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# A pinching theorem for the first eigenvalue of the Laplacian on hypersurfaces of the Euclidean space

Bruno Colbois and Jean-François Grosjean\*

**Abstract.** In this paper, we give pinching theorems for the first nonzero eigenvalue  $\lambda_1(M)$  of the Laplacian on the compact hypersurfaces of the Euclidean space. Indeed, we prove that if the volume of  $M$  is 1 then, for any  $\varepsilon > 0$ , there exists a constant  $C_\varepsilon$  depending on the dimension  $n$  of  $M$  and the  $L_\infty$ -norm of the mean curvature  $H$ , so that if the  $L_{2p}$ -norm  $\|H\|_{2p}$  ( $p \geq 2$ ) of  $H$  satisfies  $n\|H\|_{2p}^2 - C_\varepsilon < \lambda_1(M)$ , then the Hausdorff-distance between  $M$  and a round sphere of radius  $(n/\lambda_1(M))^{1/2}$  is smaller than  $\varepsilon$ . Furthermore, we prove that if  $C$  is a small enough constant depending on  $n$  and the  $L_\infty$ -norm of the second fundamental form, then the pinching condition  $n\|H\|_{2p}^2 - C < \lambda_1(M)$  implies that  $M$  is diffeomorphic to an  $n$ -dimensional sphere.

**Mathematics Subject Classification (2000).** 53A07, 53C21.

**Keywords.** Spectrum, Laplacian, pinching results, hypersurfaces.

## 1. Introduction and preliminaries

Let  $(M^n, g)$  be a compact, connected and oriented  $n$ -dimensional Riemannian manifold without boundary isometrically immersed by  $\phi$  into the  $n + 1$ -dimensional euclidean space  $(\mathbb{R}^{n+1}, can)$  (i.e.  $\phi^*can = g$ ). A well-known inequality due to Reilly ([11]) gives an extrinsic upper bound for the first nonzero eigenvalue  $\lambda_1(M)$  of the Laplacian of  $(M^n, g)$  in terms of the square of the length of the mean curvature. Indeed, we have

$$\lambda_1(M) \leq \frac{n}{V(M)} \int_M |H|^2 dv \quad (1)$$

where  $dv$  and  $V(M)$  denote respectively the Riemannian volume element and the volume of  $(M^n, g)$ . Moreover the equality holds if and only if  $(M^n, g)$  is a geodesic hypersphere of  $\mathbb{R}^{n+1}$ .

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By using Hölder inequality, we obtain some other similar estimates for the  $L_{2p}$ -norm ( $p \geq 1$ ) with  $H$  denoted by  $\|H\|_{2p}^2$

$$\lambda_1(M) \leq \frac{n}{V(M)^{1/p}} \|H\|_{2p}^2, \tag{2}$$

and as for the inequality (1), the equality case is characterized by the geodesic hyperspheres of  $\mathbb{R}^{n+1}$ .

A first natural question is to know if there exists a pinching result as the one we state now: does a constant  $C$  depending on minimum geometric invariants exist so that if we have the pinching condition

$$(P_C) \quad \frac{n}{V(M)^{1/p}} \|H\|_{2p}^2 - C < \lambda_1(M)$$

then  $M$  is close to a sphere in a certain sense?

Such questions are known for the intrinsic lower bound of Lichnerowicz–Obata ([9]) of  $\lambda_1(M)$  in terms of the lower bound of the Ricci curvature (see [4], [8], [10]). Other pinching results have been proved for Riemannian manifolds with positive Ricci curvature, with a pinching condition on the  $n + 1$ -st eigenvalue ([10]), the diameter ([5], [8], [15]), the volume or the radius (see for instance [2] and [3]).

For instance, S. Ilias proved in [8] that there exists  $\varepsilon$  depending on  $n$  and an upper bound of the sectional curvature so that if the Ricci curvature  $\text{Ric}$  of  $M$  satisfies  $\text{Ric} \geq n - 1$  and  $\lambda_1(M) \leq \lambda_1(\mathbb{S}^n) + \varepsilon$ , then  $M$  is homeomorphic to  $\mathbb{S}^n$ .

In this article, we investigate the case of hypersurfaces where, as far as we know, very little is known about pinching and stability results (see however [12], [13]).

More precisely, in our paper, the hypothesis made in [8] that  $M$  has a positive Ricci curvature is replaced by the fact that  $M$  is isometrically immersed as a hypersurface in  $\mathbb{R}^{n+1}$ , and the bound on the sectional curvature by an  $L^\infty$ -bound on the mean curvature or on the second fundamental form. Note that we do not know if such bounds are sharp, or if a bound on the  $L^q$ -norm (for some  $q$ ) of the mean curvature would be enough.

We get the following results

**Theorem 1.1.** *Let  $(M^n, g)$  be a compact, connected and oriented  $n$ -dimensional Riemannian manifold without boundary isometrically immersed by  $\phi$  in  $\mathbb{R}^{n+1}$ . Assume that  $V(M) = 1$  and let  $x_0$  be the center of mass of  $M$ . Then for any  $p \geq 2$  and for any  $\varepsilon > 0$ , there exists a constant  $C_\varepsilon$  depending only on  $n$ ,  $\varepsilon > 0$  and on the  $L_\infty$ -norm of  $H$  so that if*

$$(P_{C_\varepsilon}) \quad n\|H\|_{2p}^2 - C_\varepsilon < \lambda_1(M)$$

*then the Hausdorff-distance  $d_H$  of  $M$  to the sphere  $S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right)$  of center  $x_0$  and radius  $\sqrt{\frac{n}{\lambda_1(M)}}$  satisfies  $d_H\left(\phi(M), S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right)\right) < \varepsilon$ .*

We recall that the Hausdorff-distance between two compact subsets  $A$  and  $B$  of a metric space is given by

$$d_H(A, B) = \inf\{\eta \mid V_\eta(A) \supset B \text{ and } V_\eta(B) \supset A\}$$

where for any subset  $A$ ,  $V_\eta(A)$  is the tubular neighborhood of  $A$  defined by  $V_\eta(A) = \{x \mid \text{dist}(x, A) < \eta\}$ .

**Remark.** We will see in the proof that  $C_\varepsilon(n, \|H\|_\infty) \rightarrow 0$  when  $\|H\|_\infty \rightarrow \infty$  or  $\varepsilon \rightarrow 0$ .

In fact the previous theorem is a consequence of the above definition and the following theorem

**Theorem 1.2.** *Let  $(M^n, g)$  be a compact, connected and oriented  $n$ -dimensional Riemannian manifold without boundary isometrically immersed by  $\phi$  in  $\mathbb{R}^{n+1}$ . Assume that  $V(M) = 1$  and let  $x_0$  be the center of mass of  $M$ . Then for any  $p \geq 2$  and for any  $\varepsilon > 0$ , there exists a constant  $C_\varepsilon$  depending only on  $n, \varepsilon > 0$  and on the  $L_\infty$ -norm of  $H$  so that if*

$$(P_{C_\varepsilon}) \quad n\|H\|_{2p}^2 - C_\varepsilon < \lambda_1(M)$$

then

- (1)  $\phi(M) \subset B\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}} + \varepsilon\right) \setminus B\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}} - \varepsilon\right)$ ;
- (2)  $B(x, \varepsilon) \cap \phi(M) \neq \emptyset$  for all  $x \in S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right)$ .

In the following theorem, if the pinching is strong enough, with a control on  $n$  and the  $L_\infty$ -norm of the second fundamental form, we obtain that  $M$  is diffeomorphic to a sphere and even almost isometric with a round sphere in a sense we will make precise.

**Theorem 1.3.** *Let  $(M^n, g)$  be a compact, connected and oriented  $n$ -dimensional Riemannian manifold ( $n \geq 2$ ) without boundary isometrically immersed by  $\phi$  in  $\mathbb{R}^{n+1}$ . Assume that  $V(M) = 1$ . Then for any  $p \geq 2$ , there exists a constant  $C$  depending only on  $n$  and the  $L_\infty$ -norm of the second fundamental form  $B$  so that if*

$$(P_C) \quad n\|H\|_{2p}^2 - C < \lambda_1(M).$$

Then  $M$  is diffeomorphic to  $S^n$ .

More precisely, there exists a diffeomorphism  $F$  from  $M$  into the sphere  $S^n\left(\sqrt{\frac{n}{\lambda_1(M)}}\right)$  of radius  $\sqrt{\frac{n}{\lambda_1(M)}}$  which is a quasi-isometry. Namely, for any  $\theta$ ,

$0 < \theta < 1$ , there exists a constant  $C$  depending only on  $n$ , the  $L_\infty$ -norm of  $B$  and  $\theta$ , so that the pinching condition  $(P_C)$  implies

$$||dF_x(u)|^2 - 1| \leq \theta$$

for any  $x \in M$  and  $u \in T_x M$  so that  $|u| = 1$ .

Now we will give some preliminaries for the proof of these theorems. Throughout the paper, we consider a compact, connected and oriented  $n$ -dimensional Riemannian manifold  $(M^n, g)$  without boundary isometrically immersed by  $\phi$  into  $(\mathbb{R}^{n+1}, can)$  (i.e.  $\phi^*can = g$ ). Let  $\nu$  be the outward normal vector field. Then the second fundamental form of the immersion will be defined by  $B(X, Y) = \langle \nabla_X^0 \nu, Y \rangle$ , where  $\nabla^0$  and  $\langle \cdot, \cdot \rangle$  are respectively the Riemannian connection and the inner product of  $\mathbb{R}^{n+1}$ . Moreover the mean curvature  $H$  will be given by  $H = (1/n) \text{trace}(B)$ .

Now let  $\partial_i$  be an orthonormal frame of  $\mathbb{R}^{n+1}$  and let  $x_i : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  be the associated component functions. Putting  $X_i = x_i \circ \phi$ , a straightforward calculation shows us that

$$B \otimes \nu = - \sum_{i \leq n+1} \nabla dX_i \otimes \partial_i$$

and

$$nH\nu = \sum_{i \leq n+1} \Delta X_i \partial_i,$$

where  $\nabla$  and  $\Delta$  denote respectively the Riemannian connection and the Laplace–Beltrami operator of  $(M^n, g)$ . On the other hand, we have the well-known formula

$$\frac{1}{2} \Delta |X|^2 = nH \langle \nu, X \rangle - n \tag{3}$$

where  $X$  is the position vector given by  $X = \sum_{i \leq n+1} X_i \partial_i$ .

We recall that to prove the Reilly inequality, we use the functions  $X_i$  as test functions (cf. [11]). Indeed, doing a translation if necessary, we can assume that  $\int_M X_i dv = 0$  for all  $i \leq n + 1$  and we can apply the variational characterization of  $\lambda_1(M)$  to  $X_i$ . If the equality holds in (1) or (2), then the functions are nothing but eigenfunctions of  $\lambda_1(M)$  and from the Takahashi Theorem ([14])  $M$  is immersed isometrically in  $\mathbb{R}^{n+1}$  as a geodesic sphere of radius  $\sqrt{\frac{n}{\lambda_1(M)}}$ .

Throughout the paper we use some notations. From now on, the inner product and the norm induced by  $g$  and  $can$  on a tensor  $T$  will be denoted respectively by  $\langle \cdot, \cdot \rangle$  and  $|\cdot|^2$ , and the  $L_p$ -norm will be given by

$$\|T\|_p = \left( \int_M |T|^p dv \right)^{1/p}$$

and

$$\|T\|_\infty = \sup_M |T|.$$

We end these preliminaries by a convenient result.

**Lemma 1.1.** *Let  $(M^n, g)$  be a compact, connected and oriented  $n$ -dimensional Riemannian manifold ( $n \geq 2$ ) without boundary isometrically immersed by  $\phi$  in  $\mathbb{R}^{n+1}$ . Assume that  $V(M) = 1$ . Then there exist constants  $c_n$  and  $d_n$  depending only on  $n$  so that for any  $p \geq 2$ , if  $(P_C)$  is true with  $C < c_n$  then*

$$\frac{n}{\lambda_1(M)} \leq d_n. \quad (4)$$

*Proof.* We recall the standard Sobolev inequality (cf. [6], [7], [16] and p. 216 in [1]). If  $f$  is a smooth function and  $f \geq 0$ , then

$$\left( \int_M f^{\frac{n}{n-1}} dv \right)^{1-(1/n)} \leq K(n) \int_M (|df| + |H|f) dv \quad (5)$$

where  $K(n)$  is a constant depending on  $n$  and the volume of the unit ball in  $\mathbb{R}^n$ . Taking  $f = 1$  on  $M$ , and using the fact that  $V(M) = 1$ , we deduce that

$$\|H\|_{2p} \geq \frac{1}{K(n)}$$

and if  $(P_C)$  is satisfied and  $C \leq \frac{n}{2K(n)^2} = c_n$ , then

$$\frac{n}{\lambda_1(M)} \leq \frac{1}{n\|H\|_{2p}^2 - C} \leq 2K(n)^2 = d_n. \quad \square$$

Throughout the paper, we will assume that  $V(M) = 1$  and  $\int_M X_i dv = 0$  for all  $i \leq n+1$ . The last assertion implies that the center of mass of  $M$  is the origin of  $\mathbb{R}^{n+1}$ .

## 2. An $L^2$ -approach of the problem

A first step in the proof of Theorem 1.2 is to prove that if the pinching condition  $(P_C)$  is satisfied, then  $M$  is close to a sphere in an  $L^2$ -sense.

In the following lemma, we prove that the  $L^2$ -norm of the position vector is close to  $\sqrt{\frac{n}{\lambda_1(M)}}$ .

**Lemma 2.1.** *If we have the pinching condition  $(P_C)$  with  $C < c_n$ , then*

$$\frac{n\lambda_1(M)}{(C + \lambda_1(M))^2} \leq \|X\|_2^2 \leq \frac{n}{\lambda_1(M)} \leq d_n.$$

*Proof.* Since  $\int_M X_i dv = 0$ , we can apply the variational characterization of the eigenvalues to obtain

$$\lambda_1(M) \int_M \sum_{i \leq n+1} |X_i|^2 dv \leq \int_M \sum_{i \leq n+1} |dX_i|^2 dv = n$$

which gives the inequality of the right-hand side.

Let us prove now the inequality of the left-hand side.

$$\begin{aligned} \lambda_1(M) \int_M |X|^2 dv &\leq \frac{(\int_M \sum_{i \leq n+1} |dX_i|^2 dv)^4}{(\int_M \sum_{i \leq n+1} |dX_i|^2 dv)^3} = \frac{(\int_M \sum_{i \leq n+1} (\Delta X_i) X_i dv)^4}{n^3} \\ &\leq \frac{(\int_M \sum_{i \leq n+1} (\Delta X_i)^2 dv)^2 (\int_M |X|^2 dv)^2}{n^3} \\ &= n \left( \int_M H^2 dv \right)^2 \left( \int_M |X|^2 dv \right)^2 \end{aligned}$$

then using again the Hölder inequality, we get

$$\lambda_1(M) \leq \frac{1}{n} (n \|H\|_{2p}^2)^2 \int_M |X|^2 dv \leq \frac{(C + \lambda_1(M))^2}{n} \int_M |X|^2 dv.$$

This completes the proof.  $\square$

From now on, we will denote by  $X^T$  the orthogonal tangential projection on  $M$ . In fact, at  $x \in M$ ,  $X^T$  is nothing but the vector of  $T_x M$  defined by  $X^T = \sum_{1 \leq i \leq n} \langle X, e_i \rangle e_i$  where  $(e_i)_{1 \leq i \leq n}$  is an orthonormal basis of  $T_x M$ . In the following lemma, we will show that the condition  $(P_C)$  implies that the  $L^2$ -norm of  $X^T$  of  $X$  on  $M$  is close to 0.

**Lemma 2.2.** *If we have the pinching condition  $(P_C)$ , then*

$$\|X^T\|_2^2 \leq A(n)C.$$

*Proof.* From Lemma 2.1 and the relation (3), we have

$$\lambda_1(M) \int_M |X|^2 dv \leq n = n \left( \int_M H \langle X, \nu \rangle dv \right)^2$$

$$\begin{aligned} &\leq \left( \int_M |H| |\langle X, \nu \rangle| dv \right)^2 \leq n \|H\|_{2p}^2 \left( \int_M |\langle X, \nu \rangle|^{\frac{2p}{2p-1}} dv \right)^{\frac{2p-1}{p}} \\ &\leq n \|H\|_{2p}^2 \left( \int_M |\langle X, \nu \rangle|^2 dv \right) = n \|H\|_{2p}^2 \int_M |X|^2 dv. \end{aligned}$$

Then we deduce that

$$\begin{aligned} n \|H\|_{2p}^2 \|X^T\|_2^2 &= n \|H\|_{2p}^2 \left( \int_M (|X|^2 - |\langle X, \nu \rangle|^2) dv \right) \\ &\leq (n \|H\|_{2p}^2 - \lambda_1(M)) \|X\|_2^2 \leq d_n C \end{aligned}$$

where in the last inequality we have used the pinching condition and Lemma 2.1. □

Next we will show that the condition  $(P_C)$  implies that the component functions are almost eigenfunctions in an  $L^2$ -sense. For this, let us consider the vector field  $Y$  on  $M$  defined by

$$Y = \sum_{i \leq n+1} (\Delta X_i - \lambda_1(M) X_i) \partial_i = nH\nu - \lambda_1(M)X.$$

**Lemma 2.3.** *If  $(P_C)$  is satisfied, then*

$$\|Y\|_2^2 \leq nC.$$

*Proof.* We have

$$\int_M |Y|^2 dv = \int_M (n^2 H^2 - 2n\lambda_1(M)H \langle \nu, X \rangle + \lambda_1(M)^2 |X|^2) dv.$$

Now by integrating the relation (3) we deduce that

$$\int_M H \langle \nu, X \rangle dv = 1.$$

Furthermore, since  $\int_M X_i dv = 0$ , we can apply the variational characterization of the eigenvalues to obtain

$$\lambda_1(M) \int_M |X|^2 dv = \lambda_1(M) \int_M \sum_{i \leq n+1} |X_i|^2 dv \leq \int_M \sum_{i \leq n+1} |dX_i|^2 dv = n.$$

Then

$$\int_M |Y|^2 dv \leq n^2 \int_M |H|^2 dv - n\lambda_1(M) \leq n \left( n \|H\|_{2p}^2 - \lambda_1(M) \right) \leq nC$$

where in this last inequality we have used the Hölder inequality. □

To prove Assertion 1 of Theorem 1.2, we will show that  $\left\| |X| - \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \right\|_\infty \leq \varepsilon$ .

For this we need an  $L^2$ -upper bound on the function  $\varphi = |X| \left( |X| - \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \right)^2$ .

Before giving such estimate, we will introduce the vector field  $Z$  on  $M$  defined by

$$Z = \left( \frac{n}{\lambda_1(M)} \right)^{1/2} |X|^{1/2} H\nu - \frac{X}{|X|^{1/2}}.$$

We have

**Lemma 2.4.** *If  $(P_C)$  is satisfied with  $C < c_n$ , then*

$$\|Z\|_2^2 \leq B(n)C.$$

*Proof.* We have

$$\begin{aligned} \|Z\|_2^2 &= \left\| \left( \frac{n}{\lambda_1(M)} \right)^{1/2} |X|^{1/2} H\nu - \frac{X}{|X|^{1/2}} \right\|_2^2 \\ &= \int_M \left( \frac{n}{\lambda_1(M)} |X| H^2 - 2 \left( \frac{n}{\lambda_1(M)} \right)^{1/2} H \langle \nu, X \rangle + |X| \right) dv \\ &\leq \frac{n}{\lambda_1(M)} \left( \int_M |X|^2 dv \right)^{1/2} \left( \int_M H^4 dv \right)^{1/2} \\ &\quad - 2 \left( \frac{n}{\lambda_1(M)} \right)^{1/2} + \left( \int_M |X|^2 dv \right)^{1/2}. \end{aligned}$$

Note that we have used the relation (3). Finally for  $p \geq 2$ , we get

$$\begin{aligned} \|Z\|_2^2 &\leq \left( \int_M |X|^2 dv \right)^{1/2} \left( \frac{n}{\lambda_1(M)} \|H\|_{2p}^2 + 1 \right) - 2 \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \\ &\leq \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \left( \frac{C}{\lambda_1(M)} + 2 \right) - 2 \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \\ &= \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \frac{C}{\lambda_1(M)} \leq \frac{d_n^{3/2}}{n} C. \end{aligned}$$

This concludes the proof of the lemma. □

Now we give an  $L^2$ -upper bound of  $\varphi$ .

**Lemma 2.5.** *Let  $p \geq 2$  and  $C \leq c_n$ . If we have the pinching condition  $(P_C)$ , then*

$$\|\varphi\|_2 \leq D(n) \|\varphi\|_\infty^{3/4} C^{1/4}.$$

*Proof.* We have

$$\|\varphi\|_2 = \left( \int_M \varphi^{3/2} \varphi^{1/2} dv \right)^{1/2} \leq \|\varphi\|_\infty^{3/4} \|\varphi^{1/2}\|_1^{1/2},$$

and noting that

$$|X| \left( |X| - \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \right)^2 = \left| |X|^{1/2} X - \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \frac{X}{|X|^{1/2}} \right|^2$$

we get

$$\begin{aligned} \int_M \varphi^{1/2} dv &= \left\| |X|^{1/2} X - \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \frac{X}{|X|^{1/2}} \right\|_1 \\ &= \left\| -\frac{|X|^{1/2}}{\lambda_1(M)} Y + \frac{n}{\lambda_1(M)} |X|^{1/2} H v - \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \frac{X}{|X|^{1/2}} \right\|_1 \\ &\leq \left\| \frac{|X|^{1/2}}{\lambda_1(M)} Y \right\|_1 + \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \|Z\|_1. \end{aligned} \tag{6}$$

From Lemmas 2.3 and 1.1 we get

$$\begin{aligned} \left\| \frac{|X|^{1/2}}{\lambda_1(M)} Y \right\|_1 &\leq \frac{1}{\lambda_1(M)} \left( \int_M |X| dv \right)^{1/2} \|Y\|_2 \\ &\leq \frac{1}{\lambda_1(M)} \left( \int_M |X|^2 dv \right)^{1/4} \|Y\|_2 \leq \frac{d_n^{3/4}}{n^{1/2}} C^{1/2}. \end{aligned}$$

Moreover, using Lemmas 2.4 and 1.1 again it is easy to see that the last term of (6) is bounded by  $d_n^{1/2} B(n)^{1/2} C^{1/2}$ . Then  $\|\varphi^{1/2}\|_1^{1/2} \leq D(n) C^{1/4}$ .  $\square$

### 3. Proof of Theorem 1.2

The proof of Theorem 1.2 is immediate from the two following technical lemmas which we state below.

**Lemma 3.1.** *For  $p \geq 2$  and for any  $\eta > 0$ , there exists  $K_\eta(n, \|H\|_\infty) \leq c_n$  so that if  $(P_{K_\eta})$  is true, then  $\|\varphi\|_\infty \leq \eta$ . Moreover,  $K_\eta \rightarrow 0$  when  $\|H\|_\infty \rightarrow \infty$  or  $\eta \rightarrow 0$ .*

**Lemma 3.2.** *Let  $x_0$  be a point of the sphere  $S(O, R)$  of  $\mathbb{R}^{n+1}$  with the center at the origin and of radius  $R$ . Assume that  $x_0 = Re$  where  $e \in \mathbb{S}^n$ . Now let  $(M^n, g)$  be a compact oriented  $n$ -dimensional Riemannian manifold without boundary isometrically*

immersed by  $\phi$  in  $\mathbb{R}^{n+1}$  so that  $\phi(M) \subset (B(O, R + \eta) \setminus B(O, R - \eta)) \setminus B(x_0, \rho)$  with  $\rho = 4(2n - 1)\eta$  and suppose that there exists a point  $p \in M$  so that  $\langle X, e \rangle > 0$ . Then there exists  $y_0 \in M$  so that the mean curvature  $H(y_0)$  at  $y_0$  satisfies  $|H(y_0)| \geq \frac{1}{4n\eta}$ .

Now, let us see how to use these lemmas to prove Theorem 1.2.

*Proof of Theorem 1.2.* We consider the function  $f(t) = t \left( t - \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \right)^2$ . For  $\varepsilon > 0$  let us put

$$\begin{aligned} \eta(\varepsilon) &= \min \left( \left( \frac{1}{\|H\|_\infty} - \varepsilon \right) \varepsilon^2, \left( \frac{1}{\|H\|_\infty} + \varepsilon \right) \varepsilon^2, \frac{1}{27\|H\|_\infty^3} \right) \\ &\leq \min \left( f \left( \left( \frac{n}{\lambda_1(M)} \right)^{1/2} - \varepsilon \right), f \left( \left( \frac{n}{\lambda_1(M)} \right)^{1/2} + \varepsilon \right), \frac{1}{27\|H\|_\infty^3} \right). \end{aligned}$$

Then, as  $\eta(\varepsilon) > 0$  and from Lemma 3.1, it follows that if the pinching condition  $(P_{K_{\eta(\varepsilon)}})$  is satisfied with  $K_{\eta(\varepsilon)} \leq c_n$ , then for any  $x \in M$ , we have

$$f(|X|) \leq \eta(\varepsilon). \tag{7}$$

Now to prove Theorem 1.2, it is sufficient to assume  $\varepsilon < \frac{2}{3\|H\|_\infty}$ . Let us show that either

$$\left( \frac{n}{\lambda_1(M)} \right)^{1/2} - \varepsilon \leq |X| \leq \left( \frac{n}{\lambda_1(M)} \right)^{1/2} + \varepsilon \quad \text{or} \quad |X| < \frac{1}{3} \left( \frac{n}{\lambda_1(M)} \right)^{1/2}. \tag{8}$$

By studying the function  $f$  it is easy to see that  $f$  has a unique local maximum in  $\frac{1}{3} \left( \frac{n}{\lambda_1(M)} \right)^{1/2}$  and from the definition of  $\eta(\varepsilon)$  it follows that  $\eta(\varepsilon) < \frac{4}{27} \frac{1}{\|H\|_\infty^3} \leq \frac{4}{27} \left( \frac{n}{\lambda_1(M)} \right)^{3/2} = f \left( \frac{1}{3} \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \right)$ .

Since  $\varepsilon < \frac{2}{3\|H\|_\infty}$ , we have  $\varepsilon < \frac{2}{3} \left( \frac{n}{\lambda_1(M)} \right)^{1/2}$  and  $\frac{1}{3} \left( \frac{n}{\lambda_1(M)} \right)^{1/2} < \left( \frac{n}{\lambda_1(M)} \right)^{1/2} - \varepsilon$ . This and (7) yield (8).

Now, from Lemma 2.1 we deduce that there exists a point  $y_0 \in M$  so that  $|X(y_0)| \geq \frac{n^{1/2}\lambda_1(M)^{1/2}}{(K_{\eta(\varepsilon)} + \lambda_1(M))}$  and since  $K_{\eta(\varepsilon)} \leq c_n = \frac{n}{d_n} \leq \lambda_1(M) \leq 2\lambda_1(M)$  (see the proof of Lemma 1.1), we obtain  $|X(y_0)| \geq \frac{1}{3} \left( \frac{n}{\lambda_1(M)} \right)^{1/2}$ .

By the connectedness of  $M$ , it follows that  $\left( \frac{n}{\lambda_1(M)} \right)^{1/2} - \varepsilon \leq |X| \leq \left( \frac{n}{\lambda_1(M)} \right)^{1/2} + \varepsilon$  for any point of  $M$  and Assertion 1 of Theorem 1.2 is shown for the condition  $(P_{K_{\eta(\varepsilon)}})$ .

In order to prove the second assertion, let us consider the pinching condition  $(P_{C_\varepsilon})$  with  $C_\varepsilon = K_{\eta(\frac{\varepsilon}{4(2n-1)})}$ . Then Assertion 1 is still valid. Let  $x = \left( \frac{n}{\lambda_1(M)} \right)^{1/2} e \in S \left( O, \sqrt{\frac{n}{\lambda_1(M)}} \right)$ , with  $e \in \mathbb{S}^n$  and suppose that  $B(x, \varepsilon) \cap M = \emptyset$ . Since  $\int_M X_i \, dv = 0$

for any  $i \leq n + 1$ , there exists a point  $p \in M$  so that  $\langle X, e \rangle > 0$  and we can apply Lemma 3.2. Therefore there is a point  $y_0 \in M$  so that  $H(y_0) \geq \frac{2n-1}{n\varepsilon} > \|H\|_\infty$  since we have assumed  $\varepsilon < \frac{2}{3\|H\|_\infty} \leq \frac{2n-1}{2n\|H\|_\infty}$ . Then we obtain a contradiction which implies  $B(x, \varepsilon) \cap M \neq \emptyset$  and Assertion 2 is satisfied. Furthermore,  $C_\varepsilon \rightarrow 0$  when  $\|H\|_\infty \rightarrow \infty$  or  $\varepsilon \rightarrow 0$ .  $\square$

#### 4. Proof of Theorem 1.3

From Theorem 1.2, we know that for any  $\varepsilon > 0$ , there exists  $C_\varepsilon$  depending only on  $n$  and  $\|H\|_\infty$  so that if  $(P_{C_\varepsilon})$  is true then

$$\left| |X|_x - \sqrt{\frac{n}{\lambda_1(M)}} \right| \leq \varepsilon$$

for any  $x \in M$ . Now, since  $\sqrt{n}\|H\|_\infty \leq \|B\|_\infty$ , it is easy to see from the previous proofs that we can assume that  $C_\varepsilon$  is depending only on  $n$  and  $\|B\|_\infty$ .

The proof of Theorem 1.3 is a consequence of the following lemma on the  $L_\infty$ -norm of  $\psi = |X^T|$ .

**Lemma 4.1.** *For  $p \geq 2$  and for any  $\eta > 0$ , there exists  $K_\eta(n, \|B\|_\infty)$  so that if  $(P_{K_\eta})$  is true, then  $\|\psi\|_\infty \leq \eta$ . Moreover,  $K_\eta \rightarrow 0$  when  $\|B\|_\infty \rightarrow \infty$  or  $\eta \rightarrow 0$ .*

This lemma will be proved in the Section 5.

*Proof of Theorem 1.3.* Let  $\varepsilon < \frac{1}{2}\sqrt{\frac{n}{\|B\|_\infty}} \leq \sqrt{\frac{n}{\lambda_1(M)}}$ . From the choice of  $\varepsilon$ , we deduce that the condition  $(P_{C_\varepsilon})$  implies that  $|X_x|$  is nonzero for any  $x \in M$  (see the proof of Theorem 1.2) and we can consider the differential application

$$F: M \longrightarrow S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right),$$

$$x \longmapsto \sqrt{\frac{n}{\lambda_1(M)}} \frac{X_x}{|X_x|}.$$

We will prove that  $F$  is a quasi-isometry. Indeed, for any  $0 < \theta < 1$ , we can choose a constant  $\varepsilon(n, \|B\|_\infty, \theta)$  so that for any  $x \in M$  and any unit vector  $u \in T_x M$ , the pinching condition  $(P_{C_\varepsilon(n, \|B\|_\infty, \theta)})$  implies

$$\left| |dF_x(u)|^2 - 1 \right| \leq \theta.$$

For this, let us compute  $dF_x(u)$ . We have

$$dF_x(u) = \sqrt{\frac{n}{\lambda_1(M)}} \nabla_u^0 \left( \frac{X}{|X|} \right) \Big|_x = \sqrt{\frac{n}{\lambda_1(M)}} u \left( \frac{1}{|X|} \right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X =$$

$$\begin{aligned}
&= -\frac{1}{2} \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|^3} u(|X|^2) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} u \\
&= -\sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|^3} \langle u, X \rangle X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} u \\
&= \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \left( -\frac{\langle u, X \rangle}{|X|^2} X + u \right).
\end{aligned}$$

By a straightforward computation, we obtain

$$\begin{aligned}
| |dF_x(u)|^2 - 1 | &= \left| \frac{n}{\lambda_1(M)} \frac{1}{|X|^2} \left( 1 - \frac{\langle u, X \rangle^2}{|X|^2} \right) - 1 \right| \\
&\leq \left| \frac{n}{\lambda_1(M)} \frac{1}{|X|^2} - 1 \right| + \frac{n}{\lambda_1(M)} \frac{1}{|X|^4} \langle u, X \rangle^2.
\end{aligned} \tag{9}$$

Now

$$\begin{aligned}
\left| \frac{n}{\lambda_1(M)} \frac{1}{|X|^2} - 1 \right| &= \frac{1}{|X|^2} \left| \frac{n}{\lambda_1(M)} - |X|^2 \right| \\
&\leq \varepsilon \frac{\left| \sqrt{\frac{n}{\lambda_1(M)}} + |X| \right|}{|X|^2} \leq \varepsilon \frac{2\sqrt{\frac{n}{\lambda_1(M)}} + \varepsilon}{\left( \sqrt{\frac{n}{\lambda_1(M)}} - \varepsilon \right)^2}.
\end{aligned}$$

Let us recall that  $\frac{n}{dn} \leq \lambda_1(M) \leq \|B\|_\infty^2$  (see (4) for the first inequality). Since we assume  $\varepsilon < \frac{1}{2} \sqrt{\frac{n}{\|B\|_\infty}}$ , the right-hand side is bounded above by a constant depending only on  $n$  and  $\|B\|_\infty$  and we have

$$\left| \frac{n}{\lambda_1(M)} \frac{1}{|X|^2} - 1 \right| \leq \varepsilon \gamma(n, \|B\|_\infty). \tag{10}$$

On the other hand, since  $C_\varepsilon(n, \|B\|_\infty) \rightarrow 0$  when  $\varepsilon \rightarrow 0$ , there exists  $\varepsilon(n, \|B\|_\infty, \eta)$  so that  $C_{\varepsilon(n, \|B\|_\infty, \eta)} \leq K_\eta(n, \|B\|_\infty)$  (where  $K_\eta$  is the constant of the lemma) and then by Lemma 4.1,  $\|\psi\|_\infty^2 \leq \eta^2$ . Thus there exists a constant  $\delta$  depending only on  $n$  and  $\|B\|_\infty$  so that

$$\frac{n}{\lambda_1(M)} \frac{1}{|X|^4} \langle u, X \rangle^2 \leq \frac{n}{\lambda_1(M)} \frac{1}{|X|^4} \|\psi\|_\infty^2 \leq \eta^2 \delta(n, \|B\|_\infty), \tag{11}$$

and from (9), (10) and (11) we deduce that the condition  $(P_{C_{\varepsilon(n, \|B\|_\infty, \eta)}})$  implies

$$| |dF_x(u)|^2 - 1 | \leq \varepsilon \gamma(n, \|B\|_\infty) + \eta^2 \delta(n, \|B\|_\infty).$$

Now let us choose  $\eta = \left(\frac{\theta}{2\delta}\right)^{1/2}$ . Then we can assume that  $\varepsilon(n, \|B\|_\infty, \eta)$  is small enough in order to have  $\varepsilon(n, \|B\|_\infty, \eta)\gamma(n\|B\|_\infty) \leq \frac{\theta}{2}$ . In this case we have

$$||dF_x(u)|^2 - 1| \leq \theta.$$

Now let us fix  $\theta$ ,  $0 < \theta < 1$ . It follows that  $F$  is a local diffeomorphism from  $M$  to  $S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right)$ . Since  $S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right)$  is simply connected for  $n \geq 2$ ,  $F$  is a diffeomorphism.  $\square$

## 5. Proof of the technical lemmas

The proofs of Lemmas 3.1 and 4.1 are providing from a result stated in the following proposition using a Nirenberg–Moser type of proof.

**Proposition 5.1.** *Let  $(M^n, g)$  be a compact, connected and oriented  $n$ -dimensional Riemannian manifold without boundary isometrically immersed into the  $n+1$ -dimensional euclidean space  $(\mathbb{R}^{n+1}, \text{can})$ . Let  $\xi$  be a nonnegative continuous function so that  $\xi^k$  is smooth for  $k \geq 2$ . Let  $0 \leq r < s \leq 2$  so that*

$$\frac{1}{2}\Delta\xi^2\xi^{2k-2} \leq \delta\omega + (A_1 + kA_2)\xi^{2k-r} + (B_1 + kB_2)\xi^{2k-s}$$

where  $\delta\omega$  is the codifferential of a 1-form and  $A_1, A_2, B_1, B_2$  are nonnegative constants. Then for any  $\eta > 0$ , there exists a constant  $L(n, A_1, A_2, B_1, B_2, \|H\|_\infty, \eta)$  depending only on  $n, A_1, A_2, B_1, B_2, \|H\|_\infty$  and  $\eta$  so that if  $\|\xi\|_\infty > \eta$  then

$$\|\xi\|_\infty \leq L(n, A_1, A_2, B_1, B_2, \|H\|_\infty, \eta)\|\xi\|_2.$$

Moreover,  $L$  is bounded when  $\eta \rightarrow \infty$ , and if  $B_1 > 0$ ,  $L \rightarrow \infty$  when  $\|H\|_\infty \rightarrow \infty$  or  $\eta \rightarrow 0$ .

This proposition will be proved at the end of the paper.

Before giving the proofs of Lemmas 3.1 and 4.1, we will show that under the pinching condition  $(P_C)$  with  $C$  small enough, the  $L_\infty$ -norm of  $X$  is bounded by a constant depending only on  $n$  and  $\|H\|_\infty$ .

**Lemma 5.1.** *If we have the pinching condition  $(P_C)$  with  $C < c_n$ , then there exists  $E(n, \|H\|_\infty)$  depending only on  $n$  and  $\|H\|_\infty$  so that  $\|X\|_\infty \leq E(n, \|H\|_\infty)$ .*

*Proof.* From the relation (3), we have

$$\frac{1}{2}\Delta|X|^2|X|^{2k-2} \leq n\|H\|_\infty|X|^{2k-1}.$$

Then applying Proposition 5.1 to the function  $\xi = |X|$  with  $r = 0$  and  $s = 1$ , we obtain that if  $\|X\|_\infty > E$ , then there exists a constant  $L(n, \|H\|_\infty, E)$  depending only on  $n, \|H\|_\infty$  and  $E$  so that

$$\|X\|_\infty \leq L(n, \|H\|_\infty, E)\|X\|_2,$$

and under the pinching condition  $(P_C)$  with  $C < c_n$  we have from Lemma 2.1 that

$$\|X\|_\infty \leq L(n, \|H\|_\infty, E)d_n^{1/2}.$$

Now since  $L$  is bounded when  $E \rightarrow \infty$ , we can choose  $E = E(n, \|H\|_\infty)$  large enough so that

$$L(n, \|H\|_\infty, E)d_n^{1/2} < E.$$

In this case, we have  $\|X\|_\infty \leq E(n, \|H\|_\infty)$ . □

*Proof of Lemma 3.1.* First we compute the Laplacian of the square of  $\varphi^2$ . We have

$$\begin{aligned} \Delta\varphi^2 &= \Delta\left(|X|^4 - 2\left(\frac{n}{\lambda_1(M)}\right)^{1/2}|X|^3 + \frac{n}{\lambda_1(M)}|X|^2\right) \\ &= -2|X|^2|d|X|^2|^2 + 2|X|^2\Delta|X|^2 \\ &\quad - 2\left(\frac{n}{\lambda_1(M)}\right)^{1/2}\left(-\frac{3}{4}|X|^{-1}|d|X|^2|^2 + \frac{3}{2}|X|\Delta|X|^2\right) + \frac{n}{\lambda_1(M)}\Delta|X|^2. \end{aligned}$$

Now by a direct computation one gets  $|d|X|^2|^2 \leq 4|X|^2$ . Moreover by the relation (3) we have  $|\Delta|X|^2| \leq 2n\|H\|_\infty|X| + n$ . Then applying Lemmas 1.1 and 5.1 we get

$$\Delta\varphi^2 \leq \alpha(n, \|H\|_\infty)$$

and

$$\frac{1}{2}\Delta\varphi^2\varphi^{2k-2} \leq \alpha(n, \|H\|_\infty)\varphi^{2k-2}.$$

Now, we apply Proposition 5.1 with  $r = 0$  and  $s = 2$ . Then if  $\|\varphi\|_\infty > \eta$ , there exists a constant  $L(n, \|H\|_\infty)$  depending only on  $n$  and  $\|H\|_\infty$  so that

$$\|\varphi\|_\infty \leq L\|\varphi\|_2.$$

From Lemma 2.5, if  $C \leq c_n$  and  $(P_C)$  is true, we have  $\|\varphi\|_2 \leq D(n)\|\varphi\|_\infty^{3/4}C^{1/4}$ . Therefore

$$\|\varphi\|_\infty \leq (LD)^4C.$$

Consequently, if we choose  $C = K_\eta = \inf\left(\frac{\eta}{(LD)^4}, c_n\right)$ , then we obtain  $\|\varphi\|_\infty \leq \eta$ . □

*Proof of Lemma 4.1.* First we will prove that for any  $C < c_n$ , if  $(P_C)$  is true, then

$$\frac{1}{2}(\Delta\psi^2)\psi^{2k-2} \leq \delta\omega + (\alpha_1(n, \|B\|_\infty) + k\alpha_2(n, \|B\|_\infty))\psi^{2k-2} \tag{12}$$

where  $\delta\omega$  is the codifferential of a 1-form  $\omega$ .

First observe that the gradient  $\nabla^M|X|^2$  of  $|X|^2$  satisfies  $\nabla^M|X|^2 = 2X^T$ . Then by the Bochner formula we get

$$\begin{aligned} \frac{1}{2}\Delta|X^T|^2 &= \frac{1}{4}\langle\Delta d|X|^2, d|X|^2\rangle - \frac{1}{4}|\nabla d|X|^2|^2 - \frac{1}{4}\text{Ric}(\nabla^M|X|^2, \nabla^M|X|^2) \\ &\leq \frac{1}{4}\langle d\Delta|X|^2, d|X|^2\rangle - \frac{1}{4}\text{Ric}(\nabla^M|X|^2, \nabla^M|X|^2) \end{aligned}$$

and by the Gauss formula we obtain

$$\begin{aligned} \frac{1}{2}\Delta|X^T|^2 &\leq \frac{1}{4}\langle d\Delta|X|^2, d|X|^2\rangle - \frac{1}{4}nH\langle B\nabla^M|X|^2, \nabla^M|X|^2\rangle + \frac{1}{4}|B\nabla^M|X|^2|^2 \\ &= \frac{1}{4}\langle d\Delta|X|^2, d|X|^2\rangle - nH\langle BX^T, X^T\rangle + |BX^T|^2. \end{aligned}$$

By Lemma 5.1 we know that  $\|X\|_\infty \leq E(n, \|B\|_\infty)$  (the dependance in  $\|H\|_\infty$  can be replaced by  $\|B\|_\infty$ ). Then it follows that

$$\frac{1}{2}(\Delta\psi^2)\psi^{2k-2} \leq \frac{1}{4}\langle d\Delta|X|^2, d|X|^2\rangle\psi^{2k-2} + \alpha'(n, \|B\|_\infty)\psi^{2k-2}. \tag{13}$$

Now, let us compute the term  $\langle d\Delta|X|^2, d|X|^2\rangle\psi^{2k-2}$ . We have

$$\begin{aligned} \langle d\Delta|X|^2, d|X|^2\rangle\psi^{2k-2} &= \delta\omega + (\Delta|X|^2)^2\psi^{2k-2} - (2k-2)\Delta|X|^2\langle d|X|^2, d\psi\rangle\psi^{2k-3} \\ &= \delta\omega + (\Delta|X|^2)^2\psi^{2k-2} - 2(2k-2)\Delta|X|^2\langle X^T, \nabla^M\psi\rangle\psi^{2k-3} \end{aligned}$$

where  $\omega = -\Delta|X|^2\psi^{2k-2}d|X|^2$ . Now,

$$e_i(\psi) = \frac{e_i|X^T|^2}{2|X^T|} = \frac{e_i|X|^2 - e_i\langle X, \nu\rangle^2}{2|X^T|} = \frac{\langle e_i, X\rangle - B_{ij}\langle X, e_j\rangle\langle X, \nu\rangle}{|X^T|}.$$

Then

$$\begin{aligned} \langle d\Delta|X|^2, d|X|^2\rangle\psi^{2k-2} &= \delta\omega + (\Delta|X|^2)^2\psi^{2k-2} - 2(2k-2)\Delta|X|^2|X^T|\psi^{2k-3} \\ &\quad + 2(2k-2)\Delta|X|^2\frac{\langle BX^T, X^T\rangle}{|X^T|}\langle X, \nu\rangle\psi^{2k-3} \\ &\leq \delta\omega + (\Delta|X|^2)^2\psi^{2k-2} + 2(2k-2)|\Delta|X|^2|\psi^{2k-2} \\ &\quad + 2(2k-2)|\Delta|X|^2\|B\||X|\psi^{2k-2}. \end{aligned}$$

Now by relation (3) and Lemma 5.1 we have

$$\langle d\Delta|X|^2, d|X|^2 \rangle \psi^{2k-2} \leq \delta\omega + (\alpha_1''(n, \|B\|_\infty) + k\alpha_2''(n, \|B\|_\infty)) \psi^{2k-2}.$$

Inserting this in (13), we obtain the desired inequality (12).

Now applying again Proposition 5.1, we get that there exists  $L(n, \|B\|_\infty, \eta)$  so that if  $\|\psi\|_\infty > \eta$  then

$$\|\psi\|_\infty \leq L\|\psi\|_2.$$

From Lemma 2.2 we deduce that if the pinching condition  $(P_C)$  holds then  $\|\psi\|_2 \leq A(n)^{1/2}C^{1/2}$ . Then taking  $C = K_\eta = \inf\left(\frac{\eta}{LA^{1/2}}, c_n\right)$ , then  $\|\psi\|_\infty \leq \eta$ .  $\square$

*Proof of Lemma 3.2.* The idea of the proof consists in foliating the region  $B(O, R + \eta) \setminus B(O, R - \eta)$  with hypersurfaces of large mean curvature and to show that one of these hypersurfaces is tangent to  $\phi(M)$ . This will imply that  $\phi(M)$  has a large mean curvature at the contact point.

Consider  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$  and  $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}e$ . Let  $a, L > l > 0$  and

$$\begin{aligned} \Phi_{L,l,a} : \mathbb{S}^{n-1} \times \mathbb{S}^1 &\longrightarrow \mathbb{R}^{n+1} \\ (\xi, \theta) &\longmapsto L\xi - l \cos \theta \xi + l \sin \theta e + ae. \end{aligned}$$

Then  $\Phi_{L,l,a}$  is a family of embeddings from  $\mathbb{S}^{n-1} \times \mathbb{S}^1$  in  $\mathbb{R}^{n+1}$ . If we orient the family of hypersurfaces  $\Phi_{L,l,a}(\mathbb{S}^{n-1} \times \mathbb{S}^1)$  by the unit outward normal vector field, a straightforward computation shows that the mean curvature  $H(\theta)$  depends only on  $\theta$  and we have

$$H(\theta) = \frac{1}{n} \left( \frac{1}{l} - \frac{(n-1) \cos \theta}{L - l \cos \theta} \right) \geq \frac{1}{n} \left( \frac{1}{l} - \frac{n-1}{L-l} \right). \tag{14}$$

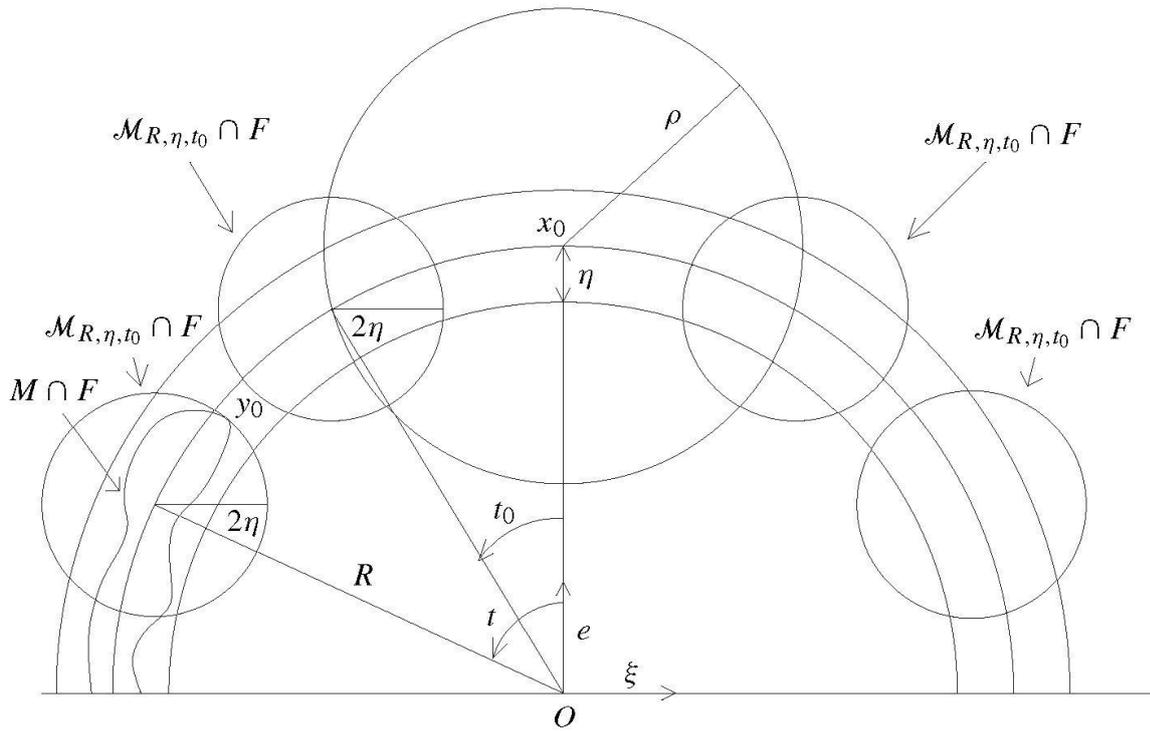
Now, let us consider the hypotheses of the lemma and for  $t_0 = 2 \arcsin\left(\frac{\rho}{2R}\right) \leq t \leq \frac{\pi}{2}$ , put  $L = R \sin t$ ,  $l = 2\eta$  and  $a = R \cos t$ . Then  $L > l$  and we can consider for  $t_0 \leq t \leq \frac{\pi}{2}$  the family  $\mathcal{M}_{R,\eta,t}$  of hypersurfaces defined by  $\mathcal{M}_{R,\eta,t} = \Phi_{R \sin t, 2\eta, R \cos t}(\mathbb{S}^{n-1} \times \mathbb{S}^1)$ .

From the relation (14), the mean curvature  $H_{R,\eta,t}$  of  $\mathcal{M}_{R,\eta,t}$  satisfies

$$\begin{aligned} H_{R,\eta,t} &\geq \frac{1}{n} \left( \frac{1}{2\eta} - \frac{n-1}{R \sin t - 2\eta} \right) \geq \frac{1}{n} \left( \frac{1}{2\eta} - \frac{n-1}{R \sin t_0 - 2\eta} \right) \\ &\geq \frac{1}{n} \left( \frac{1}{2\eta} - \frac{n-1}{R \sin(t_0/2) - 2\eta} \right) = \frac{1}{n} \left( \frac{1}{2\eta} - \frac{n-1}{\frac{\rho}{2} - 2\eta} \right) = \frac{1}{4n\eta} \end{aligned}$$

where we have used in this last equality the fact that  $\rho = 4(2n-1)\eta$ .

Since there exists a point  $p \in M$  so that  $\langle X(p), e \rangle > 0$ , we can find  $t \in [t_0, \pi/2]$  and a point  $y_0 \in M$  which is a contact point with  $\mathcal{M}_{R,\eta,t}$ . Therefore  $|H(y_0)| \geq \frac{1}{4n\eta}$ .



$F$  is the vector space spanned by  $e$  and  $\xi$ .

□

*Proof of Proposition 5.1.* Integrating by parts we have

$$\begin{aligned} \int_M \frac{1}{2} \Delta \xi^2 \xi^{2k-2} dv &= \frac{1}{2} \int_M \langle d\xi^2, d\xi^{2k-2} \rangle dv = 2 \left( \frac{k-1}{k^2} \right) \int_M |d\xi^k|^2 dv \\ &\leq (A_1 + kA_2) \int_M \xi^{2k-r} dv + (B_1 + kB_2) \int_M \xi^{2k-s} dv. \end{aligned}$$

Now, given a smooth function  $f$  and applying the Sobolev inequality (5) to  $f^2$ , we get

$$\begin{aligned} \left( \int_M f^{\frac{2n}{n-1}} dv \right)^{1-(1/n)} &\leq K(n) \int_M (2|f||df| + |H|f^2) dv \\ &\leq 2K(n) \left( \int_M f^2 dv \right)^{1/2} \left( \int_M |df|^2 dv \right)^{1/2} + K(n) \|H\|_\infty \int_M f^2 dv \\ &= K(n) \left( \int_M f^2 dv \right)^{1/2} \left( 2 \left( \int_M |df|^2 dv \right)^{1/2} + \|H\|_\infty \left( \int_M f^2 dv \right)^{1/2} \right) \end{aligned}$$

where in the second inequality, we have used the Hölder inequality. Using it again, by assuming that  $V(M) = 1$ , we have

$$\left( \int_M f^2 dv \right)^{1/2} \leq \left( \int_M f^{\frac{2n}{n-1}} dv \right)^{\frac{n-1}{2n}}.$$

And finally, we obtain

$$\|f\|_{\frac{2n}{n-1}} \leq K(n)(2\|df\|_2 + \|H\|_\infty\|f\|_2).$$

For  $k \geq 2$ ,  $\xi^k$  is smooth and we apply the above inequality to  $f = \xi^k$ . Then we get

$$\begin{aligned} \|\xi\|_{\frac{2kn}{n-1}}^k &\leq K(n) \left[ 2 \left( \int_M |d\xi^k|^2 dv \right)^{1/2} + \|H\|_\infty \left( \int_M \xi^{2k} dv \right)^{1/2} \right] \\ &\leq K(n) \left[ 2 \left( \frac{k^2}{2(k-1)} \right)^{1/2} \left( (A_1 + kA_2) \int_M \xi^{2k-r} dv \right. \right. \\ &\quad \left. \left. + (B_1 + kB_2) \int_M \xi^{2k-s} dv \right)^{1/2} + \|H\|_\infty \left( \int_M \xi^{2k} dv \right)^{1/2} \right] \\ &\leq K(n) \left[ 2 \left( \frac{k^2}{2(k-1)} \right)^{1/2} \left( (A_1 + kA_2) \|\xi\|_\infty^{2-r} \right. \right. \\ &\quad \left. \left. + (B_1 + kB_2) \|\xi\|_\infty^{2-s} \right)^{1/2} \|\xi\|_{2k-2}^{k-1} + \|H\|_\infty \|\xi\|_\infty \|\xi\|_{2k-2}^{k-1} \right] \\ &\leq K(n) \left[ 2 \left( \frac{k^2}{2(k-1)} \right)^{1/2} \left( \frac{A_1 + kA_2}{\|\xi\|_\infty^r} + \frac{B_1 + kB_2}{\|\xi\|_\infty^s} \right)^{1/2} \right. \\ &\quad \left. + \|H\|_\infty \right] \|\xi\|_\infty \|\xi\|_{2k-2}^{k-1} \\ &\leq K(n) \left[ 2 \left( \frac{k^2}{2(k-1)} \right)^{1/2} \left( \frac{A_1^{1/2} + k^{1/2}A_2^{1/2}}{\|\xi\|_\infty^{r/2}} + \frac{B_1^{1/2} + k^{1/2}B_2^{1/2}}{\|\xi\|_\infty^{s/2}} \right) \right. \\ &\quad \left. + \|H\|_\infty \right] \|\xi\|_\infty \|\xi\|_{2k-2}^{k-1}. \end{aligned}$$

If we assume that  $\|\xi\|_\infty > \eta$ , the last inequality becomes

$$\begin{aligned} \|\xi\|_{\frac{2kn}{n-1}}^k &\leq K(n) \left[ 2 \left( \frac{k^2}{2(k-1)} \right)^{1/2} \left( \frac{A_1^{1/2} + k^{1/2}A_2^{1/2}}{\eta^{r/2}} + \frac{B_1^{1/2} + k^{1/2}B_2^{1/2}}{\eta^{s/2}} \right) \right. \\ &\quad \left. + \|H\|_\infty \right] \|\xi\|_\infty \|\xi\|_{2k-2}^{k-1} \end{aligned}$$

$$= \left[ (K_1 + k^{1/2}K_2) \left( \frac{k^2}{k-1} \right)^{1/2} + K' \right] \|\xi\|_\infty \|\xi\|_{2k-2}^{k-1}.$$

Now let  $q = \frac{n}{n-1} > 1$  and for  $i \geq 0$  let  $k = q^i + 1 \geq 2$ . Then

$$\begin{aligned} \|\xi\|_{2(q^{i+1}+q)} &\leq \left( (K_1 + (q^i + 1)^{1/2}K_2) \left( \frac{q^i + 1}{q^{i/2}} \right) + K'' \right)^{\frac{1}{q^{i+1}}} \|\xi\|_\infty^{\frac{1}{q^{i+1}}} \|\xi\|_{2q^i}^{1-\frac{1}{q^{i+1}}} \\ &\leq (\tilde{K} q^i)^{\frac{1}{q^{i+1}}} \|\xi\|_\infty^{\frac{1}{q^{i+1}}} \|\xi\|_{2q^i}^{1-\frac{1}{q^{i+1}}} \end{aligned}$$

where  $\tilde{K} = 2K_1 + 2^{3/2}K_2 + K'$ . We see that  $\tilde{K}$  has a finite limit when  $\eta \rightarrow \infty$  and if  $B_1 > 0$ ,  $\tilde{K} \rightarrow \infty$  when  $\|H\|_\infty \rightarrow \infty$  or  $\eta \rightarrow 0$ . Moreover the Hölder inequality gives

$$\|\xi\|_{2q^{i+1}} \leq \|\xi\|_{2(q^{i+1}+q)}$$

which implies

$$\|\xi\|_{2q^{i+1}} \leq (\tilde{K} q^i)^{\frac{1}{q^{i+1}}} \|\xi\|_\infty^{\frac{1}{q^{i+1}}} \|\xi\|_{2q^i}^{1-\frac{1}{q^{i+1}}}.$$

Now, by iterating from 0 to  $i$ , we get

$$\begin{aligned} \|\xi\|_{2q^{i+1}} &\leq \tilde{K} \left( 1 - \prod_{k=i-j}^i \left( 1 - \frac{1}{q^{k+1}} \right) \right) q^{\sum_{k=i-j}^i \frac{k}{q^{k+1}}} \|\xi\|_\infty^{\left( 1 - \prod_{k=i-j}^i \left( 1 - \frac{1}{q^{k+1}} \right) \right)} \|\xi\|_{2q^{i-j}}^{\prod_{k=i-j}^i \left( 1 - \frac{1}{q^{k+1}} \right)} \\ &\leq \tilde{K} \left( 1 - \prod_{k=0}^i \left( 1 - \frac{1}{q^{k+1}} \right) \right) q^{\sum_{k=0}^i \frac{k}{q^{k+1}}} \|\xi\|_\infty^{\left( 1 - \prod_{k=0}^i \left( 1 - \frac{1}{q^{k+1}} \right) \right)} \|\xi\|_2^{\prod_{k=0}^i \left( 1 - \frac{1}{q^{k+1}} \right)}. \end{aligned}$$

Let  $\alpha = \sum_{k=0}^\infty \frac{k}{q^{k+1}}$  and  $\beta = \prod_{k=0}^\infty \left( 1 - \frac{1}{q^{k+1}} \right) = \prod_{k=0}^\infty \left( \frac{1}{1+(1/q)^k} \right)$ . Then

$$\|\xi\|_\infty \leq \tilde{K}^{1-\beta} q^\alpha \|\xi\|_\infty^{(1-\beta)} \|\xi\|_2^\beta,$$

and finally

$$\|\xi\|_\infty \leq L \|\xi\|_2$$

where  $L = \tilde{K}^{\frac{1-\beta}{\beta}} q^{\alpha/\beta}$  is a constant depending only on  $n, A_1, A_2, B_1, B_2, \|H\|_\infty$  and  $\eta$ . From classical methods we show that  $\beta \in [e^{-n}, e^{-n/2}]$ . In particular,  $0 < \beta < 1$  and we deduce that  $L$  is bounded when  $\eta \rightarrow \infty$  and  $L \rightarrow \infty$  when  $\|H\|_\infty \rightarrow \infty$  or  $\eta \rightarrow 0$  with  $B_1 > 0$ . □

**Remark.** In [12] and [13] Shihohama and Xu have proved that if  $(M^n, g)$  is a compact  $n$ -dimensional Riemannian manifold without boundary isometrically immersed in

$\mathbb{R}^{n+1}$  and if  $\int_M (|B|^2 - n|H|^2) < D_n$  where  $D_n$  is a constant depending on  $n$ , then all Betti numbers are zero. For  $n = 2$ ,  $D_2 = 4\pi$ , and it follows that if

$$\int_M |B|^2 dv - 4\pi < \lambda_1(M)V(M)$$

then we deduce from the Reilly inequality  $\lambda_1(M)V(M) \leq 2 \int_M H^2 dv$  that  $\int_M (|B|^2 - 2|H|^2) dv < 4\pi$  and by the result of Shihohama and Xu  $M$  is diffeomorphic to  $\mathbb{S}^2$ .

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