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Arithmetically Cohen–Macaulay bundles on hypersurfaces

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Abstract. We prove that any rank two arithmetically Cohen–Macaulay vector bundle on a general hypersurface of degree at least three in $\mathbb{P}^5$ must be split.

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1. Introduction

Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d. A vector bundle E on X is called *arithmetically Cohen–Macaulay* (ACM for short) if $\mathrm{H}^i(E(k)) = 0$ for all k and $0 < i < n-1$. By Horrock's criterion [6], this is equivalent to saying that E has a resolution

$$0 \to F_1 \to F_0 \to E \to 0,$$

where the F_i's are direct sums of line bundles on $\mathbb{P}^n$. If $d = 1$, E is a direct sum of line bundles (*op. cit*); the ACM condition is vacuous for $n = 1, 2$.

In this article, we will be interested in ACM bundles of rank two. For $n = 3$, ACM rank two bundles are ubiquitous (see Remark 1). Hence we will deal with smooth hypersurfaces X of degree $d \geq 2$ in $\mathbb{P}^n$ with $n \geq 4$ and ACM rank two bundles on X which are not direct sums of line bundles of the form $\mathcal{O}_X(k)$. By the Grothendieck–Lefschetz theorem, these bundles are the same as the indecomposable rank two ACM bundles on X.

Our main theorem is

Theorem 1.1. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \geq 2$.*

(1) *If $n \geq 6$, any ACM rank two bundle on X is a direct sum of line bundles (Kleppe* [8]*).*

(2) (a) *If $n = 5$, then for any ACM rank two bundle E on X, $\mathrm{H}^1(X, E^\vee \otimes E(k)) = 0$ for all k. In particular, E is rigid.*

(b) *If $n = 5$ and X is general (for a dense Zariski open subset of the parameter space of hypersurfaces) of degree $d \geq 3$, then any ACM bundle of rank two is a direct sum of line bundles.*

(3) *If $n = 4$ and X is general of degree $d \geq 5$, then for any ACM rank two bundle E on X, $\mathrm{H}^i(X, E^\vee \otimes E) = 0$ for $i = 1, 2$. In particular, E is rigid.*

Remarks. 1) Part (1) of the theorem is known by the work of Kleppe ([8], see Proposition 3.2) where he proves a much more general theorem in this direction. Our proof of this particular case is different and essentially falls out of some of the computations necessary for part (2) of the theorem.

2) If $n = 5$ and $d \geq 2$, there certainly exist special smooth hypersurfaces with non-split ACM bundles. Here is a fairly simple way to construct them: Let f, g, h be a regular sequence of homogeneous polynomials. Let a, b, c be non-constant homogeneous polynomials such that these six polynomials have no common zero in $\mathbb{P}^n$ and such that $F = af + bg + ch$ is homogeneous of degree $d \geq 2$. Let X be defined by $F = 0$. Then we have an exact sequence

$$0 \to E \to \mathcal{O}_X(-\deg f) \oplus \mathcal{O}_X(-\deg g) \oplus \mathcal{O}_X(-\deg h) \to I \to 0,$$

where I is the ideal generated by f, g, h in $\mathcal{O}_X$, the map to I is the obvious one and E is the kernel. One easily checks that E is an indecomposable ACM bundle on X of rank two. Any smooth quadric hypersurface in $\mathbb{P}^5$ has a Plücker equation $F = 0$ and so the above construction applies.

If $n = 5$ for $d = 3, 4, 5$ and 6, the statement (2)(b) of the theorem has been proved already by Chiantini and Madonna [5].

3) In the case of $n = 4$, the rigidity statement was proved for quintic threefolds by Chiantini and Madonna [3]. Further, it is known that indecomposable ACM bundles of rank 2 exist for any smooth hypersurface of degree d with $2 \leq d \leq 5$ ([1], [2], [7]). One way to see this is to note that any such hypersurface contains a line and hence the construction in 2) applies. On the other hand, it was shown by Chiantini and Madonna [4] that such bundles do not exist for a general sextic in $\mathbb{P}^4$ and one expects the same to be true for general hypersurfaces of degree $d \geq 6$[1] in which case our result is trivially true.

4) When $n = 3$, any smooth hypersurface contains a point and hence the construction in 2) applies in this case too.

We now give a brief sketch of the proof of Theorem 1.1 (2)(b). Suppose we have a rank two indecomposable ACM bundle E on a hypersurface X of degree d. We show that the module $N = \bigoplus_k \mathrm{H}^2(X, E^\vee \otimes E(k))$ is a non-zero graded cyclic module generated by an element of degree $-d$ (Lemma 2.3). In the $\mathbb{P}^5$ case, N is

[1] This has been recently proved by the authors and will appear in *International Mathematics Research Notices*

a Gorenstein module and the socle element is in degree $2d-6$. Thus we see that $N_k \neq 0$ for all $-d \leq k \leq 2d-6$. By means of a deformation-theoretic argument, we will show that if X is general, then the multiplication map $N_{-d} \to N_0$ by any $g \in \mathrm{H}^0(X, \mathcal{O}_X(d))$ is zero (Corollary 3.8), which by the cyclicity implies that $N_0 = 0$. This is a contradiction if d is at least three.

2. Cohomology computations

We will work over an algebraically closed field of characteristic zero, though most of the arguments will go through in characteristic not equal to two. We will assume throughout that X is a smooth hypersurface of degree $d \geq 2$ in $\mathbb{P}^n$ with $n \geq 4$ and its defining equation is $u = 0$. Let E be rank two bundle on X. By the Grothendieck–Lefschetz theorem, $\operatorname{Pic}\mathbb{P}^n \to \operatorname{Pic} X$ is an isomorphism. So by normalising E, we will assume that $c_1(E) = e$ where $e = 0$ or -1. We will now assume E is ACM. Then we have a minimal resolution

$$0 \to F_1 \to F_0 \to E \to 0,$$

where the F_i's are direct sums of line bundles on $\mathbb{P}^n$ of rank r ($E \cong \mathcal{O}_X(a) \oplus \mathcal{O}_X(b)$ for some integers a and b if and only if $r = 2$).

Dualizing the above we get,

$$0 \to F_0^\vee \to F_1^\vee \to \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(E, \mathcal{O}_{\mathbb{P}^n}) \to 0.$$

Applying $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(E, *)$ to the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \xrightarrow{u} \mathcal{O}_{\mathbb{P}^n}(d) \to \mathcal{O}_X(d) \to 0,$$

we get the long exact sequence,

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(E, \mathcal{O}_{\mathbb{P}^n}(d)) \to E^\vee(d) \to \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(E, \mathcal{O}_{\mathbb{P}^n}) \xrightarrow{u} \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(E, \mathcal{O}_{\mathbb{P}^n}(d)).$$

The first term is zero and multiplication by u is zero. Thus we get

$$E(d-e) = E^\vee(d) \xrightarrow{\cong} \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(E, \mathcal{O}_{\mathbb{P}^n}).$$

This implies that $F_1^\vee = F_0(d-e)$. Thus we can rewrite our minimal resolution as

$$0 \to F_1 = F_0{}^\vee(e-d) \xrightarrow{\phi} F_0 \to E \to 0. \tag{1}$$

In fact by [2], the map ϕ can be chosen to be skew-symmetric though we will not use this fact.

By restricting to X, we get an exact sequence

$$0 \to E(-d) \to F_1 \otimes \mathcal{O}_X = \overline{F_1} \xrightarrow{\overline{\phi}} F_0 \otimes \mathcal{O}_X = \overline{F_0} \to E \to 0. \tag{2}$$

Let the image of $\overline{\phi}$ be denoted by G. Then G is a vector bundle of rank $r-2$ on X.

Taking exterior powers of ϕ in (1), we have an exact sequence

$$0 \to \Lambda^2 F_1 \to \Lambda^2 F_0 \to \mathcal{F} \to 0 \tag{3}$$

for some cokernel sheaf $\mathcal{F}$.

Lemma 2.1. *We have an exact sequence*

$$0 \to L \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_X = \overline{\mathcal{F}} \to 0,$$

where L is a line bundle on X and $\overline{\mathcal{F}}$ is a vector bundle of rank $2r-3$ on X which fits into a natural exact sequence

$$0 \to E \otimes G \to \overline{\mathcal{F}} \to \mathcal{O}_X(e) = \Lambda^2 E \to 0.$$

Proof. We certainly have an exact sequence

$$0 \to \mathcal{I}\mathcal{F} \to \mathcal{F} \to \overline{\mathcal{F}} \to 0,$$

where $\mathcal{I}$ is the ideal sheaf defining X in $\mathbb{P}^n$. It suffices to check that $\mathcal{I}\mathcal{F}$ is a line bundle on X and $\overline{\mathcal{F}}$ is a vector bundle of rank $2r-3$ for the first part of the lemma. Both statements are local. Clearly $\mathcal{F}$ is set-theoretically supported only along X. By localising we may assume that (1) looks like

$$0 \to F_1 \to F_0 \to E \to 0.$$

where the F_i's are free of rank r and the matrix of the map $F_1 \to F_0$ is the diagonal matrix $(u, u, 1, 1, \ldots, 1)$ where $u = 0$ defines X. Then the matrix in (3) is given by the diagonal matrix $(u^2, u, \ldots u, 1, 1, \ldots, 1)$ where we have one u^2, $2(r-2)$ u's and the rest 1's. The claim about $\mathcal{I}\mathcal{F}$ and $\overline{\mathcal{F}}$ follows easily from this.

To see the final exact sequence, we restrict (3) to X to get

$$\Lambda^2 \overline{F_1} \to \Lambda^2 \overline{F_0} \to \overline{\mathcal{F}} \to 0.$$

From the exact sequence

$$0 \to G \to \overline{F_0} \to E \to 0$$

we note that

$$\operatorname{im}(\Lambda^2 \overline{F_1} \to \Lambda^2 \overline{F_0}) = \operatorname{im}(\Lambda^2 G \to \Lambda^2 \overline{F_0}).$$

This leads to the natural diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Lambda^2 G & = & \Lambda^2 G & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \Lambda^2 \overline{F_0} & \longrightarrow & \Lambda^2 E = \mathcal{O}_X(e) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & E \otimes G & \longrightarrow & \overline{\mathcal{F}} & \longrightarrow & \mathcal{O}_X(e) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0, & & & &
\end{array}$$

where $\mathcal{G}$ is defined by the diagram. □

Next we note the vanishing of various cohomologies. From (1) and (3) we have

$$\mathrm{H}^i(E(*)) = \mathrm{H}^i(\mathcal{F}(*)) = \mathrm{H}^i(L(*)) = 0, \quad 1 \le i \le n-2. \tag{4}$$

This implies from the lemma above that

$$\mathrm{H}^i(\overline{\mathcal{F}}(*)) = 0, \quad 1 \le i \le n-3. \tag{5}$$

Tensoring the exact sequence $0 \to E(-d) \to \overline{F_1} \to G \to 0$ with $E^\vee$ and taking cohomologies, we get

$$\mathrm{H}^i(E^\vee \otimes G(k)) = \mathrm{H}^{i+1}(E^\vee \otimes E(-d+k)) \quad \text{for all } k \text{ and } 1 \le i \le n-3. \tag{6}$$

Similarly, tensoring the exact sequence $0 \to G \to \overline{F_0} \to E \to 0$ with $E^\vee$ and taking cohomologies, we get

$$\mathrm{H}^i(E^\vee \otimes E(k)) = \mathrm{H}^{i+1}(E^\vee \otimes G(k)) \quad \text{for all } k \text{ and } 1 \le i \le n-3. \tag{7}$$

From the exact sequence in the lemma, using (4), (5), we get

$$\mathrm{H}^i(E \otimes G(k)) = 0 \quad \text{for all } k \text{ and } 2 \le i \le n-3. \tag{8}$$

Lemma 2.2. *The vector bundle E is a direct sum of line bundles if and only if* $\mathrm{H}^2(E^\vee \otimes E(-d)) = 0$.

Proof. If E is a direct sum of line bundles, this vanishing is clear. So assume the vanishing. From (6) we see that $\mathrm{Ext}^1(E, G) = \mathrm{H}^1(E^\vee \otimes G) = 0$ and thus we see that the exact sequence $0 \to G \to \overline{F_0} \to E \to 0$ splits. Since $\overline{F_0}$ is a direct sum of line bundles, we see that so is E. □

Corollary 2.3. *If E is an indecomposable bundle, then the finite length module $\bigoplus_k \mathrm{H}^1(E \otimes G(k))$ is a non-zero cyclic module generated by an element of degree $-e$.*

Proof. By Lemma 2.1 we have an exact sequence

$$0 \to E \otimes G \to \overline{\mathcal{F}} \to \mathcal{O}_X(e) \to 0.$$

Taking cohomologies and using (5), we get the fact that the module is cyclic generated by an element of degree $-e$. If it is zero then

$$\mathrm{H}^1(E \otimes G(-e)) = \mathrm{H}^1(E^\vee \otimes G) = 0,$$

and so by (6) $\mathrm{H}^2(E^\vee \otimes E(-d)) = 0$. By the previous lemma, the bundle would then have to be split. □

3. Deformation criteria for ACM vector bundles

We have already noted that for $n \leq 5$, given any degree $d \geq 2$, there exists a smooth hypersurface of degree d and a non-split ACM bundle of rank 2 on the hypersurface. So, in this section, we analyze the situation, when there is such a vector bundle on a general hypersurface X of degree d.

We will start with some results on vector bundles on families of varieties. We will not prove the most general results in this direction, but just what we need. Most of the arguments are similar to those used in the construction of quot schemes and are well known to experts. We are really interested in Corollary 3.5 and much of what follows consists of technical results to achieve it.

Let us start by fixing some notation. All schemes will be of finite type over the base field. Let $p\colon X \to S$ be a flat projective morphism. If $q\colon T \to S$ is any morphism, we will denote by $X_T = X \times_S T$, the fiber product and $p'\colon X_T \to T$, $q'\colon X_T \to X$, the natural S-morphisms. We start with an elementary lemma, whose proof we omit.

Lemma 3.1. *Let X, S, T be as above and let $V \subset X$ be any subset such that $q'(X_T) \cap V = \emptyset$. Then, $q'(X_T) \cap p^{-1}p(V) = \emptyset$.*

Proposition 3.2. *Let $p\colon X \to S$ be a flat projective morphism. Let F_1, F_2 be two vector bundles on X with $\mathrm{H}^1(X_s, F_2^\vee \otimes F_1 \mid_{X_s}) = 0$ for all $s \in S$. Let $m \geq 0$ be an integer. Then there exists a scheme $q\colon S' \to S$ and an exact sequence*

$$q'^* F_2 \to q'^* F_1 \to G \to 0,$$

where G is a rank m vector bundle on $X_{S'}$. Furthermore, one has the following universal property: for any reduced S-scheme T with structure morphism $f: T \to S$ and an exact sequence

$$f'^*F_2 \to f'^*F_1 \to G' \to 0$$

with G' a rank m vector bundle on X_T, there exists an S-morphism $T \to S'$ such that the second sequence is just the pull back of the first by the induced morphism $X_T \to X_{S'}$.

Proof. The hypothesis on H^1 ensures that $\mathcal{E} = p_*(F_2^\vee \otimes F_1)$ is a vector bundle on S. Let

$$\mathcal{H} = \mathbb{A}(\mathcal{E}^\vee) \xrightarrow{q} S.$$

Then $q^*\mathcal{E}$ has a section and $\mathcal{H}$ is universal with this property. We have the fiber product diagram,

$$\begin{array}{ccc} X_{\mathcal{H}} & \xrightarrow{q'} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{H} & \xrightarrow{q} & S \end{array}$$

By the flatness of q we have

$$p'_*q'^*(F_2^\vee \otimes F_1) = q^*\mathcal{E},$$

and so $q'^*(F_2^\vee \otimes F_1)$ has a section. Thus on $X_{\mathcal{H}}$ we get the universal map $q'^*F_2 \to q'^*F_1$. Let G be the cokernel. By semi-continuity, the points $x \in X_{\mathcal{H}}$ such that $\dim_{k(x)} G \otimes k(x) < m$ constitute an open set, which we denote by V. Since p' is flat $p'(V)$ is open. Let $\mathcal{H}' \subset \mathcal{H}$ be the closed subset with the reduced scheme structure which is the complement of $p'(V)$. Thus, replacing $\mathcal{H}$ with $\mathcal{H}'$, we get an exact sequence, where G has the property that at every point $x \in X_{\mathcal{H}'}$, $\dim_{k(x)} G \otimes k(x) \geq m$. Again, the set of points $x \in X_{\mathcal{H}'}$ such that $\dim_{k(x)} G \otimes k(x) > m$ is a closed subset, say Z. Since p' is proper, we may take $S' \subset \mathcal{H}'$ to be the open set which is the complement of the closed subset $p'(Z)$. It is clear that on $X_{S'}$ we have an exact sequence as claimed.

We need to check the universal property. So, let $f: T \to S$ be as in the proposition with the exact sequence as mentioned. Let us denote by $f': X_T \to X$ and $p'': X_T \to T$ the corresponding maps. The existence of the map $f'^*F_2 \to f'^*F_1$ gives a section of $p''_*f'^*(F_2^\vee \otimes F_1)$ which is equal to $f^*\mathcal{E}$ by semi-continuity. This implies that we have a morphism $g: T \to \mathcal{H}$ over S and the exact sequence

$$q''^*E_2 \to q'^*E_1 \to G \to 0$$

on $X_{\mathcal{H}}$ pulls back to the one on X_T via the induced map $g': X_T \to X_{\mathcal{H}}$. Since G' is a rank m vector bundle, we see that $g'(X_T) \cap V = \emptyset$ and thus we see that by

Lemma 3.1, $g'(X_T) \cap p'^{-1}p'(V) = \emptyset$. Since T is reduced, this implies that g factors through $\mathcal{H}'$. Next, we notice that $g'(X_T) \cap Z = \emptyset$ and by another application of Lemma 3.1, we are done. □

Given a rank two (non-split) ACM bundle E on X, we rewrite (1) with $F_0 \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(a_i)$, $a_1 \geq a_2 \geq \cdots \geq a_r$ and $r > 2$ to get

$$0 \to \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(-a_i + e - d) \xrightarrow{\phi} \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(a_i) \to E \to 0.$$

As before, we assume that $e = 0$ or $e = -1$.

Lemma 3.3. *When $n \geq 4$, for fixed d there are only finitely many possibilities for $(a_1, \ldots, a_r)$.*

Proof. Since sequence (1) is minimal, we see that all entries of ϕ must be at least of degree 1 and thus $\deg \det \phi \geq r$. But this determinant is just the square of the equation defining our hypersurface and thus it must be $2d$. So $r \leq 2d$ and is bounded. From the fact that ϕ is an inclusion, we see that $-a_r + e - d \leq a_1$. Thus, if we show that a_1 is bounded above, then it will follow that a_r is bounded below and then we will have only finitely many possibilities for the a_i's.

For this we proceed as follows. Let $a = a_1$. Since (1) is minimal, we get an inclusion $\mathcal{O}_X(a_1) \to E$ such that the quotient is torsion free. Thus, we get an exact sequence

$$0 \to \mathcal{O}_X(a) \to E \to I(e - a) \to 0,$$

where I is the ideal sheaf of a codimension two subscheme $Z \subset X$. This implies $\omega_Z \cong \mathcal{O}_Z(e - 2a + d - n - 1)$ where ω_Z is the canonical bundle of Z. Let $\pi : Z \to \mathbb{P}^{n-3}$ be a general projection, so that π is finite. Then

$$\pi_* \omega_Z = \mathcal{H}om(\pi_* \mathcal{O}_Z, \mathcal{O}_{\mathbb{P}^{n-3}}(-n + 2)).$$

Since $\mathcal{O}_{\mathbb{P}^{n-3}}$ is a direct summand of $\pi_* \mathcal{O}_Z$, $\mathrm{H}^0(\pi_* \omega_Z(n - 2)) \neq 0$ and thus $\mathrm{H}^0(\mathcal{O}_Z(e - 2a + d - 3)) \neq 0$. Since $n \geq 4$, and E is ACM we see that $\mathrm{H}^1(I(*)) = 0$ and thus $\mathrm{H}^0(\mathcal{O}_X(k)) \to \mathrm{H}^0(\mathcal{O}_Z(k))$ is onto. Since X is a smooth hypersurface, $\mathrm{H}^0(\mathcal{O}_X(k)) = 0$ for $k < 0$. Thus we see that $e - 2a + d - 3 \geq 0$ or $a \leq (e + d - 3)/2$. □

Theorem 3.4. *Let $\mathcal{P}$ denote the parameter space of all degree d smooth hypersurfaces in $\mathbb{P}^n$ with $n \geq 4$ and $\mathcal{X} \subset \mathbb{P}^n \times \mathcal{P}$ the universal hypersurface. Assume that there exist rank two ACM non-split bundles on hypersurfaces corresponding to a Zariski dense subset of $\mathcal{P}$. Then there exists a scheme $\mathcal{P}'$, a dominant morphism $\mathcal{P}' \to \mathcal{P}$ and a rank two bundle over $\mathcal{X} \times_{\mathcal{P}} \mathcal{P}'$ which is ACM and non-split for any point in $\mathcal{P}'$.*

Proof. By Lemma 3.3, for any rank two non-split ACM bundle on a hypersurface X of degree d, we have finitely many choices for r, e and the a_i's. Let us fix one of these. Then for all vector bundles E with these invariants, we have a presentation,

$$F_0^\vee(e-d) \otimes \mathcal{O}_X \to F_0 \otimes \mathcal{O}_X \to E \to 0.$$

Thus, we consider the flat projective morphism $p\colon \mathcal{X} \to \mathcal{P}$ and the vector bundles $F_0^\vee(e-d)$, F_0 pulled back to $\mathcal{X}$, which we denote by the same name. Notice that the vanishing condition on first cohomology in Proposition 3.2 holds if $n \geq 3$. Thus we get a scheme $q\colon S' = S'(e, r, a_i) \to \mathcal{P}$ as in the proposition for $m = 2$. Let G be the corresponding rank two bundle on $\mathcal{X}_{S'}$. We have a closed subset $S'' \subset S'$ where the map $F_0^\vee(e-d) \to F_0$ is minimal. We may restrict to S'' and let $p\colon \mathcal{X}_{S''} \to S''$ be the structure map.

By the relative version of Serre vanishing [9],there exists an integer m_0 such that for all $m \geq m_0$ and all $i > 0$, $\mathrm{R}^i p_* G(m) = 0$. Since G is a vector bundle and p is flat, by repeated application of semi-continuity theorems (see for example, page 41, [9]), one sees that $\mathrm{H}^i(X_s, G(m)_{|X_s}) = 0$ for all $i > 0$ and all $m \geq m_0$. By duality this is also true for all $i < n-1$ and all $m \leq m_1$ for a suitable m_1. Thus we see that there are only finitely many integers k such that $\mathrm{H}^i(X_s, G(k)\,|_{X_s}) \neq 0$ for some $s \in S''$ and some i with $0 < i < n-1$. Since the set of $s \in S''$ such that $\mathrm{H}^i(X_s, G(k)\,|_{X_s}) \neq 0$ for fixed i, k is a closed subset, we see that there is a well-defined closed subset $Z \subset S''$ such that $\mathrm{H}^i(X_s, G(k)\,|_{X_s}) \neq 0$ for some k and some i with $0 < i < n-1$ if and only if $s \in Z$. If we let $T = S'' - Z$, we see that on $\mathcal{X}_T$, the bundle G has the property that on each fibre over T, it is ACM and non-split. Let $\mathcal{P}' = \coprod T$, the union taken over all possible choices of $r > 2$, e and the a_i's. Thus, we get rank two non-split ACM bundles on all fibres of $\mathcal{X}_{\mathcal{P}'} \to \mathcal{P}'$.

To prove that $\mathcal{P}' \to \mathcal{P}$ is dominant, it suffices to show that the image of this map contains a dense set. Let $x \in \mathcal{P}$ be a point such that $\mathcal{X}_x$ supports a rank two non-split ACM bundle, say E. Let r, e, a_i be the corresponding invariants. Then by the universal property of Proposition 3.2, we see that there exists a point $y \in S' = S'(r, e, a_i)$ such that $q(y) = x$ and the pull back of the corresponding G is this bundle E. By minimality of our resolution, we see that $y \in S''$. Since E is ACM, we see that $y \in T$. This completes the proof. □

For a hypersurface $X \subset \mathbb{P}^n$ of degree d, the total infinitesimal deformation X_A of X in $\mathbb{P}^n$ is contained in $\mathbb{P}^n \times \operatorname{Spec} A$ where $A = k \oplus V^\vee$, $V = \mathrm{H}^0(\mathcal{O}_X(d))$ and $V^{\vee 2} = 0$ in the ring A. The following corollary uses characteristic zero.

Corollary 3.5. *Assume that we have non-split rank two ACM bundles on a general hypersurface of degree d in $\mathbb{P}^n$ with $n \geq 4$. Then for a general hypersurface X of degree d there exists a rank two bundle $\mathcal{E}$ on X_A such that $\mathcal{E}_{|X}$ is a non-split ACM bundle.*

Proof. By Theorem 3.4, under the hypothesis, we have a finite type scheme $\mathcal{P}'$ mapping dominantly to $\mathcal{P}$ and a rank two bundle on the 'universal hypersurface' in $\mathbb{P}^n \times \mathcal{P}'$ which is non-split and ACM on each fiber. Since $\mathcal{P}$ is integral, by replacing $\mathcal{P}'$, we may clearly assume that it is integral. Since the map is dominant, we may take a generic multi-section and thus assume that $\mathcal{P}' \to \mathcal{P}$ is generically finite. Replacing again $\mathcal{P}'$ by a suitable open set, we may assume that the map is etale since the characteristic is zero . If $x \in \mathcal{P}$ corresponding to $X \subset \mathbb{P}^n$ is the image of $y \in \mathcal{P}'$, then the tangent spaces at x and y are isomorphic. Now noting that V is the tangent space at x to $\mathcal{P}$, we are done. □

For the next theorem, we will need the following elementary lemma from homological algebra, whose proof (which we omit) follows from the construction of push-outs.

Lemma 3.6. *Let*

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{A} & \xrightarrow{i} & \mathcal{B} \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ 0 & \longrightarrow & \mathcal{A}' & \xrightarrow{i'} & \mathcal{B}' \end{array}$$

be a push-out diagram where $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$, $\mathcal{B}'$ are sheaves. Then i' splits if and only if there exists a homomorphism $\alpha\colon \mathcal{B} \to \mathcal{A}'$ such that $\alpha \circ i = j$

Before we state the next theorem, let us fix some notation. Let X be a smooth hypersurface of degree d in $\mathbb{P}^n$ with $n \geq 3$ defined by $u = 0$ and E an ACM bundle on X of arbitrary rank.

Let $v \in \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(d))$ be such that the image $\bar{v} \in V = \mathrm{H}^0(\mathcal{O}_X(d))$ is non-zero. Let $B = k[\varepsilon]$ with $\varepsilon^2 = 0$. The map $-\bar{v}\colon V^\vee \to k\varepsilon$ defines a natural surjective ring homomorphism $A \to B$ where $A = k \oplus V^\vee$ with ${V^\vee}^2 = 0$ as before. Let $X_\varepsilon \subset \mathbb{P}^n \times \operatorname{Spec} B$ be the corresponding hypersurface defined by $u - \varepsilon v = 0$. In the theorem we will assume that there is a bundle $\mathcal{E}$ on X_ε such that the restriction of $\mathcal{E}$ to $X \subset X_\varepsilon$ is E.

As always we have a resolution

$$0 \to F_1 \xrightarrow{\phi} F_0 \to E \to 0$$

with the F_i's direct sums of line bundles on $\mathbb{P}^n$. This gives as before, by restriction to X, a long exact sequence, and by splitting it to short exact sequences, an exact sequence,

$$0 \to E(-d) \to \overline{F_1} \to G \to 0$$

for some bundle G on X as in (2). Let us denote by ζ the corresponding extension class in

$$\operatorname{Ext}^1_{\mathcal{O}_X}(G, E(-d)) = \mathrm{H}^1(G^\vee \otimes E(-d)).$$

Theorem 3.7. *With the notation and assumptions as above, under the natural map*

$$\mathrm{H}^1(G^\vee \otimes E(-d)) \xrightarrow{\bar{v}} \mathrm{H}^1(G^\vee \otimes E))$$

the image of ζ is zero.

Proof. Since $\mathcal{E}$ is flat over $k[\varepsilon]$, we have,

$$E \cong \varepsilon\mathcal{E} \cong \mathcal{E}/\varepsilon\mathcal{E}.$$

We get an exact sequence

$$0 \to E = \varepsilon\mathcal{E} \xrightarrow{i} \mathcal{E} \xrightarrow{\pi} E = \mathcal{E}/\varepsilon\mathcal{E} \to 0.$$

Let $p\colon X_\varepsilon \to \mathbb{P}^n$ be the projection, which is clearly a finite map. Taking the direct image under p of the above exact sequence and noting that p restricted to $X \subset X_\varepsilon$ is a closed embedding in $\mathbb{P}^n$, we get

$$0 \to E \xrightarrow{i} p_*\mathcal{E} = \mathcal{F} \xrightarrow{\pi} E \to 0. \tag{9}$$

We want to interpret multiplication by u on $\mathcal{F}$. We know that $u - \varepsilon v$ annihilates $\mathcal{E}$, and so multiplication by u is the same as multiplication by εv. But multiplication by ε is just the composite $i \circ \pi$. Thus we see that multiplication by u is the composite

$$\mathcal{F} \xrightarrow{\pi} E \xrightarrow{\bar{v}} E(d) \xrightarrow{i} \mathcal{F}(d), \tag{10}$$

using the fact that $i \circ \bar{v} = \bar{v} \circ i\colon E \to \mathcal{F}(d)$.

The exact sequence (9) gives an element η in $\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^n}}(E, E)$. From the exact sequence

$$0 \to F_1 \to F_0 \to E \to 0$$

we get

$$\mathrm{H}^0(F_1^\vee \otimes E) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^n}}(E, E) \to \mathrm{H}^1(F_0^\vee \otimes E).$$

Since E is ACM and $n \geq 3$ the last term is zero. Thus we can lift η to an $\alpha \in \mathrm{H}^0(F_1^\vee \otimes E)$ and we get the push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \xrightarrow{\phi} & F_0 & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \| & & \\ 0 & \longrightarrow & E & \xrightarrow{i} & \mathcal{F} & \xrightarrow{\pi} & E & \longrightarrow & 0 \end{array} \tag{11}$$

Multiplication by u gives the following commutative diagram

$$\begin{array}{ccc} F_0(-d) & \xrightarrow{u} & F_0 \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathcal{F}(-d) & \xrightarrow{u} & \mathcal{F}. \end{array}$$

Since u annihilates E, the top row factors as

$$F_0(-d) \xrightarrow{\psi} F_1 \xrightarrow{\phi} F_0$$

for a suitable map ψ. Using (10) we get the following diagram, which we will show is in fact commutative:

$$\begin{array}{ccccccc} F_0(-d) & & \xrightarrow{\psi} & & F_1 & \xrightarrow{\phi} & F_0 \\ {\scriptstyle\beta}\downarrow & & & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathcal{F}(-d) & \xrightarrow{\pi} & E(-d) & \xrightarrow{\bar{v}} & E & \xrightarrow{i} & \mathcal{F}. \end{array}$$

$\beta\phi = i\alpha$ follows from (11). We have

$$i\bar{v}\pi\beta = u\beta = \beta u = \beta\phi\psi = i\alpha\psi.$$

Since i is an inclusion, this implies, $\bar{v}\pi\beta = \alpha\psi$, proving commutativity. Restricting this diagram to X, we get

$$\begin{array}{ccccccc} \overline{F_0}(-d) & & \xrightarrow{\overline{\psi}} & & \overline{F_1} & \xrightarrow{\overline{\phi}} & \overline{F_0} \\ {\scriptstyle\beta}\downarrow & & & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathcal{F}\otimes_{\mathcal{O}_{\mathbb{P}^n}}\mathcal{O}_X(-d) & \xrightarrow{\pi} & E(-d) & \xrightarrow{\bar{v}} & E & \xrightarrow{i} & \mathcal{F}\otimes_{\mathcal{O}_{\mathbb{P}^n}}\mathcal{O}_X. \end{array}$$

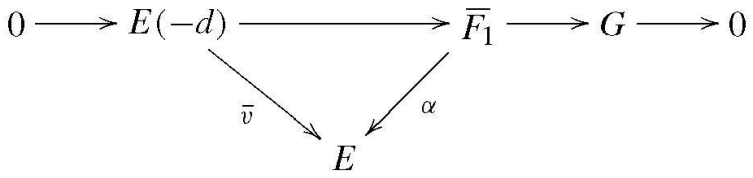
The image of $\overline{\psi}$ is just $E(-d)$ and thus we can rewrite the above diagram as

$$\begin{array}{ccccccc} \overline{F_0}(-d) & & \longrightarrow & E(-d) & \hookrightarrow \overline{F_1} & \xrightarrow{\overline{\phi}} & \overline{F_0} \\ {\scriptstyle\beta}\downarrow & & & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathcal{F}\otimes_{\mathcal{O}_{\mathbb{P}^n}}\mathcal{O}_X(-d) & \xrightarrow{\pi} & E(-d) & \xrightarrow{\bar{v}} & E & \xrightarrow{i} & \mathcal{F}\otimes_{\mathcal{O}_{\mathbb{P}^n}}\mathcal{O}_X. \end{array}$$

Restricting (11) to X and twisting by $\mathcal{O}_X(-d)$, we see that one has a commutative diagram

$$\begin{array}{ccc} E(-d) & \hookrightarrow & \overline{F_1} \\ \| & & \downarrow{\scriptstyle\alpha} \\ E(-d) & \xrightarrow{\bar{v}} & E. \end{array}$$

The extension class ζ corresponds to the top row of the following commutative diagram:



and by Lemma 3.6, under the natural map

$$\mathrm{Ext}^1(G, E(-d)) \xrightarrow{\bar{v}} \mathrm{Ext}^1(G, E),$$

ζ goes to zero. Therefore, under the map

$$\mathrm{H}^1(G^\vee \otimes E(-d)) \xrightarrow{\bar{v}} \mathrm{H}^1(G^\vee \otimes E),$$

ζ goes to zero. □

Corollary 3.8. *Assume the rank of E is two, $n \geq 4$ and that as before, E can be deformed in the direction of v. Then the natural map*

$$\mathrm{H}^2(E^\vee \otimes E(-d)) \xrightarrow{\bar{v}} \mathrm{H}^2(E^\vee \otimes E)$$

is zero.

Proof. From Theorem 3.7, under the map

$$\mathrm{H}^1(G^\vee \otimes E(-d)) \xrightarrow{\bar{v}} \mathrm{H}^1(G^\vee \otimes E),$$

ζ goes to zero. By (2), we have $G^\vee = G(d-e)$. Thus, under the map

$$\mathrm{H}^1(G \otimes E(-e)) \xrightarrow{\bar{v}} \mathrm{H}^1(G \otimes E(d-e)),$$

ζ goes to zero. If E is indecomposable, $\mathrm{H}^1(G \otimes E(-e))$ is one dimensional with ζ as basis element by Corollary 2.3. From (6) we have

$$\mathrm{H}^1(G \otimes E(-e+k)) = \mathrm{H}^2(E^\vee \otimes E(-d+k)).$$

Thus we get that the map

$$\mathrm{H}^2(E^\vee \otimes E(-d)) \xrightarrow{\bar{v}} \mathrm{H}^2(E^\vee \otimes E)$$

is zero. □

4. Proof of Theorem 1.1

Proof of Theorem 1.1 (1). For $n \geq 6$,

$$\mathrm{H}^2(E^\vee \otimes E(-d)) = \mathrm{H}^3(E^\vee \otimes G(-d))$$

by (7). This group is zero by (8). The proof now follows from Lemma 2.2. □

Proof of Theorem 1.1 (2) (a). For $n = 5$, using (7) with $i = 1$, we have

$$\mathrm{H}^1(X, E^\vee \otimes E(k)) = \mathrm{H}^2(X, E^\vee \otimes G(k)).$$

This group is zero by (8). □

Proof of Theorem 1.1 (2) (b). The proof is by contradiction. Assume that a general hypersurface X of degree $d \geq 3$ has an indecomposable ACM bundle of rank two. From Corollaries 3.5 and 3.8 we see that

$$\mathrm{H}^2(E^\vee \otimes E(-d)) \xrightarrow{\bar{v}} \mathrm{H}^2(E^\vee \otimes E)$$

is zero for any $\bar{v} \in \mathrm{H}^0(\mathcal{O}_X(d))$. By Corollary 2.3 and (6) we know that the graded module $N = \bigoplus_k \mathrm{H}^2(E^\vee \otimes E(k)) = \bigoplus_k N_k$ is cyclic and generated by a non-zero element ζ in degree $-d$. Thus N_0, the degree zero component of N, consists of multiples of ζ by elements $\bar{v} \in \mathrm{H}^0(\mathcal{O}_X(d))$. Since these are zero we get $N_i = 0$ for $i \geq 0$. As $n = 5$, we have by Serre duality that

$$\mathrm{H}^2(E^\vee \otimes E(-d)) \cong \mathrm{H}^2(E^\vee \otimes E(2d-6))^\vee.$$

Hence $N_{2d-6} \neq 0$. If $d \geq 3$, then $2d - 6 \geq 0$, and this contradiction proves the result. □

Proof of Theorem 1.1 (3). Assume that a general hypersurface X of degree $d \geq 5$ has an (indecomposable) ACM bundle of rank two. By the same arguments as in the proof above, the graded module N has $N_i = 0$ for $i \geq 0$ and thus $\mathrm{H}^2(E^\vee \otimes E) = 0$. By Serre duality we have $\mathrm{H}^1(E^\vee \otimes E) \cong \mathrm{H}^2(E^\vee \otimes E(d-5)) = N_{d-5}$ and hence is zero for $d \geq 5$. □

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