Zeitschrift:	Commentarii Mathematici Helvetici
Herausgeber:	Schweizerische Mathematische Gesellschaft
Band:	82 (2007)
Artikel:	A pinching theorem for the first eigenvalue of the Laplacian on hypersurfaces of the Euclidean space
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DOI:	https://doi.org/10.5169/seals-98881

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A pinching theorem for the first eigenvalue of the Laplacian on hypersurfaces of the Euclidean space

Bruno Colbois and Jean-François Grosjean*

Abstract. In this paper, we give pinching theorems for the first nonzero eigenvalue $\lambda_1(M)$ of the Laplacian on the compact hypersurfaces of the Euclidean space. Indeed, we prove that if the volume of M is 1 then, for any $\varepsilon > 0$, there exists a constant C_{ε} depending on the dimension n of M and the L_{∞} -norm of the mean curvature H, so that if the L_{2p} -norm $||H||_{2p}$ ($p \ge 2$) of H satisfies $n||H||_{2p}^2 - C_{\varepsilon} < \lambda_1(M)$, then the Hausdorff-distance between M and a round sphere of radius $(n/\lambda_1(M))^{1/2}$ is smaller than ε . Furthermore, we prove that if C is a small enough constant depending on n and the L_{∞} -norm of the second fundamental form, then the pinching condition $n||H||_{2p}^2 - C < \lambda_1(M)$ implies that M is diffeomorphic to an n-dimensional sphere.

Mathematics Subject Classification (2000). 53A07, 53C21.

Keywords. Spectrum, Laplacian, pinching results, hypersurfaces.

1. Introduction and preliminaries

Let (M^n, g) be a compact, connected and oriented *n*-dimensional Riemannian manifold without boundary isometrically immersed by ϕ into the n + 1-dimensional euclidean space (\mathbb{R}^{n+1}, can) (i.e. $\phi^* can = g$). A well-known inequality due to Reilly ([11]) gives an extrinsic upper bound for the first nonzero eigenvalue $\lambda_1(M)$ of the Laplacian of (M^n, g) in terms of the square of the length of the mean curvature. Indeed, we have

$$\lambda_1(M) \le \frac{n}{V(M)} \int_M |H|^2 \, dv \tag{1}$$

where dv and V(M) denote respectively the Riemannian volume element and the volume of (M^n, g) . Moreover the equality holds if and only if (M^n, g) is a geodesic hypersphere of \mathbb{R}^{n+1} .

^{*}Supported by European Commission through Human Potential Programme.

By using Hölder inequality, we obtain some other similar estimates for the L_{2p} -norm $(p \ge 1)$ with H denoted by $||H||_{2p}^2$

$$\lambda_1(M) \le \frac{n}{V(M)^{1/p}} \|H\|_{2p}^2,\tag{2}$$

and as for the inequality (1), the equality case is characterized by the geodesic hyperspheres of \mathbb{R}^{n+1} .

A first natural question is to know if there exists a pinching result as the one we state now: does a constant C depending on minimum geometric invariants exist so that if we have the pinching condition

(P_C)
$$\frac{n}{V(M)^{1/p}} \|H\|_{2p}^2 - C < \lambda_1(M)$$

then *M* is close to a sphere in a certain sense?

Such questions are known for the intrinsic lower bound of Lichnerowicz–Obata ([9]) of $\lambda_1(M)$ in terms of the lower bound of the Ricci curvature (see [4], [8], [10]). Other pinching results have been proved for Riemannian manifolds with positive Ricci curvature, with a pinching condition on the n + 1-st eigenvalue ([10]), the diameter ([5], [8], [15]), the volume or the radius (see for instance [2] and [3]).

For instance, S. Ilias proved in [8] that there exists ε depending on *n* and an upper bound of the sectional curvature so that if the Ricci curvature Ric of *M* satisfies Ric $\ge n - 1$ and $\lambda_1(M) \le \lambda_1(\mathbb{S}^n) + \varepsilon$, then *M* is homeomorphic to \mathbb{S}^n .

In this article, we investigate the case of hypersurfaces where, as far as we know, very little is known about pinching and stability results (see however [12], [13]).

More precisely, in our paper, the hypothesis made in [8] that M has a positive Ricci curvature is replaced by the fact that M is isometrically immersed as a hypersurface in \mathbb{R}^{n+1} , and the bound on the sectional curvature by an L^{∞} -bound on the mean curvature or on the second fundamental form. Note that we do not know if such bounds are sharp, or if a bound on the L^q -norm (for some q) of the mean curvature would be enough.

We get the following results

Theorem 1.1. Let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ in \mathbb{R}^{n+1} . Assume that V(M) = 1 and let x_0 be the center of mass of M. Then for any $p \ge 2$ and for any $\varepsilon > 0$, there exists a constant C_{ε} depending only on $n, \varepsilon > 0$ and on the L_{∞} -norm of H so that if

$$(\mathbf{P}_{C_{\varepsilon}}) \qquad \qquad n \|H\|_{2p}^2 - C_{\varepsilon} < \lambda_1(M)$$

then the Hausdorff-distance d_H of M to the sphere $S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right)$ of center x_0 and radius $\sqrt{\frac{n}{\lambda_1(M)}}$ satisfies $d_H\left(\phi(M), S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right)\right) < \varepsilon$.

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We recall that the Hausdorff-distance between two compact subsets A and B of a metric space is given by

$$d_H(A, B) = \inf \{ \eta | V_\eta(A) \supset B \text{ and } V_\eta(B) \supset A \}$$

where for any subset A, $V_{\eta}(A)$ is the tubular neighborhood of A defined by $V_{\eta}(A) = \{x \mid \text{dist}(x, A) < \eta\}.$

Remark. We will see in the proof that $C_{\varepsilon}(n, ||H||_{\infty}) \to 0$ when $||H||_{\infty} \to \infty$ or $\varepsilon \to 0$.

In fact the previous theorem is a consequence of the above definition and the following theorem

Theorem 1.2. Let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ in \mathbb{R}^{n+1} . Assume that V(M) = 1 and let x_0 be the center of mass of M. Then for any $p \ge 2$ and for any $\varepsilon > 0$, there exists a constant C_{ε} depending only on n, $\varepsilon > 0$ and on the L_{∞} -norm of H so that if

$$(\mathbf{P}_{C_{\varepsilon}}) \qquad \qquad n \|H\|_{2p}^2 - C_{\varepsilon} < \lambda_1(M)$$

then

(1)
$$\phi(M) \subset B\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}} + \varepsilon\right) \setminus B\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}} - \varepsilon\right);$$

(2) $B(x, \varepsilon) \cap \phi(M) \neq \emptyset$ for all $x \in S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right).$

In the following theorem, if the pinching is strong enough, with a control on n and the L_{∞} -norm of the second fundamental form, we obtain that M is diffeomorphic to a sphere and even almost isometric with a round sphere in a sense we will make precise.

Theorem 1.3. Let (M^n, g) be a compact, connected and oriented *n*-dimensional Riemannian manifold $(n \ge 2)$ without boundary isometrically immersed by ϕ in \mathbb{R}^{n+1} . Assume that V(M) = 1. Then for any $p \ge 2$, there exists a constant *C* depending only on *n* and the L_{∞} -norm of the second fundamental form *B* so that if

(P_C)
$$n \|H\|_{2p}^2 - C < \lambda_1(M).$$

Then M is diffeomorphic to \mathbb{S}^n .

More precisely, there exists a diffeomorphism F from M into the sphere $\mathbb{S}^n\left(\sqrt{\frac{n}{\lambda_1(M)}}\right)$ of radius $\sqrt{\frac{n}{\lambda_1(M)}}$ which is a quasi-isometry. Namely, for any θ ,

 $0 < \theta < 1$, there exists a constant C depending only on n, the L_{∞} -norm of B and θ , so that the pinching condition (P_C) implies

$$\left| |dF_x(u)|^2 - 1 \right| \le \theta$$

for any $x \in M$ and $u \in T_x M$ so that |u| = 1.

Now we will give some preliminaries for the proof of these theorems. Throughout the paper, we consider a compact, connected and oriented *n*-dimensional Riemannian manifold (M^n, g) without boundary isometrically immersed by ϕ into (\mathbb{R}^{n+1}, can) (i.e. $\phi^* can = g$). Let v be the outward normal vector field. Then the second fundamental form of the immersion will be defined by $B(X, Y) = \langle \nabla^0_X v, Y \rangle$, where ∇^0 and \langle , \rangle are respectively the Riemannian connection and the inner product of \mathbb{R}^{n+1} . Moreover the mean curvature H will be given by $H = (1/n) \operatorname{trace}(B)$.

Now let ∂_i be an orthonormal frame of \mathbb{R}^{n+1} and let $x_i : \mathbb{R}^{n+1} \to \mathbb{R}$ be the associated component functions. Putting $X_i = x_i \circ \phi$, a straightforward calculation shows us that

$$B \otimes \nu = -\sum_{i \le n+1} \nabla dX_i \otimes \partial_i$$

and

$$nH\nu = \sum_{i \le n+1} \Delta X_i \partial_i,$$

where ∇ and Δ denote respectively the Riemannian connection and the Laplace– Beltrami operator of (M^n, g) . On the other hand, we have the well-known formula

$$\frac{1}{2}\Delta|X|^2 = nH\langle\nu,X\rangle - n \tag{3}$$

where *X* is the position vector given by $X = \sum_{i \le n+1} X_i \partial_i$.

We recall that to prove the Reilly inequality, we use the functions X_i as test functions (cf. [11]). Indeed, doing a translation if necessary, we can assume that $\int_M X_i dv = 0$ for all $i \le n + 1$ and we can apply the variational characterization of $\lambda_1(M)$ to X_i . If the equality holds in (1) or (2), then the functions are nothing but eigenfunctions of $\lambda_1(M)$ and from the Takahashi Theorem ([14]) M is immersed isometrically in \mathbb{R}^{n+1} as a geodesic sphere of radius $\sqrt{\frac{n}{\lambda_1(M)}}$.

Throughout the paper we use some notations. From now on, the inner product and the norm induced by g and can on a tensor T will be denoted respectively by \langle , \rangle and $||^2$, and the L_p -norm will be given by

$$||T||_p = \left(\int_M |T|^p \, dv\right)^{1/p}$$

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and

$$||T||_{\infty} = \sup_{M} |T|.$$

We end these preliminaries by a convenient result.

Lemma 1.1. Let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold $(n \ge 2)$ without boundary isometrically immersed by ϕ in \mathbb{R}^{n+1} . Assume that V(M) = 1. Then there exist constants c_n and d_n depending only on n so that for any $p \ge 2$, if (\mathbf{P}_C) is true with $C < c_n$ then

$$\frac{n}{\lambda_1(M)} \le d_n. \tag{4}$$

Proof. We recall the standard Sobolev inequality (cf. [6], [7], [16] and p. 216 in [1]). If f is a smooth function and $f \ge 0$, then

$$\left(\int_{M} f^{\frac{n}{n-1}} dv\right)^{1-(1/n)} \le K(n) \int_{M} (|df| + |H|f) dv$$
(5)

where K(n) is a constant depending on *n* and the volume of the unit ball in \mathbb{R}^n . Taking f = 1 on *M*, and using the fact that V(M) = 1, we deduce that

$$\|H\|_{2p} \ge \frac{1}{K(n)}$$

and if (P_C) is satisfied and $C \leq \frac{n}{2K(n)^2} = c_n$, then

$$\frac{n}{\lambda_1(M)} \le \frac{1}{n \|H\|_{2p}^2 - C} \le 2K(n)^2 = d_n.$$

Throughout the paper, we will assume that V(M) = 1 and $\int_M X_i dv = 0$ for all $i \leq n + 1$. The last assertion implies that the center of mass of M is the origin of \mathbb{R}^{n+1} .

2. An L^2 -approach of the problem

A first step in the proof of Theorem 1.2 is to prove that if the pinching condition (P_C) is satisfied, then *M* is close to a sphere in an L^2 -sense.

In the following lemma, we prove that the L^2 -norm of the position vector is close to $\sqrt{\frac{n}{\lambda_1(M)}}$.

Lemma 2.1. If we have the pinching condition (P_C) with $C < c_n$, then

$$\frac{n\lambda_1(M)}{(C+\lambda_1(M))^2} \le \|X\|_2^2 \le \frac{n}{\lambda_1(M)} \le d_n.$$

Proof. Since $\int_M X_i dv = 0$, we can apply the variational characterization of the eigenvalues to obtain

$$\lambda_1(M) \int_M \sum_{i \le n+1} |X_i|^2 \, dv \le \int_M \sum_{i \le n+1} |dX_i|^2 \, dv = n$$

which gives the inequality of the right-hand side.

Let us prove now the inequality of the left-hand side.

$$\begin{split} \lambda_1(M) \int_M |X|^2 \, dv &\leq \frac{\left(\int_M \sum_{i \leq n+1} |dX_i|^2 \, dv\right)^4}{\left(\int_M \sum_{i \leq n+1} |dX_i|^2 \, dv\right)^3} = \frac{\left(\int_M \sum_{i \leq n+1} (\Delta X_i) X_i \, dv\right)^4}{n^3} \\ &\leq \frac{\left(\int_M \sum_{i \leq n+1} (\Delta X_i)^2 \, dv\right)^2 \left(\int_M |X|^2 \, dv\right)^2}{n^3} \\ &= n \left(\int_M H^2 \, dv\right)^2 \left(\int_M |X|^2 \, dv\right)^2 \end{split}$$

then using again the Hölder inequality, we get

$$\lambda_1(M) \leq \frac{1}{n} \left(n \|H\|_{2p}^2 \right)^2 \int_M |X|^2 \, dv \leq \frac{(C + \lambda_1(M))^2}{n} \int_M |X|^2 \, dv.$$

This completes the proof.

From now on, we will denote by X^T the orthogonal tangential projection on M. In fact, at $x \in M$, X^T is nothing but the vector of $T_x M$ defined by $X^T = \sum_{1 \le i \le n} \langle X, e_i \rangle e_i$ where $(e_i)_{1 \le i \le n}$ is an orthonormal basis of $T_x M$. In the following lemma, we will show that the condition (P_C) implies that the L^2 -norm of X^T of X on M is close to 0.

Lemma 2.2. If we have the pinching condition (P_C) , then

$$\|X^T\|_2^2 \le A(n)C.$$

Proof. From Lemma 2.1 and the relation (3), we have

$$\lambda_1(M) \int_M |X|^2 \, dv \le n = n \left(\int_M H \langle X, v \rangle \, dv \right)^2$$

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$$\leq \left(\int_{M} |H|| \langle X, v \rangle | dv\right)^{2} \leq n \|H\|_{2p}^{2} \left(\int_{M} |\langle X, v \rangle |^{\frac{2p}{2p-1}} dv\right)^{\frac{2p-1}{p}} \\ \leq n \|H\|_{2p}^{2} \left(\int_{M} |\langle X, v \rangle |^{2} dv\right) = n \|H\|_{2p}^{2} \int_{M} |X|^{2} dv.$$

Then we deduce that

$$n\|H\|_{2p}^{2}\|X^{T}\|_{2}^{2} = n\|H\|_{2p}^{2} \left(\int_{M} \left(|X|^{2} - |\langle X, \nu \rangle|^{2} \right) d\nu \right)$$

$$\leq (n\|H\|_{2p}^{2} - \lambda_{1}(M))\|X\|_{2}^{2} \leq d_{n}C$$

where in the last inequality we have used the pinching condition and Lemma 2.1. $\hfill \Box$

Next we will show that the condition (P_C) implies that the component functions are almost eigenfunctions in an L^2 -sense. For this, let us consider the vector field Y on M defined by

$$Y = \sum_{i \le n+1} \left(\Delta X_i - \lambda_1(M) X_i \right) \partial_i = n H \nu - \lambda_1(M) X.$$

Lemma 2.3. If (P_C) is satisfied, then

$$\|Y\|_2^2 \le nC.$$

Proof. We have

$$\int_{M} |Y|^2 dv = \int_{M} \left(n^2 H^2 - 2n\lambda_1(M)H \langle v, X \rangle + \lambda_1(M)^2 |X|^2 \right) dv$$

Now by integrating the relation (3) we deduce that

$$\int_M H \langle v, X \rangle \ dv = 1.$$

Furthermore, since $\int_M X_i dv = 0$, we can apply the variational characterization of the eigenvalues to obtain

$$\lambda_1(M) \int_M |X|^2 \, dv = \lambda_1(M) \int_M \sum_{i \le n+1} |X_i|^2 \, dv \le \int_M \sum_{i \le n+1} |dX_i|^2 \, dv = n.$$

Then

$$\int_{M} |Y|^2 dv \le n^2 \int_{M} |H|^2 dv - n\lambda_1(M) \le n \left(n \|H\|_{2p}^2 - \lambda_1(M) \right) \le nC$$

where in this last inequality we have used the Hölder inequality.

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To prove Assertion 1 of Theorem 1.2, we will show that $\||X| - \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\|_{\infty} \le \varepsilon$. For this we need an L^2 -upper bound on the function $\varphi = |X| \left(|X| - \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\right)^2$. Before giving such estimate, we will introduce the vector field Z on M defined

by

$$Z = \left(\frac{n}{\lambda_1(M)}\right)^{1/2} |X|^{1/2} H\nu - \frac{X}{|X|^{1/2}}.$$

We have

Lemma 2.4. If (P_C) is satisfied with $C < c_n$, then

$$\|Z\|_2^2 \le B(n)C$$

Proof. We have

$$\begin{split} \|Z\|_{2}^{2} &= \left\| \left(\frac{n}{\lambda_{1}(M)} \right)^{1/2} |X|^{1/2} H v - \frac{X}{|X|^{1/2}} \right\|_{2}^{2} \\ &= \int_{M} \left(\frac{n}{\lambda_{1}(M)} |X| H^{2} - 2 \left(\frac{n}{\lambda_{1}(M)} \right)^{1/2} H \langle v, X \rangle + |X| \right) dv \\ &\leq \frac{n}{\lambda_{1}(M)} \left(\int_{M} |X|^{2} dv \right)^{1/2} \left(\int_{M} H^{4} dv \right)^{1/2} \\ &- 2 \left(\frac{n}{\lambda_{1}(M)} \right)^{1/2} + \left(\int_{M} |X|^{2} dv \right)^{1/2}. \end{split}$$

Note that we have used the relation (3). Finally for $p \ge 2$, we get

$$\begin{split} \|Z\|_{2}^{2} &\leq \left(\int_{M} |X|^{2} dv\right)^{1/2} \left(\frac{n}{\lambda_{1}(M)} \|H\|_{2p}^{2} + 1\right) - 2\left(\frac{n}{\lambda_{1}(M)}\right)^{1/2} \\ &\leq \left(\frac{n}{\lambda_{1}(M)}\right)^{1/2} \left(\frac{C}{\lambda_{1}(M)} + 2\right) - 2\left(\frac{n}{\lambda_{1}(M)}\right)^{1/2} \\ &= \left(\frac{n}{\lambda_{1}(M)}\right)^{1/2} \frac{C}{\lambda_{1}(M)} \leq \frac{d_{n}^{3/2}}{n} C. \end{split}$$

This concludes the proof of the lemma.

Now we give an L^2 -upper bound of φ .

Lemma 2.5. Let $p \ge 2$ and $C \le c_n$. If we have the pinching condition (P_C), then

$$\|\varphi\|_2 \le D(n) \|\varphi\|_{\infty}^{3/4} C^{1/4}.$$

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Proof. We have

$$\|\varphi\|_{2} = \left(\int_{M} \varphi^{3/2} \varphi^{1/2} \, dv\right)^{1/2} \le \|\varphi\|_{\infty}^{3/4} \|\varphi^{1/2}\|_{1}^{1/2},$$

and noting that

$$|X| \left(|X| - \left(\frac{n}{\lambda_1(M)}\right)^{1/2} \right)^2 = \left| |X|^{1/2} X - \left(\frac{n}{\lambda_1(M)}\right)^{1/2} \frac{X}{|X|^{1/2}} \right|^2$$

we get

$$\int_{M} \varphi^{1/2} dv = \left\| |X|^{1/2} X - \left(\frac{n}{\lambda_{1}(M)}\right)^{1/2} \frac{X}{|X|^{1/2}} \right\|_{1}$$

$$= \left\| -\frac{|X|^{1/2}}{\lambda_{1}(M)} Y + \frac{n}{\lambda_{1}(M)} |X|^{1/2} Hv - \left(\frac{n}{\lambda_{1}(M)}\right)^{1/2} \frac{X}{|X|^{1/2}} \right\|_{1}$$

$$\leq \left\| \frac{|X|^{1/2}}{\lambda_{1}(M)} Y \right\|_{1} + \left(\frac{n}{\lambda_{1}(M)}\right)^{1/2} \|Z\|_{1}.$$
(6)

From Lemmas 2.3 and 1.1 we get

$$\begin{split} \left\| \frac{|X|^{1/2}}{\lambda_1(M)} Y \right\|_1 &\leq \frac{1}{\lambda_1(M)} \left(\int_M |X| \, dv \right)^{1/2} \|Y\|_2 \\ &\leq \frac{1}{\lambda_1(M)} \left(\int_M |X|^2 \, dv \right)^{1/4} \|Y\|_2 \leq \frac{d_n^{3/4}}{n^{1/2}} C^{1/2}. \end{split}$$

Moreover, using Lemmas 2.4 and 1.1 again it is easy to see that the last term of (6) is bounded by $d_n^{1/2} B(n)^{1/2} C^{1/2}$. Then $\|\varphi^{1/2}\|_1^{1/2} \leq D(n) C^{1/4}$.

3. Proof of Theorem 1.2

The proof of Theorem 1.2 is immediate from the two following technical lemmas which we state below.

Lemma 3.1. For $p \ge 2$ and for any $\eta > 0$, there exists $K_{\eta}(n, ||H||_{\infty}) \le c_n$ so that if $(P_{K_{\eta}})$ is true, then $\|\varphi\|_{\infty} \le \eta$. Moreover, $K_{\eta} \to 0$ when $\|H\|_{\infty} \to \infty$ or $\eta \to 0$.

Lemma 3.2. Let x_0 be a point of the sphere S(O, R) of \mathbb{R}^{n+1} with the center at the origin and of radius R. Assume that $x_0 = Re$ where $e \in \mathbb{S}^n$. Now let (M^n, g) be a compact oriented n-dimensional Riemannian manifold without boundary isometrically

immersed by ϕ in \mathbb{R}^{n+1} so that $\phi(M) \subset (B(O, R + \eta) \setminus B(O, R - \eta)) \setminus B(x_0, \rho)$ with $\rho = 4(2n-1)\eta$ and suppose that there exists a point $p \in M$ so that $\langle X, e \rangle > 0$. Then there exists $y_0 \in M$ so that the mean curvature $H(y_0)$ at y_0 satisfies $|H(y_0)| \ge \frac{1}{4n\eta}$.

Now, let us see how to use these lemmas to prove Theorem 1.2.

Proof of Theorem 1.2. We consider the function $f(t) = t \left(t - \left(\frac{n}{\lambda_1(M)} \right)^{1/2} \right)^2$. For $\varepsilon > 0$ let us put

$$\begin{split} \eta(\varepsilon) &= \min\left(\left(\frac{1}{\|H\|_{\infty}} - \varepsilon\right)\varepsilon^{2}, \left(\frac{1}{\|H\|_{\infty}} + \varepsilon\right)\varepsilon^{2}, \frac{1}{27\|H\|_{\infty}^{3}}\right) \\ &\leq \min\left(f\left(\left(\frac{n}{\lambda_{1}(M)}\right)^{1/2} - \varepsilon\right), f\left(\left(\frac{n}{\lambda_{1}(M)}\right)^{1/2} + \varepsilon\right), \frac{1}{27\|H\|_{\infty}^{3}}\right). \end{split}$$

Then, as $\eta(\varepsilon) > 0$ and from Lemma 3.1, it follows that if the pinching condition $(P_{K_{\eta(\varepsilon)}})$ is satisfied with $K_{\eta(\varepsilon)} \leq c_n$, then for any $x \in M$, we have

$$f(|X|) \le \eta(\varepsilon). \tag{7}$$

Now to prove Theorem 1.2, it is sufficient to assume $\varepsilon < \frac{2}{3\|H\|_{\infty}}$. Let us show that either

$$\left(\frac{n}{\lambda_1(M)}\right)^{1/2} - \varepsilon \le |X| \le \left(\frac{n}{\lambda_1(M)}\right)^{1/2} + \varepsilon \quad \text{or} \quad |X| < \frac{1}{3} \left(\frac{n}{\lambda_1(M)}\right)^{1/2}.$$
 (8)

By studying the function f it is easy to see that f has a unique local maximum in $\frac{1}{3} \left(\frac{n}{\lambda_1(M)}\right)^{1/2}$ and from the definition of $\eta(\varepsilon)$ it follows that $\eta(\varepsilon) < \frac{4}{27} \frac{1}{\|H\|_{\infty}^3} \leq \frac{4}{27} \left(\frac{n}{\lambda_1(M)}\right)^{3/2} = f\left(\frac{1}{3} \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\right).$ Since $\varepsilon < \frac{2}{3\|H\|_{\infty}}$, we have $\varepsilon < \frac{2}{3} \left(\frac{n}{\lambda_1(M)}\right)^{1/2}$ and $\frac{1}{3} \left(\frac{n}{\lambda_1(M)}\right)^{1/2} < \left(\frac{n}{\lambda_1(M)}\right)^{1/2} - \varepsilon$. This and (7) yield (8).

Now, from Lemma 2.1 we deduce that there exists a point $y_0 \in M$ so that $|X(y_0)| \geq \frac{n^{1/2}\lambda_1(M)^{1/2}}{(K_{\eta(\varepsilon)} + \lambda_1(M))}$ and since $K_{\eta(\varepsilon)} \leq c_n = \frac{n}{d_n} \leq \lambda_1(M) \leq 2\lambda_1(M)$ (see the proof of Lemma 1.1), we obtain $|X(y_0)| \geq \frac{1}{3} \left(\frac{n}{\lambda_1(M)}\right)^{1/2}$.

By the connectedness of M, it follows that $\left(\frac{n}{\lambda_1(M)}\right)^{1/2} - \varepsilon \le |X| \le \left(\frac{n}{\lambda_1(M)}\right)^{1/2} + \varepsilon$ for any point of M and Assertion 1 of Theorem 1.2 is shown for the condition $(P_{K_{\eta(\varepsilon)}})$.

In order to prove the second assertion, let us consider the pinching condition $(P_{C_{\varepsilon}})$ with $C_{\varepsilon} = K_{\eta(\frac{\varepsilon}{4(2n-1)})}$. Then Assertion 1 is still valid. Let $x = \left(\frac{n}{\lambda_1(M)}\right)^{1/2} e \in S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right)$, with $e \in \mathbb{S}^n$ and suppose that $B(x, \varepsilon) \cap M = \emptyset$. Since $\int_M X_i \, dv = 0$

for any $i \le n + 1$, there exists a point $p \in M$ so that $\langle X, e \rangle > 0$ and we can apply Lemma 3.2. Therefore there is a point $y_0 \in M$ so that $H(y_0) \ge \frac{2n-1}{n\varepsilon} > ||H||_{\infty}$ since we have assumed $\varepsilon < \frac{2}{3||H||_{\infty}} \le \frac{2n-1}{2n||H||_{\infty}}$. Then we obtain a contradiction which implies $B(x, \varepsilon) \cap M \ne \emptyset$ and Assertion 2 is satisfied. Furthermore, $C_{\varepsilon} \to 0$ when $||H||_{\infty} \to \infty$ or $\varepsilon \to 0$.

4. Proof of Theorem 1.3

From Theorem 1.2, we know that for any $\varepsilon > 0$, there exists C_{ε} depending only on n and $||H||_{\infty}$ so that if $(P_{C_{\varepsilon}})$ is true then

$$\left||X|_x - \sqrt{\frac{n}{\lambda_1(M)}}\right| \le \varepsilon$$

for any $x \in M$. Now, since $\sqrt{n} ||H||_{\infty} \le ||B||_{\infty}$, it is easy to see from the previous proofs that we can assume that C_{ε} is depending only on *n* and $||B||_{\infty}$.

The proof of Theorem 1.3 is a consequence of the following lemma on the L_{∞} -norm of $\psi = |X^T|$.

Lemma 4.1. For $p \ge 2$ and for any $\eta > 0$, there exists $K_{\eta}(n, ||B||_{\infty})$ so that if $(P_{K_{\eta}})$ is true, then $\|\psi\|_{\infty} \le \eta$. Moreover, $K_{\eta} \to 0$ when $\|B\|_{\infty} \to \infty$ or $\eta \to 0$.

This lemma will be proved in the Section 5.

Proof of Theorem 1.3. Let $\varepsilon < \frac{1}{2}\sqrt{\frac{n}{\|B\|_{\infty}}} \le \sqrt{\frac{n}{\lambda_1(M)}}$. From the choice of ε , we deduce that the condition $(P_{C_{\varepsilon}})$ implies that $|X_x|$ is nonzero for any $x \in M$ (see the proof of Theorem 1.2) and we can consider the differential application

$$F: M \longrightarrow S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right),$$
$$x \longmapsto \sqrt{\frac{n}{\lambda_1(M)}} \frac{X_x}{|X_x|}.$$

We will prove that *F* is a quasi-isometry. Indeed, for any $0 < \theta < 1$, we can choose a constant $\varepsilon(n, ||B||_{\infty}, \theta)$ so that for any $x \in M$ and any unit vector $u \in T_x M$, the pinching condition $(P_{C_{\varepsilon(n, ||B||_{\infty}, \theta)}})$ implies

$$\left| |dF_x(u)|^2 - 1 \right| \le \theta.$$

For this, let us compute $dF_x(u)$. We have

$$dF_x(u) = \sqrt{\frac{n}{\lambda_1(M)}} \nabla_u^0 \left(\frac{X}{|X|}\right) \Big|_x = \sqrt{\frac{n}{\lambda_1(M)}} u\left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X = \frac{1}{\lambda_1(M)} \left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \left(\frac{1}{|X|}\right) X + \sqrt{\frac$$

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$$\begin{split} &= -\frac{1}{2}\sqrt{\frac{n}{\lambda_1(M)}}\frac{1}{|X|^3}u(|X|^2)X + \sqrt{\frac{n}{\lambda_1(M)}}\frac{1}{|X|}u\\ &= -\sqrt{\frac{n}{\lambda_1(M)}}\frac{1}{|X|^3}\left\langle u, X\right\rangle X + \sqrt{\frac{n}{\lambda_1(M)}}\frac{1}{|X|}u\\ &= \sqrt{\frac{n}{\lambda_1(M)}}\frac{1}{|X|}\left(-\frac{\langle u, X\rangle}{|X|^2}X + u\right). \end{split}$$

By a straightforward computation, we obtain

$$\left| |dF_{x}(u)|^{2} - 1 \right| = \left| \frac{n}{\lambda_{1}(M)} \frac{1}{|X|^{2}} \left(1 - \frac{\langle u, X \rangle^{2}}{|X|^{2}} \right) - 1 \right|$$

$$\leq \left| \frac{n}{\lambda_{1}(M)} \frac{1}{|X|^{2}} - 1 \right| + \frac{n}{\lambda_{1}(M)} \frac{1}{|X|^{4}} \langle u, X \rangle^{2}.$$
(9)

Now

$$\begin{split} \left| \frac{n}{\lambda_1(M)} \frac{1}{|X|^2} - 1 \right| &= \frac{1}{|X|^2} \left| \frac{n}{\lambda_1(M)} - |X|^2 \right| \\ &\leq \varepsilon \frac{\left| \sqrt{\frac{n}{\lambda_1(M)}} + |X| \right|}{|X|^2} \leq \varepsilon \frac{2\sqrt{\frac{n}{\lambda_1(M)}} + \varepsilon}{\left(\sqrt{\frac{n}{\lambda_1(M)}} - \varepsilon\right)^2}. \end{split}$$

Let us recall that $\frac{n}{d_n} \leq \lambda_1(M) \leq \|B\|_{\infty}^2$ (see (4) for the first inequality). Since we assume $\varepsilon < \frac{1}{2}\sqrt{\frac{n}{\|B\|_{\infty}}}$, the right-hand side is bounded above by a constant depending only on *n* and $\|B\|_{\infty}$ and we have

$$\left|\frac{n}{\lambda_1(M)}\frac{1}{|X|^2} - 1\right| \le \varepsilon \gamma(n, \|B\|_{\infty}).$$
(10)

On the other hand, since $C_{\varepsilon}(n, \|B\|_{\infty}) \to 0$ when $\varepsilon \to 0$, there exists $\varepsilon(n, \|B\|_{\infty}, \eta)$ so that $C_{\varepsilon_{(n,\|B\|_{\infty},\eta)}} \leq K_{\eta}(n, \|B\|_{\infty})$ (where K_{η} is the constant of the lemma) and then by Lemma 4.1, $\|\psi\|_{\infty}^2 \leq \eta^2$. Thus there exists a constant δ depending only on *n* and $\|B\|_{\infty}$ so that

$$\frac{n}{\lambda_1(M)} \frac{1}{|X|^4} \langle u, X \rangle^2 \le \frac{n}{\lambda_1(M)} \frac{1}{|X|^4} \|\psi\|_{\infty}^2 \le \eta^2 \delta(n, \|B\|_{\infty}), \tag{11}$$

and from (9), (10) and (11) we deduce that the condition $(P_{C_{\varepsilon(n, ||B||_{\infty}, \eta)}})$ implies

$$\left| |dF_x(u)|^2 - 1 \right| \le \varepsilon \gamma(n, \|B\|_{\infty}) + \eta^2 \delta(n, \|B\|_{\infty}).$$

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Now let us choose $\eta = \left(\frac{\theta}{2\delta}\right)^{1/2}$. Then we can assume that $\varepsilon(n, \|B\|_{\infty}, \eta)$ is small enough in order to have $\varepsilon(n, \|B\|_{\infty}, \eta)\gamma(n\|B\|_{\infty}) \leq \frac{\theta}{2}$. In this case we have

$$\left||dF_x(u)|^2-1\right|\leq\theta.$$

Now let us fix θ , $0 < \theta < 1$. It follows that F is a local diffeomorphism from M to $S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right)$. Since $S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right)$ is simply connected for $n \ge 2$, F is a diffeomorphism.

5. Proof of the technical lemmas

The proofs of Lemmas 3.1 and 4.1 are providing from a result stated in the following proposition using a Nirenberg–Moser type of proof.

Proposition 5.1. Let (M^n, g) be a compact, connected and oriented *n*-dimensional Riemannian manifold without boundary isometrically immersed into the *n*+1-dimensional euclidean space (\mathbb{R}^{n+1}, can) . Let ξ be a nonnegative continuous function so that ξ^k is smooth for $k \ge 2$. Let $0 \le r < s \le 2$ so that

$$\frac{1}{2}\Delta\xi^2\xi^{2k-2} \le \delta\omega + (A_1 + kA_2)\xi^{2k-r} + (B_1 + kB_2)\xi^{2k-s}$$

where $\delta \omega$ is the codifferential of a 1-form and A_1 , A_2 , B_1 , B_2 are nonnegative constants. Then for any $\eta > 0$, there exists a constant $L(n, A_1, A_2, B_1, B_2, ||H||_{\infty}, \eta)$ depending only on n, A_1 , A_2 , B_1 , B_2 , $||H||_{\infty}$ and η so that if $||\xi||_{\infty} > \eta$ then

$$\|\xi\|_{\infty} \le L(n, A_1, A_2, B_1, B_2, \|H\|_{\infty}, \eta) \|\xi\|_2.$$

Moreover, L is bounded when $\eta \to \infty$, and if $B_1 > 0$, $L \to \infty$ when $||H||_{\infty} \to \infty$ or $\eta \to 0$.

This proposition will be proved at the end of the paper.

Before giving the proofs of Lemmas 3.1 and 4.1, we will show that under the pinching condition (P_C) with C small enough, the L_{∞} -norm of X is bounded by a constant depending only on n and $||H||_{\infty}$.

Lemma 5.1. If we have the pinching condition (P_C) with $C < c_n$, then there exists $E(n, ||H||_{\infty})$ depending only on n and $||H||_{\infty}$ so that $||X||_{\infty} \leq E(n, ||H||_{\infty})$.

Proof. From the relation (3), we have

$$\frac{1}{2}\Delta |X|^2 |X|^{2k-2} \le n ||H||_{\infty} |X|^{2k-1}.$$

Then applying Proposition 5.1 to the function $\xi = |X|$ with r = 0 and s = 1, we obtain that if $||X||_{\infty} > E$, then there exists a constant $L(n, ||H||_{\infty}, E)$ depending only on n, $||H||_{\infty}$ and E so that

$$||X||_{\infty} \le L(n, ||H||_{\infty}, E) ||X||_{2},$$

and under the pinching condition (P_C) with $C < c_n$ we have from Lemma 2.1 that

$$||X||_{\infty} \leq L(n, ||H||_{\infty}, E)d_n^{1/2}.$$

Now since L is bounded when $E \to \infty$, we can choose $E = E(n, ||H||_{\infty})$ large enough so that

$$L(n, ||H||_{\infty}, E)d_n^{1/2} < E$$

In this case, we have $||X||_{\infty} \leq E(n, ||H||_{\infty})$.

Proof of Lemma 3.1. First we compute the Laplacian of the square of φ^2 . We have

$$\begin{split} \Delta \varphi^2 &= \Delta \left(|X|^4 - 2 \left(\frac{n}{\lambda_1(M)} \right)^{1/2} |X|^3 + \frac{n}{\lambda_1(M)} |X|^2 \right) \\ &= -2|X|^2 |d|X|^2|^2 + 2|X|^2 \Delta |X|^2 \\ &- 2 \left(\frac{n}{\lambda_1(M)} \right)^{1/2} \left(-\frac{3}{4} |X|^{-1} |d|X|^2|^2 + \frac{3}{2} |X| \Delta |X|^2 \right) + \frac{n}{\lambda_1(M)} \Delta |X|^2. \end{split}$$

Now by a direct computation one gets $|d|X|^2|^2 \le 4|X|^2$. Moreover by the relation (3) we have $|\Delta|X|^2| \le 2n ||H||_{\infty} |X| + n$. Then applying Lemmas 1.1 and 5.1 we get

$$\Delta \varphi^2 \le \alpha(n, \|H\|_{\infty})$$

and

$$\frac{1}{2}\Delta\varphi^2\varphi^{2k-2} \le \alpha(n, \|H\|_{\infty})\varphi^{2k-2}.$$

Now, we apply Proposition 5.1 with r = 0 and s = 2. Then if $\|\varphi\|_{\infty} > \eta$, there exists a constant $L(n, \|H\|_{\infty})$ depending only on *n* and $\|H\|_{\infty}$ so that

$$\|\varphi\|_{\infty} \le L \|\varphi\|_2.$$

From Lemma 2.5, if $C \leq c_n$ and (P_C) is true, we have $\|\varphi\|_2 \leq D(n) \|\varphi\|_{\infty}^{3/4} C^{1/4}$. Therefore

$$\|\varphi\|_{\infty} \le (LD)^4 C.$$

Consequently, if we choose $C = K_{\eta} = \inf \left(\frac{\eta}{(LD)^4}, c_n \right)$, then we obtain $\|\varphi\|_{\infty} \leq \eta$.

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Proof of Lemma 4.1. First we will prove that for any $C < c_n$, if (P_C) is true, then

$$\frac{1}{2}(\Delta\psi^2)\psi^{2k-2} \le \delta\omega + (\alpha_1(n, \|B\|_{\infty}) + k\alpha_2(n, \|B\|_{\infty}))\psi^{2k-2}$$
(12)

where $\delta \omega$ is the codifferential of a 1-form ω .

First observe that the gradient $\nabla^M |X|^2$ of $|X|^2$ satisfies $\nabla^M |X|^2 = 2X^T$. Then by the Bochner formula we get

$$\begin{split} \frac{1}{2}\Delta |X^{T}|^{2} &= \frac{1}{4} \left\langle \Delta d |X|^{2}, d |X|^{2} \right\rangle - \frac{1}{4} |\nabla d |X|^{2}|^{2} - \frac{1}{4} \operatorname{Ric}(\nabla^{M} |X|^{2}, \nabla^{M} |X|^{2}) \\ &\leq \frac{1}{4} \left\langle d\Delta |X|^{2}, d |X|^{2} \right\rangle - \frac{1}{4} \operatorname{Ric}(\nabla^{M} |X|^{2}, \nabla^{M} |X|^{2}) \end{split}$$

and by the Gauss formula we obtain

$$\begin{split} \frac{1}{2}\Delta |X^T|^2 &\leq \frac{1}{4} \langle d\Delta |X|^2, d|X|^2 \rangle - \frac{1}{4} n H \langle B \nabla^M |X|^2, \nabla^M |X|^2 \rangle + \frac{1}{4} |B \nabla^M |X|^2|^2 \\ &= \frac{1}{4} \langle d\Delta |X|^2, d|X|^2 \rangle - n H \langle B X^T, X^T \rangle + |B X^T|^2. \end{split}$$

By Lemma 5.1 we know that $||X||_{\infty} \leq E(n, ||B||_{\infty})$ (the dependance in $||H||_{\infty}$ can be replaced by $||B||_{\infty}$). Then it follows that

$$\frac{1}{2}(\Delta\psi^2)\psi^{2k-2} \le \frac{1}{4} \left\langle d\Delta |X|^2, \, d|X|^2 \right\rangle \psi^{2k-2} + \alpha'(n, \|B\|_{\infty})\psi^{2k-2}. \tag{13}$$

Now, let us compute the term $\langle d\Delta | X |^2, d | X |^2 \rangle \psi^{2k-2}$. We have

$$\langle d\Delta | X |^{2}, d | X |^{2} \rangle \psi^{2k-2}$$

= $\delta \omega + (\Delta | X |^{2})^{2} \psi^{2k-2} - (2k-2)\Delta | X |^{2} \langle d | X |^{2}, d\psi \rangle \psi^{2k-3}$
= $\delta \omega + (\Delta | X |^{2})^{2} \psi^{2k-2} - 2(2k-2)\Delta | X |^{2} \langle X^{T}, \nabla^{M} \psi \rangle \psi^{2k-3}$

where $\omega = -\Delta |X|^2 \psi^{2k-2} d|X|^2$. Now,

$$e_i(\psi) = \frac{e_i |X^T|^2}{2|X^T|} = \frac{e_i |X|^2 - e_i \langle X, \nu \rangle^2}{2|X^T|} = \frac{\langle e_i, X \rangle - B_{ij} \langle X, e_j \rangle \langle X, \nu \rangle}{|X^T|}.$$

Then

$$\begin{split} \left\langle d\Delta |X|^{2}, d|X|^{2} \right\rangle \psi^{2k-2} &= \delta \omega + (\Delta |X|^{2})^{2} \psi^{2k-2} - 2(2k-2)\Delta |X|^{2} |X^{T}| \psi^{2k-3} \\ &+ 2(2k-2)\Delta |X|^{2} \frac{\left\langle BX^{T}, X^{T} \right\rangle}{|X^{T}|} \left\langle X, \upsilon \right\rangle \psi^{2k-3} \\ &\leq \delta \omega + (\Delta |X|^{2})^{2} \psi^{2k-2} + 2(2k-2) |\Delta |X|^{2} |\psi^{2k-2} \\ &+ 2(2k-2) |\Delta |X|^{2} ||B| |X| \psi^{2k-2}. \end{split}$$

Now by relation (3) and Lemma 5.1 we have

$$\langle d\Delta |X|^2, d|X|^2 \rangle \psi^{2k-2} \le \delta\omega + (\alpha_1''(n, ||B||_{\infty}) + k\alpha_2''(n, ||B||_{\infty})) \psi^{2k-2}$$

Inserting this in (13), we obtain the desired inequality (12).

Now applying again Proposition 5.1, we get that there exists $L(n, ||B||_{\infty}, \eta)$ so that if $||\psi||_{\infty} > \eta$ then

$$\|\psi\|_{\infty} \le L \|\psi\|_2.$$

From Lemma 2.2 we deduce that if the pinching condition (P_C) holds then $\|\psi\|_2 \le A(n)^{1/2}C^{1/2}$. Then taking $C = K_\eta = \inf\left(\frac{\eta}{LA^{1/2}}, c_n\right)$, then $\|\psi\|_{\infty} \le \eta$. \Box

Proof of Lemma 3.2. The idea of the proof consists in foliating the region $B(O, R + \eta) \setminus B(O, R - \eta)$ with hypersurfaces of large mean curvature and to show that one of these hypersurfaces is tangent to $\phi(M)$. This will imply that $\phi(M)$ has a large mean curvature at the contact point.

Consider
$$\mathbb{S}^{n-1} \subset \mathbb{R}^n$$
 and $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}e$. Let $a, L > l > 0$ and
 $\Phi_{L,l,a} : \mathbb{S}^{n-1} \times \mathbb{S}^1 \longrightarrow \mathbb{R}^{n+1}$
 $(\xi, \theta) \longmapsto L\xi - l\cos\theta\xi + l\sin\theta e + ae.$

Then $\Phi_{L,l,a}$ is a family of embeddings from $\mathbb{S}^{n-1} \times \mathbb{S}^1$ in \mathbb{R}^{n+1} . If we orient the family of hypersurfaces $\Phi_{L,l,a}(\mathbb{S}^{n-1} \times \mathbb{S}^1)$ by the unit outward normal vector field, a straightforward computation shows that the mean curvature $H(\theta)$ depends only on θ and we have

$$H(\theta) = \frac{1}{n} \left(\frac{1}{l} - \frac{(n-1)\cos\theta}{L - l\cos\theta} \right) \ge \frac{1}{n} \left(\frac{1}{l} - \frac{n-1}{L - l} \right).$$
(14)

Now, let us consider the hypotheses of the lemma and for $t_0 = 2 \arcsin\left(\frac{\rho}{2R}\right) \le t \le \frac{\pi}{2}$, put $L = R \sin t$, $l = 2\eta$ and $a = R \cos t$. Then L > l and we can consider for $t_0 \le t \le \frac{\pi}{2}$ the family $\mathcal{M}_{R,\eta,t}$ of hypersurfaces defined by $\mathcal{M}_{R,\eta,t} = \Phi_{R \sin t, 2\eta, R \cos t}(\mathbb{S}^{n-1} \times \mathbb{S}^1)$.

From the relation (14), the mean curvature $H_{R,\eta,t}$ of $\mathcal{M}_{R,\eta,t}$ satisfies

$$H_{R,\eta,t} \ge \frac{1}{n} \left(\frac{1}{2\eta} - \frac{n-1}{R\sin t - 2\eta} \right) \ge \frac{1}{n} \left(\frac{1}{2\eta} - \frac{n-1}{R\sin t_0 - 2\eta} \right)$$
$$\ge \frac{1}{n} \left(\frac{1}{2\eta} - \frac{n-1}{R\sin(t_0/2) - 2\eta} \right) = \frac{1}{n} \left(\frac{1}{2\eta} - \frac{n-1}{\frac{\rho}{2} - 2\eta} \right) = \frac{1}{4n\eta}$$

where we have used in this last equality the fact that $\rho = 4(2n-1)\eta$.

Since there exists a point $p \in M$ so that $\langle X(p), e \rangle > 0$, we can find $t \in [t_0, \pi/2]$ and a point $y_0 \in M$ which is a contact point with $\mathcal{M}_{R,\eta,t}$. Therefore $|H(y_0)| \ge \frac{1}{4n\eta}$.



F is the vector space spanned by e and ξ .

Proof of Proposition 5.1. Integrating by parts we have

$$\int_{M} \frac{1}{2} \Delta \xi^{2} \xi^{2k-2} dv = \frac{1}{2} \int_{M} \left\langle d\xi^{2}, d\xi^{2k-2} \right\rangle dv = 2 \left(\frac{k-1}{k^{2}} \right) \int_{M} |d\xi^{k}|^{2} dv$$
$$\leq (A_{1} + kA_{2}) \int_{M} \xi^{2k-r} dv + (B_{1} + kB_{2}) \int_{M} \xi^{2k-s} dv.$$

Now, given a smooth function f and applying the Sobolev inequality (5) to f^2 , we get

$$\begin{split} \left(\int_{M} f^{\frac{2n}{n-1}} dv \right)^{1-(1/n)} &\leq K(n) \int_{M} \left(2|f| |df| + |H| f^{2} \right) dv \\ &\leq 2K(n) \left(\int_{M} f^{2} dv \right)^{1/2} \left(\int_{M} |df|^{2} dv \right)^{1/2} + K(n) ||H||_{\infty} \int_{M} f^{2} dv \\ &= K(n) \left(\int_{M} f^{2} dv \right)^{1/2} \left(2 \left(\int_{M} |df|^{2} dv \right)^{1/2} + ||H||_{\infty} \left(\int_{M} f^{2} dv \right)^{1/2} \right) \end{split}$$

where in the second inequality, we have used the Hölder inequality. Using it again, by assuming that V(M) = 1, we have

$$\left(\int_M f^2 \, dv\right)^{1/2} \le \left(\int_M f^{\frac{2n}{n-1}} \, dv\right)^{\frac{n-1}{2n}}.$$

And finally, we obtain

$$\|f\|_{\frac{2n}{n-1}} \le K(n)(2\|df\|_2 + \|H\|_{\infty}\|f\|_2).$$

For $k \ge 2$, ξ^k is smooth and we apply the above inequality to $f = \xi^k$. Then we get

$$\begin{split} \|\xi\|_{\frac{2kn}{n-1}}^{k} &\leq K(n) \left[2 \left(\int_{M} |d\xi^{k}|^{2} dv \right)^{1/2} + \|H\|_{\infty} \left(\int_{M} \xi^{2k} dv \right)^{1/2} \right] \\ &\leq K(n) \left[2 \left(\frac{k^{2}}{2(k-1)} \right)^{1/2} \left((A_{1} + kA_{2}) \int_{M} \xi^{2k-r} dv + (B_{1} + kB_{2}) \int_{M} \xi^{2k-s} dv \right)^{1/2} + \|H\|_{\infty} \left(\int_{M} \xi^{2k} dv \right)^{1/2} \right] \\ &\leq K(n) \left[2 \left(\frac{k^{2}}{2(k-1)} \right)^{1/2} \left((A_{1} + kA_{2}) \|\xi\|_{\infty}^{k-1} + \|H\|_{\infty} \|\xi\|_{\infty} \|\xi\|_{2k-2}^{k-1} \right] \\ &\leq K(n) \left[2 \left(\frac{k^{2}}{2(k-1)} \right)^{1/2} \left(\frac{A_{1} + kA_{2}}{\|\xi\|_{\infty}^{r}} + \frac{B_{1} + kB_{2}}{\|\xi\|_{\infty}^{s}} \right)^{1/2} + \|H\|_{\infty} \|\|\xi\|_{2k-2}^{k-1} \right] \\ &\leq K(n) \left[2 \left(\frac{k^{2}}{2(k-1)} \right)^{1/2} \left(\frac{A_{1}^{1/2} + k^{1/2}A_{2}^{1/2}}{\|\xi\|_{\infty}^{r}} + \frac{B_{1}^{1/2} + k^{1/2}B_{2}^{1/2}}{\|\xi\|_{\infty}^{s/2}} \right) \\ &+ \|H\|_{\infty} \right] \|\xi\|_{\infty} \|\xi\|_{2k-2}^{k-1}. \end{split}$$

If we assume that $\|\xi\|_{\infty} > \eta$, the last inequality becomes

$$\begin{split} \|\xi\|_{\frac{2kn}{n-1}}^k &\leq K(n) \left[2\left(\frac{k^2}{2(k-1)}\right)^{1/2} \left(\frac{A_1^{1/2} + k^{1/2}A_2^{1/2}}{\eta^{r/2}} + \frac{B_1^{1/2} + k^{1/2}B_2^{1/2}}{\eta^{s/2}}\right) \\ &+ \|H\|_{\infty} \right] \|\xi\|_{\infty} \|\xi\|_{2k-2}^{k-1} \end{split}$$

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$$= \left[(K_1 + k^{1/2} K_2) \left(\frac{k^2}{k-1} \right)^{1/2} + K' \right] \|\xi\|_{\infty} \|\xi\|_{2k-2}^{k-1}.$$

Now let $q = \frac{n}{n-1} > 1$ and for $i \ge 0$ let $k = q^i + 1 \ge 2$. Then

$$\begin{split} \|\xi\|_{2(q^{i+1}+q)} &\leq \left(\left(K_1 + (q^i+1)^{1/2}K_2\right) \left(\frac{q^i+1}{q^{i/2}}\right) + K'' \right)^{\frac{1}{q^i+1}} \|\xi\|_{\infty}^{\frac{1}{q^i+1}} \|\xi\|_{2q^i}^{1-\frac{1}{q^i+1}} \\ &\leq \left(\tilde{K}q^i\right)^{\frac{1}{q^i+1}} \|\xi\|_{\infty}^{\frac{1}{q^i+1}} \|\xi\|_{2q^i}^{1-\frac{1}{q^i+1}} \end{split}$$

where $\tilde{K} = 2K_1 + 2^{3/2}K_2 + K'$. We see that \tilde{K} has a finite limit when $\eta \to \infty$ and if $B_1 > 0$, $\tilde{K} \to \infty$ when $||H||_{\infty} \to \infty$ or $\eta \to 0$. Moreover the Hölder inequality gives

$$\|\xi\|_{2q^{i+1}} \le \|\xi\|_{2(q^{i+1}+q)}$$

which implies

$$\|\xi\|_{2q^{i+1}} \le \left(\tilde{K}q^{i}\right)^{\frac{1}{q^{i+1}}} \|\xi\|_{\infty}^{\frac{1}{q^{i+1}}} \|\xi\|_{2q^{i}}^{1-\frac{1}{q^{i+1}}}.$$

Now, by iterating from 0 to i, we get

$$\begin{split} \|\xi\|_{2q^{i+1}} &\leq \tilde{K}^{\left(1-\prod_{k=i-j}^{i}\left(1-\frac{1}{q^{k}+1}\right)\right)} q^{\sum_{k=i-j}^{i}\frac{k}{q^{k}+1}} \|\xi\|_{\infty}^{\left(1-\prod_{k=i-j}^{i}\left(1-\frac{1}{q^{k}+1}\right)\right)} \|\xi\|_{2q^{i-j}}^{\prod_{k=i-j}^{i}\left(1-\frac{1}{q^{k}+1}\right)} \\ &\leq \tilde{K}^{\left(1-\prod_{k=0}^{i}\left(1-\frac{1}{q^{k}+1}\right)\right)} q^{\sum_{k=0}^{i}\frac{k}{q^{k}+1}} \|\xi\|_{\infty}^{\left(1-\prod_{k=0}^{i}\left(1-\frac{1}{q^{k}+1}\right)\right)} \|\xi\|_{2}^{\prod_{k=0}^{i}\left(1-\frac{1}{q^{k}+1}\right)}. \\ \text{Let } \alpha &= \sum_{k=0}^{\infty}\frac{k}{q^{k}+1} \text{ and } \beta = \prod_{k=0}^{\infty}\left(1-\frac{1}{q^{k}+1}\right) = \prod_{k=0}^{\infty}\left(\frac{1}{1+(1/q)^{k}}\right). \text{ Then} \\ &\|\xi\|_{\infty} \leq \tilde{K}^{1-\beta}q^{\alpha}\|\xi\|_{\infty}^{(1-\beta)}\|\xi\|_{2}^{\beta}, \end{split}$$

and finally

 $\|\xi\|_{\infty} \le L \|\xi\|_2$

where $L = \tilde{K}^{\frac{1-\beta}{\beta}} q^{\alpha/\beta}$ is a constant depending only on $n, A_1, A_2, B_1, B_2, ||H||_{\infty}$ and η . From classical methods we show that $\beta \in [e^{-n}, e^{-n/2}]$. In particular, $0 < \beta < 1$ and we deduce that L is bounded when $\eta \to \infty$ and $L \to \infty$ when $||H||_{\infty} \to \infty$ or $\eta \to 0$ with $B_1 > 0$.

Remark. In [12] and [13] Shihohama and Xu have proved that if (M^n, g) is a compact *n*-dimensional Riemannian manifold without boundary isometrically immersed in

 \mathbb{R}^{n+1} and if $\int_M (|B|^2 - n|H|^2) < D_n$ where D_n is a constant depending on n, then all Betti numbers are zero. For n = 2, $D_2 = 4\pi$, and it follows that if

$$\int_M |B|^2 \, dv - 4\pi < \lambda_1(M) V(M)$$

then we deduce from the Reilly inequality $\lambda_1(M)V(M) \leq 2\int_M H^2 dv$ that $\int_M (|B|^2 - 2|H|^2) dv < 4\pi$ and by the result of Shihohama and Xu M is diffeomorphic to \mathbb{S}^2 .

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Received October 21, 2003; revised November 4, 2005

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