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A pinching theorem for the first eigenvalue of the Laplacian on hypersurfaces of the Euclidean space

Bruno Colbois and Jean-François Grosjean*

Abstract. In this paper, we give pinching theorems for the first nonzero eigenvalue $\lambda_1(M)$ of the Laplacian on the compact hypersurfaces of the Euclidean space. Indeed, we prove that if the volume of M is 1 then, for any $\varepsilon > 0$, there exists a constant C_ε depending on the dimension n of M and the L_∞ -norm of the mean curvature H , so that if the L_{2p} -norm $\|H\|_{2p}$ ($p \geq 2$) of H satisfies $n\|H\|_{2p}^2 - C_\varepsilon < \lambda_1(M)$, then the Hausdorff-distance between M and a round sphere of radius $(n/\lambda_1(M))^{1/2}$ is smaller than ε . Furthermore, we prove that if C is a small enough constant depending on n and the L_∞ -norm of the second fundamental form, then the pinching condition $n\|H\|_{2p}^2 - C < \lambda_1(M)$ implies that M is diffeomorphic to an n -dimensional sphere.

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1. Introduction and preliminaries

Let (M^n, g) be a compact, connected and oriented n -dimensional Riemannian manifold without boundary isometrically immersed by ϕ into the $n + 1$ -dimensional euclidean space (\mathbb{R}^{n+1}, can) (i.e. $\phi^*can = g$). A well-known inequality due to Reilly ([11]) gives an extrinsic upper bound for the first nonzero eigenvalue $\lambda_1(M)$ of the Laplacian of (M^n, g) in terms of the square of the length of the mean curvature. Indeed, we have

$$\lambda_1(M) \leq \frac{n}{V(M)} \int_M |H|^2 dv \quad (1)$$

where dv and $V(M)$ denote respectively the Riemannian volume element and the volume of (M^n, g) . Moreover the equality holds if and only if (M^n, g) is a geodesic hypersphere of \mathbb{R}^{n+1} .

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By using Hölder inequality, we obtain some other similar estimates for the L_{2p} -norm ($p \geq 1$) with H denoted by $\|H\|_{2p}^2$

$$\lambda_1(M) \leq \frac{n}{V(M)^{1/p}} \|H\|_{2p}^2, \quad (2)$$

and as for the inequality (1), the equality case is characterized by the geodesic hyperspheres of \mathbb{R}^{n+1} .

A first natural question is to know if there exists a pinching result as the one we state now: does a constant C depending on minimum geometric invariants exist so that if we have the pinching condition

$$(P_C) \quad \frac{n}{V(M)^{1/p}} \|H\|_{2p}^2 - C < \lambda_1(M)$$

then M is close to a sphere in a certain sense?

Such questions are known for the intrinsic lower bound of Lichnerowicz–Obata ([9]) of $\lambda_1(M)$ in terms of the lower bound of the Ricci curvature (see [4], [8], [10]). Other pinching results have been proved for Riemannian manifolds with positive Ricci curvature, with a pinching condition on the $n + 1$ -st eigenvalue ([10]), the diameter ([5], [8], [15]), the volume or the radius (see for instance [2] and [3]).

For instance, S. Ilias proved in [8] that there exists ε depending on n and an upper bound of the sectional curvature so that if the Ricci curvature Ric of M satisfies $\text{Ric} \geq n - 1$ and $\lambda_1(M) \leq \lambda_1(\mathbb{S}^n) + \varepsilon$, then M is homeomorphic to \mathbb{S}^n .

In this article, we investigate the case of hypersurfaces where, as far as we know, very little is known about pinching and stability results (see however [12], [13]).

More precisely, in our paper, the hypothesis made in [8] that M has a positive Ricci curvature is replaced by the fact that M is isometrically immersed as a hypersurface in \mathbb{R}^{n+1} , and the bound on the sectional curvature by an L^∞ -bound on the mean curvature or on the second fundamental form. Note that we do not know if such bounds are sharp, or if a bound on the L^q -norm (for some q) of the mean curvature would be enough.

We get the following results

Theorem 1.1. *Let (M^n, g) be a compact, connected and oriented n -dimensional Riemannian manifold without boundary isometrically immersed by ϕ in \mathbb{R}^{n+1} . Assume that $V(M) = 1$ and let x_0 be the center of mass of M . Then for any $p \geq 2$ and for any $\varepsilon > 0$, there exists a constant C_ε depending only on n , $\varepsilon > 0$ and on the L_∞ -norm of H so that if*

$$(P_{C_\varepsilon}) \quad n \|H\|_{2p}^2 - C_\varepsilon < \lambda_1(M)$$

then the Hausdorff-distance d_H of M to the sphere $S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right)$ of center x_0 and radius $\sqrt{\frac{n}{\lambda_1(M)}}$ satisfies $d_H\left(\phi(M), S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right)\right) < \varepsilon$.

We recall that the Hausdorff-distance between two compact subsets A and B of a metric space is given by

$$d_H(A, B) = \inf\{\eta \mid V_\eta(A) \supset B \text{ and } V_\eta(B) \supset A\}$$

where for any subset A , $V_\eta(A)$ is the tubular neighborhood of A defined by $V_\eta(A) = \{x \mid \text{dist}(x, A) < \eta\}$.

Remark. We will see in the proof that $C_\varepsilon(n, \|H\|_\infty) \rightarrow 0$ when $\|H\|_\infty \rightarrow \infty$ or $\varepsilon \rightarrow 0$.

In fact the previous theorem is a consequence of the above definition and the following theorem

Theorem 1.2. *Let (M^n, g) be a compact, connected and oriented n -dimensional Riemannian manifold without boundary isometrically immersed by ϕ in \mathbb{R}^{n+1} . Assume that $V(M) = 1$ and let x_0 be the center of mass of M . Then for any $p \geq 2$ and for any $\varepsilon > 0$, there exists a constant C_ε depending only on n , $\varepsilon > 0$ and on the L_∞ -norm of H so that if*

$$(P_{C_\varepsilon}) \quad n\|H\|_{2p}^2 - C_\varepsilon < \lambda_1(M)$$

then

- (1) $\phi(M) \subset B\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}} + \varepsilon\right) \setminus B\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}} - \varepsilon\right)$;
- (2) $B(x, \varepsilon) \cap \phi(M) \neq \emptyset$ for all $x \in S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right)$.

In the following theorem, if the pinching is strong enough, with a control on n and the L_∞ -norm of the second fundamental form, we obtain that M is diffeomorphic to a sphere and even almost isometric with a round sphere in a sense we will make precise.

Theorem 1.3. *Let (M^n, g) be a compact, connected and oriented n -dimensional Riemannian manifold ($n \geq 2$) without boundary isometrically immersed by ϕ in \mathbb{R}^{n+1} . Assume that $V(M) = 1$. Then for any $p \geq 2$, there exists a constant C depending only on n and the L_∞ -norm of the second fundamental form B so that if*

$$(P_C) \quad n\|H\|_{2p}^2 - C < \lambda_1(M).$$

Then M is diffeomorphic to S^n .

More precisely, there exists a diffeomorphism F from M into the sphere $S^n\left(\sqrt{\frac{n}{\lambda_1(M)}}\right)$ of radius $\sqrt{\frac{n}{\lambda_1(M)}}$ which is a quasi-isometry. Namely, for any θ ,

$0 < \theta < 1$, there exists a constant C depending only on n , the L_∞ -norm of B and θ , so that the pinching condition (P_C) implies

$$||dF_x(u)|^2 - 1| \leq \theta$$

for any $x \in M$ and $u \in T_x M$ so that $|u| = 1$.

Now we will give some preliminaries for the proof of these theorems. Throughout the paper, we consider a compact, connected and oriented n -dimensional Riemannian manifold (M^n, g) without boundary isometrically immersed by ϕ into (\mathbb{R}^{n+1}, can) (i.e. $\phi^* can = g$). Let ν be the outward normal vector field. Then the second fundamental form of the immersion will be defined by $B(X, Y) = \langle \nabla_X^0 \nu, Y \rangle$, where ∇^0 and $\langle \cdot, \cdot \rangle$ are respectively the Riemannian connection and the inner product of \mathbb{R}^{n+1} . Moreover the mean curvature H will be given by $H = (1/n) \text{trace}(B)$.

Now let ∂_i be an orthonormal frame of \mathbb{R}^{n+1} and let $x_i : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be the associated component functions. Putting $X_i = x_i \circ \phi$, a straightforward calculation shows us that

$$B \otimes \nu = - \sum_{i \leq n+1} \nabla dX_i \otimes \partial_i$$

and

$$nH\nu = \sum_{i \leq n+1} \Delta X_i \partial_i,$$

where ∇ and Δ denote respectively the Riemannian connection and the Laplace–Beltrami operator of (M^n, g) . On the other hand, we have the well-known formula

$$\frac{1}{2} \Delta |X|^2 = nH \langle \nu, X \rangle - n \quad (3)$$

where X is the position vector given by $X = \sum_{i \leq n+1} X_i \partial_i$.

We recall that to prove the Reilly inequality, we use the functions X_i as test functions (cf. [11]). Indeed, doing a translation if necessary, we can assume that $\int_M X_i dv = 0$ for all $i \leq n+1$ and we can apply the variational characterization of $\lambda_1(M)$ to X_i . If the equality holds in (1) or (2), then the functions are nothing but eigenfunctions of $\lambda_1(M)$ and from the Takahashi Theorem ([14]) M is immersed isometrically in \mathbb{R}^{n+1} as a geodesic sphere of radius $\sqrt{\frac{n}{\lambda_1(M)}}$.

Throughout the paper we use some notations. From now on, the inner product and the norm induced by g and can on a tensor T will be denoted respectively by $\langle \cdot, \cdot \rangle$ and $|\cdot|^2$, and the L_p -norm will be given by

$$\|T\|_p = \left(\int_M |T|^p dv \right)^{1/p}$$

and

$$\|T\|_\infty = \sup_M |T|.$$

We end these preliminaries by a convenient result.

Lemma 1.1. *Let (M^n, g) be a compact, connected and oriented n -dimensional Riemannian manifold ($n \geq 2$) without boundary isometrically immersed by ϕ in \mathbb{R}^{n+1} . Assume that $V(M) = 1$. Then there exist constants c_n and d_n depending only on n so that for any $p \geq 2$, if (P_C) is true with $C < c_n$ then*

$$\frac{n}{\lambda_1(M)} \leq d_n. \quad (4)$$

Proof. We recall the standard Sobolev inequality (cf. [6], [7], [16] and p. 216 in [1]). If f is a smooth function and $f \geq 0$, then

$$\left(\int_M f^{\frac{n}{n-1}} dv \right)^{1-(1/n)} \leq K(n) \int_M (|df| + |H|f) dv \quad (5)$$

where $K(n)$ is a constant depending on n and the volume of the unit ball in \mathbb{R}^n . Taking $f = 1$ on M , and using the fact that $V(M) = 1$, we deduce that

$$\|H\|_{2p} \geq \frac{1}{K(n)}$$

and if (P_C) is satisfied and $C \leq \frac{n}{2K(n)^2} = c_n$, then

$$\frac{n}{\lambda_1(M)} \leq \frac{1}{n\|H\|_{2p}^2 - C} \leq 2K(n)^2 = d_n. \quad \square$$

Throughout the paper, we will assume that $V(M) = 1$ and $\int_M X_i dv = 0$ for all $i \leq n+1$. The last assertion implies that the center of mass of M is the origin of \mathbb{R}^{n+1} .

2. An L^2 -approach of the problem

A first step in the proof of Theorem 1.2 is to prove that if the pinching condition (P_C) is satisfied, then M is close to a sphere in an L^2 -sense.

In the following lemma, we prove that the L^2 -norm of the position vector is close to $\sqrt{\frac{n}{\lambda_1(M)}}$.

Lemma 2.1. *If we have the pinching condition (P_C) with $C < c_n$, then*

$$\frac{n\lambda_1(M)}{(C + \lambda_1(M))^2} \leq \|X\|_2^2 \leq \frac{n}{\lambda_1(M)} \leq d_n.$$

Proof. Since $\int_M X_i dv = 0$, we can apply the variational characterization of the eigenvalues to obtain

$$\lambda_1(M) \int_M \sum_{i \leq n+1} |X_i|^2 dv \leq \int_M \sum_{i \leq n+1} |dX_i|^2 dv = n$$

which gives the inequality of the right-hand side.

Let us prove now the inequality of the left-hand side.

$$\begin{aligned} \lambda_1(M) \int_M |X|^2 dv &\leq \frac{(\int_M \sum_{i \leq n+1} |dX_i|^2 dv)^4}{(\int_M \sum_{i \leq n+1} |dX_i|^2 dv)^3} = \frac{(\int_M \sum_{i \leq n+1} (\Delta X_i) X_i dv)^4}{n^3} \\ &\leq \frac{(\int_M \sum_{i \leq n+1} (\Delta X_i)^2 dv)^2 (\int_M |X|^2 dv)^2}{n^3} \\ &= n \left(\int_M H^2 dv \right)^2 \left(\int_M |X|^2 dv \right)^2 \end{aligned}$$

then using again the Hölder inequality, we get

$$\lambda_1(M) \leq \frac{1}{n} (n \|H\|_{2p}^2)^2 \int_M |X|^2 dv \leq \frac{(C + \lambda_1(M))^2}{n} \int_M |X|^2 dv.$$

This completes the proof. \square

From now on, we will denote by X^T the orthogonal tangential projection on M . In fact, at $x \in M$, X^T is nothing but the vector of $T_x M$ defined by $X^T = \sum_{1 \leq i \leq n} \langle X, e_i \rangle e_i$ where $(e_i)_{1 \leq i \leq n}$ is an orthonormal basis of $T_x M$. In the following lemma, we will show that the condition (P_C) implies that the L^2 -norm of X^T of X on M is close to 0.

Lemma 2.2. *If we have the pinching condition (P_C) , then*

$$\|X^T\|_2^2 \leq A(n)C.$$

Proof. From Lemma 2.1 and the relation (3), we have

$$\lambda_1(M) \int_M |X|^2 dv \leq n = n \left(\int_M H \langle X, v \rangle dv \right)^2$$

$$\begin{aligned}
&\leq \left(\int_M |H| |\langle X, v \rangle| dv \right)^2 \leq n \|H\|_{2p}^2 \left(\int_M |\langle X, v \rangle|^{\frac{2p}{2p-1}} dv \right)^{\frac{2p-1}{p}} \\
&\leq n \|H\|_{2p}^2 \left(\int_M |\langle X, v \rangle|^2 dv \right) = n \|H\|_{2p}^2 \int_M |X|^2 dv.
\end{aligned}$$

Then we deduce that

$$\begin{aligned}
n \|H\|_{2p}^2 \|X^T\|_2^2 &= n \|H\|_{2p}^2 \left(\int_M (|X|^2 - |\langle X, v \rangle|^2) dv \right) \\
&\leq (n \|H\|_{2p}^2 - \lambda_1(M)) \|X\|_2^2 \leq d_n C
\end{aligned}$$

where in the last inequality we have used the pinching condition and Lemma 2.1. \square

Next we will show that the condition (P_C) implies that the component functions are almost eigenfunctions in an L^2 -sense. For this, let us consider the vector field Y on M defined by

$$Y = \sum_{i \leq n+1} (\Delta X_i - \lambda_1(M) X_i) \partial_i = n H v - \lambda_1(M) X.$$

Lemma 2.3. *If (P_C) is satisfied, then*

$$\|Y\|_2^2 \leq nC.$$

Proof. We have

$$\int_M |Y|^2 dv = \int_M (n^2 H^2 - 2n \lambda_1(M) H \langle v, X \rangle + \lambda_1(M)^2 |X|^2) dv.$$

Now by integrating the relation (3) we deduce that

$$\int_M H \langle v, X \rangle dv = 1.$$

Furthermore, since $\int_M X_i dv = 0$, we can apply the variational characterization of the eigenvalues to obtain

$$\lambda_1(M) \int_M |X|^2 dv = \lambda_1(M) \int_M \sum_{i \leq n+1} |X_i|^2 dv \leq \int_M \sum_{i \leq n+1} |dX_i|^2 dv = n.$$

Then

$$\int_M |Y|^2 dv \leq n^2 \int_M |H|^2 dv - n \lambda_1(M) \leq n (n \|H\|_{2p}^2 - \lambda_1(M)) \leq nC$$

where in this last inequality we have used the Hölder inequality. \square

To prove Assertion 1 of Theorem 1.2, we will show that $\left\| |X| - \left(\frac{n}{\lambda_1(M)} \right)^{1/2} \right\|_\infty \leq \varepsilon$.

For this we need an L^2 -upper bound on the function $\varphi = |X| \left(|X| - \left(\frac{n}{\lambda_1(M)} \right)^{1/2} \right)^2$.

Before giving such estimate, we will introduce the vector field Z on M defined by

$$Z = \left(\frac{n}{\lambda_1(M)} \right)^{1/2} |X|^{1/2} H\nu - \frac{X}{|X|^{1/2}}.$$

We have

Lemma 2.4. *If (P_C) is satisfied with $C < c_n$, then*

$$\|Z\|_2^2 \leq B(n)C.$$

Proof. We have

$$\begin{aligned} \|Z\|_2^2 &= \left\| \left(\frac{n}{\lambda_1(M)} \right)^{1/2} |X|^{1/2} H\nu - \frac{X}{|X|^{1/2}} \right\|_2^2 \\ &= \int_M \left(\frac{n}{\lambda_1(M)} |X| H^2 - 2 \left(\frac{n}{\lambda_1(M)} \right)^{1/2} H \langle \nu, X \rangle + |X| \right) dv \\ &\leq \frac{n}{\lambda_1(M)} \left(\int_M |X|^2 dv \right)^{1/2} \left(\int_M H^4 dv \right)^{1/2} \\ &\quad - 2 \left(\frac{n}{\lambda_1(M)} \right)^{1/2} + \left(\int_M |X|^2 dv \right)^{1/2}. \end{aligned}$$

Note that we have used the relation (3). Finally for $p \geq 2$, we get

$$\begin{aligned} \|Z\|_2^2 &\leq \left(\int_M |X|^2 dv \right)^{1/2} \left(\frac{n}{\lambda_1(M)} \|H\|_{2p}^2 + 1 \right) - 2 \left(\frac{n}{\lambda_1(M)} \right)^{1/2} \\ &\leq \left(\frac{n}{\lambda_1(M)} \right)^{1/2} \left(\frac{C}{\lambda_1(M)} + 2 \right) - 2 \left(\frac{n}{\lambda_1(M)} \right)^{1/2} \\ &= \left(\frac{n}{\lambda_1(M)} \right)^{1/2} \frac{C}{\lambda_1(M)} \leq \frac{d_n^{3/2}}{n} C. \end{aligned}$$

This concludes the proof of the lemma. □

Now we give an L^2 -upper bound of φ .

Lemma 2.5. *Let $p \geq 2$ and $C \leq c_n$. If we have the pinching condition (P_C) , then*

$$\|\varphi\|_2 \leq D(n) \|\varphi\|_\infty^{3/4} C^{1/4}.$$

Proof. We have

$$\|\varphi\|_2 = \left(\int_M \varphi^{3/2} \varphi^{1/2} dv \right)^{1/2} \leq \|\varphi\|_\infty^{3/4} \|\varphi^{1/2}\|_1^{1/2},$$

and noting that

$$|X| \left(|X| - \left(\frac{n}{\lambda_1(M)} \right)^{1/2} \right)^2 = \left| |X|^{1/2} X - \left(\frac{n}{\lambda_1(M)} \right)^{1/2} \frac{X}{|X|^{1/2}} \right|^2$$

we get

$$\begin{aligned} \int_M \varphi^{1/2} dv &= \left\| |X|^{1/2} X - \left(\frac{n}{\lambda_1(M)} \right)^{1/2} \frac{X}{|X|^{1/2}} \right\|_1 \\ &= \left\| -\frac{|X|^{1/2}}{\lambda_1(M)} Y + \frac{n}{\lambda_1(M)} |X|^{1/2} H v - \left(\frac{n}{\lambda_1(M)} \right)^{1/2} \frac{X}{|X|^{1/2}} \right\|_1 \\ &\leq \left\| \frac{|X|^{1/2}}{\lambda_1(M)} Y \right\|_1 + \left(\frac{n}{\lambda_1(M)} \right)^{1/2} \|Z\|_1. \end{aligned} \quad (6)$$

From Lemmas 2.3 and 1.1 we get

$$\begin{aligned} \left\| \frac{|X|^{1/2}}{\lambda_1(M)} Y \right\|_1 &\leq \frac{1}{\lambda_1(M)} \left(\int_M |X| dv \right)^{1/2} \|Y\|_2 \\ &\leq \frac{1}{\lambda_1(M)} \left(\int_M |X|^2 dv \right)^{1/4} \|Y\|_2 \leq \frac{d_n^{3/4}}{n^{1/2}} C^{1/2}. \end{aligned}$$

Moreover, using Lemmas 2.4 and 1.1 again it is easy to see that the last term of (6) is bounded by $d_n^{1/2} B(n)^{1/2} C^{1/2}$. Then $\|\varphi^{1/2}\|_1^{1/2} \leq D(n) C^{1/4}$. \square

3. Proof of Theorem 1.2

The proof of Theorem 1.2 is immediate from the two following technical lemmas which we state below.

Lemma 3.1. *For $p \geq 2$ and for any $\eta > 0$, there exists $K_\eta(n, \|H\|_\infty) \leq c_n$ so that if (P_{K_η}) is true, then $\|\varphi\|_\infty \leq \eta$. Moreover, $K_\eta \rightarrow 0$ when $\|H\|_\infty \rightarrow \infty$ or $\eta \rightarrow 0$.*

Lemma 3.2. *Let x_0 be a point of the sphere $S(O, R)$ of \mathbb{R}^{n+1} with the center at the origin and of radius R . Assume that $x_0 = Re$ where $e \in \mathbb{S}^n$. Now let (M^n, g) be a compact oriented n -dimensional Riemannian manifold without boundary isometrically*

immersed by ϕ in \mathbb{R}^{n+1} so that $\phi(M) \subset (B(O, R + \eta) \setminus B(O, R - \eta)) \setminus B(x_0, \rho)$ with $\rho = 4(2n - 1)\eta$ and suppose that there exists a point $p \in M$ so that $\langle X, e \rangle > 0$. Then there exists $y_0 \in M$ so that the mean curvature $H(y_0)$ at y_0 satisfies $|H(y_0)| \geq \frac{1}{4n\eta}$.

Now, let us see how to use these lemmas to prove Theorem 1.2.

Proof of Theorem 1.2. We consider the function $f(t) = t \left(t - \left(\frac{n}{\lambda_1(M)} \right)^{1/2} \right)^2$. For $\varepsilon > 0$ let us put

$$\begin{aligned} \eta(\varepsilon) &= \min \left(\left(\frac{1}{\|H\|_\infty} - \varepsilon \right) \varepsilon^2, \left(\frac{1}{\|H\|_\infty} + \varepsilon \right) \varepsilon^2, \frac{1}{27\|H\|_\infty^3} \right) \\ &\leq \min \left(f \left(\left(\frac{n}{\lambda_1(M)} \right)^{1/2} - \varepsilon \right), f \left(\left(\frac{n}{\lambda_1(M)} \right)^{1/2} + \varepsilon \right), \frac{1}{27\|H\|_\infty^3} \right). \end{aligned}$$

Then, as $\eta(\varepsilon) > 0$ and from Lemma 3.1, it follows that if the pinching condition $(P_{K_{\eta(\varepsilon)}})$ is satisfied with $K_{\eta(\varepsilon)} \leq c_n$, then for any $x \in M$, we have

$$f(|X|) \leq \eta(\varepsilon). \quad (7)$$

Now to prove Theorem 1.2, it is sufficient to assume $\varepsilon < \frac{2}{3\|H\|_\infty}$. Let us show that either

$$\left(\frac{n}{\lambda_1(M)} \right)^{1/2} - \varepsilon \leq |X| \leq \left(\frac{n}{\lambda_1(M)} \right)^{1/2} + \varepsilon \quad \text{or} \quad |X| < \frac{1}{3} \left(\frac{n}{\lambda_1(M)} \right)^{1/2}. \quad (8)$$

By studying the function f it is easy to see that f has a unique local maximum in $\frac{1}{3} \left(\frac{n}{\lambda_1(M)} \right)^{1/2}$ and from the definition of $\eta(\varepsilon)$ it follows that $\eta(\varepsilon) < \frac{4}{27} \frac{1}{\|H\|_\infty^3} \leq \frac{4}{27} \left(\frac{n}{\lambda_1(M)} \right)^{3/2} = f \left(\frac{1}{3} \left(\frac{n}{\lambda_1(M)} \right)^{1/2} \right)$.

Since $\varepsilon < \frac{2}{3\|H\|_\infty}$, we have $\varepsilon < \frac{2}{3} \left(\frac{n}{\lambda_1(M)} \right)^{1/2}$ and $\frac{1}{3} \left(\frac{n}{\lambda_1(M)} \right)^{1/2} < \left(\frac{n}{\lambda_1(M)} \right)^{1/2} - \varepsilon$. This and (7) yield (8).

Now, from Lemma 2.1 we deduce that there exists a point $y_0 \in M$ so that $|X(y_0)| \geq \frac{n^{1/2}\lambda_1(M)^{1/2}}{(K_{\eta(\varepsilon)} + \lambda_1(M))}$ and since $K_{\eta(\varepsilon)} \leq c_n = \frac{n}{d_n} \leq \lambda_1(M) \leq 2\lambda_1(M)$ (see the proof of Lemma 1.1), we obtain $|X(y_0)| \geq \frac{1}{3} \left(\frac{n}{\lambda_1(M)} \right)^{1/2}$.

By the connectedness of M , it follows that $\left(\frac{n}{\lambda_1(M)} \right)^{1/2} - \varepsilon \leq |X| \leq \left(\frac{n}{\lambda_1(M)} \right)^{1/2} + \varepsilon$ for any point of M and Assertion 1 of Theorem 1.2 is shown for the condition $(P_{K_{\eta(\varepsilon)}})$.

In order to prove the second assertion, let us consider the pinching condition (P_{C_ε}) with $C_\varepsilon = K_{\eta(\frac{\varepsilon}{4(2n-1)})}$. Then Assertion 1 is still valid. Let $x = \left(\frac{n}{\lambda_1(M)} \right)^{1/2} e \in S \left(O, \sqrt{\frac{n}{\lambda_1(M)}} \right)$, with $e \in \mathbb{S}^n$ and suppose that $B(x, \varepsilon) \cap M = \emptyset$. Since $\int_M X_i dv = 0$

for any $i \leq n+1$, there exists a point $p \in M$ so that $\langle X, e \rangle > 0$ and we can apply Lemma 3.2. Therefore there is a point $y_0 \in M$ so that $H(y_0) \geq \frac{2n-1}{n\varepsilon} > \|H\|_\infty$ since we have assumed $\varepsilon < \frac{2}{3\|H\|_\infty} \leq \frac{2n-1}{2n\|H\|_\infty}$. Then we obtain a contradiction which implies $B(x, \varepsilon) \cap M \neq \emptyset$ and Assertion 2 is satisfied. Furthermore, $C_\varepsilon \rightarrow 0$ when $\|H\|_\infty \rightarrow \infty$ or $\varepsilon \rightarrow 0$. \square

4. Proof of Theorem 1.3

From Theorem 1.2, we know that for any $\varepsilon > 0$, there exists C_ε depending only on n and $\|H\|_\infty$ so that if (P_{C_ε}) is true then

$$\left| |X|_x - \sqrt{\frac{n}{\lambda_1(M)}} \right| \leq \varepsilon$$

for any $x \in M$. Now, since $\sqrt{n}\|H\|_\infty \leq \|B\|_\infty$, it is easy to see from the previous proofs that we can assume that C_ε is depending only on n and $\|B\|_\infty$.

The proof of Theorem 1.3 is a consequence of the following lemma on the L_∞ -norm of $\psi = |X^T|$.

Lemma 4.1. *For $p \geq 2$ and for any $\eta > 0$, there exists $K_\eta(n, \|B\|_\infty)$ so that if (P_{K_η}) is true, then $\|\psi\|_\infty \leq \eta$. Moreover, $K_\eta \rightarrow 0$ when $\|B\|_\infty \rightarrow \infty$ or $\eta \rightarrow 0$.*

This lemma will be proved in the Section 5.

Proof of Theorem 1.3. Let $\varepsilon < \frac{1}{2}\sqrt{\frac{n}{\|B\|_\infty}} \leq \sqrt{\frac{n}{\lambda_1(M)}}$. From the choice of ε , we deduce that the condition (P_{C_ε}) implies that $|X_x|$ is nonzero for any $x \in M$ (see the proof of Theorem 1.2) and we can consider the differential application

$$F: M \longrightarrow S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right),$$

$$x \longmapsto \sqrt{\frac{n}{\lambda_1(M)}} \frac{X_x}{|X_x|}.$$

We will prove that F is a quasi-isometry. Indeed, for any $0 < \theta < 1$, we can choose a constant $\varepsilon(n, \|B\|_\infty, \theta)$ so that for any $x \in M$ and any unit vector $u \in T_x M$, the pinching condition $(P_{C_{\varepsilon(n, \|B\|_\infty, \theta)}})$ implies

$$||dF_x(u)|^2 - 1| \leq \theta.$$

For this, let us compute $dF_x(u)$. We have

$$dF_x(u) = \sqrt{\frac{n}{\lambda_1(M)}} \nabla_u^0 \left(\frac{X}{|X|} \right) \Big|_x = \sqrt{\frac{n}{\lambda_1(M)}} u \left(\frac{1}{|X|} \right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X =$$

$$\begin{aligned}
&= -\frac{1}{2} \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|^3} u(|X|^2) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} u \\
&= -\sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|^3} \langle u, X \rangle X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} u \\
&= \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \left(-\frac{\langle u, X \rangle}{|X|^2} X + u \right).
\end{aligned}$$

By a straightforward computation, we obtain

$$\begin{aligned}
|dF_x(u)|^2 - 1 &= \left| \frac{n}{\lambda_1(M)} \frac{1}{|X|^2} \left(1 - \frac{\langle u, X \rangle^2}{|X|^2} \right) - 1 \right| \\
&\leq \left| \frac{n}{\lambda_1(M)} \frac{1}{|X|^2} - 1 \right| + \frac{n}{\lambda_1(M)} \frac{1}{|X|^4} \langle u, X \rangle^2.
\end{aligned} \tag{9}$$

Now

$$\begin{aligned}
\left| \frac{n}{\lambda_1(M)} \frac{1}{|X|^2} - 1 \right| &= \frac{1}{|X|^2} \left| \frac{n}{\lambda_1(M)} - |X|^2 \right| \\
&\leq \varepsilon \frac{\left| \sqrt{\frac{n}{\lambda_1(M)}} + |X| \right|}{|X|^2} \leq \varepsilon \frac{2\sqrt{\frac{n}{\lambda_1(M)}} + \varepsilon}{\left(\sqrt{\frac{n}{\lambda_1(M)}} - \varepsilon \right)^2}.
\end{aligned}$$

Let us recall that $\frac{n}{d_n} \leq \lambda_1(M) \leq \|B\|_\infty^2$ (see (4) for the first inequality). Since we assume $\varepsilon < \frac{1}{2} \sqrt{\frac{n}{\|B\|_\infty}}$, the right-hand side is bounded above by a constant depending only on n and $\|B\|_\infty$ and we have

$$\left| \frac{n}{\lambda_1(M)} \frac{1}{|X|^2} - 1 \right| \leq \varepsilon \gamma(n, \|B\|_\infty). \tag{10}$$

On the other hand, since $C_\varepsilon(n, \|B\|_\infty) \rightarrow 0$ when $\varepsilon \rightarrow 0$, there exists $\varepsilon(n, \|B\|_\infty, \eta)$ so that $C_{\varepsilon(n, \|B\|_\infty, \eta)} \leq K_\eta(n, \|B\|_\infty)$ (where K_η is the constant of the lemma) and then by Lemma 4.1, $\|\psi\|_\infty^2 \leq \eta^2$. Thus there exists a constant δ depending only on n and $\|B\|_\infty$ so that

$$\frac{n}{\lambda_1(M)} \frac{1}{|X|^4} \langle u, X \rangle^2 \leq \frac{n}{\lambda_1(M)} \frac{1}{|X|^4} \|\psi\|_\infty^2 \leq \eta^2 \delta(n, \|B\|_\infty), \tag{11}$$

and from (9), (10) and (11) we deduce that the condition $(P_{C_{\varepsilon(n, \|B\|_\infty, \eta)}})$ implies

$$|dF_x(u)|^2 - 1 \leq \varepsilon \gamma(n, \|B\|_\infty) + \eta^2 \delta(n, \|B\|_\infty).$$

Now let us choose $\eta = \left(\frac{\theta}{2\delta}\right)^{1/2}$. Then we can assume that $\varepsilon(n, \|B\|_\infty, \eta)$ is small enough in order to have $\varepsilon(n, \|B\|_\infty, \eta)\gamma(n\|B\|_\infty) \leq \frac{\theta}{2}$. In this case we have

$$||dF_x(u)|^2 - 1| \leq \theta.$$

Now let us fix θ , $0 < \theta < 1$. It follows that F is a local diffeomorphism from M to $S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right)$. Since $S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right)$ is simply connected for $n \geq 2$, F is a diffeomorphism. \square

5. Proof of the technical lemmas

The proofs of Lemmas 3.1 and 4.1 are providing from a result stated in the following proposition using a Nirenberg–Moser type of proof.

Proposition 5.1. *Let (M^n, g) be a compact, connected and oriented n -dimensional Riemannian manifold without boundary isometrically immersed into the $n+1$ -dimensional euclidean space $(\mathbb{R}^{n+1}, \text{can})$. Let ξ be a nonnegative continuous function so that ξ^k is smooth for $k \geq 2$. Let $0 \leq r < s \leq 2$ so that*

$$\frac{1}{2}\Delta\xi^2\xi^{2k-2} \leq \delta\omega + (A_1 + kA_2)\xi^{2k-r} + (B_1 + kB_2)\xi^{2k-s}$$

where $\delta\omega$ is the codifferential of a 1-form and A_1, A_2, B_1, B_2 are nonnegative constants. Then for any $\eta > 0$, there exists a constant $L(n, A_1, A_2, B_1, B_2, \|H\|_\infty, \eta)$ depending only on $n, A_1, A_2, B_1, B_2, \|H\|_\infty$ and η so that if $\|\xi\|_\infty > \eta$ then

$$\|\xi\|_\infty \leq L(n, A_1, A_2, B_1, B_2, \|H\|_\infty, \eta)\|\xi\|_2.$$

Moreover, L is bounded when $\eta \rightarrow \infty$, and if $B_1 > 0$, $L \rightarrow \infty$ when $\|H\|_\infty \rightarrow \infty$ or $\eta \rightarrow 0$.

This proposition will be proved at the end of the paper.

Before giving the proofs of Lemmas 3.1 and 4.1, we will show that under the pinching condition (P_C) with C small enough, the L_∞ -norm of X is bounded by a constant depending only on n and $\|H\|_\infty$.

Lemma 5.1. *If we have the pinching condition (P_C) with $C < c_n$, then there exists $E(n, \|H\|_\infty)$ depending only on n and $\|H\|_\infty$ so that $\|X\|_\infty \leq E(n, \|H\|_\infty)$.*

Proof. From the relation (3), we have

$$\frac{1}{2}\Delta|X|^2|X|^{2k-2} \leq n\|H\|_\infty|X|^{2k-1}.$$

Then applying Proposition 5.1 to the function $\xi = |X|$ with $r = 0$ and $s = 1$, we obtain that if $\|X\|_\infty > E$, then there exists a constant $L(n, \|H\|_\infty, E)$ depending only on n , $\|H\|_\infty$ and E so that

$$\|X\|_\infty \leq L(n, \|H\|_\infty, E)\|X\|_2,$$

and under the pinching condition (P_C) with $C < c_n$ we have from Lemma 2.1 that

$$\|X\|_\infty \leq L(n, \|H\|_\infty, E)d_n^{1/2}.$$

Now since L is bounded when $E \rightarrow \infty$, we can choose $E = E(n, \|H\|_\infty)$ large enough so that

$$L(n, \|H\|_\infty, E)d_n^{1/2} < E.$$

In this case, we have $\|X\|_\infty \leq E(n, \|H\|_\infty)$. □

Proof of Lemma 3.1. First we compute the Laplacian of the square of φ^2 . We have

$$\begin{aligned} \Delta\varphi^2 &= \Delta \left(|X|^4 - 2 \left(\frac{n}{\lambda_1(M)} \right)^{1/2} |X|^3 + \frac{n}{\lambda_1(M)} |X|^2 \right) \\ &= -2|X|^2 |d|X|^2|^2 + 2|X|^2 \Delta|X|^2 \\ &\quad - 2 \left(\frac{n}{\lambda_1(M)} \right)^{1/2} \left(-\frac{3}{4}|X|^{-1} |d|X|^2|^2 + \frac{3}{2}|X| \Delta|X|^2 \right) + \frac{n}{\lambda_1(M)} \Delta|X|^2. \end{aligned}$$

Now by a direct computation one gets $|d|X|^2|^2 \leq 4|X|^2$. Moreover by the relation (3) we have $|\Delta|X|^2| \leq 2n\|H\|_\infty|X| + n$. Then applying Lemmas 1.1 and 5.1 we get

$$\Delta\varphi^2 \leq \alpha(n, \|H\|_\infty)$$

and

$$\frac{1}{2}\Delta\varphi^2\varphi^{2k-2} \leq \alpha(n, \|H\|_\infty)\varphi^{2k-2}.$$

Now, we apply Proposition 5.1 with $r = 0$ and $s = 2$. Then if $\|\varphi\|_\infty > \eta$, there exists a constant $L(n, \|H\|_\infty)$ depending only on n and $\|H\|_\infty$ so that

$$\|\varphi\|_\infty \leq L\|\varphi\|_2.$$

From Lemma 2.5, if $C \leq c_n$ and (P_C) is true, we have $\|\varphi\|_2 \leq D(n)\|\varphi\|_\infty^{3/4}C^{1/4}$. Therefore

$$\|\varphi\|_\infty \leq (LD)^4C.$$

Consequently, if we choose $C = K_\eta = \inf \left(\frac{\eta}{(LD)^4}, c_n \right)$, then we obtain $\|\varphi\|_\infty \leq \eta$. □

Proof of Lemma 4.1. First we will prove that for any $C < c_n$, if (P_C) is true, then

$$\frac{1}{2}(\Delta\psi^2)\psi^{2k-2} \leq \delta\omega + (\alpha_1(n, \|B\|_\infty) + k\alpha_2(n, \|B\|_\infty))\psi^{2k-2} \quad (12)$$

where $\delta\omega$ is the codifferential of a 1-form ω .

First observe that the gradient $\nabla^M|X|^2$ of $|X|^2$ satisfies $\nabla^M|X|^2 = 2X^T$. Then by the Bochner formula we get

$$\begin{aligned} \frac{1}{2}\Delta|X^T|^2 &= \frac{1}{4}\langle\Delta d|X|^2, d|X|^2\rangle - \frac{1}{4}|\nabla d|X|^2|^2 - \frac{1}{4}\text{Ric}(\nabla^M|X|^2, \nabla^M|X|^2) \\ &\leq \frac{1}{4}\langle d\Delta|X|^2, d|X|^2\rangle - \frac{1}{4}\text{Ric}(\nabla^M|X|^2, \nabla^M|X|^2) \end{aligned}$$

and by the Gauss formula we obtain

$$\begin{aligned} \frac{1}{2}\Delta|X^T|^2 &\leq \frac{1}{4}\langle d\Delta|X|^2, d|X|^2\rangle - \frac{1}{4}nH\langle B\nabla^M|X|^2, \nabla^M|X|^2\rangle + \frac{1}{4}|B\nabla^M|X|^2|^2 \\ &= \frac{1}{4}\langle d\Delta|X|^2, d|X|^2\rangle - nH\langle BX^T, X^T\rangle + |BX^T|^2. \end{aligned}$$

By Lemma 5.1 we know that $\|X\|_\infty \leq E(n, \|B\|_\infty)$ (the dependance in $\|H\|_\infty$ can be replaced by $\|B\|_\infty$). Then it follows that

$$\frac{1}{2}(\Delta\psi^2)\psi^{2k-2} \leq \frac{1}{4}\langle d\Delta|X|^2, d|X|^2\rangle\psi^{2k-2} + \alpha'(n, \|B\|_\infty)\psi^{2k-2}. \quad (13)$$

Now, let us compute the term $\langle d\Delta|X|^2, d|X|^2\rangle\psi^{2k-2}$. We have

$$\begin{aligned} \langle d\Delta|X|^2, d|X|^2\rangle\psi^{2k-2} &= \delta\omega + (\Delta|X|^2)^2\psi^{2k-2} - (2k-2)\Delta|X|^2\langle d|X|^2, d\psi\rangle\psi^{2k-3} \\ &= \delta\omega + (\Delta|X|^2)^2\psi^{2k-2} - 2(2k-2)\Delta|X|^2\langle X^T, \nabla^M\psi\rangle\psi^{2k-3} \end{aligned}$$

where $\omega = -\Delta|X|^2\psi^{2k-2}d|X|^2$. Now,

$$e_i(\psi) = \frac{e_i|X^T|^2}{2|X^T|} = \frac{e_i|X|^2 - e_i\langle X, v\rangle^2}{2|X^T|} = \frac{\langle e_i, X\rangle - B_{ij}\langle X, e_j\rangle\langle X, v\rangle}{|X^T|}.$$

Then

$$\begin{aligned} \langle d\Delta|X|^2, d|X|^2\rangle\psi^{2k-2} &= \delta\omega + (\Delta|X|^2)^2\psi^{2k-2} - 2(2k-2)\Delta|X|^2|X^T|\psi^{2k-3} \\ &\quad + 2(2k-2)\Delta|X|^2\frac{\langle BX^T, X^T\rangle}{|X^T|}\langle X, v\rangle\psi^{2k-3} \\ &\leq \delta\omega + (\Delta|X|^2)^2\psi^{2k-2} + 2(2k-2)|\Delta|X|^2|\psi^{2k-2} \\ &\quad + 2(2k-2)|\Delta|X|^2\|B\||X|\psi^{2k-2}. \end{aligned}$$

Now by relation (3) and Lemma 5.1 we have

$$\langle d\Delta|X|^2, d|X|^2 \rangle \psi^{2k-2} \leq \delta\omega + (\alpha_1''(n, \|B\|_\infty) + k\alpha_2''(n, \|B\|_\infty)) \psi^{2k-2}.$$

Inserting this in (13), we obtain the desired inequality (12).

Now applying again Proposition 5.1, we get that there exists $L(n, \|B\|_\infty, \eta)$ so that if $\|\psi\|_\infty > \eta$ then

$$\|\psi\|_\infty \leq L\|\psi\|_2.$$

From Lemma 2.2 we deduce that if the pinching condition (P_C) holds then $\|\psi\|_2 \leq A(n)^{1/2}C^{1/2}$. Then taking $C = K_\eta = \inf\left(\frac{\eta}{LA^{1/2}}, c_n\right)$, then $\|\psi\|_\infty \leq \eta$. \square

Proof of Lemma 3.2. The idea of the proof consists in foliating the region $B(O, R + \eta) \setminus B(O, R - \eta)$ with hypersurfaces of large mean curvature and to show that one of these hypersurfaces is tangent to $\phi(M)$. This will imply that $\phi(M)$ has a large mean curvature at the contact point.

Consider $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}e$. Let $a, L > l > 0$ and

$$\begin{aligned} \Phi_{L,l,a} : \mathbb{S}^{n-1} \times \mathbb{S}^1 &\longrightarrow \mathbb{R}^{n+1} \\ (\xi, \theta) &\longmapsto L\xi - l \cos \theta \xi + l \sin \theta e + ae. \end{aligned}$$

Then $\Phi_{L,l,a}$ is a family of embeddings from $\mathbb{S}^{n-1} \times \mathbb{S}^1$ in \mathbb{R}^{n+1} . If we orient the family of hypersurfaces $\Phi_{L,l,a}(\mathbb{S}^{n-1} \times \mathbb{S}^1)$ by the unit outward normal vector field, a straightforward computation shows that the mean curvature $H(\theta)$ depends only on θ and we have

$$H(\theta) = \frac{1}{n} \left(\frac{1}{l} - \frac{(n-1) \cos \theta}{L - l \cos \theta} \right) \geq \frac{1}{n} \left(\frac{1}{l} - \frac{n-1}{L-l} \right). \quad (14)$$

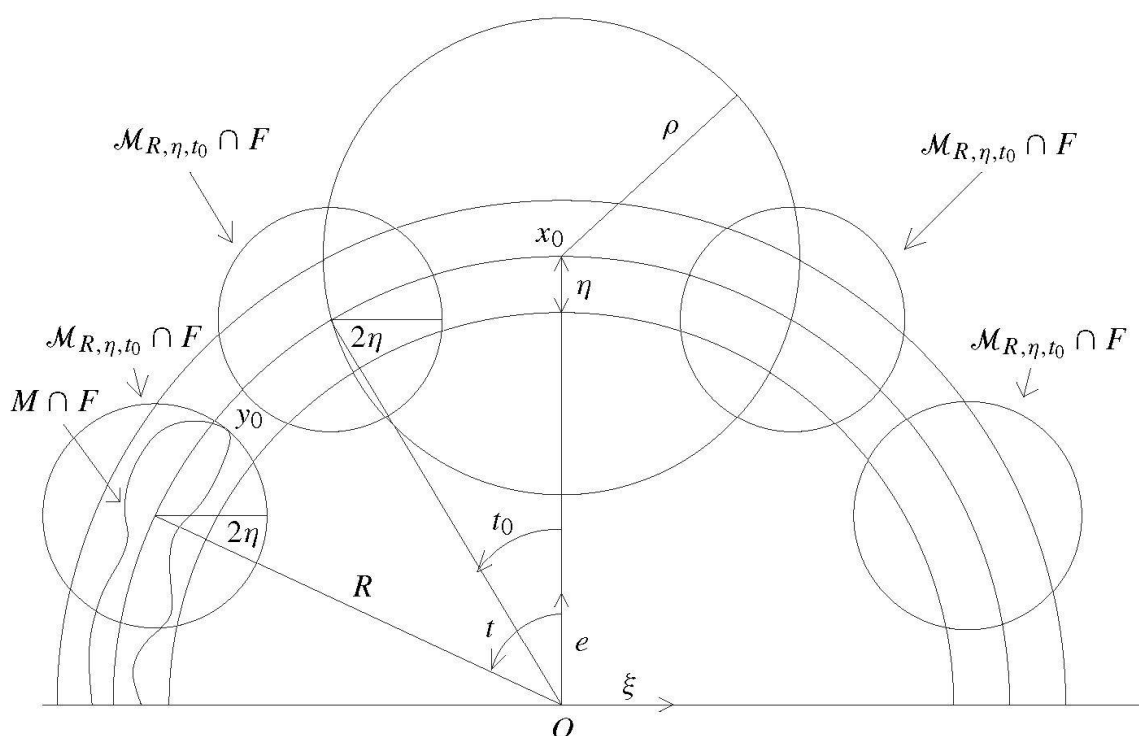
Now, let us consider the hypotheses of the lemma and for $t_0 = 2 \arcsin\left(\frac{\rho}{2R}\right) \leq t \leq \frac{\pi}{2}$, put $L = R \sin t$, $l = 2\eta$ and $a = R \cos t$. Then $L > l$ and we can consider for $t_0 \leq t \leq \frac{\pi}{2}$ the family $\mathcal{M}_{R,\eta,t}$ of hypersurfaces defined by $\mathcal{M}_{R,\eta,t} = \Phi_{R \sin t, 2\eta, R \cos t}(\mathbb{S}^{n-1} \times \mathbb{S}^1)$.

From the relation (14), the mean curvature $H_{R,\eta,t}$ of $\mathcal{M}_{R,\eta,t}$ satisfies

$$\begin{aligned} H_{R,\eta,t} &\geq \frac{1}{n} \left(\frac{1}{2\eta} - \frac{n-1}{R \sin t - 2\eta} \right) \geq \frac{1}{n} \left(\frac{1}{2\eta} - \frac{n-1}{R \sin t_0 - 2\eta} \right) \\ &\geq \frac{1}{n} \left(\frac{1}{2\eta} - \frac{n-1}{R \sin(t_0/2) - 2\eta} \right) = \frac{1}{n} \left(\frac{1}{2\eta} - \frac{\frac{\rho}{2}}{2\eta} \right) = \frac{1}{4n\eta} \end{aligned}$$

where we have used in this last equality the fact that $\rho = 4(2n-1)\eta$.

Since there exists a point $p \in M$ so that $\langle X(p), e \rangle > 0$, we can find $t \in [t_0, \pi/2]$ and a point $y_0 \in M$ which is a contact point with $\mathcal{M}_{R,\eta,t}$. Therefore $|H(y_0)| \geq \frac{1}{4n\eta}$.



F is the vector space spanned by e and ξ .

□

Proof of Proposition 5.1. Integrating by parts we have

$$\begin{aligned} \int_M \frac{1}{2} \Delta \xi^2 \xi^{2k-2} dv &= \frac{1}{2} \int_M \langle d\xi^2, d\xi^{2k-2} \rangle dv = 2 \left(\frac{k-1}{k^2} \right) \int_M |d\xi^k|^2 dv \\ &\leq (A_1 + kA_2) \int_M \xi^{2k-r} dv + (B_1 + kB_2) \int_M \xi^{2k-s} dv. \end{aligned}$$

Now, given a smooth function f and applying the Sobolev inequality (5) to f^2 , we get

$$\begin{aligned} \left(\int_M f^{\frac{2n}{n-1}} dv \right)^{1-(1/n)} &\leq K(n) \int_M (2|f||df| + |H|f^2) dv \\ &\leq 2K(n) \left(\int_M f^2 dv \right)^{1/2} \left(\int_M |df|^2 dv \right)^{1/2} + K(n) \|H\|_\infty \int_M f^2 dv \\ &= K(n) \left(\int_M f^2 dv \right)^{1/2} \left(2 \left(\int_M |df|^2 dv \right)^{1/2} + \|H\|_\infty \left(\int_M f^2 dv \right)^{1/2} \right) \end{aligned}$$

where in the second inequality, we have used the Hölder inequality. Using it again, by assuming that $V(M) = 1$, we have

$$\left(\int_M f^2 dv \right)^{1/2} \leq \left(\int_M f^{\frac{2n}{n-1}} dv \right)^{\frac{n-1}{2n}}.$$

And finally, we obtain

$$\|f\|_{\frac{2n}{n-1}} \leq K(n)(2\|df\|_2 + \|H\|_\infty \|f\|_2).$$

For $k \geq 2$, ξ^k is smooth and we apply the above inequality to $f = \xi^k$. Then we get

$$\begin{aligned} \|\xi\|_{\frac{2kn}{n-1}}^k &\leq K(n) \left[2 \left(\int_M |d\xi^k|^2 dv \right)^{1/2} + \|H\|_\infty \left(\int_M \xi^{2k} dv \right)^{1/2} \right] \\ &\leq K(n) \left[2 \left(\frac{k^2}{2(k-1)} \right)^{1/2} \left((A_1 + kA_2) \int_M \xi^{2k-r} dv \right. \right. \\ &\quad \left. \left. + (B_1 + kB_2) \int_M \xi^{2k-s} dv \right)^{1/2} + \|H\|_\infty \left(\int_M \xi^{2k} dv \right)^{1/2} \right] \\ &\leq K(n) \left[2 \left(\frac{k^2}{2(k-1)} \right)^{1/2} ((A_1 + kA_2) \|\xi\|_\infty^{2-r} \right. \\ &\quad \left. + (B_1 + kB_2) \|\xi\|_\infty^{2-s})^{1/2} \|\xi\|_{2k-2}^{k-1} + \|H\|_\infty \|\xi\|_\infty \|\xi\|_{2k-2}^{k-1} \right] \\ &\leq K(n) \left[2 \left(\frac{k^2}{2(k-1)} \right)^{1/2} \left(\frac{A_1 + kA_2}{\|\xi\|_\infty^r} + \frac{B_1 + kB_2}{\|\xi\|_\infty^s} \right)^{1/2} \right. \\ &\quad \left. + \|H\|_\infty \right] \|\xi\|_\infty \|\xi\|_{2k-2}^{k-1} \\ &\leq K(n) \left[2 \left(\frac{k^2}{2(k-1)} \right)^{1/2} \left(\frac{A_1^{1/2} + k^{1/2} A_2^{1/2}}{\|\xi\|_\infty^{r/2}} + \frac{B_1^{1/2} + k^{1/2} B_2^{1/2}}{\|\xi\|_\infty^{s/2}} \right) \right. \\ &\quad \left. + \|H\|_\infty \right] \|\xi\|_\infty \|\xi\|_{2k-2}^{k-1}. \end{aligned}$$

If we assume that $\|\xi\|_\infty > \eta$, the last inequality becomes

$$\begin{aligned} \|\xi\|_{\frac{2kn}{n-1}}^k &\leq K(n) \left[2 \left(\frac{k^2}{2(k-1)} \right)^{1/2} \left(\frac{A_1^{1/2} + k^{1/2} A_2^{1/2}}{\eta^{r/2}} + \frac{B_1^{1/2} + k^{1/2} B_2^{1/2}}{\eta^{s/2}} \right) \right. \\ &\quad \left. + \|H\|_\infty \right] \|\xi\|_\infty \|\xi\|_{2k-2}^{k-1} \end{aligned}$$

$$= \left[(K_1 + k^{1/2} K_2) \left(\frac{k^2}{k-1} \right)^{1/2} + K' \right] \|\xi\|_\infty \|\xi\|_{2k-2}^{k-1}.$$

Now let $q = \frac{n}{n-1} > 1$ and for $i \geq 0$ let $k = q^i + 1 \geq 2$. Then

$$\begin{aligned} \|\xi\|_{2(q^{i+1}+q)} &\leq \left((K_1 + (q^i + 1)^{1/2} K_2) \left(\frac{q^i + 1}{q^{i/2}} \right) + K'' \right)^{\frac{1}{q^{i+1}}} \|\xi\|_\infty^{\frac{1}{q^{i+1}}} \|\xi\|_{2q^i}^{1 - \frac{1}{q^{i+1}}} \\ &\leq (\tilde{K} q^i)^{\frac{1}{q^{i+1}}} \|\xi\|_\infty^{\frac{1}{q^{i+1}}} \|\xi\|_{2q^i}^{1 - \frac{1}{q^{i+1}}} \end{aligned}$$

where $\tilde{K} = 2K_1 + 2^{3/2} K_2 + K'$. We see that \tilde{K} has a finite limit when $\eta \rightarrow \infty$ and if $B_1 > 0$, $\tilde{K} \rightarrow \infty$ when $\|H\|_\infty \rightarrow \infty$ or $\eta \rightarrow 0$. Moreover the Hölder inequality gives

$$\|\xi\|_{2q^{i+1}} \leq \|\xi\|_{2(q^{i+1}+q)}$$

which implies

$$\|\xi\|_{2q^{i+1}} \leq (\tilde{K} q^i)^{\frac{1}{q^{i+1}}} \|\xi\|_\infty^{\frac{1}{q^{i+1}}} \|\xi\|_{2q^i}^{1 - \frac{1}{q^{i+1}}}.$$

Now, by iterating from 0 to i , we get

$$\begin{aligned} \|\xi\|_{2q^{i+1}} &\leq \tilde{K}^{(1 - \prod_{k=i-j}^i (1 - \frac{1}{q^{k+1}}))} q^{\sum_{k=i-j}^i \frac{k}{q^{k+1}}} \|\xi\|_\infty^{(1 - \prod_{k=i-j}^i (1 - \frac{1}{q^{k+1}}))} \|\xi\|_{2q^{i-j}}^{\prod_{k=i-j}^i (1 - \frac{1}{q^{k+1}})} \\ &\leq \tilde{K}^{(1 - \prod_{k=0}^i (1 - \frac{1}{q^{k+1}}))} q^{\sum_{k=0}^i \frac{k}{q^{k+1}}} \|\xi\|_\infty^{(1 - \prod_{k=0}^i (1 - \frac{1}{q^{k+1}}))} \|\xi\|_2^{\prod_{k=0}^i (1 - \frac{1}{q^{k+1}})}. \end{aligned}$$

Let $\alpha = \sum_{k=0}^\infty \frac{k}{q^{k+1}}$ and $\beta = \prod_{k=0}^\infty (1 - \frac{1}{q^{k+1}}) = \prod_{k=0}^\infty (\frac{1}{1 + (1/q)^k})$. Then

$$\|\xi\|_\infty \leq \tilde{K}^{1-\beta} q^\alpha \|\xi\|_\infty^{(1-\beta)} \|\xi\|_2^\beta,$$

and finally

$$\|\xi\|_\infty \leq L \|\xi\|_2$$

where $L = \tilde{K}^{\frac{1-\beta}{\beta}} q^{\alpha/\beta}$ is a constant depending only on $n, A_1, A_2, B_1, B_2, \|H\|_\infty$ and η . From classical methods we show that $\beta \in [e^{-n}, e^{-n/2}]$. In particular, $0 < \beta < 1$ and we deduce that L is bounded when $\eta \rightarrow \infty$ and $L \rightarrow \infty$ when $\|H\|_\infty \rightarrow \infty$ or $\eta \rightarrow 0$ with $B_1 > 0$. \square

Remark. In [12] and [13] Shihohama and Xu have proved that if (M^n, g) is a compact n -dimensional Riemannian manifold without boundary isometrically immersed in

\mathbb{R}^{n+1} and if $\int_M (|B|^2 - n|H|^2) < D_n$ where D_n is a constant depending on n , then all Betti numbers are zero. For $n = 2$, $D_2 = 4\pi$, and it follows that if

$$\int_M |B|^2 dv - 4\pi < \lambda_1(M)V(M)$$

then we deduce from the Reilly inequality $\lambda_1(M)V(M) \leq 2 \int_M H^2 dv$ that $\int_M (|B|^2 - 2|H|^2) dv < 4\pi$ and by the result of Shihohama and Xu M is diffeomorphic to \mathbb{S}^2 .

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