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Asymptotic non-degeneracy of the solution to the Liouville–Gel'fand problem in two dimensions

Tomohiko Sato and Takashi Suzuki

Abstract. In this paper we study the asymptotic non-degeneracy of the solution to the Liouville–Gel'fand problem

$$-\Delta v = \lambda V(x) e^{v} \ \text{ in } \Omega, \quad v = 0 \ \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, $V(x)$ is a positive-valued $C^1(\overline{\Omega})$ function, and $\lambda > 0$ is a constant.

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1. Introduction

The purpose of the present paper is to study the asymptotic non-degeneracy of the solution to the Liouville–Gel'fand problem

$$-\Delta v = \lambda V(x) e^{v} \ \text{ in } \Omega, \quad v = 0 \ \text{ on } \partial\Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $V = V(x) > 0$ is a C^1 function defined on $\overline{\Omega}$, and $\lambda > 0$ is a constant. We shall extend a result of Gladiali–Grossi [5], which is valid for the homogeneous case of $V(x) \equiv 1$,

$$-\Delta v = \lambda e^{v} \ \text{ in } \Omega, \quad v = 0 \ \text{ on } \partial\Omega \tag{2}$$

based on the following fact [8].

Theorem 1.1. *If (λ_k, v_k) $(k = 1, 2, \dots)$ is a solution sequence for (2) satisfying $\lambda_k \to 0$, then we have a subsequence (denoted by the same symbol) such that $\Sigma_k = \int_\Omega \lambda_k e^{v_k} \to 8\pi m$ for some $m = 0, 1, 2, \dots, +\infty$. According to this value of m, we have the following.*

(1) *If $m = 0$, then it holds that $\|v_k\|_\infty \to 0$.*

(2) *If* $0 < m < +\infty$, *then the blowup set of* v_k ($k = 1, 2, \dots$), *defined by*

$$\mathcal{S} = \left\{x_0 \in \overline{\Omega} \mid \text{there exists } x_k \to x_0 \text{ such that } v_k(x_k) \to +\infty\right\},$$

is composed of m*-interior points, and* $v_k \to 8\pi \sum_{x_0 \in \mathcal{S}} G(\,\cdot\,, x_0)$ *locally uniformly in* $\overline{\Omega} \setminus \mathcal{S}$, *where* $G = G(x, y)$ *denotes the Green's function of* $-\Delta$ *in* Ω *with* $\cdot|_{\partial\Omega} = 0$. *We have* $-\Delta v_k(x)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} 8\pi \delta_{x_0}(dx)$ *in the sense of measure on* $\overline{\Omega}$. *Furthermore, it holds that*

$$\frac{1}{2}\nabla R(x_0) + \sum_{x_0' \in \mathcal{S} \setminus \{x_0\}} \nabla_x G(x_0, x_0') = 0 \tag{3}$$

for each $x_0 \in \mathcal{S}$, *where* $R(x) = \left[G(x, y) + \frac{1}{2\pi} \log|x - y|\right]_{y=x}$ *is the Robin function.*

(3) *If* $m = +\infty$, *then* $v_k \to +\infty$ *locally uniformly in* Ω.

Gladiali and Grossi [5] are concerned with the case $m = 1$, and study the non-degeneracy of (λ_k, v_k) for large k. From the above theorem, we have $\mathcal{S} = \{x_0\}$ if $m = 1$ and this $x_0 \in \Omega$ is a critical point of the Robin function. What they obtained is the following theorem, motivated by the study of the detailed bifurcation diagram for (2).

Theorem 1.2. *If* $m = 1$ *holds in the previous theorem and* $x_0 \in \mathcal{S}$ *is a non-degenerate critical point of* $R(x)$, *then the solution* (λ_k, v_k) *is non-degenerate for large* k, *that is, the linearized operator* $-\Delta - \lambda_k e^{v_k}$ *in* Ω *with* $\cdot|_{\partial\Omega} = 0$ *is invertible.*

Theorem 1.1, on the other hand, has an extension to (1). Although the results of Ma–Wei [7] are presented in the mean field formulation,

$$-\Delta v = \frac{\lambda V(x) e^v}{\int_\Omega V(x) e^v} \quad \text{in } \Omega, \qquad v = 0 \ \text{ on } \partial\Omega,$$

it is easy to translate them into the following theorem on (1). (See also [9].)

Theorem 1.3. *All the results stated in Theorem* 1.1 *continue to hold for* (1), *provided that* Σ_k *and* (3) *are replaced by* $\Sigma_k = \int_\Omega \lambda_k V(x) e^{v_k}$ *and*

$$\frac{1}{2}\nabla R(x_0) + \sum_{x_0' \in \mathcal{S} \setminus \{x_0\}} \nabla_x G(x_0, x_0') + \frac{1}{8\pi} \nabla \log V(x_0) = 0, \tag{4}$$

respectively.

In the case of $m = 1$ again, equation (4) means that $x_0 \in \Omega$ is a critical point of $R(x) + \frac{1}{4\pi}\log V(x)$. From this point of view, it is natural to extend Theorem 1.2 as follows.

Theorem 1.4. *In Theorem 1.3, if $m = 1$, $V(x)$ is C^2 near $x_0 \in \mathcal{S}$, and x_0 is a non-degenerate critical point of $R(x) + \frac{1}{4\pi}\log V(x)$, then the solution (λ_k, v_k) is non-degenerate for large k, that is, the linearized operator $-\Delta - \lambda_k V(x)e^{v_k}$ in Ω with $\cdot|_{\partial\Omega} = 0$ is invertible.*

To prove the above theorem, we follow the argument of [5], namely, the existence of $w_k = w_k(x)$ $(k = 1, 2, \dots)$ satisfying

$$\begin{aligned} &-\Delta w_k = \lambda_k V(x)e^{v_k} w_k \ \text{ in } \Omega, \quad w_k = 0 \ \text{ on } \partial\Omega, \\ &\|w_k\|_\infty = 1, \end{aligned} \tag{5}$$

implies a contradiction. The next section is devoted to examine the validity of the blowup analysis [5] to (1), originally developed for (2). In the latter case, $w_k' = \frac{\partial v_k}{\partial x_i}$ $(i = 1, 2)$ solves the linearized equation

$$-\Delta w_k' = \lambda_k e^{v_k} w_k' \quad \text{in } \Omega$$

(except for the boundary condition). This structure is useful to prove Theorem 1.2, but obviously does not hold in (1). In the final section, we complete the proof of Theorem 1.4, providing new arguments to compensate this obstruction.

2. Preliminaries

In this section, we confirm that several assertions for (2) presented in [5] are still valid for (1). Henceforth, (λ_k, v_k) $(k = 1, 2, \dots)$ is a solution sequence for (1) satisfying

$$\Sigma_k = \int_\Omega \lambda_k V(x)e^{v_k} \to 8\pi, \quad \lambda_k \to 0, \tag{6}$$

and $x_k \in \Omega$ denotes a maximum point of v_k:

$$v_k(x_k) = \|v_k\|_\infty .$$

Then we have $x_k \to x_0$ with $\mathcal{S} = \{x_0\}$, and this blowup point $x_0 \in \Omega$ is a critical point of $R(x) + \frac{1}{4\pi}\log V(x)$.

The first lemma corresponds to Theorem 6 of [5].

Lemma 2.1. *There is a constant $C_1 > 0$ such that*

$$\left| v_k(x) - \log \frac{e^{v_k(x_k)}}{\left\{1 + \frac{1}{8}\lambda_k V(x_k)e^{v_k(x_k)}\,|x - x_k|^2\right\}^2} \right| \le C_1 \tag{7}$$

for any $x \in \overline{\Omega}$ and $k = 1, 2, \dots$.

Proof. Putting $u_k = v_k + \log\lambda_k$, we obtain

$$-\Delta u_k = V(x)e^{u_k} \ \text{ in } \Omega, \qquad u_k = \log\lambda_k \ \text{ on } \partial\Omega,$$
$$\int_\Omega e^{u_k} = O(1).$$

Passing to a subsequence, we shall show that $u_k(x_k) \to +\infty$ holds. Then, Theorem 0.3 of Y. Y. Li [6] guarantees the existence of $C_1 > 0$ such that

$$\left| u_k(x) - \log \frac{e^{u_k(x_k)}}{\left\{1 + \frac{1}{8}V(x_k)e^{u_k(x_k)}\,|x - x_k|^2\right\}^2} \right| \le C_1$$

for any $x \in \overline{\Omega}$ and $k = 1, 2, \ldots$, or, equivalently, (7).

In fact, if $u_k(x_k) \to +\infty$ does not occur, then we may assume either $u_k(x_k) \to -\infty$ or $u_k(x_k) \to c \in \mathbb{R}$. In the first alternative, we have

$$\int_\Omega \lambda_k e^{v_k} \to 0,$$

which is impossible by (6), because there are $a, b > 0$ such that

$$a \le V(x) \le b \quad (x \in \overline{\Omega}).$$

In the second alternative, on the other hand, the sequence $\{u_k\}$ is locally uniformly bounded in Ω by Brezis–Merle [1], while Theorem 1.3 guarantees $u_k = v_k + \log\lambda_k \to -\infty$ locally uniformly in $\overline{\Omega} \setminus \{x_0\}$. Again, we have a contradiction, and the proof is complete. □

Now we define $\delta_k > 0$ by

$$\delta_k^2 \lambda_k e^{v_k(x_k)} = 1. \tag{8}$$

The next lemma corresponds to Lemma 5 of [5].

Lemma 2.2. *It holds that $\delta_k \to 0$.*

Proof. Inequality (7) reads

$$\left| v_k(x) - v_k(x_k) + \log\left\{1 + \frac{V(x_k)}{8\delta_k^2}\,|x - x_k|^2\right\}^2 \right| \le C_1$$

for $x \in \overline{\Omega}$ and $k = 1, 2, \ldots$, and we have $v_k \to 8\pi G(\,\cdot\,, x_0)$ locally uniformly in $\overline{\Omega} \setminus \{x_0\}$, $V(x_k) \to V(x_0)$, and $v_k(x_k) \to +\infty$. These imply $\delta_k \to 0$, because otherwise we have a contradiction. □

We assume the existence of $w_k = w_k(x)$ satisfying (5) and derive a contradiction. For this purpose, we put

$$\begin{aligned}\tilde{v}_k(x) &= v_k(x_k + \delta_k x) - v_k(x_k),\\ \tilde{w}_k(x) &= w_k(x_k + \delta_k x),\\ \tilde{V}_k(x) &= V(x_k + \delta_k x),\end{aligned}$$

where $x \in \tilde{\Omega}_k$ for $\tilde{\Omega}_k = \left\{x \in \mathbb{R}^2 \mid x_k + \delta_k x \in \Omega\right\}$. We have

$$\begin{aligned}&-\Delta \tilde{v}_k = \tilde{V}_k e^{\tilde{v}_k}, \quad \tilde{v}_k \leq 0 = \tilde{v}_k(0) \text{ in } \tilde{\Omega}_k,\\ &\int_{\tilde{\Omega}_k} e^{\tilde{v}_k} = \int_\Omega \lambda_k e^{v_k} \leq C_2\end{aligned}$$

with a constant $C_2 > 0$ independent of k, and

$$\begin{aligned}&-\Delta \tilde{w}_k = \tilde{V}_k e^{\tilde{v}_k} \tilde{w}_k \text{ in } \tilde{\Omega}_k, \quad \tilde{w}_k = 0 \text{ on } \partial\tilde{\Omega}_k,\\ &\|\tilde{w}_k\|_\infty = 1.\end{aligned}$$

Concerning $\tilde{v}_k$, we can apply [1]. Thus, passing to a subsequence, we obtain $\tilde{v}_k \to \tilde{v}_0$ in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$ for $0 < \alpha < 1$, with $\tilde{v}_0 = \tilde{v}_0(x)$ satisfying

$$-\Delta \tilde{v}_0 = V(x_0) e^{\tilde{v}_0}, \quad \tilde{v}_0 \leq 0 = \tilde{v}_0(0) \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^{\tilde{v}_0} < +\infty,$$

and therefore

$$\tilde{v}_0(x) = \log \frac{1}{\left\{1 + \frac{1}{8} V(x_0)\,|x|^2\right\}^2}$$

by [4]. This implies $\tilde{w}_k \to \tilde{w}_0$ in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$ for a subsequence, with $\tilde{w}_0 = \tilde{w}_0(x)$ satisfying

$$\begin{aligned}&-\Delta \tilde{w}_0 = V(x_0) e^{\tilde{v}_0} \tilde{w}_0 = \frac{V(x_0)}{\left\{1 + \frac{1}{8} V(x_0)\,|x|^2\right\}^2} \tilde{w}_0 \quad \text{in } \mathbb{R}^2,\\ &\|\tilde{w}_0\|_\infty \leq 1.\end{aligned} \tag{9}$$

We shall show $\tilde{w}_0 = 0$ in $\mathbb{R}^2$. In fact, if this is the case, then it holds that $|y_k| \to +\infty$, where $y_k \in \tilde{\Omega}_k$ denotes a maximum point of $\tilde{w}_k = \tilde{w}_k(x)$; $\tilde{w}_k(y_k) = \|\tilde{w}_k\|_\infty = 1$. We make the Kelvin transformation

$$\hat{v}_k(x) = \tilde{v}_k\left(\frac{x}{|x|^2}\right), \quad \hat{w}_k(x) = \tilde{w}_k\left(\frac{x}{|x|^2}\right),$$

and obtain

$$\|\hat{w}_k\|_\infty = \hat{w}_k\left(\frac{y_k}{|y_k|^2}\right) = 1,$$
$$-\Delta\hat{w}_k = \frac{1}{|x|^4}\tilde{V}_k\left(\frac{x}{|x|^2}\right)e^{\hat{v}_k}\hat{w}_k \quad \text{in } B_1(0)\setminus\{0\}$$

for large k. On the other hand, inequality (7) reads

$$\left|\tilde{v}_k(x) + \log\left\{1 + \frac{1}{8}V(x_k)\,|x|^2\right\}^2\right| \le C_1 \tag{10}$$

for $x \in \tilde{\Omega}_k$ and $k = 1, 2, \dots$, and we have $e^{\tilde{v}_k(x)} = O\left(\frac{1}{|x|^4}\right)$ uniformly in k. This means $\frac{1}{|x|^4}e^{\hat{v}_k(x)} = O(1)$ uniformly in k, and therefore $x = 0$ is a removable singularity of $\hat{w}_k$,

$$-\Delta\hat{w}_k = a_k(x)\hat{w}_k \quad \text{in } B_1(0)$$

with $a_k = a_k(x)$ satisfying $\|a_k\|_{L^\infty(B_1(0))} = O(1)$. Then, the local elliptic estimate guarantees $1 = \|\hat{w}_k\|_{L^\infty(B_{1/2}(0))} \le C\|\hat{w}_k\|_{L^2(B_1(0))}$, where the right-hand side converges to 0 by the dominated convergence theorem. This is a contradiction and we obtain the proof of Theorem 1.4.

To prove $\tilde{w}_0 = 0$ in $\mathbb{R}^2$, we put $c = V(x_0) > 0$ and $v(x) = \tilde{w}_0(x/\sqrt{c})$ in (9). Then, this $v = v(x) \in L^\infty(\mathbb{R}^2)$ satisfies

$$-\Delta v = \frac{v}{\left\{1 + \frac{1}{8}|x|^2\right\}^2} \quad \text{in } \mathbb{R}^2$$

and hence it holds that

$$v(x) = \sum_{i=1}^{2}\frac{a_i x_i}{8 + |x|^2} + b\cdot\frac{8 - |x|^2}{8 + |x|^2}$$

by [2], where $a_i, b \in \mathbb{R}$. Thus, we only have to derive $a_i = b = 0$ in

$$\tilde{w}_0(x) = \sum_{i=1}^{2}\frac{a_i x_i}{\frac{8}{c} + |x|^2} + b\cdot\frac{\frac{8}{c} - |x|^2}{\frac{8}{c} + |x|^2}.$$

We note that $a_i/\sqrt{c}$ (a_i in the formula for $v(x)$) is newly denoted by a_i.

To show $a_i = 0$, we use the following lemma, proven similarly to (3.13) in [5].

Lemma 2.3. *In case $(a_1, a_2) \neq (0, 0)$, it holds that*

$$\delta_k^{-1}w_k(x) = 2\pi\sum_{j=1}^{2}a_j\frac{\partial G}{\partial y_j}(x, x_0) + o(1) \tag{11}$$

locally uniformly in $x \in \overline{\Omega}\setminus\{x_0\}$.

Proof. In fact, we have

$$\begin{aligned} w_k(x) &= \int_\Omega G(x,y)\lambda_k V(y)e^{v_k(y)}w_k(y)dy \\ &= \int_{\tilde\Omega_k} G(x, x_k+\delta_k y')\tilde V_k(y')e^{\tilde v_k(y')}\tilde w_k(y')dy' = \mathrm{I}_{1,k}(x)+\mathrm{I}_{2,k}(x), \end{aligned}$$

where

$$\begin{aligned} \mathrm{I}_{1,k}(x) &= \int_{\tilde\Omega_k} G(x, x_k+\delta_k y')\cdot f_k(y')dy' \\ \mathrm{I}_{2,k}(x) &= \int_{\tilde\Omega_k} G(x, x_k+\delta_k y')\cdot \frac{64b}{c}\cdot\frac{\frac{8}{c}-|y'|^2}{\left(\frac{8}{c}+|y'|^2\right)^3}dy' \end{aligned}$$

with

$$f_k(y) = \tilde V_k(y)e^{\tilde v_k(y)}\tilde w_k(y) - \frac{64b}{c}\cdot\frac{\frac{8}{c}-|y|^2}{\left(\frac{8}{c}+|y|^2\right)^3}.$$

We have

$$\tilde V_k(y)e^{\tilde v_k(y)}\tilde w_k(y) \to c\cdot\frac{1}{\left(1+\frac{c}{8}|y|^2\right)^2}\cdot\left(\sum_{i=1}^2\frac{a_i y_i}{\frac{8}{c}+|y|^2}+b\cdot\frac{\frac{8}{c}-|y|^2}{\frac{8}{c}+|y|^2}\right),$$

or equivalently,

$$f_k(y)\to f_0(y) = \frac{64}{c}\sum_{i=1}^2\frac{a_i y_i}{\left(\frac{8}{c}+|y|^2\right)^3},$$

locally uniformly in $y\in\mathbb{R}^2$.

We have, on the other hand, $f_k(y) = O\left(\frac{1}{|y|^4}\right)$ uniformly in $k=1,2,\dots$ by (10), and therefore $g_k(y)\to g_0(y)$ locally uniformly in $y\in\mathbb{R}^2$ by the dominated convergence theorem, where

$$g_k(y_1,y_2) = -\int_{\frac{a_1y_1+a_2y_2}{a_1^2+a_2^2}}^{+\infty} f_k\left(a_1t+\frac{a_2^2y_1-a_1a_2y_2}{a_1^2+a_2^2}, a_2t-\frac{a_1a_2y_1-a_1^2y_2}{a_1^2+a_2^2}\right)dt$$

for $k=0,1,2,\dots$. This g_k, introduced in Lemma 6 of [5], satisfies

$$a_1\frac{\partial g_k}{\partial y_1}+a_2\frac{\partial g_k}{\partial y_2} = f_k,$$

and therefore it holds that

$$
\begin{aligned}
\mathrm{I}_{1,k}(x) &= \int_{\tilde{\Omega}_k} G(x, x_k + \delta_k y') f_k(y') dy' \\
&= \int_{\tilde{\Omega}_k} G(x, x_k + \delta_k y') \cdot \sum_{j=1}^{2} a_j \frac{\partial g_k}{\partial y'_j}(y') dy' \\
&= -\delta_k \sum_{j=1}^{2} a_j \int_{\tilde{\Omega}_k} \frac{\partial G}{\partial y_j}(x, x_k + \delta_k y') \cdot g_k(y') dy' \\
&= \delta_k \Big\{ \sum_{j=1}^{2} a_j \frac{\partial G}{\partial y_j}(x, x_0) \int_{\mathbb{R}^2} \frac{16}{c} \cdot \frac{1}{\left(\frac{8}{c} + |y'|^2\right)^2} dy' + o(1) \Big\} \\
&= \delta_k \Big\{ 2\pi \sum_{j=1}^{2} a_j \frac{\partial G}{\partial y_j}(x, x_0) + o(1) \Big\}
\end{aligned}
$$

locally uniformly in $x \in \overline{\Omega} \setminus \{x_0\}$ by the dominated convergence theorem.

To study $\mathrm{I}_{2,k}(x)$, we note that $u(y) = \log \frac{64}{c} \cdot \frac{1}{\left(\frac{8}{c}+|y|^2\right)^2}$ satisfies

$$
\frac{\partial}{\partial y_1}\left(y_1 e^u\right) + \frac{\partial}{\partial y_2}\left(y_2 e^u\right) = \frac{128}{c} \cdot \frac{\frac{8}{c} - |y|^2}{\left(\frac{8}{c} + |y|^2\right)^3},
$$

and in this case we obtain

$$
\begin{aligned}
\mathrm{I}_{2,k}(x) &= \frac{b}{2} \int_{\tilde{\Omega}_k} G(x, x_k + \delta_k y') \cdot \sum_{j=1}^{2} \frac{\partial}{\partial y_j}\left(y_j e^{u(y)}\right)\Big|_{y=y'} dy' \\
&= -\delta_k \frac{b}{2} \sum_{j=1}^{2} \int_{\tilde{\Omega}_k} \frac{\partial G}{\partial y_j}(x, x_k + \delta_k y') \cdot y'_j e^{u(y')} dy' \\
&= -\delta_k \frac{b}{2} \Big\{ \sum_{j=1}^{2} \frac{\partial G}{\partial y_j}(x, x_0) \cdot \int_{\mathbb{R}^2} y'_j e^{u(y')} dy' + o(1) \Big\} = o\left(\delta_k\right)
\end{aligned}
$$

locally uniformly in $x \in \overline{\Omega} \setminus \{x_0\}$, again by the dominated convergence theorem. Thus, the proof of (11) is complete. □

3. Proof of Theorem 1.4

We prove the following lemma, using new arguments.

Lemma 3.1. *If $V(x)$ is C^2 near $x = x_0 \in \Omega$ and x_0 is a non-degenerate critical point of $R(x) + \frac{1}{4\pi} \log V(x)$, then it holds that $a_1 = a_2 = 0$.*

Proof. We suppose the contrary, and then obtain (11) locally uniformly in $x \in \overline{\Omega}\setminus\{x_0\}$. We note

$$-\Delta \frac{\partial v_k}{\partial x_i} = \lambda_k V e^{v_k} \frac{\partial v_k}{\partial x_i} + \lambda_k V e^{v_k} \frac{\partial \log V}{\partial x_i} \quad \text{in } \Omega$$

and define $h_{i,k} = h_{i,k}(x)$ by

$$-\Delta h_{i,k} = \frac{\partial \log V}{\partial x_i} \cdot \lambda_k V e^{v_k} \ \text{ in } \Omega, \quad h_{i,k} = 0 \ \text{ on } \partial\Omega,$$

where $i = 1, 2$. Then it follows that

$$w_k \Delta \left(\frac{\partial v_k}{\partial x_i} - h_{i,k} \right) - \Delta w_k \cdot \frac{\partial v_k}{\partial x_i} = 0 \quad \text{in } \Omega$$

by (5), and therefore we have

$$\int_{\partial\Omega} \left\{ w_k \frac{\partial}{\partial \nu} \left(\frac{\partial v_k}{\partial x_i} - h_{i,k} \right) - \frac{\partial w_k}{\partial \nu} \cdot \left(\frac{\partial v_k}{\partial x_i} - h_{i,k} \right) \right\} = \int_{\Omega} h_{i,k} \Delta w_k.$$

Here and henceforth, ν denotes the outer unit normal vector on $\partial\Omega$. Since $w_k = h_{i,k} = 0$ on $\partial\Omega$, the above equation is reduced to

$$\begin{aligned} \delta_k^{-1} \int_{\partial\Omega} \frac{\partial v_k}{\partial x_i} \frac{\partial w_k}{\partial \nu} &= -\delta_k^{-1} \int_{\Omega} h_{i,k} \Delta w_k = -\delta_k^{-1} \int_{\Omega} \Delta h_{i,k} \cdot w_k \\ &= \delta_k^{-1} \int_{\Omega} \frac{\partial \log V}{\partial x_i} \cdot \lambda_k V e^{v_k} \cdot w_k. \end{aligned} \tag{12}$$

We have

$$\begin{aligned} v_k &\to 8\pi G(\,\cdot\,, x_0) && \text{in } C^{2,\alpha}_{\mathrm{loc}}(\overline{\Omega} \setminus \{x_0\}), \\ \delta_k^{-1} w_k &\to 2\pi \sum_{j=1}^{2} a_j \frac{\partial G}{\partial y_j}(\,\cdot\,, x_0) && \text{in } C^{2,\alpha}_{\mathrm{loc}}(\overline{\Omega} \setminus \{x_0\}) \end{aligned}$$

by Theorem 1.3 and the elliptic estimate, and therefore the left-hand side of (12) converges to

$$16\pi^2 \sum_{j=1}^{2} a_j \int_{\partial\Omega} \frac{\partial G}{\partial x_i}(x, x_0) \frac{\partial^2 G}{\partial y_j \partial \nu_x}(x, x_0).$$

Now we apply Lemma 7 of [5]:

$$\int_{\partial\Omega} \frac{\partial G}{\partial x_i}(x, x_0) \frac{\partial^2 G}{\partial y_j \partial \nu_x}(x, x_0) = -\frac{1}{2} \frac{\partial^2 R}{\partial x_i \partial x_j}(x_0), \tag{13}$$

and then obtain

$$\lim_{k\to+\infty} \delta_k^{-1} \int_{\partial\Omega} \frac{\partial v_k}{\partial x_i} \frac{\partial w_k}{\partial \nu} = -8\pi^2 \sum_{j=1}^{2} a_j \frac{\partial^2 R}{\partial x_i \partial x_j}(x_0).$$

We note here that (13) is shown by the Pohozaev identity [10].

Therefore, if we can show

$$\lim_{k\to+\infty} \delta_k^{-1} \int_{\Omega} \frac{\partial \log V}{\partial x_i} \cdot \lambda_k V e^{v_k} \cdot w_k = 2\pi \sum_{j=1}^{2} a_j \frac{\partial^2 \log V}{\partial x_i \partial x_j}(x_0), \tag{14}$$

then

$$\sum_{j=1}^{2} a_j \left\{ \frac{\partial^2 R}{\partial x_i \partial x_j}(x_0) + \frac{1}{4\pi} \frac{\partial^2 \log V}{\partial x_i \partial x_j}(x_0) \right\} = 0$$

follows for $i = 1, 2$, and hence $a_1 = a_2 = 0$ from the assumption.

For this purpose, we use the Taylor expansion around $x_k = (x_{k1}, x_{k2})$ for large k and obtain

$$\begin{aligned} \frac{\partial \log V}{\partial x_i}(x) = \frac{\partial \log V}{\partial x_i}(x_k) + \left[(x_1 - x_{k1}) \frac{\partial}{\partial x_1} + (x_2 - x_{k2}) \frac{\partial}{\partial x_2} \right] \\ \cdot \frac{\partial \log V}{\partial x_i}(x_k) + R_k(x) \, |x - x_k| \end{aligned} \tag{15}$$

for $x = (x_1, x_2)$ with $|R_k(x)| \le r(x, x_k)$, where $r(\,\cdot\,, x_k)$ is uniformly bounded on $\overline{\Omega}$, and near x_0,

$$r(x, x_k) = \sup_{y \in B(x_k, |x - x_k|)} \sum_{i,j} \left| \frac{\partial^2 \log V}{\partial x_i \partial x_j}(y) - \frac{\partial^2 \log V}{\partial x_i \partial x_j}(x_k) \right|.$$

Therefore, this $r(\,\cdot\,, x_k)$ is continuous there, satisfying $r(x_k, x_k) = 0$ and converging to $r(\,\cdot\,, x_0)$ uniformly. We shall show that there exists $C_3 > 0$ such that

$$\delta_k^{-1} \, |(x - x_k) w_k(x)| \le C_3 \tag{16}$$

for any $x \in \overline{\Omega}$ and $k = 1, 2, \ldots$. Then, we have

$$\left| \int_{\Omega} R_k(x) \, |x - x_k| \, \lambda_k V e^{v_k} \delta_k^{-1} w_k \right| \le C_3 \int_{\Omega} r(x, x_k) \lambda_k V e^{v_k} \to 0$$

by $\lambda_k V e^{v_k} dx \rightharpoonup 8\pi \delta_{x_0}(dx)$ and $r(x_0, x_0) = 0$, and therefore the contribution of the residual term of (15) is neglected in the limit of (12).

To show (16), we use

$$w_k(x) = \mathrm{I}_{1,k}(x) + \mathrm{I}_{2,k}(x)$$

with

$$\delta_k^{-1}\mathrm{I}_{1,k}(x) = -\sum_{j=1}^{2} a_j \int_{\tilde{\Omega}_k} \frac{\partial G}{\partial y_j}(x, x_k + \delta_k y') \cdot g_k(y')dy',$$

$$\delta_k^{-1}\mathrm{I}_{2,k}(x) = -\frac{b}{2}\sum_{j=1}^{2} \int_{\tilde{\Omega}_k} \frac{\partial G}{\partial y_j}(x, x_k + \delta_k y') \cdot y_j' e^{u(y')}dy'.$$

There is $C_4 > 0$ such that

$$\left|\frac{\partial G}{\partial y_j}(x, y)\right| \leq C_4 \left|x - y\right|^{-1}$$

for any $(x, y) \in \overline{\Omega} \times \overline{\Omega}$, and therefore

$$\delta_k^{-1}\left|w_k(x)\right| \leq C_4\left(a_1 + a_2 + \frac{b}{2}\right) \cdot \int_{\tilde{\Omega}_k} \left|x - \delta_k y' - x_k\right|^{-1}\left(\left|g_k(y')\right| + \left|y_j'\right| e^{u(y')}\right)dy'$$

holds true. It is obvious that

$$\left|g_k(y)\right| + \left|y_j\right|e^{u(y)} \leq C_5\left(1 + |y|^2\right)^{-\frac{3}{2}}$$

with $C_5 > 0$ independent of $y \in \mathbb{R}^2$ and $k = 1, 2, \ldots$, and hence

$$\delta_k^{-1}\left|w_k(x)\right| \leq C_4C_5\left(a_1 + a_2 + \frac{b}{2}\right)\int_{\tilde{\Omega}_k} \left|x - \delta_k y' - x_k\right|^{-1}\left(1 + \left|y'\right|^2\right)^{-\frac{3}{2}} dy'.$$

This implies

$$\delta_k^{-1}\left|\left(\delta_k x'\right) w_k(x_k + \delta_k x')\right| \leq C_4C_5\left(a_1 + a_2 + \frac{b}{2}\right)\int_{\mathbb{R}^2} \frac{|x'|}{|x' - y'|}\left(1 + \left|y'\right|^2\right)^{-\frac{3}{2}} dy',$$

but we have

$$\int_{\mathbb{R}^2} \frac{|x'|}{|x' - y'|}\left(1 + \left|y'\right|^2\right)^{-\frac{3}{2}} dy' = \int_0^{2\pi} d\theta \int_0^{\infty} \left|x'\right|\left(1 + \left|x' + re^{\imath\theta}\right|^2\right)^{-\frac{3}{2}} dr \leq C_6$$

with $C_6 > 0$ independent of $x' \in \mathbb{R}^2$. Hence (16) follows for $x \in \overline{\Omega}$ and $k = 1, 2, \ldots$. Thus, we have proven that the limit of the right-hand side of (12) is reduced to

$$\lim_{k\to+\infty} \delta_k^{-1}\int_{\Omega} \frac{\partial \log V}{\partial x_i} \cdot \lambda_k V e^{v_k} \cdot w_k = \lim_{k\to+\infty}\left\{\mathrm{II}_{0,k} + \mathrm{II}_{1,k} + \mathrm{II}_{2,k}\right\},$$

where

$$\mathrm{II}_{0,k} = \frac{\partial \log V}{\partial x_i}(x_k) \int_\Omega \lambda_k V e^{v_k} \cdot \delta_k^{-1} w_k,$$
$$\mathrm{II}_{1,k} = \frac{\partial^2 \log V}{\partial x_1 \partial x_i}(x_k) \int_\Omega (x_1 - x_{k1}) \cdot \lambda_k V e^{v_k} \cdot \delta_k^{-1} w_k,$$
$$\mathrm{II}_{2,k} = \frac{\partial^2 \log V}{\partial x_2 \partial x_i}(x_k) \int_\Omega (x_2 - x_{k2}) \, \lambda_k V e^{v_k} \cdot \delta_k^{-1} w_k.$$

First, we have

$$\begin{aligned}\mathrm{II}_{0,k} &= -\frac{\partial \log V}{\partial x_i}(x_k) \int_\Omega \delta_k^{-1} \Delta w_k = -\frac{\partial \log V}{\partial x_i}(x_k) \int_{\partial\Omega} \delta_k^{-1} \frac{\partial w_k}{\partial \nu} \\ &\to -\frac{\partial \log V}{\partial x_i}(x_0) \cdot 2\pi \sum_{j=1}^{2} a_j \int_{\partial\Omega} \frac{\partial^2 G}{\partial \nu_x \partial y_j}(\,\cdot\,, x_0)\end{aligned}$$

and

$$\int_{\partial\Omega} \frac{\partial^2 G}{\partial \nu_x \partial y_j}(\,\cdot\,, x_0) = \int_{\partial B_r(x_0)} \frac{\partial^2 G}{\partial \nu_x \partial y_j}(\,\cdot\,, x_0) = \int_{\partial B_r(x_0)} \frac{\partial^2 G_0}{\partial \nu_x \partial y_j}(\,\cdot\,, x_0) + o(1)$$

as $r \downarrow 0$, where $G_0(x, y) = \frac{1}{2\pi} \log \frac{1}{|x-y|}$. Then it holds that

$$\frac{\partial^2 G_0}{\partial \nu_x \partial y_j}(x, x_0) = -\frac{1}{2\pi} \frac{x_j - x_{0j}}{|x - x_0|^3}$$

for $x \in \partial B_r(x_0)$, and therefore

$$\int_{\partial B_r(x_0)} \frac{\partial^2 G_0}{\partial \nu_x \partial y_j}(\,\cdot\,, x_0) = 0.$$

Thus, we have proven $\lim_{k\to+\infty} \mathrm{II}_{0,k} = 0$.

Next, we have

$$\begin{aligned}&\int_\Omega (x_\ell - x_{k\ell}) \cdot \lambda_k V e^{v_k} \cdot w_k = -\int_\Omega (x_\ell - x_{k\ell}) \, \Delta w_k \\ &\quad = \int_\Omega \frac{\partial w_k}{\partial x_\ell} - \int_{\partial\Omega} (x_\ell - x_{k\ell}) \frac{\partial w_k}{\partial \nu} = \int_{\partial\Omega} \left\{ \nu_\ell w_k - (x_\ell - x_{k\ell}) \frac{\partial w_k}{\partial \nu} \right\} \\ &\quad = -\int_{\partial\Omega} (x_\ell - x_{k\ell}) \frac{\partial w_k}{\partial \nu}\end{aligned}$$

for $\ell = 1, 2$, and this implies

$$\begin{aligned}
\mathrm{II}_{\ell,k} &= -\frac{\partial^2 \log V}{\partial x_\ell \partial x_i}(x_k) \int_{\partial\Omega} (x_\ell - x_{k\ell})\, \delta_k^{-1} \frac{\partial w_k}{\partial \nu} \\
&\to -\frac{\partial^2 \log V}{\partial x_\ell \partial x_i}(x_0) \cdot 2\pi \sum_{j=1}^{2} a_j \int_{\partial\Omega} (x_\ell - x_{0\ell})\, \frac{\partial^2 G}{\partial \nu_x \partial y_j}(\,\cdot\,, x_0).
\end{aligned}$$

Here, we have

$$\begin{aligned}
&\int_{\partial\Omega} (x_\ell - x_{0\ell})\, \frac{\partial^2 G}{\partial \nu_x \partial y_j}(x, x_0) = \int_{\partial\Omega} \frac{\partial}{\partial \nu_x} \left\{ (x_\ell - x_{0\ell})\, \frac{\partial G}{\partial y_j}(x, x_0) \right\} \\
&\quad = \int_{\partial B_r(x_0)} \frac{\partial}{\partial \nu_x} \left\{ (x_\ell - x_{0\ell})\, \frac{\partial G}{\partial y_j}(x, x_0) \right\} \\
&\qquad + \int_{\Omega \setminus B_r(x_0)} \Delta \left[(x_\ell - x_{0\ell})\, \frac{\partial G}{\partial y_j}(x, x_0) \right] \\
&\quad = \int_{\partial B_r(x_0)} \frac{\partial}{\partial \nu_x} \left\{ (x_\ell - x_{0\ell})\, \frac{\partial G}{\partial y_j}(x, x_0) \right\} + 2 \int_{\Omega \setminus B_r(x_0)} \frac{\partial^2 G}{\partial x_\ell \partial y_j}(x, x_0) \\
&\quad = \int_{\partial B_r(x_0)} \frac{\partial}{\partial \nu_x} \left\{ (x_\ell - x_{0\ell})\, \frac{\partial G}{\partial y_j}(x, x_0) \right\} - 2 \int_{\partial B_r(x_0)} \nu_\ell \frac{\partial G}{\partial y_j}(x, x_0) \\
&\quad = \int_{\partial B_r(x_0)} \frac{\partial}{\partial \nu_x} \left\{ (x_\ell - x_{0\ell})\, \frac{\partial G_0}{\partial y_j}(x, x_0) \right\} - 2 \int_{\partial B_r(x_0)} \nu_\ell \frac{\partial G_0}{\partial y_j}(x, x_0) + o(1)
\end{aligned}$$

as $r \downarrow 0$, and the first term of the right-hand side is equal to 0 because

$$\frac{\partial}{\partial \nu_x} \left\{ (x_\ell - x_{0\ell})\, \frac{\partial G_0}{\partial y_j}(x, x_0) \right\} = \frac{x_\ell - x_{0\ell}}{r} \left[\frac{\partial G_0}{\partial y_j}(x, x_0) + r \frac{\partial^2 G_0}{\partial r \partial y_j}(x, x_0) \right] = 0$$

in terms of $r = |x - x_0|$. On the other hand, the second term is equal to

$$-\frac{1}{\pi} \int_{\partial B_r(x_0)} \frac{(x_\ell - x_{0\ell})\left(x_j - x_{0j}\right)}{r^3} = -\delta_{j\ell} = \begin{cases} -1 & (\ell = j), \\ 0 & (\ell \neq j), \end{cases}$$

and therefore

$$\lim_{k \to +\infty} \mathrm{II}_{\ell,k} = 2\pi a_\ell \frac{\partial^2 \log V}{\partial x_\ell \partial x_i}(x_0)$$

holds for $\ell = 1, 2$. We obtain (14), and the proof is complete. □

Once $a_1 = a_2 = 0$ is obtained, then the proof of $b = 0$ is similar to [5]. For the sake of completeness, we confirm the following lemma and conclude the proof of Theorem 1.4.

Lemma 3.2. *Under the assumptions of the previous lemma, it holds that $b = 0$.*

Proof. By Lemma 3.1, we have

$$\tilde{w}_k(x) \longrightarrow b\frac{\frac{8}{c} - |x|^2}{\frac{8}{c} + |x|^2} \quad \text{in } C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^2).$$

We assume $b \neq 0$ and note the equalities

$$-w_k \Delta v_k = \lambda_k V e^{v_k} w_k \quad \text{and} \quad -v_k \Delta w_k = \lambda_k V e^{v_k} v_k w_k$$

in Ω and also

$$\int_\Omega (w_k \Delta v_k - v_k \Delta w_k) = \int_{\partial\Omega} \left(w_k \frac{\partial v_k}{\partial \nu} - v_k \frac{\partial w_k}{\partial \nu} \right) = 0.$$

Then we have

$$\lambda_k \int_\Omega V e^{v_k} w_k = \lambda_k \int_\Omega V e^{v_k} v_k w_k. \tag{17}$$

We also have

$$\begin{aligned} \lambda_k \int_\Omega V e^{v_k} v_k w_k &= \int_{\tilde{\Omega}_k} \tilde{V}_k e^{\tilde{v}_k} \tilde{v}_k \tilde{w}_k + \|v_k\|_\infty \lambda_k \int_\Omega V e^{v_k} w_k \\ &= \int_{\mathbb{R}^2} \frac{c}{\left(1 + \frac{c}{8}|x|^2\right)^2} \cdot \log \frac{1}{\left(1 + \frac{c}{8}|x|^2\right)^2} \cdot b\frac{\frac{8}{c} - |x|^2}{\frac{8}{c} + |x|^2} dx \\ &\quad + \|v_k\|_\infty \lambda_k \int_\Omega V e^{v_k} w_k + o(1) \\ &= 8\pi b + \|v_k\|_\infty \lambda_k \int_\Omega V e^{v_k} w_k + o(1) \end{aligned}$$

by (7), and therefore

$$8\pi b = (1 - \|v_k\|_\infty) \lambda_k \int_\Omega V e^{v_k} w_k + o(1) \tag{18}$$

by (17).

We shall show

$$\frac{\partial w_k}{\partial x_i} = o(\delta_k) \quad \text{locally uniformly in } \overline{\Omega} \setminus \{x_0\} \tag{19}$$

for $i = 1, 2$ and

$$\|v_k\|_\infty = -2\log \lambda_k + 2\log \frac{8}{c} - 8\pi R(x_0) + o(1). \tag{20}$$

In fact, if this is the case we obtain $\lambda_k \sim \delta_k^2$ by (8), and therefore

$$\|v_k\|_\infty \lambda_k \int_\Omega V e^{v_k} w_k = - \|v_k\|_\infty \int_{\partial\Omega} \frac{\partial w_k}{\partial \nu} = o\left(\delta_k \log \lambda_k\right) = o(1).$$

Then, $b = 0$ follows from (18).

Proof of (19). In fact, we have

$$\frac{\partial w_k}{\partial x_i} = \lambda_k \int_\Omega \frac{\partial G}{\partial x_i}(x, y) \cdot V(y) e^{v_k(y)} w_k(y) dy = \int_{\tilde{\Omega}_k} \frac{\partial G}{\partial x_i}(x, x_k + \delta_k y') h_k(y') dy'$$

with

$$h_k(y) = \tilde{V}_k(y) e^{\tilde{v}_k(y)} \tilde{w}_k(y) = O\left(\frac{1}{|y|^4}\right)$$

uniformly in k and

$$h_k(y) \to h_0(y) = 64bc \frac{8 - c\,|y|^2}{\left(8 + c\,|y|^2\right)^3}$$

locally uniformly in $y \in \mathbb{R}^2$. Therefore $\zeta_k(y) \to \zeta_0(y)$ locally uniformly in $y \in \mathbb{R}^2$ for $\zeta_k = \zeta_k(y)$ defined in Lemma 6 of [5]:

$$\zeta_k(y_1, y_2) = \log \left[\frac{1}{y_1^2 + y_2^2} \int_{-\infty}^{\sqrt{y_1^2 + y_2^2}} t h_k \left(\frac{t y_1}{\sqrt{y_1^2 + y_2^2}}, \frac{t y_2}{\sqrt{y_1^2 + y_2^2}} \right) dt \right].$$

Here we have

$$\left(y_1 \frac{\partial \zeta_k}{\partial y_1} + y_2 \frac{\partial \zeta_k}{\partial y_2} + 2 \right) e^{\zeta_k} = h_k$$

and

$$\zeta_0(y) = \log \frac{32bc}{\left(8 + c\,|y|^2\right)^2},$$

and the dominated convergence theorem guarantees

$$\begin{aligned}
\frac{\partial w_k}{\partial x_i}(x) &= \int_{\tilde{\Omega}_k} \frac{\partial G}{\partial x_i}(x, x_k + \delta_k y') \cdot \left(y_1 \frac{\partial \zeta_k}{\partial y_1} + y_2 \frac{\partial \zeta_k}{\partial y_2} + 2 \right) e^{\zeta_k} \Big|_{y = y'} dy' \\
&= - \sum_{j=1}^{2} \delta_k \int_{\tilde{\Omega}_k} \frac{\partial^2 G}{\partial x_i \partial y_j}(x, x_k + \delta_k y') \cdot y_j' e^{\zeta_k(y')} dy' \\
&= -\delta_k \left\{ \sum_{j=1}^{2} \frac{\partial^2 G}{\partial x_i \partial y_j}(x, x_0) \int_{\mathbb{R}^2} \frac{32bc y_j'}{\left(8 + c|y'|^2\right)^2} dy' + o(1) \right\} = o\left(\delta_k\right)
\end{aligned}$$

and hence (19).

Proof of (20). We have $G(x,y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} + K(x,y)$ with $K \in C^{2,\alpha}(\Omega\times\overline{\Omega})$, and therefore it follows that

$$\|v_k\|_\infty = v_k(x_k) = \mathrm{III}_{1,k} + \mathrm{III}_{2,k},$$

where

$$\mathrm{III}_{1,k} = -\frac{\lambda_k}{2\pi}\int_\Omega \log|x_k - y|\cdot V(y)e^{v_k(y)}dy,$$
$$\mathrm{III}_{2,k} = \lambda_k\int_\Omega K(x_k,y)V(y)e^{v_k(y)}dy.$$

We have $\lambda_k V e^{v_k}dx \rightharpoonup 8\pi\delta_{x_0}(dx)$, and therefore

$$\mathrm{III}_{2,k} = 8\pi K(x_0,x_0) + o(1) = 8\pi R(x_0) + o(1).$$

For the first term, on the other hand, we have

$$\begin{aligned}\mathrm{III}_{1,k} &= -\frac{1}{2\pi}\int_{\tilde\Omega_k}\log\left|\delta_k y'\right|\cdot\tilde V_k(y')e^{\tilde v_k(y')}dy' \\ &= \frac{1}{4\pi}\left(\log\lambda_k + \|v_k\|_\infty\right)\int_{\tilde\Omega_k}\tilde V_k(y')e^{\tilde v_k(y')}dy' \\ &\quad -\frac{1}{2\pi}\int_{\tilde\Omega_k}\log\left|y'\right|\cdot\tilde V_k(y')e^{\tilde v_k(y')}dy'\end{aligned}$$

by (8), and therefore an asymptotic formula of [3] guarantees

$$\int_{\tilde\Omega_k}\tilde V_k(y')e^{\tilde v_k(y')}dy' = \int_\Omega V_k(y)e^{v_k(y)}dy = 8\pi + O\left(\lambda_k\left|\log\lambda_k\right|\right).$$

Thus we obtain

$$\begin{aligned}\mathrm{III}_{1,k} &= \left(2\log\lambda_k + 2\left\|v_k\right\|_\infty\right)\left(1 + O\left(\lambda_k\left|\log\lambda_k\right|\right)\right) \\ &\quad -\frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\log\left|y'\right|\right)\cdot\frac{c}{\left(1+\frac{c}{8}\left|y'\right|^2\right)^2}dy' + o(1) \\ &= \left(2\log\lambda_k + 2\left\|v_k\right\|_\infty\right)\left(1 + O\left(\lambda_k\left|\log\lambda_k\right|\right)\right) - 2\log\frac{8}{c} + o(1).\end{aligned}$$

These results are summarized as

$$\begin{aligned}&\left\|v_k\right\|_\infty\left(1 + O\left(\lambda_k\left|\log\lambda_k\right|\right)\right) \\ &\quad = -2(\log\lambda_k)\cdot\left(1 + O\left(\lambda_k\left|\log\lambda_k\right|\right)\right) + 2\log\frac{8}{c} - 8\pi R(x_0) + o(1),\end{aligned}$$

or (20). □

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