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A pinching theorem for the first eigenvalue of the Laplacian on hypersurfaces of the Euclidean space

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Abstract. In this paper, we give pinching theorems for the first nonzero eigenvalue $\lambda_1(M)$ of the Laplacian on the compact hypersurfaces of the Euclidean space. Indeed, we prove that if the volume of M is 1 then, for any $\varepsilon > 0$, there exists a constant C_ε depending on the dimension n of M and the L_∞-norm of the mean curvature H, so that if the L_{2p}-norm $\|H\|_{2p}$ $(p \geq 2)$ of H satisfies $n\|H\|_{2p}^2 - C_\varepsilon < \lambda_1(M)$, then the Hausdorff-distance between M and a round sphere of radius $(n/\lambda_1(M))^{1/2}$ is smaller than ε. Furthermore, we prove that if C is a small enough constant depending on n and the L_∞-norm of the second fundamental form, then the pinching condition $n\|H\|_{2p}^2 - C < \lambda_1(M)$ implies that M is diffeomorphic to an n-dimensional sphere.

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1. Introduction and preliminaries

Let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ into the $n+1$-dimensional euclidean space $(\mathbb{R}^{n+1}, can)$ (i.e. $\phi^\star can = g$). A well-known inequality due to Reilly ([11]) gives an extrinsic upper bound for the first nonzero eigenvalue $\lambda_1(M)$ of the Laplacian of (M^n, g) in terms of the square of the length of the mean curvature. Indeed, we have

$$\lambda_1(M) \leq \frac{n}{V(M)} \int_M |H|^2 \, dv \tag{1}$$

where dv and $V(M)$ denote respectively the Riemannian volume element and the volume of (M^n, g). Moreover the equality holds if and only if (M^n, g) is a geodesic hypersphere of $\mathbb{R}^{n+1}$.

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By using Hölder inequality, we obtain some other similar estimates for the L_{2p}-norm ($p \geq 1$) with H denoted by $\|H\|_{2p}^2$

$$\lambda_1(M) \leq \frac{n}{V(M)^{1/p}} \|H\|_{2p}^2, \tag{2}$$

and as for the inequality (1), the equality case is characterized by the geodesic hyperspheres of $\mathbb{R}^{n+1}$.

A first natural question is to know if there exists a pinching result as the one we state now: does a constant C depending on minimum geometric invariants exist so that if we have the pinching condition

$$(\mathrm{P}_C) \qquad \frac{n}{V(M)^{1/p}} \|H\|_{2p}^2 - C < \lambda_1(M)$$

then M is close to a sphere in a certain sense?

Such questions are known for the intrinsic lower bound of Lichnerowicz–Obata ([9]) of $\lambda_1(M)$ in terms of the lower bound of the Ricci curvature (see [4], [8], [10]). Other pinching results have been proved for Riemannian manifolds with positive Ricci curvature, with a pinching condition on the $n+1$-st eigenvalue ([10]), the diameter ([5], [8], [15]), the volume or the radius (see for instance [2] and [3]).

For instance, S. Ilias proved in [8] that there exists ε depending on n and an upper bound of the sectional curvature so that if the Ricci curvature Ric of M satisfies $\mathrm{Ric} \geq n-1$ and $\lambda_1(M) \leq \lambda_1(\mathbb{S}^n) + \varepsilon$, then M is homeomorphic to $\mathbb{S}^n$.

In this article, we investigate the case of hypersurfaces where, as far as we know, very little is known about pinching and stability results (see however [12], [13]).

More precisely, in our paper, the hypothesis made in [8] that M has a positive Ricci curvature is replaced by the fact that M is isometrically immersed as a hypersurface in $\mathbb{R}^{n+1}$, and the bound on the sectional curvature by an L^∞-bound on the mean curvature or on the second fundamental form. Note that we do not know if such bounds are sharp, or if a bound on the L^q-norm (for some q) of the mean curvature would be enough.

We get the following results

Theorem 1.1. *Let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ in $\mathbb{R}^{n+1}$. Assume that $V(M) = 1$ and let x_0 be the center of mass of M. Then for any $p \geq 2$ and for any $\varepsilon > 0$, there exists a constant C_ε depending only on n, $\varepsilon > 0$ and on the L_∞-norm of H so that if*

$$(\mathrm{P}_{C_\varepsilon}) \qquad n\|H\|_{2p}^2 - C_\varepsilon < \lambda_1(M)$$

then the Hausdorff-distance d_H of M to the sphere $S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right)$ of center x_0 and radius $\sqrt{\frac{n}{\lambda_1(M)}}$ satisfies $d_H\left(\phi(M), S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right)\right) < \varepsilon$.

We recall that the Hausdorff-distance between two compact subsets A and B of a metric space is given by

$$d_H(A, B) = \inf\{\eta | V_\eta(A) \supset B \text{ and } V_\eta(B) \supset A\}$$

where for any subset A, $V_\eta(A)$ is the tubular neighborhood of A defined by $V_\eta(A) = \{x \mid \operatorname{dist}(x, A) < \eta\}$.

Remark. We will see in the proof that $C_\varepsilon(n, \|H\|_\infty) \to 0$ when $\|H\|_\infty \to \infty$ or $\varepsilon \to 0$.

In fact the previous theorem is a consequence of the above definition and the following theorem

Theorem 1.2. *Let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ in $\mathbb{R}^{n+1}$. Assume that $V(M) = 1$ and let x_0 be the center of mass of M. Then for any $p \geq 2$ and for any $\varepsilon > 0$, there exists a constant C_ε depending only on n, $\varepsilon > 0$ and on the L_∞-norm of H so that if*

$$(\mathrm{P}_{C_\varepsilon}) \qquad n\|H\|_{2p}^2 - C_\varepsilon < \lambda_1(M)$$

then

(1) $\phi(M) \subset B\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}} + \varepsilon\right) \setminus B\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}} - \varepsilon\right)$;

(2) $B(x, \varepsilon) \cap \phi(M) \neq \emptyset$ *for all* $x \in S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right)$.

In the following theorem, if the pinching is strong enough, with a control on n and the L_∞-norm of the second fundamental form, we obtain that M is diffeomorphic to a sphere and even almost isometric with a round sphere in a sense we will make precise.

Theorem 1.3. *Let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold ($n \geq 2$) without boundary isometrically immersed by ϕ in $\mathbb{R}^{n+1}$. Assume that $V(M) = 1$. Then for any $p \geq 2$, there exists a constant C depending only on n and the L_∞-norm of the second fundamental form B so that if*

$$(\mathrm{P}_C) \qquad n\|H\|_{2p}^2 - C < \lambda_1(M).$$

Then M is diffeomorphic to $\mathbb{S}^n$.

More precisely, there exists a diffeomorphism F from M into the sphere $\mathbb{S}^n\left(\sqrt{\frac{n}{\lambda_1(M)}}\right)$ of radius $\sqrt{\frac{n}{\lambda_1(M)}}$ which is a quasi-isometry. Namely, for any θ,

$0 < \theta < 1$, *there exists a constant C depending only on n, the L_∞-norm of B and θ, so that the pinching condition* (P_C) *implies*

$$\left| |dF_x(u)|^2 - 1 \right| \le \theta$$

for any $x \in M$ and $u \in T_xM$ so that $|u| = 1$.

Now we will give some preliminaries for the proof of these theorems. Throughout the paper, we consider a compact, connected and oriented n-dimensional Riemannian manifold (M^n, g) without boundary isometrically immersed by ϕ into $(\mathbb{R}^{n+1}, can)$ (i.e. $\phi^\star can = g$). Let ν be the outward normal vector field. Then the second fundamental form of the immersion will be defined by $B(X, Y) = \left\langle \nabla^0_X \nu, Y \right\rangle$, where ∇^0 and $\langle\ ,\ \rangle$ are respectively the Riemannian connection and the inner product of $\mathbb{R}^{n+1}$. Moreover the mean curvature H will be given by $H = (1/n)\,\mathrm{trace}(B)$.

Now let ∂_i be an orthonormal frame of $\mathbb{R}^{n+1}$ and let $x_i : \mathbb{R}^{n+1} \to \mathbb{R}$ be the associated component functions. Putting $X_i = x_i \circ \phi$, a straightforward calculation shows us that

$$B \otimes \nu = -\sum_{i \le n+1} \nabla dX_i \otimes \partial_i$$

and

$$nH\nu = \sum_{i \le n+1} \Delta X_i \partial_i,$$

where ∇ and Δ denote respectively the Riemannian connection and the Laplace–Beltrami operator of (M^n, g). On the other hand, we have the well-known formula

$$\frac{1}{2}\Delta |X|^2 = nH \langle \nu, X \rangle - n \tag{3}$$

where X is the position vector given by $X = \sum_{i \le n+1} X_i \partial_i$.

We recall that to prove the Reilly inequality, we use the functions X_i as test functions (cf. [11]). Indeed, doing a translation if necessary, we can assume that $\int_M X_i \, dv = 0$ for all $i \le n+1$ and we can apply the variational characterization of $\lambda_1(M)$ to X_i. If the equality holds in (1) or (2), then the functions are nothing but eigenfunctions of $\lambda_1(M)$ and from the Takahashi Theorem ([14]) M is immersed isometrically in $\mathbb{R}^{n+1}$ as a geodesic sphere of radius $\sqrt{\frac{n}{\lambda_1(M)}}$.

Throughout the paper we use some notations. From now on, the inner product and the norm induced by g and can on a tensor T will be denoted respectively by $\langle\ ,\ \rangle$ and $|\ |^2$, and the L_p-norm will be given by

$$\|T\|_p = \left(\int_M |T|^p \, dv \right)^{1/p}$$

and

$$\|T\|_\infty = \sup_M |T|.$$

We end these preliminaries by a convenient result.

Lemma 1.1. *Let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold ($n \geq 2$) without boundary isometrically immersed by ϕ in $\mathbb{R}^{n+1}$. Assume that $V(M) = 1$. Then there exist constants c_n and d_n depending only on n so that for any $p \geq 2$, if* (P_C) *is true with $C < c_n$ then*

$$\frac{n}{\lambda_1(M)} \leq d_n. \tag{4}$$

Proof. We recall the standard Sobolev inequality (cf. [6], [7], [16] and p. 216 in [1]). If f is a smooth function and $f \geq 0$, then

$$\left(\int_M f^{\frac{n}{n-1}}\, dv\right)^{1-(1/n)} \leq K(n) \int_M (|df| + |H| f)\, dv \tag{5}$$

where $K(n)$ is a constant depending on n and the volume of the unit ball in $\mathbb{R}^n$. Taking $f = 1$ on M, and using the fact that $V(M) = 1$, we deduce that

$$\|H\|_{2p} \geq \frac{1}{K(n)}$$

and if (P_C) is satisfied and $C \leq \frac{n}{2K(n)^2} = c_n$, then

$$\frac{n}{\lambda_1(M)} \leq \frac{1}{n\|H\|_{2p}^2 - C} \leq 2K(n)^2 = d_n. \qquad \square$$

Throughout the paper, we will assume that $V(M) = 1$ and $\int_M X_i\, dv = 0$ for all $i \leq n+1$. The last assertion implies that the center of mass of M is the origin of $\mathbb{R}^{n+1}$.

2. An L^2-approach of the problem

A first step in the proof of Theorem 1.2 is to prove that if the pinching condition (P_C) is satisfied, then M is close to a sphere in an L^2-sense.

In the following lemma, we prove that the L^2-norm of the position vector is close to $\sqrt{\frac{n}{\lambda_1(M)}}$.

Lemma 2.1. *If we have the pinching condition* (P_C) *with* $C < c_n$, *then*

$$\frac{n\lambda_1(M)}{(C+\lambda_1(M))^2} \leq \|X\|_2^2 \leq \frac{n}{\lambda_1(M)} \leq d_n.$$

Proof. Since $\int_M X_i \, dv = 0$, we can apply the variational characterization of the eigenvalues to obtain

$$\lambda_1(M) \int_M \sum_{i \leq n+1} |X_i|^2 \, dv \leq \int_M \sum_{i \leq n+1} |dX_i|^2 \, dv = n$$

which gives the inequality of the right-hand side.

Let us prove now the inequality of the left-hand side.

$$\begin{aligned}
\lambda_1(M) \int_M |X|^2 \, dv &\leq \frac{\left(\int_M \sum_{i\leq n+1} |dX_i|^2 \, dv\right)^4}{\left(\int_M \sum_{i\leq n+1} |dX_i|^2 \, dv\right)^3} = \frac{\left(\int_M \sum_{i\leq n+1} (\Delta X_i) X_i \, dv\right)^4}{n^3} \\
&\leq \frac{\left(\int_M \sum_{i\leq n+1} (\Delta X_i)^2 \, dv\right)^2 \left(\int_M |X|^2 \, dv\right)^2}{n^3} \\
&= n \left(\int_M H^2 \, dv\right)^2 \left(\int_M |X|^2 \, dv\right)^2
\end{aligned}$$

then using again the Hölder inequality, we get

$$\lambda_1(M) \leq \frac{1}{n} \left(n \|H\|_{2p}^2\right)^2 \int_M |X|^2 \, dv \leq \frac{(C+\lambda_1(M))^2}{n} \int_M |X|^2 \, dv.$$

This completes the proof. □

From now on, we will denote by X^T the orthogonal tangential projection on M. In fact, at $x \in M$, X^T is nothing but the vector of T_xM defined by $X^T = \sum_{1\leq i\leq n} \langle X, e_i \rangle e_i$ where $(e_i)_{1\leq i\leq n}$ is an orthonormal basis of T_xM. In the following lemma, we will show that the condition (P_C) implies that the L^2-norm of X^T of X on M is close to 0.

Lemma 2.2. *If we have the pinching condition* (P_C), *then*

$$\|X^T\|_2^2 \leq A(n)C.$$

Proof. From Lemma 2.1 and the relation (3), we have

$$\lambda_1(M) \int_M |X|^2 \, dv \leq n = n \left(\int_M H \langle X, \nu \rangle \, dv\right)^2$$

$$\leq \left(\int_M |H| \, |\langle X, \nu\rangle| \, dv\right)^2 \leq n\|H\|_{2p}^2 \left(\int_M |\langle X, \nu\rangle|^{\frac{2p}{2p-1}} \, dv\right)^{\frac{2p-1}{p}}$$
$$\leq n\|H\|_{2p}^2 \left(\int_M |\langle X, \nu\rangle|^2 \, dv\right) = n\|H\|_{2p}^2 \int_M |X|^2 \, dv.$$

Then we deduce that

$$n\|H\|_{2p}^2 \|X^T\|_2^2 = n\|H\|_{2p}^2 \left(\int_M \left(|X|^2 - |\langle X, \nu\rangle|^2\right) dv\right)$$
$$\leq (n\|H\|_{2p}^2 - \lambda_1(M))\|X\|_2^2 \leq d_n C$$

where in the last inequality we have used the pinching condition and Lemma 2.1. □

Next we will show that the condition (P_C) implies that the component functions are almost eigenfunctions in an L^2-sense. For this, let us consider the vector field Y on M defined by

$$Y = \sum_{i \leq n+1} (\Delta X_i - \lambda_1(M) X_i) \, \partial_i = nH\nu - \lambda_1(M) X.$$

Lemma 2.3. *If* (P_C) *is satisfied, then*

$$\|Y\|_2^2 \leq nC.$$

Proof. We have

$$\int_M |Y|^2 \, dv = \int_M \left(n^2 H^2 - 2n\lambda_1(M) H \langle \nu, X\rangle + \lambda_1(M)^2 |X|^2\right) dv.$$

Now by integrating the relation (3) we deduce that

$$\int_M H \langle \nu, X\rangle \, dv = 1.$$

Furthermore, since $\int_M X_i \, dv = 0$, we can apply the variational characterization of the eigenvalues to obtain

$$\lambda_1(M) \int_M |X|^2 \, dv = \lambda_1(M) \int_M \sum_{i \leq n+1} |X_i|^2 \, dv \leq \int_M \sum_{i \leq n+1} |dX_i|^2 \, dv = n.$$

Then

$$\int_M |Y|^2 \, dv \leq n^2 \int_M |H|^2 \, dv - n\lambda_1(M) \leq n \left(n\|H\|_{2p}^2 - \lambda_1(M)\right) \leq nC$$

where in this last inequality we have used the Hölder inequality. □

To prove Assertion 1 of Theorem 1.2, we will show that $\left\| |X| - \left(\frac{n}{\lambda_1(M)}\right)^{1/2} \right\|_\infty \leq \varepsilon$. For this we need an L^2-upper bound on the function $\varphi = |X|\Big(|X| - \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\Big)^2$.

Before giving such estimate, we will introduce the vector field Z on M defined by

$$Z = \left(\frac{n}{\lambda_1(M)}\right)^{1/2} |X|^{1/2} H\nu - \frac{X}{|X|^{1/2}}.$$

We have

Lemma 2.4. *If* (P_C) *is satisfied with* $C < c_n$*, then*

$$\|Z\|_2^2 \leq B(n)C.$$

Proof. We have

$$\begin{aligned}
\|Z\|_2^2 &= \left\| \left(\frac{n}{\lambda_1(M)}\right)^{1/2} |X|^{1/2} H\nu - \frac{X}{|X|^{1/2}} \right\|_2^2 \\
&= \int_M \left(\frac{n}{\lambda_1(M)} |X| H^2 - 2\left(\frac{n}{\lambda_1(M)}\right)^{1/2} H\, \langle \nu, X\rangle + |X| \right) dv \\
&\leq \frac{n}{\lambda_1(M)} \left(\int_M |X|^2\, dv\right)^{1/2} \left(\int_M H^4\, dv\right)^{1/2} \\
&\quad - 2\left(\frac{n}{\lambda_1(M)}\right)^{1/2} + \left(\int_M |X|^2\, dv\right)^{1/2}.
\end{aligned}$$

Note that we have used the relation (3). Finally for $p \geq 2$, we get

$$\begin{aligned}
\|Z\|_2^2 &\leq \left(\int_M |X|^2\, dv\right)^{1/2} \left(\frac{n}{\lambda_1(M)} \|H\|_{2p}^2 + 1\right) - 2\left(\frac{n}{\lambda_1(M)}\right)^{1/2} \\
&\leq \left(\frac{n}{\lambda_1(M)}\right)^{1/2} \left(\frac{C}{\lambda_1(M)} + 2\right) - 2\left(\frac{n}{\lambda_1(M)}\right)^{1/2} \\
&= \left(\frac{n}{\lambda_1(M)}\right)^{1/2} \frac{C}{\lambda_1(M)} \leq \frac{d_n^{3/2}}{n} C.
\end{aligned}$$

This concludes the proof of the lemma. □

Now we give an L^2-upper bound of φ.

Lemma 2.5. *Let* $p \geq 2$ *and* $C \leq c_n$*. If we have the pinching condition* (P_C)*, then*

$$\|\varphi\|_2 \leq D(n) \|\varphi\|_\infty^{3/4} C^{1/4}.$$

Proof. We have

$$\|\varphi\|_2 = \left(\int_M \varphi^{3/2}\varphi^{1/2}\,dv\right)^{1/2} \le \|\varphi\|_\infty^{3/4}\|\varphi^{1/2}\|_1^{1/2},$$

and noting that

$$|X|\left(|X| - \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\right)^2 = \left||X|^{1/2}X - \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\frac{X}{|X|^{1/2}}\right|^2$$

we get

$$\begin{aligned}
\int_M \varphi^{1/2}\,dv &= \left\||X|^{1/2}X - \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\frac{X}{|X|^{1/2}}\right\|_1 \\
&= \left\| -\frac{|X|^{1/2}}{\lambda_1(M)}Y + \frac{n}{\lambda_1(M)}|X|^{1/2}H\nu - \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\frac{X}{|X|^{1/2}}\right\|_1 \\
&\le \left\|\frac{|X|^{1/2}}{\lambda_1(M)}Y\right\|_1 + \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\|Z\|_1 . \qquad (6)
\end{aligned}$$

From Lemmas 2.3 and 1.1 we get

$$\begin{aligned}
\left\|\frac{|X|^{1/2}}{\lambda_1(M)}Y\right\|_1 &\le \frac{1}{\lambda_1(M)}\left(\int_M |X|\,dv\right)^{1/2}\|Y\|_2 \\
&\le \frac{1}{\lambda_1(M)}\left(\int_M |X|^2\,dv\right)^{1/4}\|Y\|_2 \le \frac{d_n^{3/4}}{n^{1/2}}C^{1/2}.
\end{aligned}$$

Moreover, using Lemmas 2.4 and 1.1 again it is easy to see that the last term of (6) is bounded by $d_n^{1/2}B(n)^{1/2}C^{1/2}$. Then $\|\varphi^{1/2}\|_1^{1/2} \le D(n)C^{1/4}$. □

3. Proof of Theorem 1.2

The proof of Theorem 1.2 is immediate from the two following technical lemmas which we state below.

Lemma 3.1. *For $p \ge 2$ and for any $\eta > 0$, there exists $K_\eta(n, \|H\|_\infty) \le c_n$ so that if (P_{K_η}) is true, then $\|\varphi\|_\infty \le \eta$. Moreover, $K_\eta \to 0$ when $\|H\|_\infty \to \infty$ or $\eta \to 0$.*

Lemma 3.2. *Let x_0 be a point of the sphere $S(O, R)$ of $\mathbb{R}^{n+1}$ with the center at the origin and of radius R. Assume that $x_0 = Re$ where $e \in \mathbb{S}^n$. Now let (M^n, g) be a compact oriented n-dimensional Riemannian manifold without boundary isometrically* *immersed by ϕ in $\mathbb{R}^{n+1}$ so that $\phi(M) \subset (B(O, R+\eta)\backslash B(O, R-\eta)) \backslash B(x_0, \rho)$ with $\rho = 4(2n-1)\eta$ and suppose that there exists a point $p \in M$ so that $\langle X, e\rangle > 0$. Then there exists $y_0 \in M$ so that the mean curvature $H(y_0)$ at y_0 satisfies $|H(y_0)| \geq \frac{1}{4n\eta}$.*

Now, let us see how to use these lemmas to prove Theorem 1.2.

Proof of Theorem 1.2. We consider the function $f(t) = t\left(t - \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\right)^2$. For $\varepsilon > 0$ let us put

$$\begin{aligned}\eta(\varepsilon) &= \min\left(\left(\frac{1}{\|H\|_\infty} - \varepsilon\right)\varepsilon^2, \left(\frac{1}{\|H\|_\infty} + \varepsilon\right)\varepsilon^2, \frac{1}{27\|H\|_\infty^3}\right)\\ &\leq \min\left(f\left(\left(\frac{n}{\lambda_1(M)}\right)^{1/2} - \varepsilon\right), f\left(\left(\frac{n}{\lambda_1(M)}\right)^{1/2} + \varepsilon\right), \frac{1}{27\|H\|_\infty^3}\right).\end{aligned}$$

Then, as $\eta(\varepsilon) > 0$ and from Lemma 3.1, it follows that if the pinching condition $(\mathrm{P}_{K_{\eta(\varepsilon)}})$ is satisfied with $K_{\eta(\varepsilon)} \leq c_n$, then for any $x \in M$, we have

$$f(|X|) \leq \eta(\varepsilon). \tag{7}$$

Now to prove Theorem 1.2, it is sufficient to assume $\varepsilon < \frac{2}{3\|H\|_\infty}$. Let us show that either

$$\left(\frac{n}{\lambda_1(M)}\right)^{1/2} - \varepsilon \leq |X| \leq \left(\frac{n}{\lambda_1(M)}\right)^{1/2} + \varepsilon \quad \text{or} \quad |X| < \frac{1}{3}\left(\frac{n}{\lambda_1(M)}\right)^{1/2}. \tag{8}$$

By studying the function f it is easy to see that f has a unique local maximum in $\frac{1}{3}\left(\frac{n}{\lambda_1(M)}\right)^{1/2}$ and from the definition of $\eta(\varepsilon)$ it follows that $\eta(\varepsilon) < \frac{4}{27}\frac{1}{\|H\|_\infty^3} \leq \frac{4}{27}\left(\frac{n}{\lambda_1(M)}\right)^{3/2} = f\left(\frac{1}{3}\left(\frac{n}{\lambda_1(M)}\right)^{1/2}\right)$.

Since $\varepsilon < \frac{2}{3\|H\|_\infty}$, we have $\varepsilon < \frac{2}{3}\left(\frac{n}{\lambda_1(M)}\right)^{1/2}$ and $\frac{1}{3}\left(\frac{n}{\lambda_1(M)}\right)^{1/2} < \left(\frac{n}{\lambda_1(M)}\right)^{1/2} - \varepsilon$. This and (7) yield (8).

Now, from Lemma 2.1 we deduce that there exists a point $y_0 \in M$ so that $|X(y_0)| \geq \frac{n^{1/2}\lambda_1(M)^{1/2}}{(K_{\eta(\varepsilon)}+\lambda_1(M))}$ and since $K_{\eta(\varepsilon)} \leq c_n = \frac{n}{d_n} \leq \lambda_1(M) \leq 2\lambda_1(M)$ (see the proof of Lemma 1.1), we obtain $|X(y_0)| \geq \frac{1}{3}\left(\frac{n}{\lambda_1(M)}\right)^{1/2}$.

By the connectedness of M, it follows that $\left(\frac{n}{\lambda_1(M)}\right)^{1/2} - \varepsilon \leq |X| \leq \left(\frac{n}{\lambda_1(M)}\right)^{1/2} + \varepsilon$ for any point of M and Assertion 1 of Theorem 1.2 is shown for the condition $(\mathrm{P}_{K_{\eta(\varepsilon)}})$.

In order to prove the second assertion, let us consider the pinching condition $(\mathrm{P}_{C_\varepsilon})$ with $C_\varepsilon = K_{\eta(\frac{\varepsilon}{4(2n-1)})}$. Then Assertion 1 is still valid. Let $x = \left(\frac{n}{\lambda_1(M)}\right)^{1/2} e \in S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right)$, with $e \in \mathbb{S}^n$ and suppose that $B(x, \varepsilon) \cap M = \emptyset$. Since $\int_M X_i\, dv = 0$

for any $i \leq n+1$, there exists a point $p \in M$ so that $\langle X, e\rangle > 0$ and we can apply Lemma 3.2. Therefore there is a point $y_0 \in M$ so that $H(y_0) \geq \frac{2n-1}{n\varepsilon} > \|H\|_\infty$ since we have assumed $\varepsilon < \frac{2}{3\|H\|_\infty} \leq \frac{2n-1}{2n\|H\|_\infty}$. Then we obtain a contradiction which implies $B(x,\varepsilon) \cap M \neq \emptyset$ and Assertion 2 is satisfied. Furthermore, $C_\varepsilon \to 0$ when $\|H\|_\infty \to \infty$ or $\varepsilon \to 0$. □

4. Proof of Theorem 1.3

From Theorem 1.2, we know that for any $\varepsilon > 0$, there exists C_ε depending only on n and $\|H\|_\infty$ so that if $(\mathrm{P}_{C_\varepsilon})$ is true then

$$\left| |X|_x - \sqrt{\frac{n}{\lambda_1(M)}} \right| \leq \varepsilon$$

for any $x \in M$. Now, since $\sqrt{n}\|H\|_\infty \leq \|B\|_\infty$, it is easy to see from the previous proofs that we can assume that C_ε is depending only on n and $\|B\|_\infty$.

The proof of Theorem 1.3 is a consequence of the following lemma on the L_∞-norm of $\psi = |X^T|$.

Lemma 4.1. *For $p \geq 2$ and for any $\eta > 0$, there exists $K_\eta(n, \|B\|_\infty)$ so that if (P_{K_η}) is true, then $\|\psi\|_\infty \leq \eta$. Moreover, $K_\eta \to 0$ when $\|B\|_\infty \to \infty$ or $\eta \to 0$.*

This lemma will be proved in the Section 5.

Proof of Theorem 1.3. Let $\varepsilon < \frac{1}{2}\sqrt{\frac{n}{\|B\|_\infty}} \leq \sqrt{\frac{n}{\lambda_1(M)}}$. From the choice of ε, we deduce that the condition $(\mathrm{P}_{C_\varepsilon})$ implies that $|X_x|$ is nonzero for any $x \in M$ (see the proof of Theorem 1.2) and we can consider the differential application

$$\begin{aligned} F\colon M &\longrightarrow S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right), \\ x &\longmapsto \sqrt{\frac{n}{\lambda_1(M)}} \frac{X_x}{|X_x|}. \end{aligned}$$

We will prove that F is a quasi-isometry. Indeed, for any $0 < \theta < 1$, we can choose a constant $\varepsilon(n, \|B\|_\infty, \theta)$ so that for any $x \in M$ and any unit vector $u \in T_x M$, the pinching condition $(\mathrm{P}_{C_{\varepsilon(n,\|B\|_\infty,\theta)}})$ implies

$$\left| |dF_x(u)|^2 - 1 \right| \leq \theta.$$

For this, let us compute $dF_x(u)$. We have

$$dF_x(u) = \sqrt{\frac{n}{\lambda_1(M)}} \nabla^0_u \left(\frac{X}{|X|}\right)\Big|_x = \sqrt{\frac{n}{\lambda_1(M)}} u\left(\frac{1}{|X|}\right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla^0_u X =$$

$$
\begin{aligned}
&= -\frac{1}{2}\sqrt{\frac{n}{\lambda_1(M)}}\frac{1}{|X|^3}u(|X|^2)X + \sqrt{\frac{n}{\lambda_1(M)}}\frac{1}{|X|}u \\
&= -\sqrt{\frac{n}{\lambda_1(M)}}\frac{1}{|X|^3}\langle u, X\rangle X + \sqrt{\frac{n}{\lambda_1(M)}}\frac{1}{|X|}u \\
&= \sqrt{\frac{n}{\lambda_1(M)}}\frac{1}{|X|}\left(-\frac{\langle u, X\rangle}{|X|^2}X + u\right).
\end{aligned}
$$

By a straightforward computation, we obtain

$$
\begin{aligned}
\big||dF_x(u)|^2 - 1\big| &= \left|\frac{n}{\lambda_1(M)}\frac{1}{|X|^2}\left(1 - \frac{\langle u, X\rangle^2}{|X|^2}\right) - 1\right| \\
&\le \left|\frac{n}{\lambda_1(M)}\frac{1}{|X|^2} - 1\right| + \frac{n}{\lambda_1(M)}\frac{1}{|X|^4}\langle u, X\rangle^2 .
\end{aligned}
\tag{9}
$$

Now

$$
\begin{aligned}
\left|\frac{n}{\lambda_1(M)}\frac{1}{|X|^2} - 1\right| &= \frac{1}{|X|^2}\left|\frac{n}{\lambda_1(M)} - |X|^2\right| \\
&\le \varepsilon\frac{\left|\sqrt{\frac{n}{\lambda_1(M)}} + |X|\right|}{|X|^2} \le \varepsilon\frac{2\sqrt{\frac{n}{\lambda_1(M)}} + \varepsilon}{\left(\sqrt{\frac{n}{\lambda_1(M)}} - \varepsilon\right)^2}.
\end{aligned}
$$

Let us recall that $\frac{n}{d_n} \le \lambda_1(M) \le \|B\|_\infty^2$ (see (4) for the first inequality). Since we assume $\varepsilon < \frac{1}{2}\sqrt{\frac{n}{\|B\|_\infty}}$, the right-hand side is bounded above by a constant depending only on n and $\|B\|_\infty$ and we have

$$
\left|\frac{n}{\lambda_1(M)}\frac{1}{|X|^2} - 1\right| \le \varepsilon\gamma(n, \|B\|_\infty). \tag{10}
$$

On the other hand, since $C_\varepsilon(n, \|B\|_\infty) \to 0$ when $\varepsilon \to 0$, there exists $\varepsilon(n, \|B\|_\infty, \eta)$ so that $C_{\varepsilon(n,\|B\|_\infty,\eta)} \le K_\eta(n, \|B\|_\infty)$ (where K_η is the constant of the lemma) and then by Lemma 4.1, $\|\psi\|_\infty^2 \le \eta^2$. Thus there exists a constant δ depending only on n and $\|B\|_\infty$ so that

$$
\frac{n}{\lambda_1(M)}\frac{1}{|X|^4}\langle u, X\rangle^2 \le \frac{n}{\lambda_1(M)}\frac{1}{|X|^4}\|\psi\|_\infty^2 \le \eta^2\delta(n, \|B\|_\infty), \tag{11}
$$

and from (9), (10) and (11) we deduce that the condition $(\mathrm{P}_{C_{\varepsilon(n,\|B\|_\infty,\eta)}})$ implies

$$
\big||dF_x(u)|^2 - 1\big| \le \varepsilon\gamma(n, \|B\|_\infty) + \eta^2\delta(n, \|B\|_\infty).
$$

Now let us choose $\eta = \left(\frac{\theta}{2\delta}\right)^{1/2}$. Then we can assume that $\varepsilon(n, \|B\|_\infty, \eta)$ is small enough in order to have $\varepsilon(n, \|B\|_\infty, \eta)\gamma(n\|B\|_\infty) \leq \frac{\theta}{2}$. In this case we have

$$\left||dF_x(u)|^2 - 1\right| \leq \theta.$$

Now let us fix θ, $0 < \theta < 1$. It follows that F is a local diffeomorphism from M to $S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right)$. Since $S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right)$ is simply connected for $n \geq 2$, F is a diffeomorphism. □

5. Proof of the technical lemmas

The proofs of Lemmas 3.1 and 4.1 are providing from a result stated in the following proposition using a Nirenberg–Moser type of proof.

Proposition 5.1. *Let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold without boundary isometrically immersed into the $n+1$-dimensional euclidean space $(\mathbb{R}^{n+1}, can)$. Let ξ be a nonnegative continuous function so that ξ^k is smooth for $k \geq 2$. Let $0 \leq r < s \leq 2$ so that*

$$\frac{1}{2}\Delta\xi^2\xi^{2k-2} \leq \delta\omega + (A_1 + kA_2)\xi^{2k-r} + (B_1 + kB_2)\xi^{2k-s}$$

where $\delta\omega$ is the codifferential of a 1-form and A_1, A_2, B_1, B_2 are nonnegative constants. Then for any $\eta > 0$, there exists a constant $L(n, A_1, A_2, B_1, B_2, \|H\|_\infty, \eta)$ depending only on n, A_1, A_2, B_1, B_2, $\|H\|_\infty$ and η so that if $\|\xi\|_\infty > \eta$ then

$$\|\xi\|_\infty \leq L(n, A_1, A_2, B_1, B_2, \|H\|_\infty, \eta)\|\xi\|_2.$$

Moreover, L is bounded when $\eta \to \infty$, and if $B_1 > 0$, $L \to \infty$ when $\|H\|_\infty \to \infty$ or $\eta \to 0$.

This proposition will be proved at the end of the paper.

Before giving the proofs of Lemmas 3.1 and 4.1, we will show that under the pinching condition (P_C) with C small enough, the L_∞-norm of X is bounded by a constant depending only on n and $\|H\|_\infty$.

Lemma 5.1. *If we have the pinching condition (P_C) with $C < c_n$, then there exists $E(n, \|H\|_\infty)$ depending only on n and $\|H\|_\infty$ so that $\|X\|_\infty \leq E(n, \|H\|_\infty)$.*

Proof. From the relation (3), we have

$$\frac{1}{2}\Delta|X|^2|X|^{2k-2} \leq n\|H\|_\infty|X|^{2k-1}.$$

Then applying Proposition 5.1 to the function $\xi = |X|$ with $r = 0$ and $s = 1$, we obtain that if $\|X\|_\infty > E$, then there exists a constant $L(n, \|H\|_\infty, E)$ depending only on n, $\|H\|_\infty$ and E so that

$$\|X\|_\infty \leq L(n, \|H\|_\infty, E)\|X\|_2,$$

and under the pinching condition (P_C) with $C < c_n$ we have from Lemma 2.1 that

$$\|X\|_\infty \leq L(n, \|H\|_\infty, E) d_n^{1/2}.$$

Now since L is bounded when $E \to \infty$, we can choose $E = E(n, \|H\|_\infty)$ large enough so that

$$L(n, \|H\|_\infty, E) d_n^{1/2} < E.$$

In this case, we have $\|X\|_\infty \leq E(n, \|H\|_\infty)$. □

Proof of Lemma 3.1. First we compute the Laplacian of the square of φ^2. We have

$$\begin{aligned}
\Delta\varphi^2 &= \Delta\left(|X|^4 - 2\left(\frac{n}{\lambda_1(M)}\right)^{1/2}|X|^3 + \frac{n}{\lambda_1(M)}|X|^2\right) \\
&= -2|X|^2|d|X|^2|^2 + 2|X|^2\Delta|X|^2 \\
&\quad - 2\left(\frac{n}{\lambda_1(M)}\right)^{1/2}\left(-\frac{3}{4}|X|^{-1}|d|X|^2|^2 + \frac{3}{2}|X|\Delta|X|^2\right) + \frac{n}{\lambda_1(M)}\Delta|X|^2.
\end{aligned}$$

Now by a direct computation one gets $|d|X|^2|^2 \leq 4|X|^2$. Moreover by the relation (3) we have $|\Delta|X|^2| \leq 2n\|H\|_\infty|X| + n$. Then applying Lemmas 1.1 and 5.1 we get

$$\Delta\varphi^2 \leq \alpha(n, \|H\|_\infty)$$

and

$$\frac{1}{2}\Delta\varphi^2\varphi^{2k-2} \leq \alpha(n, \|H\|_\infty)\varphi^{2k-2}.$$

Now, we apply Proposition 5.1 with $r = 0$ and $s = 2$. Then if $\|\varphi\|_\infty > \eta$, there exists a constant $L(n, \|H\|_\infty)$ depending only on n and $\|H\|_\infty$ so that

$$\|\varphi\|_\infty \leq L\|\varphi\|_2.$$

From Lemma 2.5, if $C \leq c_n$ and (P_C) is true, we have $\|\varphi\|_2 \leq D(n)\|\varphi\|_\infty^{3/4}C^{1/4}$. Therefore

$$\|\varphi\|_\infty \leq (LD)^4 C.$$

Consequently, if we choose $C = K_\eta = \inf\left(\frac{\eta}{(LD)^4}, c_n\right)$, then we obtain $\|\varphi\|_\infty \leq \eta$. □

Proof of Lemma 4.1. First we will prove that for any $C < c_n$, if (P_C) is true, then

$$\frac{1}{2}(\Delta\psi^2)\psi^{2k-2} \leq \delta\omega + (\alpha_1(n, \|B\|_\infty) + k\alpha_2(n, \|B\|_\infty))\, \psi^{2k-2} \tag{12}$$

where $\delta\omega$ is the codifferential of a 1-form ω.

First observe that the gradient $\nabla^M|X|^2$ of $|X|^2$ satisfies $\nabla^M|X|^2 = 2X^T$. Then by the Bochner formula we get

$$\begin{aligned}\frac{1}{2}\Delta|X^T|^2 &= \frac{1}{4}\left\langle\Delta d|X|^2, d|X|^2\right\rangle - \frac{1}{4}|\nabla d|X|^2|^2 - \frac{1}{4}\operatorname{Ric}(\nabla^M|X|^2, \nabla^M|X|^2)\\ &\leq \frac{1}{4}\left\langle d\Delta|X|^2, d|X|^2\right\rangle - \frac{1}{4}\operatorname{Ric}(\nabla^M|X|^2, \nabla^M|X|^2)\end{aligned}$$

and by the Gauss formula we obtain

$$\begin{aligned}\frac{1}{2}\Delta|X^T|^2 &\leq \frac{1}{4}\left\langle d\Delta|X|^2, d|X|^2\right\rangle - \frac{1}{4}nH\left\langle B\nabla^M|X|^2, \nabla^M|X|^2\right\rangle + \frac{1}{4}|B\nabla^M|X|^2|^2\\ &= \frac{1}{4}\left\langle d\Delta|X|^2, d|X|^2\right\rangle - nH\left\langle BX^T, X^T\right\rangle + |BX^T|^2.\end{aligned}$$

By Lemma 5.1 we know that $\|X\|_\infty \leq E(n, \|B\|_\infty)$ (the dependance in $\|H\|_\infty$ can be replaced by $\|B\|_\infty$). Then it follows that

$$\frac{1}{2}(\Delta\psi^2)\psi^{2k-2} \leq \frac{1}{4}\left\langle d\Delta|X|^2, d|X|^2\right\rangle\psi^{2k-2} + \alpha'(n, \|B\|_\infty)\psi^{2k-2}. \tag{13}$$

Now, let us compute the term $\left\langle d\Delta|X|^2, d|X|^2\right\rangle\psi^{2k-2}$. We have

$$\begin{aligned}&\left\langle d\Delta|X|^2, d|X|^2\right\rangle\psi^{2k-2}\\ &\qquad= \delta\omega + (\Delta|X|^2)^2\psi^{2k-2} - (2k-2)\Delta|X|^2\left\langle d|X|^2, d\psi\right\rangle\psi^{2k-3}\\ &\qquad= \delta\omega + (\Delta|X|^2)^2\psi^{2k-2} - 2(2k-2)\Delta|X|^2\left\langle X^T, \nabla^M\psi\right\rangle\psi^{2k-3}\end{aligned}$$

where $\omega = -\Delta|X|^2\psi^{2k-2}d|X|^2$. Now,

$$e_i(\psi) = \frac{e_i|X^T|^2}{2|X^T|} = \frac{e_i|X|^2 - e_i\langle X, \nu\rangle^2}{2|X^T|} = \frac{\langle e_i, X\rangle - B_{ij}\left\langle X, e_j\right\rangle\langle X, \nu\rangle}{|X^T|}.$$

Then

$$\begin{aligned}\left\langle d\Delta|X|^2, d|X|^2\right\rangle\psi^{2k-2} &= \delta\omega + (\Delta|X|^2)^2\psi^{2k-2} - 2(2k-2)\Delta|X|^2|X^T|\psi^{2k-3}\\ &\quad+ 2(2k-2)\Delta|X|^2\frac{\left\langle BX^T, X^T\right\rangle}{|X^T|}\langle X, \nu\rangle\,\psi^{2k-3}\\ &\leq \delta\omega + (\Delta|X|^2)^2\psi^{2k-2} + 2(2k-2)|\Delta|X|^2|\psi^{2k-2}\\ &\quad+ 2(2k-2)|\Delta|X|^2|\,|B|\,|X|\psi^{2k-2}.\end{aligned}$$

Now by relation (3) and Lemma 5.1 we have

$$\left\langle d\Delta|X|^2, d|X|^2\right\rangle \psi^{2k-2} \leq \delta\omega + \left(\alpha_1''(n, \|B\|_\infty) + k\alpha_2''(n, \|B\|_\infty)\right)\psi^{2k-2}.$$

Inserting this in (13), we obtain the desired inequality (12).

Now applying again Proposition 5.1, we get that there exists $L(n, \|B\|_\infty, \eta)$ so that if $\|\psi\|_\infty > \eta$ then

$$\|\psi\|_\infty \leq L\|\psi\|_2.$$

From Lemma 2.2 we deduce that if the pinching condition (P_C) holds then $\|\psi\|_2 \leq A(n)^{1/2}C^{1/2}$. Then taking $C = K_\eta = \inf\left(\frac{\eta}{LA^{1/2}}, c_n\right)$, then $\|\psi\|_\infty \leq \eta$. □

Proof of Lemma 3.2. The idea of the proof consists in foliating the region $B(O, R+\eta)\backslash B(O, R-\eta)$ with hypersurfaces of large mean curvature and to show that one of these hypersurfaces is tangent to $\phi(M)$. This will imply that $\phi(M)$ has a large mean curvature at the contact point.

Consider $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}e$. Let a, $L > l > 0$ and

$$\begin{aligned}\Phi_{L,l,a}: \mathbb{S}^{n-1} \times \mathbb{S}^1 &\longrightarrow \mathbb{R}^{n+1}\\ (\xi, \theta) &\longmapsto L\xi - l\cos\theta\xi + l\sin\theta e + ae.\end{aligned}$$

Then $\Phi_{L,l,a}$ is a family of embeddings from $\mathbb{S}^{n-1} \times \mathbb{S}^1$ in $\mathbb{R}^{n+1}$. If we orient the family of hypersurfaces $\Phi_{L,l,a}(\mathbb{S}^{n-1} \times \mathbb{S}^1)$ by the unit outward normal vector field, a straightforward computation shows that the mean curvature $H(\theta)$ depends only on θ and we have

$$H(\theta) = \frac{1}{n}\left(\frac{1}{l} - \frac{(n-1)\cos\theta}{L - l\cos\theta}\right) \geq \frac{1}{n}\left(\frac{1}{l} - \frac{n-1}{L-l}\right). \tag{14}$$

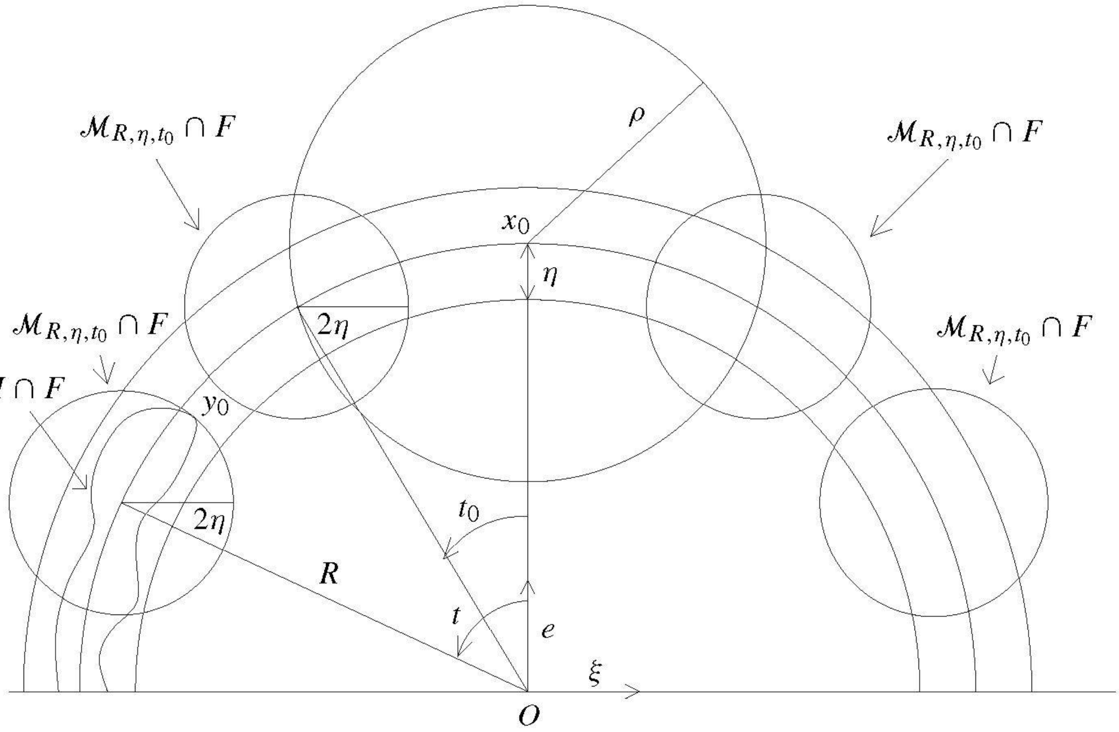
Now, let us consider the hypotheses of the lemma and for $t_0 = 2\arcsin\left(\frac{\rho}{2R}\right) \leq t \leq \frac{\pi}{2}$, put $L = R\sin t$, $l = 2\eta$ and $a = R\cos t$. Then $L > l$ and we can consider for $t_0 \leq t \leq \frac{\pi}{2}$ the family $\mathcal{M}_{R,\eta,t}$ of hypersurfaces defined by $\mathcal{M}_{R,\eta,t} = \Phi_{R\sin t, 2\eta, R\cos t}(\mathbb{S}^{n-1} \times \mathbb{S}^1)$.

From the relation (14), the mean curvature $H_{R,\eta,t}$ of $\mathcal{M}_{R,\eta,t}$ satisfies

$$\begin{aligned}H_{R,\eta,t} &\geq \frac{1}{n}\left(\frac{1}{2\eta} - \frac{n-1}{R\sin t - 2\eta}\right) \geq \frac{1}{n}\left(\frac{1}{2\eta} - \frac{n-1}{R\sin t_0 - 2\eta}\right)\\ &\geq \frac{1}{n}\left(\frac{1}{2\eta} - \frac{n-1}{R\sin(t_0/2) - 2\eta}\right) = \frac{1}{n}\left(\frac{1}{2\eta} - \frac{n-1}{\frac{\rho}{2} - 2\eta}\right) = \frac{1}{4n\eta}\end{aligned}$$

where we have used in this last equality the fact that $\rho = 4(2n-1)\eta$.

Since there exists a point $p \in M$ so that $\langle X(p), e\rangle > 0$, we can find $t \in [t_0, \pi/2]$ and a point $y_0 \in M$ which is a contact point with $\mathcal{M}_{R,\eta,t}$. Therefore $|H(y_0)| \geq \frac{1}{4n\eta}$.



F is the vector space spanned by e and ξ.

□

Proof of Proposition 5.1. Integrating by parts we have

$$\begin{aligned}\int_M \frac{1}{2}\Delta\xi^2\xi^{2k-2}\,dv &= \frac{1}{2}\int_M \left\langle d\xi^2, d\xi^{2k-2}\right\rangle dv = 2\left(\frac{k-1}{k^2}\right)\int_M |d\xi^k|^2\,dv \\ &\le (A_1+kA_2)\int_M \xi^{2k-r}\,dv + (B_1+kB_2)\int_M \xi^{2k-s}\,dv.\end{aligned}$$

Now, given a smooth function f and applying the Sobolev inequality (5) to f^2, we get

$$\begin{aligned}\left(\int_M f^{\frac{2n}{n-1}}\,dv\right)^{1-(1/n)} &\le K(n)\int_M \left(2|f||df| + |H|f^2\right) dv \\ &\le 2K(n)\left(\int_M f^2\,dv\right)^{1/2}\left(\int_M |df|^2\,dv\right)^{1/2} + K(n)\|H\|_\infty \int_M f^2\,dv \\ &= K(n)\left(\int_M f^2\,dv\right)^{1/2}\left(2\left(\int_M |df|^2\,dv\right)^{1/2} + \|H\|_\infty\left(\int_M f^2\,dv\right)^{1/2}\right)\end{aligned}$$

where in the second inequality, we have used the Hölder inequality. Using it again, by assuming that $V(M)=1$, we have

$$\left(\int_M f^2\,dv\right)^{1/2} \le \left(\int_M f^{\frac{2n}{n-1}}\,dv\right)^{\frac{n-1}{2n}}.$$

And finally, we obtain

$$\|f\|_{\frac{2n}{n-1}} \le K(n)(2\|df\|_2 + \|H\|_\infty \|f\|_2).$$

For $k\ge 2$, ξ^k is smooth and we apply the above inequality to $f=\xi^k$. Then we get

$$\begin{aligned}
\|\xi\|^k_{\frac{2kn}{n-1}} &\le K(n)\left[2\left(\int_M |d\xi^k|^2\,dv\right)^{1/2} + \|H\|_\infty\left(\int_M \xi^{2k}\,dv\right)^{1/2}\right]\\
&\le K(n)\left[2\left(\frac{k^2}{2(k-1)}\right)^{1/2}\left((A_1+kA_2)\int_M \xi^{2k-r}\,dv\right.\right.\\
&\qquad\left.\left. + (B_1+kB_2)\int_M \xi^{2k-s}\,dv\right)^{1/2} + \|H\|_\infty\left(\int_M \xi^{2k}\,dv\right)^{1/2}\right]\\
&\le K(n)\left[2\left(\frac{k^2}{2(k-1)}\right)^{1/2}\left((A_1+kA_2)\|\xi\|_\infty^{2-r}\right.\right.\\
&\qquad\left.\left. + (B_1+kB_2)\|\xi\|_\infty^{2-s}\right)^{1/2}\|\xi\|^{k-1}_{2k-2} + \|H\|_\infty\|\xi\|_\infty\|\xi\|^{k-1}_{2k-2}\right]\\
&\le K(n)\left[2\left(\frac{k^2}{2(k-1)}\right)^{1/2}\left(\frac{A_1+kA_2}{\|\xi\|_\infty^r} + \frac{B_1+kB_2}{\|\xi\|_\infty^s}\right)^{1/2}\right.\\
&\qquad\left. + \|H\|_\infty\right]\|\xi\|_\infty\|\xi\|^{k-1}_{2k-2}\\
&\le K(n)\left[2\left(\frac{k^2}{2(k-1)}\right)^{1/2}\left(\frac{A_1^{1/2}+k^{1/2}A_2^{1/2}}{\|\xi\|_\infty^{r/2}} + \frac{B_1^{1/2}+k^{1/2}B_2^{1/2}}{\|\xi\|_\infty^{s/2}}\right)\right.\\
&\qquad\left. + \|H\|_\infty\right]\|\xi\|_\infty\|\xi\|^{k-1}_{2k-2}.
\end{aligned}$$

If we assume that $\|\xi\|_\infty > \eta$, the last inequality becomes

$$\begin{aligned}
\|\xi\|^k_{\frac{2kn}{n-1}} &\le K(n)\left[2\left(\frac{k^2}{2(k-1)}\right)^{1/2}\left(\frac{A_1^{1/2}+k^{1/2}A_2^{1/2}}{\eta^{r/2}} + \frac{B_1^{1/2}+k^{1/2}B_2^{1/2}}{\eta^{s/2}}\right)\right.\\
&\qquad\left. + \|H\|_\infty\right]\|\xi\|_\infty\|\xi\|^{k-1}_{2k-2}
\end{aligned}$$

$$= \left[(K_1 + k^{1/2}K_2)\left(\frac{k^2}{k-1}\right)^{1/2} + K'\right] \|\xi\|_\infty \|\xi\|_{2k-2}^{k-1}.$$

Now let $q = \frac{n}{n-1} > 1$ and for $i \geq 0$ let $k = q^i + 1 \geq 2$. Then

$$\begin{aligned}\|\xi\|_{2(q^{i+1}+q)} &\leq \left(\left(K_1 + (q^i+1)^{1/2}K_2\right)\left(\frac{q^i+1}{q^{i/2}}\right) + K''\right)^{\frac{1}{q^i+1}} \|\xi\|_\infty^{\frac{1}{q^i+1}} \|\xi\|_{2q^i}^{1-\frac{1}{q^i+1}} \\ &\leq \left(\tilde{K}q^i\right)^{\frac{1}{q^i+1}} \|\xi\|_\infty^{\frac{1}{q^i+1}} \|\xi\|_{2q^i}^{1-\frac{1}{q^i+1}}\end{aligned}$$

where $\tilde{K} = 2K_1 + 2^{3/2}K_2 + K'$. We see that $\tilde{K}$ has a finite limit when $\eta \to \infty$ and if $B_1 > 0$, $\tilde{K} \to \infty$ when $\|H\|_\infty \to \infty$ or $\eta \to 0$. Moreover the Hölder inequality gives

$$\|\xi\|_{2q^{i+1}} \leq \|\xi\|_{2(q^{i+1}+q)}$$

which implies

$$\|\xi\|_{2q^{i+1}} \leq \left(\tilde{K}q^i\right)^{\frac{1}{q^i+1}} \|\xi\|_\infty^{\frac{1}{q^i+1}} \|\xi\|_{2q^i}^{1-\frac{1}{q^i+1}}.$$

Now, by iterating from 0 to i, we get

$$\begin{aligned}&\|\xi\|_{2q^{i+1}} \\ &\leq \tilde{K}^{\left(1-\prod_{k=i-j}^{i}\left(1-\frac{1}{q^k+1}\right)\right)} q^{\sum_{k=i-j}^{i}\frac{k}{q^k+1}} \|\xi\|_\infty^{\left(1-\prod_{k=i-j}^{i}\left(1-\frac{1}{q^k+1}\right)\right)} \|\xi\|_{2q^{i-j}}^{\prod_{k=i-j}^{i}\left(1-\frac{1}{q^k+1}\right)} \\ &\leq \tilde{K}^{\left(1-\prod_{k=0}^{i}\left(1-\frac{1}{q^k+1}\right)\right)} q^{\sum_{k=0}^{i}\frac{k}{q^k+1}} \|\xi\|_\infty^{\left(1-\prod_{k=0}^{i}\left(1-\frac{1}{q^k+1}\right)\right)} \|\xi\|_2^{\prod_{k=0}^{i}\left(1-\frac{1}{q^k+1}\right)}.\end{aligned}$$

Let $\alpha = \sum_{k=0}^{\infty}\frac{k}{q^k+1}$ and $\beta = \prod_{k=0}^{\infty}\left(1-\frac{1}{q^k+1}\right) = \prod_{k=0}^{\infty}\left(\frac{1}{1+(1/q)^k}\right)$. Then

$$\|\xi\|_\infty \leq \tilde{K}^{1-\beta} q^\alpha \|\xi\|_\infty^{(1-\beta)} \|\xi\|_2^\beta,$$

and finally

$$\|\xi\|_\infty \leq L\|\xi\|_2$$

where $L = \tilde{K}^{\frac{1-\beta}{\beta}} q^{\alpha/\beta}$ is a constant depending only on n, A_1, A_2, B_1, B_2, $\|H\|_\infty$ and η. From classical methods we show that $\beta \in [e^{-n}, e^{-n/2}]$. In particular, $0 < \beta < 1$ and we deduce that L is bounded when $\eta \to \infty$ and $L \to \infty$ when $\|H\|_\infty \to \infty$ or $\eta \to 0$ with $B_1 > 0$. □

Remark. In [12] and [13] Shihohama and Xu have proved that if (M^n, g) is a compact n-dimensional Riemannian manifold without boundary isometrically immersed in

$\mathbb{R}^{n+1}$ and if $\int_M \left(|B|^2 - n|H|^2\right) < D_n$ where D_n is a constant depending on n, then all Betti numbers are zero. For $n = 2$, $D_2 = 4\pi$, and it follows that if

$$\int_M |B|^2 \, dv - 4\pi < \lambda_1(M)V(M)$$

then we deduce from the Reilly inequality $\lambda_1(M)V(M) \leq 2\int_M H^2 \, dv$ that $\int_M(|B|^2 - 2|H|^2)\, dv < 4\pi$ and by the result of Shihohama and Xu M is diffeomorphic to $\mathbb{S}^2$.

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