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# Non-vanishing for Koszul cohomology of curves

M. Aprodu and J. Nagel

**Abstract.** We study the relationship between rank  $p+2$  Koszul classes and rank 2 vector bundles on a smooth curve. We show that every rank  $p+2$  Koszul class is obtained from a rank 2 vector bundle and give an explicit nonvanishing theorem for Koszul classes arising in this way.

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## 1. Introduction

Let  $X$  be a smooth complex projective variety. The geometry of  $X$  is reflected in the behaviour of the Koszul cohomology groups  $K_{p,q}(X, L)$  introduced by Green [4], more specifically the vanishing/nonvanishing of certain Koszul cohomology groups. The fundamental result in this direction is the nonvanishing theorem of Green–Lazarsfeld [5]. This theorem states that if a line bundle  $L$  admits a decomposition  $L = L_1 \otimes L_2$  with  $r_i = h^0(X, L_i) - 1 \geq 1$  ( $i = 1, 2$ ) then  $K_{r_1+r_2-1,1}(X, L) \neq 0$ . Voisin [9, (1.1)] has given a different proof of this result under the hypothesis that  $L_1$  and  $L_2$  are globally generated.

The aim of this note is to give a more geometric approach to this type of problems. The starting point is the following construction due to Voisin. Given a rank two vector bundle  $E$  on  $X$  with determinant  $L$ , Voisin [11, (2.22)] defined a homomorphism

$$\varphi: S^p H^0(X, E) \otimes \bigwedge^{p+2} H^0(X, E) \rightarrow \bigwedge^p H^0(X, L) \otimes H^0(X, L).$$

By [11, Lemma 5], this homomorphism produces elements of  $K_{p,1}(X, L)$ . If we take  $E = L_1 \oplus L_2$ , we get back the classes constructed by Green and Lazarsfeld. As one of the referees pointed out to us, Koh and Stillman [7] had generalised the Green–Lazarsfeld construction before from a different point of view.

Recall that the *rank* of a Koszul class  $\gamma \in K_{p,1}(X, L)$  is the minimal dimension of a linear subspace  $W \subset H^0(X, L)$  such that  $\gamma$  is represented by an element in  $\bigwedge^p W \otimes H^0(X, L)$ ; cf. [6, Definition 2.2]. (Note that the subspace  $W$  is uniquely

determined if  $p \geq 2$ .) By definition, the Koszul classes constructed in this paper are of rank  $p + 2$  if the vector bundle  $E$  is indecomposable.

Section 3 contains the main results of this paper. We first give a necessary and sufficient condition for nonvanishing of Koszul classes on smooth curves obtained from rank 2 vector bundles (Theorem 3.1). This result generalises the nonvanishing theorem of Green–Lazarsfeld in the case of curves. Our second main result, Theorem 3.4, states that every rank  $p + 2$  Koszul class on a smooth curve comes from a rank two vector bundle. This theorem is a generalisation of [6, Theorem 6.7].

## 2. Preliminaries

**2.1. The method of Voisin.** Let  $E$  be a rank two vector bundle on a smooth projective variety  $X$  defined over an algebraically closed field  $k$  of characteristic zero. Write  $L = \det E$  and  $V = H^0(X, L)$ , and let

$$d: \bigwedge^2 H^0(X, E) \rightarrow V$$

be the determinant map. Given  $t \in H^0(X, E)$ , define a linear map

$$d_t: H^0(X, E) \rightarrow V$$

by  $d_t(u) = d(t \wedge u)$ , and choose a subspace  $U \subset H^0(X, E)$  with  $U \cap \ker(d_t) = 0$ . Suppose that  $\dim(U) = p + 2$  with  $p \geq 1$ , and put  $W = d_t(U) \cong U$ . The restriction of  $d$  to  $\bigwedge^2 U$  defines a map  $\bigwedge^2 U \rightarrow V$ , which we can view as an element of

$$\bigwedge^2 U^\vee \otimes V \cong \bigwedge^p U \otimes V.$$

Let

$$\gamma \in \bigwedge^p W \otimes V \subset \bigwedge^p V \otimes V$$

be the image of this element under the map  $d_t$ .

Following Voisin [11, (2.22)], we prove that  $\gamma$  defines a Koszul class in  $K_{p,1}(X, L)$ . To this end, we make the previous construction explicit using coordinates. If we choose a basis  $\{e_1, \dots, e_{p+3}\}$  of  $\langle t \rangle \oplus U \subset H^0(X, E)$  such that  $e_1 = t$ , we have

$$\begin{aligned} \gamma = \sum_{i < j} (-1)^{i+j} d(t \wedge e_2) \wedge \cdots \wedge \widehat{d(t \wedge e_i)} \wedge \cdots \\ \cdots \wedge \widehat{d(t \wedge e_j)} \wedge \cdots \wedge d(t \wedge e_{p+3}) \otimes d(e_i \wedge e_j). \end{aligned} \tag{1}$$

As in [11] one shows that the image of the  $\gamma$  by the Koszul differential

$$\delta: \bigwedge^p V \otimes H^0(X, L) \rightarrow \bigwedge^{p-1} V \otimes S^2 H^0(X, L)$$

equals

$$\begin{aligned} & \sum_{i < j < k} (-1)^{i+j+k} d(t \wedge e_2) \wedge \cdots \wedge \widehat{d(t \wedge e_i)} \wedge \cdots \\ & \cdots \wedge \widehat{d(t \wedge e_j)} \wedge \cdots \wedge \widehat{d(t \wedge e_k)} \wedge \cdots \wedge d(t \wedge e_{p+3}) \\ & \otimes \{d(t \wedge e_i)d(e_j \wedge e_k) - d(t \wedge e_j)d(e_i \wedge e_k) + d(t \wedge e_k)d(e_i \wedge e_j)\}. \end{aligned} \quad (2)$$

**Lemma 2.1** (Voisin). *Given four elements  $w_1, w_2, w_3, w \in H^0(X, E)$  we have the relation*

$$d(w \wedge w_1)d(w_2 \wedge w_3) - d(w \wedge w_2)d(w_1 \wedge w_3) + d(w \wedge w_3)d(w_1 \wedge w_2) = 0$$

in  $H^0(X, L^2)$ .

*Proof.* See [11, Lemma 5]. □

The previous lemma shows that  $\gamma$  belongs to the kernel of the Koszul differential

$$\delta_X: \bigwedge^p V \otimes H^0(X, L) \rightarrow \bigwedge^{p-1} V \otimes H^0(X, L^2).$$

Hence  $\gamma$  defines a Koszul class  $[\gamma] = \gamma(U, t) \in K_{p,1}(X, L, W) \subseteq K_{p,1}(X, L)$ .

**Remark 2.2.** If  $U' \subset \langle t \rangle \oplus U \subset d_t^{-1}(W)$  is another lifting of  $W$ , then  $\gamma(U, t) = \gamma(U', t)$ . In particular, if  $\ker(d_t) = \mathbb{C}.t$  the given class only depends on  $t$  and  $W$ ; we write  $[\gamma] = \gamma(W, t)$  in this case.

**2.2. The method of Green–Lazarsfeld.** Let  $L_1, L_2$  be two line bundles on a smooth projective variety  $X$  such that  $r_i = h^0(X, L_i) - 1 \geq 1$  ( $i = 1, 2$ ). Write  $L_i = M_i + F_i$  with  $M_i$  the mobile part and  $F_i$  the fixed part. Let  $B$  be the divisorial part of  $F_1 \cap F_2$ . It is possible to choose  $s_i \in H^0(X, L_i)$  such that  $V(s_1, s_2) = B \cup Z$  with  $\text{codim}(Z) \geq 2$ . Set  $L = L_1 \otimes L_2$ , and put  $t = (s_1, s_2) \in H^0(X, L_1 \oplus L_2)$ ,  $W = \text{im}(d_t) \subset H^0(X, L(-B))$ . By construction  $h^0(X, \mathcal{O}_X(B)) = 1$ , hence  $\ker(d_t) = \mathbb{C}.t$  and  $\dim W = r_1 + r_2 + 1$ . By the previous discussion, we obtain a Koszul class  $\gamma(W, t) \in K_{r_1+r_2-1,1}(X, L)$ . We call such classes *Green–Lazarsfeld classes*.

Note that the rank of a Green–Lazarsfeld class is either  $p + 1$  or  $p + 2$ . Classes of rank  $p + 1$  are of scrollar type; see e.g. [8] or [6, Corollary 5.2].

**Definition 2.3.** Given a nonnegative integer  $k \geq 0$ , let  $K_{k,1}(X, L)_{\text{GL}} \subseteq K_{k,1}(X, L)$  be the subspace generated by Green–Lazarsfeld classes for all decompositions  $L = L_1 \otimes L_2$  with  $k = r_1 + r_2 - 1$ , ( $r_1 \geq 1, r_2 \geq 1$ ).

**2.3. The method of Koh–Stillman.** Voisin’s method produces syzygies of rank  $\leq p + 2$ . As we have seen in the previous subsection, rank  $p + 1$  syzygies are Green–Lazarsfeld syzygies of scollar type. Rank  $p + 2$  syzygies can be obtained in the following way. Suppose that  $L$  is a globally generated line bundle on a projective variety  $X$ , and let  $[\gamma] \in K_{p,1}(X, L)$  be a nonzero class represented by an element  $\gamma \in \bigwedge^p W \otimes V$  with  $\dim W = p + 2$ . We view  $\gamma$  as an element in  $\bigwedge^2 W^\vee \otimes V \cong \text{Hom}(\bigwedge^2 W, V)$ . Following [6, Proof of Theorem 6.1] we consider the map

$$\gamma': \bigwedge^2(\mathbb{C} \oplus W) = W \oplus \bigwedge^2 W \rightarrow V$$

defined by taking the direct sum of  $\gamma$  and the inclusion  $W \hookrightarrow V$ . If we choose a generator  $e_1$  for the first summand and a basis  $\{e_2, \dots, e_{p+3}\}$  for  $W$ , we obtain a skew-symmetric  $(p + 3) \times (p + 3)$  matrix  $A$  by setting

$$a_{ij} = \gamma'(e_i \wedge e_j).$$

By construction, the inclusion  $W \rightarrow V$  corresponds to the map  $\gamma'(e_1 \wedge -)$ . This allows us to identify  $a_{1j}$  and  $e_j$ ,  $2 \leq j \leq p + 3$ . Let  $\alpha$  be the image of  $\gamma$  under the Koszul differential

$$\delta: \bigwedge^p V \otimes V \rightarrow \bigwedge^{p-1} V \otimes S^2 V.$$

Writing this out, we obtain

$$\alpha = \sum_{i < j < k} (-1)^{i+j+k} a_{12} \wedge \cdots \wedge \widehat{a_{1,i}} \wedge \cdots \wedge \widehat{a_{1,j}} \wedge \cdots \wedge \widehat{a_{1,k}} \wedge \cdots \wedge a_{1,p+3} \otimes \text{Pf}_{1ijk}(A). \quad (3)$$

As the elements  $\{a_{12}, \dots, a_{1,p+3}\} = \{e_2, \dots, e_{p+3}\}$  are linearly independent, this expression is nonzero if and only if at least one of the Pfaffians  $\text{Pf}_{1ijk}(A)$  is nonzero. Furthermore, since  $\alpha$  maps to zero in  $\bigwedge^{p-1} V \otimes H^0(X, L^2)$  the Pfaffians  $\text{Pf}_{1ijk}(A)$  have to vanish on the image of  $X$ .

The preceding discussion shows that every rank  $p + 2$  syzygy arises from a skew-symmetric  $(p + 3) \times (p + 3)$  matrix  $A$  such that

- (i) the elements  $\{a_{12}, \dots, a_{1,p+3}\}$  are linearly independent;
- (ii) there exists a nonzero Pfaffian  $\text{Pf}_{1ijk}(A)$ ;
- (iii) the Pfaffians  $\text{Pf}_{1ijk}(A)$  vanish on the image of  $X$  in  $\mathbb{P}(V^\vee)$ .

This is exactly the method used by Koh and Stillman to produce syzygies; see [7, Lemma 1.3].

**Remark 2.4.** In the geometric setting of Section 2.1, let  $Y$  be the image of  $X$  in  $\mathbb{P}(V^\vee)$ . The expression (2) shows that the canonical isomorphism

$$K_{p,1}(X, L) \cong K_{p-1,2}(\mathbb{P}^r, \mathcal{I}_Y, \mathcal{O}_{\mathbb{P}}(1))$$

maps the class  $[\gamma]$  to the element  $\alpha$  defined in (3). Moreover, if  $d$  does not vanish on decomposable elements then  $[\gamma] \neq 0$ . Indeed, this condition is satisfied if and only if the matrix  $A$  has no generalised zero; cf. [7, Definition (1.1)]. One then applies [loc. cit., Remark p. 122].

### 3. Main results

**Theorem 3.1.** *Let  $X$  be a smooth curve, let  $L$  be a base-point free line bundle on  $X$  and let  $W \subset H^0(X, L)$  be a linear subspace. Put  $B = \text{Bs}(W)$ , and let  $t$  be a section of  $H^0(X, \mathcal{O}_X(B))$  vanishing on  $B$ . Consider an extension*

$$0 \rightarrow \mathcal{O}_X(B) \rightarrow E \rightarrow L(-B) \rightarrow 0 \quad (4)$$

such that

$$W \subset (\ker H^0(X, L(-B)) \xrightarrow{\delta} H^1(X, \mathcal{O}_X(B))).$$

Then the Koszul classes  $\gamma(U, t)$  defined in Section 2.1 are nonzero for all liftings  $U$  of  $W$  if and only if the extension (4) is non-split.

*Proof.* The proof proceeds in several steps. We use the notation of Section 2.1.

“Only if”. Suppose that the extension (4) splits, hence  $W \subset H^0(X, E)$  canonically. We then put  $U = W$ . In this case, one readily verifies that  $d$  vanishes identically on  $\bigwedge^2 U$ . The formula (1) then shows that  $\gamma(U, t) = 0$ .

“If”. Suppose there exists  $U$  such that  $\gamma(U, t) = 0$ . We proceed in several steps.

*Step 1.* There exists a linear map  $h: U \rightarrow \mathbb{C}$  such that

$$d(u_1 \wedge u_2) = h(u_2)d_t(u_1) - h(u_1)d_t(u_2) \quad (5)$$

for all  $u_1, u_2 \in U$ .

Indeed, suppose that there exists a nonzero element  $\tilde{\gamma} \in \bigwedge^{p+1} W \cong W^\vee$  such that  $\gamma$  is the image of  $\tilde{\gamma}$  under the Koszul differential. Then  $\gamma$  coincides with the composition of maps

$$\bigwedge^2 W \xrightarrow{\delta} W \otimes W \xrightarrow{\tilde{\gamma} \otimes \text{id}} W \hookrightarrow V.$$

Since

$$\begin{aligned} d(u_1 \wedge u_2) &= \gamma(d_t(u_1) \wedge d_t(u_2)) \\ &= \tilde{\gamma}(d_t(u_2))d_t(u_1) - \tilde{\gamma}(d_t(u_1))d_t(u_2), \end{aligned}$$

condition (5) is satisfied with  $h = \tilde{\gamma} \circ d_t: U \rightarrow \mathbb{C}$ .

*Step 2.* Let  $u_1, u_2 \in \langle t \rangle \oplus U$  be two sections such that  $d_t(u_1)$  and  $d_t(u_2)$  generate  $L(-B)$ . If  $d(u_1 \wedge u_2) = 0$ , the extension (4) splits.

To prove this assertion, put  $s_i = d_t(u_i)$  ( $i = 1, 2$ ) and consider the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{O}_X(B) & \longrightarrow & E & \longrightarrow & L(-B) & \longrightarrow 0 \\ & & & \uparrow \text{ev}_2 & & \uparrow \text{ev}_1 & \\ & & & 0 \longrightarrow \langle u_1, u_2 \rangle \otimes \mathcal{O}_X & \xrightarrow{\sim} & \langle s_1, s_2 \rangle \otimes \mathcal{O}_X & \longrightarrow 0. \end{array}$$

Put  $M = \ker(\text{ev}_1)$ , and note that  $\ker(\text{ev}_2) \cong L^{-1}(B)$  since  $\text{ev}_2$  is surjective. By the Snake Lemma we obtain an exact sequence

$$0 \rightarrow M \rightarrow L^{-1}(B) \rightarrow \mathcal{O}_X(B) \rightarrow \text{coker}(\text{ev}_1) \rightarrow 0.$$

Note that

$$d(u_1 \wedge u_2) = 0 \iff \text{rank im}(\langle u_1, u_2 \rangle \otimes \mathcal{O}_X \rightarrow E) = 1 \iff \text{rank } M = 1$$

where the first equivalence follows from [10, p. 380]. If  $d(u_1 \wedge u_2) = 0$  the above exact sequence shows that  $M \cong L^{-1}(B)$ , hence the isomorphism  $\langle u_1, u_2 \rangle \otimes \mathcal{O}_X \xrightarrow{\sim} \langle s_1, s_2 \rangle \otimes \mathcal{O}_X$  induces an isomorphism  $\text{im}(\text{ev}_1) \cong L(-B)$ . The inverse of this isomorphism provides a splitting of the extension (4).

*Step 3.* By Step 1, there exists a linear map  $h: U \rightarrow \mathbb{C}$  verifying the relation (5). If  $h$  is identically zero, then we can apply Step 1 and Step 2 to conclude. Suppose  $h \neq 0$ . Consider the morphism

$$\pi: X \rightarrow \mathbb{P}(W^\vee)$$

defined by the base-point free linear system  $W \subset H^0(X, L(-B))$ , and choose a linear subspace  $\Lambda \subset \mathbb{P}(W^\vee)$  of codimension two such that  $\Lambda \cap \pi(X) = \emptyset$ . The hyperplane  $\ker(h) \subset W$  corresponds to a point  $p \in \mathbb{P}(W^\vee)$ . Put  $H_1 = \langle \Lambda, p \rangle$  and choose a hyperplane  $H_2 \subset \mathbb{P}(W^\vee)$  containing  $\Lambda$  such that  $p \notin H_2$ . Let  $u_1, u_2$  be the sections corresponding to  $H_1, H_2$ . Then  $d_t(u_1)$  and  $d_t(u_2)$  generate  $L(-B)$  and  $u_1 \in \ker(h), u_2 \notin \ker(h)$ . Equation (5) yields the identity

$$d(u_1 \wedge u_2) = h(u_2)d_t(u_1).$$

Rewriting this identity, we obtain  $d(u_1 \wedge (u_2 + h(u_2)t)) = 0$ . Since the pair  $\{d_t(u_1), d_t(u_2 + h(u_2)t)\} = \{d_t(u_1), d_t(u_2)\}$  generates  $L(-B)$ , Step 2 implies that the extension (4) splits.  $\square$

**Remark 3.2.** In the statement of Theorem 3.1 it is not necessary to suppose that  $L$  is globally generated, since  $K_{p,1}(X, L(-\text{Bs}(L))) \cong K_{p,1}(X, L)$ .

Theorem 3.1 yields a short, geometric proof of the Green–Lazarsfeld nonvanishing theorem for curves.

**Theorem 3.3** (Green–Lazarsfeld). *Let  $X$  be a smooth curve, and let  $L$  be a line bundle on  $X$  that admits a decomposition  $L = L_1 \otimes L_2$  with  $r_i = \dim |L_i| \geq 1$  for  $i = 1, 2$ . Then  $K_{r_1+r_2-1,1}(X, L) \neq 0$ .*

*Proof.* We define  $s_1, s_2, t, W, B$  and  $\gamma(W, t)$  as in Section 2.2. Let  $C$  be the base locus of  $W$ , seen as a subspace of  $H^0(X, L(-B))$ . We prove that  $\gamma(W, t) \neq 0$ . Suppose that  $\gamma(W, t) = 0$ . Consider the extension

$$0 \rightarrow \mathcal{O}_X(B) \rightarrow L_1 \oplus L_2 \rightarrow L(-B) \rightarrow 0.$$

Pulling back this extension along the injective homomorphism  $L(-B - C) \rightarrow L(-B)$ , we obtain an induced extension

$$0 \rightarrow \mathcal{O}_X(B) \rightarrow E \rightarrow L(-B - C) \rightarrow 0.$$

Applying Theorem 3.1 to the line bundle  $L(-C)$ , we find that this extension splits. Hence there exists an injective homomorphism

$$\mathcal{O}_X(B) \oplus L(-B - C) \rightarrow L_1 \oplus L_2.$$

In particular there exists  $i \in \{1, 2\}$  such that  $\text{Hom}(L(-B - C), L_i) \neq 0$ . This implies that

$$r_i + 1 = h^0(X, L_i) \geq h^0(X, L(-B - C)) \geq \dim W = r_1 + r_2 + 1,$$

and this is impossible since  $r_1 \geq 1$  and  $r_2 \geq 1$ .  $\square$

**Theorem 3.4.** *Let  $X$  be a smooth curve, and let  $\alpha \neq 0 \in K_{p,1}(X, L)$  be a Koszul class of rank  $p + 2$  represented by an element of  $\bigwedge^p W \otimes H^0(X, L)$  with  $\dim W = p + 2$ . There exist a rank 2 vector bundle  $E$  on  $X$ , a section  $t \in H^0(X, E)$  and a subspace  $W \cong U \subset H^0(X, E)$  such that  $\alpha = \gamma(U, t)$ .*

*Proof.* Put  $T = \mathbb{C} \oplus W$ , and choose a basis  $\{e_1, \dots, e_{p+3}\}$  of  $T$  such that  $t = e_1$  is the generator of the first summand. Writing  $z_{ij} = e_i \wedge e_j$ , we obtain a skew-symmetric matrix  $Z = (z_{ij})$  and coordinates  $(z_{ij})_{1 \leq i < j \leq p+3}$  on  $\mathbb{P}(\bigwedge^2 T^\vee)$ . Consider the Grassmannian  $G = G(2, T)$  of 2-dimensional quotients of  $T$ . The ideal of  $G$  under the Plücker embedding  $G \subset \mathbb{P}(\bigwedge^2 T^\vee)$  is generated by the  $4 \times 4$  Pfaffians  $\text{Pf}_{ijkl}(Z)$  of the matrix  $Z$ . Taking exterior powers in the exact sequence

$$0 \rightarrow \langle t \rangle \rightarrow T \rightarrow W \rightarrow 0$$



we obtain an exact sequence

$$0 \rightarrow \langle t \rangle \otimes W \rightarrow \bigwedge^2 T \rightarrow \bigwedge^2 W \rightarrow 0.$$

The linear subspace  $\mathbb{P}(\bigwedge^2 W^\vee) \subset \mathbb{P}(\bigwedge^2 T^\vee)$  is defined by the vanishing of the linear forms  $z_{1j}$ ,  $j = 2, \dots, p+3$ . A straightforward computation then shows that the ideal of the union

$$G(2, T) \cup \mathbb{P}(\bigwedge^2 W^\vee) \subset \mathbb{P}(\bigwedge^2 T^\vee)$$

is generated by the Pfaffians  $\text{Pf}_{1ijk}(Z)$ . The tautological exact sequence

$$0 \rightarrow S \rightarrow T \otimes \mathcal{O}_G \rightarrow Q \rightarrow 0$$

induces an isomorphism  $T \cong H^0(G, Q)$ . Under this isomorphism, we have  $G(2, W) = V(t)$ .

As in Section 2.3 we associate to the Koszul class  $\alpha$  a matrix  $A = (a_{ij})$  of linear forms such that

- (a) the linear forms in the first row of  $A$  span  $W$ ;
- (b) there exists a nonzero  $4 \times 4$  Pfaffian of  $A$  involving the first row and column;
- (c) the  $4 \times 4$  Pfaffians involving the first row and column of  $A$  vanish on the image of  $X$  in  $\mathbb{P}H^0(X, L)^\vee$ .

Let  $C$  be the base locus of the image of  $A$ . Replacing  $L$  by  $L(-C)$  if necessary ( $W$  is obviously contained in the image of  $A$ ) we can suppose that  $C$  is empty, hence the matrix  $A$  defines a morphism

$$\psi: X \rightarrow \mathbb{P}(\bigwedge^2 T^\vee).$$

Condition (c) implies that the image  $Y = \psi(X)$  is contained in the union  $G(2, T) \cup \mathbb{P}(\bigwedge^2 W^\vee)$ , and condition (a) shows that  $Y$  is not contained in  $\mathbb{P}(\bigwedge^2 W^\vee)$ . As  $Y$  is irreducible, this implies that  $Y$  is contained in  $G(2, T)$ .

Put  $E = \psi^* Q$ . Twisting the exact sequence

$$0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{O}_G \rightarrow \psi_* \mathcal{O}_X \rightarrow 0$$

by the universal quotient bundle  $Q$  and taking global sections, we obtain an exact sequence

$$0 \rightarrow H^0(G, Q \otimes \mathcal{I}_Y) \rightarrow H^0(G, Q) \xrightarrow{\psi^*} H^0(G, \psi_* \mathcal{O}_X \otimes Q) \cong H^0(X, E).$$

Condition (a) implies that  $Y$  is not contained in  $G(2, W) = G(2, T) \cap \mathbb{P}(\bigwedge^2 W^\vee)$ , hence  $t$  does not vanish identically on  $X$  and defines a global section of  $E$ . The zero locus of this section is given by the equations  $a_{12} = \dots = a_{1,p+3} = 0$ , hence

it coincides with the base locus  $B$  of the sublinear system of  $|L|$  induced by  $W$ . Consequently the line bundle  $E$  is given by an extension

$$0 \rightarrow \mathcal{O}_X(B) \rightarrow E \rightarrow L(-B) \rightarrow 0. \quad (6)$$

Consider the commutative diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ H^0(G, \mathcal{O}_G) & \longrightarrow & H^0(X, \mathcal{O}_X(B)) \\ \downarrow \cdot t & & \downarrow \cdot \psi^*(t) \\ H^0(G, Q) & \xrightarrow{\psi^*} & H^0(X, E) \\ \downarrow & & \downarrow d_t \\ W & \xrightarrow{i} & H^0(X, L(-B)). \end{array}$$

Note that  $\ker i = W \cap H^0(G, \mathcal{O}_G(1) \otimes \mathcal{I}_Y) = 0$  by condition (a). As the map  $H^0(G, Q) \rightarrow W$  is surjective, we find that  $W$  is contained in the image of the map  $d_t: H^0(X, E) \rightarrow H^0(X, L(-B))$ . The embedding  $W \subset H^0(G, Q) = \langle t \rangle \oplus W$  composed with  $\psi^*$  is a section of  $d_t$ . Put  $U = \psi^*(W)$ . By construction we obtain  $\gamma = \gamma(U, t)$ .  $\square$

**Remark 3.5.** The union  $G(2, T) \cup \mathbb{P}(\bigwedge^2 W^\vee)$  is a generic syzygy scheme; see [6, Theorem 6.1]. In [loc. cit., Theorem 6.7] it was shown that a rank  $p+2$  syzygy gives rise to a rank 2 vector bundle if  $L$  is very ample and the ideal of  $X$  is generated by quadrics.

The condition of Theorem 3.1 can be reinterpreted in terms of surjectivity of a natural multiplication map.

**Proposition 3.6.** *Let  $X$  be a smooth curve, and let  $W \subset H^0(X, L)$  be a linear subspace. We put  $B = \text{Bs}(W)$  and view  $W$  as a base-point free linear subspace of  $H^0(X, L(-B))$ . Let*

$$\mu: W \otimes H^0(X, K_X(-B)) \rightarrow H^0(K_X \otimes L(-2B))$$

*be the multiplication map. The following conditions are equivalent.*

- (i) *The map  $\mu$  is not surjective.*
- (ii) *There exists a non-split extension  $0 \rightarrow \mathcal{O}_X(B) \rightarrow E \rightarrow L(-B) \rightarrow 0$  such that  $W$  is contained in the kernel of the map  $\delta: H^0(X, L(-B)) \rightarrow H^1(X, \mathcal{O}_X(B))$ .*

*Proof.* We first show that (i) implies (ii). Since  $\mu$  is not surjective, there exists a hyperplane  $H \subset H^0(X, K_X \otimes L(-2B))$  that contains  $\text{im}(\mu)$ . Let  $\eta$  be a linear functional defining  $H$ . Put  $0 \neq \xi = \eta^\vee \in H^1(X, L^{-1}(2B))$ , and let

$$0 \rightarrow \mathcal{O}_X(B) \rightarrow E \rightarrow L(-B) \rightarrow 0$$

be the corresponding non-split extension. Given  $w \in W$  and  $v \in H^0(X, K_X(-B))$ , the formula

$$\delta(w)(v) = (\eta \circ \mu)(w \otimes v) \quad (7)$$

shows that  $W$  is contained in the kernel of  $\delta$ .

For the converse, note that formula (7) implies that  $\eta|_{\text{im } \mu} \equiv 0$ .  $\square$

**Remark 3.7.** If  $B$  is a fixed divisor, the result of the previous Proposition follows from Green's duality theorem [4, Corollary (2.c.10)]. Indeed,

$$\text{coker } \mu \cong K_{0,1}(X, K_X(-B), L(-B), W) \cong K_{p,1}(X, B, L(-B), W)^\vee \quad (8)$$

and since  $h^0(X, \mathcal{O}_X(B)) = 1$  we have an injection

$$K_{p,1}(X, B, L(-B), W) \hookrightarrow K_{p,1}(X, L).$$

Theorem 3.4 shows that Voisin's method may produce nontrivial Koszul classes that are not contained in the space  $K_{p,1}(X, L)_{\text{GL}}$  spanned by Green–Lazarsfeld classes.

**Example 3.8.** By [2, Theorem 3.6 and Theorem 4.3] there exists a smooth curve of genus 14 and Clifford index 5 whose Clifford index is computed by a unique line bundle  $L$  such that  $L^2 = K_X$ . The line bundle  $L$  embeds  $X$  in  $\mathbb{P}^4$  as a projectively normal curve of degree 13 which is not contained in any quadric of rank  $\leq 4$ , and the ideal of  $X$  is generated by the  $4 \times 4$  Pfaffians of a skew-symmetric matrix  $(a_{ij})_{1 \leq i, j \leq 5}$  with

$$\deg(a_{ij}) = \begin{cases} 2 & \text{if } i = 1 \text{ or } j = 1 \\ 1 & \text{if } i \geq 2 \text{ and } j \geq 2 \end{cases}$$

such that the quadric  $Q = a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34}$  has rank 5.

By [loc.cit.] the group  $K_{1,1}(X, L)$  is generated by  $[Q]$ , hence  $I_X$  contains no quadrics of rank  $\leq 4$ . If  $K_{1,1}(X, L)$  contains a Green–Lazarsfeld class this class would be of scrollar type, since it necessarily comes from two pencils  $|L_1|$ ,  $|L_2|$ . This is impossible, since classes of scrollar type give rise to quadrics of rank  $\leq 4$ .

The Koszul class  $[Q] \in K_{1,1}(X, L)$  has rank 3, since it is represented by the linear subspace  $W = \langle a_{23}, a_{24}, a_{25} \rangle$ . Hence  $[Q]$  comes from Voisin's method by Theorem 3.4.

**Remark 3.9.** A more geometric description of a subspace  $W$  representing  $[Q]$  is the following. A smooth intersection of the quadric  $V(Q) \subset \mathbb{P}H^0(X, L)^\vee$  with one of the cubic Pfaffians is a  $K3$  surface in  $\mathbb{P}H^0(X, L)^\vee$  containing a line  $\ell$  which is disjoint from  $X$  by [2, Proposition 4.1]. The line  $\ell$  corresponds to a 3-dimensional linear subspace  $W \subset H^0(X, L)$ , which is base-point-free since  $\ell$  does not meet  $X$ .

One could ask whether the syzygies constructed in Section 2.1 span  $K_{p,1}(X, L)$ . In principle it may be possible to obtain higher rank syzygies as linear combinations of rank  $p + 2$  syzygies. However, if  $K_{p,1}(X, L)$  is spanned by a single syzygy of rank  $\geq p + 3$  this is not possible.

**Example 3.10** (Eusen–Schreyer). Eusen and Schreyer [3, Theorem 1.7 (a)] have constructed a smooth curve  $X \subset \mathbb{P}^5$  of genus 7 and Clifford index 3 embedded by the linear system  $|K_X(-x)|$  such that  $K_{2,1}(X, K_X(-x)) \cong \mathbb{C}$  is spanned by a syzygy  $s_0$ . The explicit expression for  $s_0$  given on p. 8 of [loc. cit.] shows that  $s_0$  is a rank 5 syzygy. Hence  $s_0$  cannot be obtained by the Green–Lazarsfeld construction or the method of Section 2.1.

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