \mathcal{H}] = \mathcal{O}_S$  and we are done.  $\square$

## 6. Chern class computations

In this section we will compute the length of the determinantal subscheme  $\mathcal{D} \subset \mathcal{N}_{X_1}$  by evaluating the Chern class  $c_5(\mathcal{F}_0 - \mathcal{F}_1)$  – see Proposition 5.1 (b).

Let  $l$  be a prime number different from  $p$ . We have to recall some properties of the cohomology ring  $H^*(X_1 \times JX_1 \times Z, \mathbb{Z}_l)$  (see also [LN]). In the sequel we identify all classes of  $H^*(X_1, \mathbb{Z}_l)$ ,  $H^*(JX_1, \mathbb{Z}_l)$  etc. with their preimages in  $H^*(X_1 \times JX_1 \times Z, \mathbb{Z}_l)$  under the natural pull-back maps.

Let  $\Theta \in H^2(JX_1, \mathbb{Z}_l)$  denote the class of the theta divisor in  $JX_1$ . Let  $f$  denote a positive generator of  $H^2(X_1, \mathbb{Z}_l)$ . The cup product  $H^1(X_1, \mathbb{Z}_l) \times H^1(X_1, \mathbb{Z}_l) \rightarrow H^2(X_1, \mathbb{Z}_l) \simeq \mathbb{Z}_l$  gives a symplectic structure on  $H^1(X_1, \mathbb{Z}_l)$ . Choose a symplectic basis  $e_1, e_2, e_3, e_4$  of  $H^1(X_1, \mathbb{Z}_l)$  such that  $e_1 e_3 = e_2 e_4 = -f$  and all other products  $e_i e_j = 0$ . We can then normalize the Poincaré bundle  $\mathcal{L}$  on  $X_1 \times JX_1$  so that

$$c(\mathcal{L}) = 1 + \xi_1 \quad (9)$$

where  $\xi_1 \in H^1(X_1, \mathbb{Z}_l) \otimes H^1(JX_1, \mathbb{Z}_l) \subset H^2(X_1 \times JX_1, \mathbb{Z}_l)$  can be written as

$$\xi_1 = \sum_{i=1}^4 e_i \otimes \varphi_i$$

with  $\varphi_i \in H^1(JX_1, \mathbb{Z}_l)$ . Moreover, we have by the same reasoning, applying [ACGH], p. 335 and p. 21,

$$\xi_1^2 = -2\Theta f \quad \text{and} \quad \Theta^2[JX_1] = 2. \quad (10)$$

Since the variety  $\mathcal{M}_{X_1}(x)$  is a smooth intersection of 2 quadrics in  $\mathbb{P}^5$ , one can work out that the  $l$ -adic cohomology groups  $H^i(\mathcal{M}_{X_1}(x), \mathbb{Z}_l)$  for  $i = 0, \dots, 6$  are (see e.g. [Re], p. 0.19)

$$\mathbb{Z}_l, \quad 0, \quad \mathbb{Z}_l, \quad \mathbb{Z}_l^4, \quad \mathbb{Z}_l, \quad 0, \quad \mathbb{Z}_l.$$

In particular  $H^2(\mathcal{M}_{X_1}(x), \mathbb{Z}_l)$  is free of rank 1 and, if  $\alpha$  denotes a positive generator of it, then

$$\alpha^3[\mathcal{M}_{X_1}(x)] = 4. \quad (11)$$

According to [N2] p. 338 and applying reduction mod  $p$  and a comparison theorem, the Chern classes of the universal bundle  $\mathcal{U}$  are of the form

$$c_1(\mathcal{U}) = \alpha + f \quad \text{and} \quad c_2(\mathcal{U}) = \chi + \xi_2 + \alpha f \quad (12)$$

with  $\chi \in H^4(\mathcal{M}_{X_1}(x), \mathbb{Z}_l)$  and  $\xi_2 \in H^1(X_1, \mathbb{Z}_l) \otimes H^3(\mathcal{M}_{X_1}(x), \mathbb{Z}_l)$ . As in [N2] and [KN] we write

$$\beta = \alpha^2 - 4\chi \quad \text{and} \quad \xi_2^2 = \gamma f \quad \text{with} \quad \gamma \in H^6(\mathcal{M}_{X_1}(x), \mathbb{Z}_l). \quad (13)$$

Then the relations of [KN] give

$$\alpha^2 + \beta = 0 \quad \text{and} \quad \alpha^3 + 5\alpha\beta + 4\gamma = 0.$$

Hence  $\beta = -\alpha^2$ ,  $\gamma = \alpha^3$ . Together with (12) and (13) this gives

$$c_2(\mathcal{U}) = \frac{\alpha^2}{2} + \xi_2 + \alpha f \quad \text{and} \quad \xi_2^2 = \alpha^3 f \quad (14)$$

Define  $\Lambda \in H^1(JX_1, \mathbb{Z}_l) \otimes H^3(\mathcal{M}_{X_1}(x), \mathbb{Z}_l)$  by

$$\xi_1 \xi_2 = \Lambda f. \quad (15)$$

Then we have for dimensional reasons and noting that  $H^5(\mathcal{M}_{X_1}(x), \mathbb{Z}_l) = 0$ , that the following classes are all zero:

$$f^2, \quad \xi_1^3, \quad \alpha^4, \quad \xi_1 f, \quad \xi_2 f, \quad \alpha \xi_2, \quad \alpha \Lambda, \quad \Theta^2 \Lambda, \quad \Theta^3. \quad (16)$$

Finally,  $Z$  is the  $\mathbb{P}^1$ -bundle associated to the vector bundle  $\mathcal{U}_x$  on  $\mathcal{M}_{X_1}(x)$ . Let  $H \in H^2(Z, \mathbb{Z}_l)$  denote the first Chern class of the tautological line bundle on  $Z$ . We have, using the definition of the Chern classes  $c_i(\mathcal{U})$  and (11),

$$H^2 = \alpha H - \frac{\alpha^2}{2}, \quad H^4 = 0, \quad \alpha^3 H[Z] = 4 \quad (17)$$

and we get for the “universal” bundle  $\mathcal{V}$ ,

$$c_1(\mathcal{V}) = \alpha \quad \text{and} \quad c_2(\mathcal{V}) = \frac{\alpha^2}{2} + \xi_2 + Hf. \quad (18)$$

**Lemma 6.1.** (a) *The cohomology class  $\alpha \cdot c_5(\mathcal{F}_0 - \mathcal{F}_1) \in H^{12}(JX_1 \times Z, \mathbb{Z}_l)$  is a multiple of the class  $\alpha^3 H \Theta^2$ .*

(b) *The pull-back under the map  $\varphi: Z \rightarrow \mathcal{M}_{X_1} \cong \mathbb{P}^3$  of the class of a point is the class  $H^3 = \frac{\alpha^2}{2} H - \frac{\alpha^3}{2}$ .*

*Proof.* For part (a) it is enough to note that all other relevant cohomology classes vanish, since  $\alpha^4 = 0$  and  $\alpha \Lambda = 0$ .

As for part (b), it suffices to show that  $c_1(\varphi^* \mathcal{O}_{\mathbb{P}^3}(1)) = H$ . The line bundle  $\mathcal{O}_{\mathbb{P}^3}(1)$  is the inverse of the determinant line bundle [KM] over the moduli space  $\mathcal{M}_{X_1}$ . Since the formation of the determinant line bundle commutes with any base change (see [KM]), the pull-back  $\varphi^* \mathcal{O}_{\mathbb{P}^3}(1)$  is the inverse of the determinant line bundle associated to the family  $\mathcal{V} \otimes \pi_{X_1}^* N$  for any line bundle  $N$  of degree 1 over  $X_1$ . Hence the first Chern class of  $\varphi^* \mathcal{O}_{\mathbb{P}^3}(1)$  can be computed by the Grothendieck–Riemann–Roch theorem applied to the sheaf  $\mathcal{V} \otimes \pi_{X_1}^* N$  over  $X_1 \times Z$  and the morphism

$\pi_Z: X_1 \times Z \rightarrow Z$ . We have

$$\begin{aligned} ch(\mathcal{V} \otimes \pi_{X_1}^* N) \cdot \pi_{X_1}^* td(X_1) &= (2 + \alpha + (-\xi_2 - Hf) + \text{h.o.t.})(1 + f)(1 - f) \\ &= 2 + \alpha + (-\xi_2 - Hf) + \text{h.o.t.}, \end{aligned}$$

and therefore G-R-R implies that  $c_1(\varphi^* \mathcal{O}_{\mathbb{P}^3}(1)) = H$  – note that  $\pi_{Z*}(\xi_2) = 0$ .  $\square$

**Proposition 6.2.** *We have*

$$l(\mathcal{D}) = \frac{1}{24} p^3 (p^2 - 1).$$

*Proof.* Let  $\lambda$  denote the length of the subscheme  $m^{-1}(\mathcal{D}) \subset JX_1 \times \mathcal{M}_{X_1}$ . Since the map  $m^s$  is étale of degree 16, we obviously have the relation  $\lambda = 16 \cdot l(\mathcal{D})$ . According to Lemma 6.1 (b) we have in  $H^{10}(JX_1 \times Z, \mathbb{Z}_l)$

$$[(\text{id} \times \varphi)^{-1}(pt)] = H^3 \cdot \frac{\Theta^2}{2} = \frac{1}{4} \alpha^2 H \Theta^2 - \frac{1}{4} \alpha^3 \Theta^2,$$

where  $pt$  denotes the class of a point in  $JX_1 \times \mathcal{M}_{X_1}$ . Using Proposition 5.6 we obtain that the class  $\delta = c_5(\mathcal{F}_0 - \mathcal{F}_1) \in H^{10}(JX_1 \times Z, \mathbb{Z}_l)$  equals  $\lambda \cdot (\frac{1}{4} \alpha^2 H \Theta^2 - \frac{1}{4} \alpha^3 \Theta^2)$ . Intersecting with  $\alpha$  we obtain with Lemma 6.1 (a) and (16)

$$\alpha \cdot c_5(\mathcal{F}_0 - \mathcal{F}_1) = \frac{\lambda}{4} \alpha^3 H \Theta^2. \tag{19}$$

So we have to compute the class  $\alpha \cdot c_5(\mathcal{F}_0 - \mathcal{F}_1)$ . By (9) and (10),

$$ch(\mathcal{L}) = 1 + \xi_1 - \Theta f$$

whereas by (14), (16) and (18),

$$ch(\mathcal{V}) = 2 + \alpha + (-\xi_2 - Hf) + \frac{1}{12}(-\alpha^3 - 6\alpha Hf) + \frac{1}{12}(\alpha^3 f - \alpha^2 Hf).$$

Moreover

$$ch(\pi_{X_1}^*(F_*(\theta^{-1}) \otimes \omega_{X_1})) \cdot \pi_{X_1}^* td(X_1) = p + (2p - 2)f.$$

So using (14), (15) and (16),

$$\begin{aligned} ch(\mathcal{V}^* \otimes \mathcal{L}^* \otimes \pi_{X_1}^*(F_*(\theta^{-1}) \otimes \omega_{X_1})) \cdot \pi_{X_1}^* td(X_1) \\ = 2p + [(4p - 4)f - p\alpha - 2p\xi_1] \\ + \left[ p\alpha\xi_1 - 2p\Theta f - (2p - 2)\alpha f - p\xi_2 - pHf \right] \\ + \left[ \frac{p}{12}\alpha^3 + \frac{p}{2}\alpha Hf + p\Lambda f + p\alpha\Theta f \right] \\ + \left[ \frac{3p - 2}{12}\alpha^3 f - \frac{p}{12}\alpha^3 \xi_1 - \frac{p}{12}\alpha^2 Hf \right] + \left[ -\frac{p}{12}\alpha^3 \Theta f \right]. \end{aligned}$$

Hence by Grothendieck–Riemann–Roch for the morphism  $q$  we get

$$\begin{aligned} ch(\mathcal{F}_1) = & 4p - 4 + [-(2p - 2)\alpha - 2p\Theta - pH] + \left[ \frac{p}{2}\alpha H + p\Lambda + p\alpha\Theta \right] \\ & + \left[ \frac{3p - 2}{12}\alpha^3 - \frac{p}{12}\alpha^2 H \right] + \left[ -\frac{p}{12}\alpha^3\Theta \right]. \end{aligned}$$

From (10) and (18) we easily obtain

$$ch(\mathcal{F}_0) = 4p - 2p\alpha + \frac{p}{6}\alpha^3.$$

So

$$\begin{aligned} ch(\mathcal{F}_0 - \mathcal{F}_1) = & 4 + [2p\Theta - 2\alpha + pH] + \left[ -\frac{p}{2}\alpha H - p\Lambda - p\alpha\Theta \right] \\ & + \left[ -\frac{p + 1}{12}\alpha^3 + \frac{p}{12}\alpha^2 H \right] + \left[ \frac{p}{12}\alpha^3\Theta \right]. \end{aligned}$$

Defining  $p_n := n! \cdot ch_n(\mathcal{F}_0 - \mathcal{F}_1)$  we have according to Newton's recursive formula ([F] p. 56),

$$c_5(\mathcal{F}_0 - \mathcal{F}_1) = \frac{1}{5} \left( p_5 - \frac{5}{6}p_2p_3 - \frac{5}{4}p_1p_4 + \frac{5}{6}p_1^2p_3 + \frac{5}{8}p_1p_2^2 - \frac{5}{12}p_1^3p_2 + \frac{1}{24}p_1^5 \right)$$

with

$$\begin{aligned} p_1 = & 2p\Theta - 2\alpha + pH, \quad p_2 = -p(\alpha H + 2\Lambda + 2\alpha\Theta), \\ p_3 = & \frac{1}{2}(-(p + 1)\alpha^3 + p\alpha^2 H), \quad p_4 = 2p\alpha^3\Theta, \quad p_5 = 0. \end{aligned}$$

Now an immediate computation using (16) and (17) gives

$$\alpha \cdot c_5(\mathcal{F}_0 - \mathcal{F}_1) = \frac{p^3(p^2 - 1)}{6}\alpha^3 H\Theta^2.$$

We conclude from (19) that  $\lambda = \frac{2}{3}p^3(p^2 - 1)$  and we are done.  $\square$

**Remark 6.3.** If  $k = \mathbb{C}$ , the number of maximal subbundles of a general vector bundle has recently been computed by Y. Holla by using Gromov–Witten invariants [Ho]. His formula ([Ho], Corollary 4.6) coincides with ours.

## 7. Proof of Theorem 2

The proof of Theorem 2 is now straightforward. It suffices to combine Corollary 4.9, Proposition 5.7 and Proposition 6.2 to obtain the length  $l(\mathcal{B})$ .

The fact that  $\mathcal{B}$  is a local complete intersection follows from the isomorphism  $\mathcal{B}_\theta = \mathcal{Q}_0$  (Proposition 4.6) and Proposition 4.1.  $\square$

## 8. Questions and remarks

- (1) Is the rank- $p$  vector bundle  $F_*L$  very stable, i.e.  $F_*L$  has no nilpotent  $\omega_{X_1}$ -valued endomorphisms, for a general line bundle?
- (2) Is  $F_*(\theta^{-1})$  very stable for a general curve  $X$ ? Note that very-stability of  $F_*(\theta^{-1})$  implies reducedness of  $\mathcal{B}$  (see e.g. [LN], Lemma 3.3).
- (3) If  $g = 2$ , we have shown that for a general stable  $E \in \mathcal{M}_X$  the fibre  $V^{-1}(E)$  consists of  $\frac{1}{3}p(p^2 + 2)$  stable vector bundles  $E_1 \in \mathcal{M}_{X_1}$ , i.e. bundles  $E_1$  such that  $F^*E_1 \cong E$  or equivalently (via adjunction)  $E_1 \subset F_*E$ . The Quot-scheme parametrizing rank-2 subbundles of degree 0 of the rank- $2p$  vector bundle  $F_*E$  has expected dimension 0, contains the fibre  $V^{-1}(E)$ , but it also has a 1-dimensional component arising from Frobenius-destabilized bundles as follows: for any  $M \in \text{Pic}^1(X)$  with  $\text{Hom}(M^{-1}, E) \neq 0$  consider a stable degree 0 rank-2 bundle  $E_1$  such that  $F^*E_1$  has a nonzero map to  $M^{-1}$ .
- (4) If  $p = 3$  the base locus  $\mathcal{B}$  consists of 16 reduced points, which correspond to the 16 nodes of the Kummer surface associated to  $JX$  (see [LP2], Corollary 6.6). For general  $p$ , does the configuration of points determined by  $\mathcal{B}$  have some geometric significance?

## Appendix on base loci and substack of non-semistable vector bundles.

For lack of a suitable reference, we include a detailed proof of the following fact, which was used in Lemma 4.5. We use the notation of Lemma 4.5.

**Proposition A.** *Let  $X$  be a smooth curve of genus 2. The closed substack  $\mathfrak{M}_X^1$  equals the base locus  $\text{Bs}|\mathcal{O}(1)|$  of the linear system  $|\mathcal{O}(1)|$  over the moduli stack  $\mathfrak{M}_X^{\leq 1}$ .*

*Proof.* Let  $E$  be a rank-2 vector bundle with trivial determinant over  $X$ . It follows from [R], Proposition 1.6.2, that  $E$  is semistable if and only if there exists a line bundle  $M$  of degree 1 such that  $h^0(X, E \otimes M) = h^1(X, E \otimes M) = 0$ . Consider the determinant divisor  $\theta_M$  associated to  $M$ . Then  $\theta_M \in |\mathcal{O}(1)|$  and for an  $S$ -valued point  $\mathcal{E}$  of  $\mathfrak{M}_X^{\leq 1}$

$$\text{supp}(\theta_M) = \{s \in S \mid h^0(X, \mathcal{E}_s \otimes M) > 0\}.$$

We know (see e.g. [B1], Proposition 2.5) that the linear system  $|\mathcal{O}(1)|$  is linearly generated by the divisors  $\theta_M$  when  $M$  varies in  $\text{Pic}^1(X)$ . The previous equivalence implies that the open complements of the closed substacks  $\text{Bs}|\mathcal{O}(1)|$  and  $\mathfrak{M}_X^1$  coincide. To conclude the proposition it remains to show that the base locus  $\text{Bs}|\mathcal{O}(1)|$  is a reduced substack of  $\mathfrak{M}_X^{\leq 1}$ .

The normal bundle  $N$  of the closed substack  $\mathfrak{M}_X^1$  in  $\mathfrak{M}_X^{\leq 1}$  can be described as follows (e.g. [He], Behauptung 2.1.12, p. 44 or [VL], exposé 4, Théorème 4, p. 90): let  $\mathcal{E}$  denote the universal bundle over  $X \times \mathfrak{M}_X$  restricted to  $X \times \mathfrak{M}_X^1$ . There is a canonical inclusion

$$\text{End}_0(\mathcal{E})^{\text{filt}} \subset \text{End}_0(\mathcal{E}),$$

where  $\text{End}_0(\mathcal{E})^{\text{filt}}$  denotes the sheaf of tracefree endomorphisms preserving the Harder–Narasimhan filtration. We denote by  $\text{End}'_0(\mathcal{E})$  the quotient. Then the normal bundle  $N$  equals  $R^1 p_* \text{End}'_0(\mathcal{E})$ , where  $p$  denotes projection onto  $\mathfrak{M}_X^1$ . In the rank-2 case the universal Harder–Narasimhan filtration over  $X \times \mathfrak{M}_X^1$  is of the form

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}^{-1} \longrightarrow 0,$$

where  $\mathcal{L}$  is a degree 1 line bundle. In that case we have  $\text{End}'_0(\mathcal{E}) = \text{Hom}(\mathcal{L}, \mathcal{L}^{-1})$  and therefore  $N = R^1 p_* \mathcal{L}^{-2}$ .

Consider an  $S$ -point  $\mathcal{E} \in \mathfrak{M}_X^{\leq 1}(S)$  and  $x \in S$  such that the vector bundle  $\mathcal{E}_x = E \in \mathfrak{M}_X^1(k)$ , i.e.,  $E$  is destabilized by  $L$  of degree 1. Consider a line bundle  $M$  of degree 1 and its associated determinant divisor  $\theta_M$ . Then the divisor  $\theta_M$  contains the closed substack  $\mathfrak{M}_X^1$ . The Kodaira–Spencer map at the point  $x \in S$  associated to  $\mathcal{E}$  is a  $k$ -linear map

$$\kappa : T_x S \longrightarrow H^1(X, \text{End}_0(E)).$$

Note that we consider bundles with trivial determinant, hence  $\kappa$  takes values in  $H^1(X, \text{End}_0(E))$ . By [Las], Sections II and III, the linear form on  $T_x S$  defining the tangent space  $T_x \theta_M$  to the determinant divisor  $\theta_M$  is the map  $\Phi \circ \kappa$ , where  $\Phi$  is given by cup product

$$\Phi : H^1(X, \text{End}_0(E)) \longrightarrow \text{Hom}(H^0(X, E \otimes M), H^1(X, E \otimes M)), \quad e \mapsto \cup e.$$

Using Serre duality we identify  $H^1(X, \text{End}_0(E))^*$  with  $H^0(X, \text{End}_0(E) \otimes \omega)$  and  $H^1(X, E \otimes M)$  with  $H^0(X, E \otimes \omega M^{-1})^*$ . The dual of  $\Phi$  equals the symmetrized multiplication map of global sections (note that  $\text{End}_0(E) = \text{Sym}^2 E$  and  $E = E^*$ )

$$\mu : H^0(X, E \otimes M) \otimes H^0(X, E \otimes \omega M^{-1}) \longrightarrow H^0(X, \text{End}_0(E) \otimes \omega).$$

Note that both spaces on the left have dimension equal to 1 for general  $M$  and that  $H^0(X, E \otimes M) = H^0(X, L \otimes M)$  and  $H^0(X, E \otimes \omega M^{-1}) = H^0(X, L \otimes \omega M^{-1})$  for general  $M$ . This implies that  $\dim \text{im}(\mu) = 1$  and

$$\text{im}(\mu) \subset H^0(X, L^2 \omega) \subset H^0(X, \text{End}_0(E) \otimes \omega).$$

We denote by  $h$  a generator of  $\text{im}(\mu)$ . We obtain that for general  $M$  the conormal vector defined by  $T_x \theta_M$  is given (up to a scalar) by

$$h \in H^0(X, L^2 \omega) = H^1(X, L^{-2})^* = N_x^*.$$

The corresponding rational map

$$\mathrm{Pic}^1(X) \longrightarrow \mathbb{P}H^0(X, L^2\omega) = \mathbb{P}^2, \quad M \mapsto h,$$

is easily seen to be dominant. In particular its image is non degenerate. This shows that the point  $E$  is a reduced point of  $\mathrm{Bs}|\mathcal{O}(1)|$ , because the linear span of the family of conormal vectors defined by  $T_x\theta_M$  when  $M$  varies in an open set of  $\mathrm{Pic}^1(X)$  equals the full space  $N_x^*$ .  $\square$

## References

- [ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris, *Geometry of Algebraic Curves*. Grundlehren Math. Wiss. 267, Springer-Verlag, New York 1985. [Zbl 0559.14017](#) [MR 0770932](#)
- [B1] A. Beauville, Fibrés de rang 2 sur les courbes, fibré déterminant et fonctions thêta. *Bull. Soc. Math. France* **116** (1988), 431–448. [Zbl 0691.14016](#) [MR 1005388](#)
- [B2] A. Beauville, On the stability of the direct image of a generic vector bundle. Preprint. <http://math.unice.fr/~beauvill/pubs/imdir.pdf>
- [BL] A. Beauville, Y. Laszlo, Conformal blocks and generalized theta functions. *Comm. Math. Phys.* **164** (1994), 385–419. [Zbl 0815.14015](#) [MR 1289330](#)
- [E] D. Eisenbud, *Commutative Algebra*. Grad. Texts in Math. 150, Springer-Verlag, Berlin 1994. [Zbl 0819.13001](#) [MR 1322960](#)
- [F] W. Fulton, *Intersection Theory*. Ergeb. Math. Grenzgeb. (3) 2, Springer-Verlag, Berlin 1984. [Zbl 0541.14005](#) [MR R0732620](#)
- [G] A. Grothendieck, Fondements de la Géométrie Algébrique, IV, Les schémas de Hilbert. Séminaire Bourbaki, t. 13, 1960/61, n. 221. [Zbl 0239.14002](#) [MR 146040](#)
- [Ha] R. Hartshorne, *Algebraic Geometry*. Grad. Texts in Math. 52, Springer-Verlag, New York, Heidelberg, Berlin 1977. [Zbl 0367.14001](#) [MR 0463157](#)
- [Hir] A. Hirschowitz, Problèmes de Brill-Noether en rang supérieur. *C. R. Acad. Sci.* **307** (1988), 153–156. [Zbl 0654.14017](#) [MR 0956606](#)
- [He] J. Heinloth, Über den Modulstack der Vektorbündel auf Kurven, Diploma Thesis. <http://www.uni-due.de/~hm0002/>
- [Ho] Y. I. Holla, Counting maximal subbundles via Gromov-Witten invariants. *Math. Ann.* **328** (2004), 121–133. [Zbl 1065.14042](#) [MR 2030371](#)
- [HL] D. Huybrechts, M. Lehn, *The geometry of moduli spaces of sheaves*. Aspects Math E31, Vieweg & Sohn, Braunschweig 1997. [Zbl 0872.14002](#) [MR 1450870](#)
- [JRXY] K. Joshi, S. Ramanan, E.Z. Xia, J.-K. Yu, On vector bundles destabilized by Frobenius pull-back. *Compositio Math.* **142** (2006), 616–630. [Zbl 1101.14049](#) [MR 2231194](#)
- [K] N. Katz, Nilpotent connections and the monodromy theorem: Applications of a result of Turittin. *Inst. Hautes Études Sci. Publ. Math.* **39** (1970), 175–232. [Zbl 0221.14007](#) [MR 0291177](#)

- [KN] A. King, P. E. Newstead, On the cohomology ring of the moduli space of rank 2 vector bundles on a curve. *Topology* **37** (1998), 407–418. [Zbl 0913.14008](#) [MR 1489212](#)
- [KM] F. Knudsen, D. Mumford, The projectivity of the moduli space of stable curves I. *Math. Scand.* **39** (1976), 19–55. [Zbl 0343.14008](#) [MR 0437541](#)
- [L] H. Lange, Some geometrical aspects of vector bundles on curves. In *Topics in algebraic geometry*, Aportaciones Mat. Notas Investigación 5, Soc. Mat. Mexicana, México 1992, 53–74. [Zbl 0899.14012](#) [MR 1308330](#)
- [LS] H. Lange, U. Stuhler, Vektorbündel auf Kurven und Darstellungen der algebraischen Fundamentalgruppe. *Math. Z.* **156** (1977), 73–83. [Zbl 0349.14018](#) [MR 0472827](#)
- [LN] H. Lange, P. Newstead, Maximal subbundles and Gromov-Witten invariants. In *A Tribute to C. S. Seshadri*, Trends Math., Birkhäuser, Basel 2003, 310–322. [Zbl 1071.14036](#) [MR 2017590](#)
- [La1] A. Langer, Moduli spaces of sheaves in mixed characteristic. *Duke Math. J.* **124** (2004), 571–586. [Zbl 02113314](#) [MR 2085175](#)
- [La2] A. Langer, Moduli spaces and Castelnuovo-Mumford regularity of sheaves on surfaces. *Amer. J. Math.* **128** (2006), 373–417. [Zbl 1102.14030](#) [MR 2214897](#)
- [Las] Y. Laszlo, Un théorème de Riemann pour les diviseurs thêta sur les espaces de modules de fibrés stables sur une courbe. *Duke Math. J.* **64** (1991), 333–347. [Zbl 0753.14023](#) [MR 1136379](#)
- [LP1] Y. Laszlo, C. Pauly, The action of the Frobenius map on rank 2 vector bundles in characteristic 2. *J. Algebraic Geom.* **11** (2002), 219–243. [Zbl 1080.14527](#) [MR 1874113](#)
- [LP2] Y. Laszlo, C. Pauly, The Frobenius map, rank 2 vector bundles and Kummer’s quartic surface in characteristic 2 and 3. *Adv. Math.* **185** (2004), 246–269. [Zbl 1055.14038](#) [MR 2060469](#)
- [MS] V. B. Mehta, S. Subramanian, Nef line bundles which are not ample. *Math. Z.* **219** (1995), 235–244. [Zbl 0826.14009](#) [MR 1337219](#)
- [Mo] S. Mochizuki, *Foundations of  $p$ -adic Teichmüller Theory*. AMS/IP Stud. Adv. Math. 11, Amer. Math. Soc., Providence, RI, 1999 [Zbl 0969.14013](#) [MR 1700772](#)
- [Mu] S. Mukai, Non-Abelian Brill-Noether theory and Fano 3-folds. *Sugaku* **49** (1997), 1–24 (in Japanese; English transl. *Sugaku Expositions* **14** (2001), 125–153. [Zbl 0929.14021](#) [MR 1478148](#)
- [MuSa] S. Mukai, F. Sakai, Maximal subbundles of vector bundles on a curve. *Manuscripta Math.* **52** (1985), 251–256. [Zbl 0572.14008](#) [MR 0790801](#)
- [Mum] D. Mumford, *The red book of varieties and schemes*. Lecture Notes in Math. 1358, Springer-Verlag, Berlin 1999. [Zbl 0658.14001](#) [MR 1748380](#)
- [NR1] M. S. Narasimhan, S. Ramanan, Deformations of the moduli space of vector bundles over an algebraic curve. *Ann. of Math. (2)* **101** (1975), 391–417. [Zbl 0314.14004](#) [MR 0384797](#)
- [NR2] M. S. Narasimhan, S. Ramanan, Moduli of vector bundles on a compact Riemann surface. *Ann. of Math. (2)* **89** (1969), 14–51. [Zbl 0186.54902](#) [MR 0242185](#)
- [N1] P. E. Newstead, Stable bundles of rank 2 and odd degree over a curve of genus 2. *Topology* **7** (1968), 205–215. [Zbl 0174.52901](#) [MR 0237500](#)

- [N2] P. E. Newstead, Characteristic classes of stable bundles of rank 2 over an algebraic curve. *Trans. Amer. Math. Soc.* **169** (1972), 337–345. [Zbl 0256.14008](#) [MR 0316452](#)
- [O1] B. Osserman, The generalized Verschiebung map for curves of genus 2. *Math. Ann.* **336** (2006), 963–986. [Zbl 1111.14031](#) [MR 2255181](#)
- [O2] B. Osserman, Mochizuki’s crys-stable bundles: A lexicon and applications. *Publ. Res. Inst. Math. Sci.* **43** (2007), 95–119. [MR 2317114](#)
- [R] M. Raynaud, Sections des fibrés vectoriels sur une courbe. *Bull. Soc. Math. France* **110** (1982), 103–125. [Zbl 0505.14011](#) [MR 0662131](#)
- [Re] M. Reid, The complete intersection of two or more quadrics, Doctoral Thesis, Cambridge University, Cambridge 1972.
- [S] C. S. Seshadri, Vector bundles on curves. In *Linear algebraic groups and their representations*, Contemp. Math. 153, Amer. Math. Soc., Providence, RI, 1993, 163–200. [Zbl 0799.14013](#) [MR 1247504](#)
- [Sh] S. Shatz, The decomposition and specialization of algebraic families of vector bundles. *Compositio Math.* **35** (1977), 163–187. [Zbl 0371.14010](#) [MR 0498573](#)
- [SB] N. I. Shepherd-Barron, Semi-stability and reduction mod  $p$ . *Topology* **37** (1998), 659–664. [Zbl 0926.14021](#) [MR 1604907](#)
- [VL] J.-L. Verdier, J. Le Potier (eds.), *Module des fibrés stables sur les courbes algébriques*. Progr. Math. 54, Birkhäuser, Boston, MA, 1985 [Zbl 0546.00011](#) [MR 0790317](#)

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