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Quasi-states and symplectic intersections

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Abstract. We establish a link between symplectic topology and a recently emerged branch of functional analysis called the theory of quasi-states and quasi-measures (also known as topological measures). In the symplectic context quasi-states can be viewed as an algebraic way of packaging certain information contained in Floer theory, and in particular in spectral invariants of Hamiltonian diffeomorphisms introduced recently by Yong-Geun Oh. As a consequence we prove a number of new results on rigidity of intersections in symplectic manifolds. This work is a part of a joint project with Paul Biran.

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1. Introduction

Rigidity of intersections is a class of phenomena in symplectic topology meaning that certain subsets of a symplectic manifold intersect each other in more points than dictated by algebraic and differential topology (see [11] for an excellent survey). In this paper we show that such rigidity phenomena in a closed symplectic manifold M

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sometimes formally follow from the existence of real-valued functionals with some interesting algebraic properties on the Poisson algebra $C^\infty(M)$.

On the one hand, these functionals are related to the notions of *quasi-state* and *quasi-measure* (which have been recently called *topological measures*) on M (see Section 3) which originate in quantum mechanics [1], [2] and have been a subject of intensive study in recent years following the paper [3] by J. F. Aarnes.

On the other hand, they are linked to a group-theoretic notion of *quasi-morphism* (see e.g. [29]) which already appeared in the context of symplectic topology in [20], [13]. The symplectic quasi-states on the Poisson-Lie algebra of functions on certain symplectic manifolds M considered below arise as an infinitesimal version of the Calabi quasi-morphism introduced in [20]. This quasi-morphism is defined on the universal cover $\widetilde{\text{Ham}}(M)$ of the group $\text{Ham}(M)$ of Hamiltonian diffeomorphisms of M .

All the above-mentioned functionals are constructed by means of Floer theory for Hamiltonian flows on M and can be viewed as an algebraic way of packaging certain information contained in that theory.

Throughout the paper M always stands for a closed connected symplectic manifold with a symplectic form ω . For technical reasons we assume that M is *rational*, i.e. the image of $\pi_2(M)$ under the cohomology class of ω is a discrete subgroup of \mathbb{R} . Furthermore, we assume that M is *strongly semi-positive*, that is

$$2 - n \leq c_1(A) < 0 \implies \omega(A) \leq 0, \quad \text{for any } A \in \pi_2(M), \quad (1)$$

where c_1 stands for the 1st Chern class of (M, ω) . For instance, every symplectic 4-manifold is strongly semi-positive. Another interesting class of examples is given by *spherically monotone* symplectic manifolds, which means that $[\omega]|_{\pi_2(M)}$ is a positive multiple of $c_1|_{\pi_2(M)}$. Note that this condition automatically implies strong semi-positivity and rationality of M .

Organization of the paper. The next section contains our main results on symplectic intersections. In Section 3 we focus on a special class of symplectic manifolds M which, for instance, includes monotone products of complex projective spaces. After a brief review of quasi-states and quasi-measures, we introduce symplectic quasi-states on the algebra $C(M)$ which turn out to be useful for symplectic intersections in M . In Section 4 we present a weaker notion of a partial symplectic quasi-state and its applications to non-displaceability phenomenon on more general symplectic manifolds. In Section 5 we review spectral invariants of Hamiltonian diffeomorphisms introduced recently by Y.-G. Oh. In Sections 6 and 7 these invariants are used in order to construct the above-mentioned (partial) symplectic quasi-states. In Section 8 we discuss symplectic quasi-states on surfaces. The reader will see that some innocently looking basic questions in this direction require more advanced tools of the theory of quasi-states and quasi-measures. Section 9 contains some applications (in the spirit

of our paper [13] with P. Biran) of our results to the Lagrangian intersection problem. In Section 10 we discuss the history and the physical meaning of quasi-states. In addition, in Sections 8–10 we present a number of open problems.

2. Results on symplectic intersections

We say that a subset X of M is *displaceable* if there exists a Hamiltonian diffeomorphism $\phi \in \text{Ham}(M)$ so that

$$\phi(X) \cap \text{Closure}(X) = \emptyset.$$

Otherwise, we call X *non-displaceable*. For instance, an open hemisphere in \mathbb{S}^2 with the Euclidean area form is displaceable while the closed hemisphere is not.

A linear subspace $\mathcal{A} \subset C^\infty(M)$ is called *Poisson-commutative*, if $\{F, G\} = 0$ for all $F, G \in \mathcal{A}$, where $\{\cdot, \cdot\}$ stand for the Poisson brackets. Given a finite-dimensional Poisson-commutative subspace $\mathcal{A} \subset C^\infty(M)$, its *moment map* $\Phi_{\mathcal{A}}: M \rightarrow \mathcal{A}^*$ is defined as

$$\langle \Phi_{\mathcal{A}}(x), F \rangle = F(x).$$

Non-empty subsets of the form $\Phi_{\mathcal{A}}^{-1}(p)$, $p \in \mathcal{A}^*$, are called *fibers* of \mathcal{A} .

Theorem 2.1. *Any finite-dimensional Poisson-commutative subspace of $C^\infty(M)$ has at least one non-displaceable fiber. Moreover, if every fiber has a finite number of connected components, there exists a fiber with a non-displaceable connected component.*

Poisson-commutative subspaces naturally appear when M is equipped with the structure of a (singular) Lagrangian fibration. In this case Theorem 2.1 shows that the fibration has at least one non-displaceable fiber. For instance, we have the following corollary, where the fibration is given by the moment map of a Hamiltonian torus action.

Corollary 2.2. *Assume that M^{2n} is equipped with a Hamiltonian action of \mathbb{T}^n . Then at least one Lagrangian orbit of this action is non-displaceable.*

Proof. Let \mathcal{A} be the span of the coordinate functions associated to the moment map of the action. Every fiber of \mathcal{A} is a fiber of the moment map: it is either a Lagrangian torus, or an isotropic torus of dimension less than n . The latter are displaceable (see e.g. [12]). Hence the result follows immediately from Theorem 2.1. \square

Definition 2.3. A closed subset $X \subset M$ is called a *stem*, if there exists a finite-dimensional Poisson-commutative subspace $\mathcal{A} \subset C^\infty(M)$ so that X is a fiber of \mathcal{A} and each fiber of \mathcal{A} , other than X , is displaceable.

Note that the image of a stem under *any* symplectomorphism of M is again a stem.

Theorem 2.1 guarantees that *every stem is non-displaceable*. This result can be strengthened for a special class of symplectic manifolds as follows.

Theorem 2.4. *Suppose M is one of the following symplectic manifolds: \mathbb{CP}^n , a complex Grassmannian, $\mathbb{CP}^{n_1} \times \cdots \times \mathbb{CP}^{n_k}$ with a monotone product symplectic structure, the monotone symplectic blow-up of \mathbb{CP}^2 at one point. Then any two stems in M have an non-empty intersection.*

In particular, a stem in such an M cannot be displaced from itself by any (not necessarily Hamiltonian) symplectomorphism.

Here is a sample corollary of this theorem. Consider the 2-sphere \mathbb{S}^2 with a symplectic form ω of total area 1. Define a class $\mathcal{G}_{\mathbb{S}^2}$ of closed subsets $\Gamma \subset \mathbb{S}^2$ with the following property: The complement $\mathbb{S}^2 \setminus \Gamma$ has a finite number of connected components, and each of them is homeomorphic to a disc and has area $\leq \frac{1}{2}$. For instance, one can take an equator, or the 1-skeleton of a piecewise smooth triangulation of \mathbb{S}^2 with small enough 2-dimensional faces.

Corollary 2.5. *Let M be the direct product of m copies of (\mathbb{S}^2, ω) and let $\Gamma_i, \Gamma'_i \in \mathcal{G}_{\mathbb{S}^2}$, $i = 1, \dots, m$. Then the subsets $\Gamma_1 \times \cdots \times \Gamma_m$ and $\phi(\Gamma'_1 \times \cdots \times \Gamma'_m)$ have a non-empty intersection for every symplectomorphism ϕ of M .*

Proof. Note that a direct product of stems is a stem. Hence it suffices to verify that every $\Gamma \in \mathcal{G}_{\mathbb{S}^2}$ is a stem. Let U_1, \dots, U_d be the connected components of $\mathbb{S}^2 \setminus \Gamma$.

Take smooth functions H_1, \dots, H_d as follows: H_i vanishes on $\mathbb{S}^2 \setminus U_i$ and H_i is strictly positive on U_i . The existence of such H_1, \dots, H_d follows easily from the fact that any closed subset of \mathbb{R}^2 is the zero-level set of some smooth real-valued function on \mathbb{R}^2 (see e.g. [35], Lemma 1.4.13).

Put $\mathcal{A} = \text{Span}_{\mathbb{R}}(H_1, \dots, H_d)$. Clearly \mathcal{A} is Poisson-commutative and $\Gamma = \Phi_{\mathcal{A}}^{-1}(0)$ is its fiber. All other fibers are closed subsets of one of the U_i 's, and hence are displaceable. Therefore Γ is a stem and the result follows from Theorem 2.4. \square

Here is another corollary of Theorem 2.1. Let \mathbb{T}^2 be a torus with coordinates $p, q \in \mathbb{R}/\mathbb{Z}$ and the symplectic form $dp \wedge dq$. Equip $M \times \mathbb{T}^2$ with the product symplectic structure and assume that the resulting symplectic manifold is strongly semi-positive and rational. Denote by S a meridian $p = \text{const}$ of \mathbb{T}^2 .

Corollary 2.6. *Assume $X \subset M$ is a stem. Then $X \times S \subset M \times \mathbb{T}^2$ is non-displaceable.*

Proof. Let $\mathcal{A} \subset C^\infty(M)$ be a finite-dimensional Poisson-commutative subspace such that the stem X is its only non-displaceable fiber. Lift to $M \times \mathbb{T}^2$ the functions on M that belong to \mathcal{A} as well as the functions $\sin 2\pi p, \cos 2\pi p$ on \mathbb{T}^2 . All these lifts

together span a Poisson-commutative subspace $\mathcal{A}' \subset C^\infty(M \times \mathbb{T}^2)$ such that each of its fibers is a direct product of a fiber of \mathcal{A} and a meridian of \mathbb{T}^2 .

Theorem 2.1 says that \mathcal{A}' must have a non-displaceable fiber Y . Since X is the only non-displaceable fiber of \mathcal{A} , the fiber Y has to have the form $Y = X \times S'$ for some meridian S' of \mathbb{T}^2 . But any two meridians of \mathbb{T}^2 can be mapped into each other by a symplectomorphism of \mathbb{T}^2 – hence the products of these meridians with X can be mapped into each other by a symplectomorphism of $M \times \mathbb{T}^2$. Thus if $X \times S'$ is non-displaceable, then $X \times S$ has to be non-displaceable as well. \square

3. Quasi-states and quasi-measures

Write $C(M)$ for the commutative (with respect to multiplication) Banach algebra of all continuous functions on M endowed with the uniform norm. For a function $F \in C(M)$ denote by \mathcal{A}_F the uniform closure of the set of functions of the form $p \circ F$, where p is a real polynomial. A (not necessarily linear) functional $\zeta : C(M) \rightarrow \mathbb{R}$ is called a *quasi-state* [3], if it satisfies the following axioms:

Quasi-linearity. ζ is linear on \mathcal{A}_F for every $F \in C(M)$ (in particular ζ is homogeneous).

Monotonicity. $\zeta(F) \leq \zeta(G)$ for $F \leq G$.

Normalization. $\zeta(1) = 1$.

A quasi-state is called *symplectic*, if it has the following additional properties:

Strong quasi-additivity. $\zeta(F + G) = \zeta(F) + \zeta(G)$ for all smooth functions F, G which commute with respect to the Poisson bracket: $\{F, G\} = 0$.

Vanishing. $\zeta(F) = 0$, provided $\text{supp } F$ is displaceable.

Symplectic invariance. $\zeta(F) = \zeta(F \circ f)$ for every symplectic diffeomorphism $f \in \text{Symp}_0(M)$ (here $\text{Symp}_0(M)$ stands for the identity component of the group $\text{Symp}(M)$ of symplectomorphisms).

Note that strong quasi-additivity together with homogeneity yields quasi-linearity. Indeed, if F is smooth, $\{p_1 \circ F, p_2 \circ F\} = 0$ for every pair of polynomials p_1 and p_2 . Observing that ζ is continuous in the uniform topology because of the monotonicity and normalization axioms, one can easily extend the result for a general continuous F .

Theorem 3.1. *Suppose M is one of the following symplectic manifolds: \mathbb{CP}^n , a complex Grassmannian, $\mathbb{CP}^{n_1} \times \cdots \times \mathbb{CP}^{n_k}$ with a monotone product symplectic structure, the monotone symplectic blow-up of \mathbb{CP}^2 at one point. Then $C(M)$ admits a symplectic quasi-state.*

In [3] Aarnes proved a generalized Riesz representation theorem which associates to each quasi-state ζ a *quasi-measure* τ_ζ , that is a “measure” which is finitely additive but not necessarily sub-additive. More precisely, denote by \mathcal{S} the collection of all subsets of M which are *either open or closed*. A *quasi-measure* (recently called a *topological measure* in the literature) on M is a $[0, 1]$ -valued set-function τ on \mathcal{S} such that

- 1) $\tau(M) = 1$;
- 2) $X_1 \subset X_2 \Rightarrow \tau(X_1) \leq \tau(X_2)$ for all $X_1, X_2 \in \mathcal{S}$;
- 3) $\tau(X_1 \sqcup \dots \sqcup X_k) = \tau(X_1) + \dots + \tau(X_k)$ for all $X_1, \dots, X_k \in \mathcal{S}$ with $X_1 \sqcup \dots \sqcup X_k \in \mathcal{S}$;
- 4) for every open subset X one has $\tau(X) = \sup \tau(A)$, where the supremum is taken over all closed subsets $A \subset X$.

The relation between a quasi-state ζ and the corresponding quasi-measure τ_ζ is the following ([3]). Given a closed $X \subset M$, consider the set \mathcal{F}_X of smooth functions $M \rightarrow [0, 1]$ which are identically equal to 1 on X . A quasi-state ζ is bounded on \mathcal{F}_X by 0 and 1 and therefore one can define

$$\tau_\zeta(X) := \inf_{F \in \mathcal{F}_X} \zeta(F). \quad (2)$$

Intuitively, $\tau_\zeta(X)$ is the “value” of the functional ζ on the (discontinuous) characteristic function of X . For an open subset Y put $\tau_\zeta(Y) = 1 - \tau_\zeta(M \setminus Y)$.

Lemma 3.2. *Assume a closed connected symplectic manifold M admits a symplectic quasi-state ζ . Denote by τ the corresponding quasi-measure. Then $\tau(X) = 1$ for every stem $X \subset M$.*

Proof. Let $\mathcal{A} \subset C^\infty(M)$ be a finitely generated Poisson-commutative subspace. Denote by $\Delta \subset \mathcal{A}^*$ the image of the moment map $\Phi_{\mathcal{A}}$. Write $C_0^\infty(\mathcal{A}^*)$ for the space of all smooth compactly supported functions on \mathcal{A}^* . Note that the functional

$$I: C_0^\infty(\mathcal{A}^*) \rightarrow \mathbb{R}, \quad G \mapsto \zeta(\Phi_{\mathcal{A}}^* G),$$

is a positive distribution¹ (use the strong quasi-additivity and the monotonicity axioms of ζ). Hence it defines a measure σ on \mathcal{A}^* so that $I(G) = \int_{\mathcal{A}^*} G d\sigma$ (see e.g. [22], Ch. 2, Sec. 2). By the normalization axiom, σ is a probability measure. Obviously, $\text{supp } \sigma \subset \Delta$. The vanishing axiom yields that if $\Phi_{\mathcal{A}}^{-1}(p)$ is displaceable for some $p \in \Delta$, then $p \notin \text{supp } \sigma$. Thus, if $X = \Phi_{\mathcal{A}}^{-1}(p_0)$ is a stem associated to \mathcal{A} , the measure σ must be the Dirac measure at p_0 . Using this and considering in the definition of $\tau(X)$ the functions $F \in \mathcal{F}_X$ of the form $F = \Phi_{\mathcal{A}}^* G$, $G \in C_0^\infty(\mathcal{A}^*)$, one readily gets $\tau(X) = 1$. \square

¹Recall that a distribution (that is a continuous linear functional) on $C_0^\infty(\mathbb{R}^N)$ is called *positive* if it takes non-negative values on non-negative functions.

The proof of the lemma shows that if τ is a quasi-measure defined by a symplectic quasi-state, and $\mathcal{A} \subset C^\infty(M)$ is a finitely generated Poisson-commutative subspace, the push-forward of τ by the moment map $\Phi_{\mathcal{A}}$ is a *genuine measure* on the image of $\Phi_{\mathcal{A}}$. In case when τ comes from a quasi-state which is not strongly quasi-additive (and thus not symplectic), this may no longer be true and moreover such a quasi-measure may vanish on a stem – see Remark 8.4.

Proof of Theorem 2.4 (assuming Theorem 3.1). According to Theorem 3.1, any M mentioned in the hypothesis of Theorem 2.4 admits a symplectic quasi-state. Let τ be the corresponding quasi-measure and let $X, Y \subset M$ be stems. Lemma 3.2 implies that $\tau(X) = \tau(Y) = 1$. If X and Y do not intersect, we have $\tau(X \cup Y) = \tau(X) + \tau(Y) = 1 + 1 = 2$, and we get a contradiction with $\tau(X \cup Y) \leq \tau(M) = 1$. \square

4. What happens on more general symplectic manifolds?

Let $\zeta : C(M) \rightarrow \mathbb{R}$ be a (not necessarily quasi-linear) functional which satisfies monotonicity, normalization, vanishing and invariance axioms from the previous section. Assume that it has two additional properties:

Partial additivity. If $F_1, F_2 \in C^\infty(M)$, $\{F_1, F_2\} = 0$ and the support of F_2 is displaceable, then $\zeta(F_1 + F_2) = \zeta(F_1)$.

Semi-homogeneity. $\zeta(\lambda F) = \lambda \zeta(F)$ for any F and any $\lambda \in \mathbb{R}_{\geq 0}$.

We call ζ a *partial symplectic quasi-state*.

Theorem 4.1. *Let M be a strongly semi-positive and rational closed connected symplectic manifold. Then $C(M)$ admits a partial symplectic quasi-state.*

Theorem 4.1 will be proved in Section 7.

Proof of Theorem 2.1 (assuming Theorem 4.1). Let ζ be a partial symplectic quasi-state. Assume on the contrary that all fibers of \mathcal{A} are displaceable. Choose an open covering $U := \{U_1, \dots, U_d\}$ of the image Δ of the moment map $\Phi_{\mathcal{A}}$ so that the preimages $\Phi^{-1}(U_i)$ are displaceable. Let ρ_1, \dots, ρ_d be a partition of unity associated to U , that is $\text{supp } \rho_i \subset U_i$ and $\sum_{i=1}^d \rho_i|_{\Delta} = 1$. Note that $\zeta(\Phi^* \rho_i) = 0$ by vanishing property. Using the normalization and the partial additivity, we get

$$1 = \zeta(1) = \zeta\left(\sum_{i=1}^d \Phi^* \rho_i\right) = \sum_{i=1}^d \zeta(\Phi^* \rho_i) = 0,$$

and we get a contradiction.

A similar argument shows that, if any fiber of \mathcal{A} has a finite number of connected components, then at least one connected component of some fiber of \mathcal{A} has to be non-displaceable. \square

5. Spectral numbers – review

We review a few basic facts about the spectral numbers of Hamiltonian diffeomorphisms introduced by Yong-Geun Oh [36] (see also [42], [41] for earlier versions of this theory). For the precise definitions and further details see [36], [20] and [34]. We assume here that M is strongly semi-positive and rational. The strong semi-positivity of M is needed to guarantee that the moduli spaces of pseudo-holomorphic curves involved in the definitions of Floer and quantum homology and the isomorphism between them are well-behaved. In view of the developments [21], [30], [31], [32], concerning Floer theory for general symplectic manifolds, it is likely that the strong semi-positivity of M is not essential for the existence of spectral numbers. The assumption that M is rational is needed to guarantee the spectrality property below², though it is likely that eventually this assumption will also be removed, see [37].

By $\overline{\text{spec}}(H)$ we denote the action spectrum of a Hamiltonian H . Recall that it is the set of critical values of the action functional defined by H on the universal cover of the space of free contractible loops in M .

A time-dependent Hamiltonian $H : M \times \mathbb{S}^1 \rightarrow \mathbb{R}$ is called *normalized* if

$$\int_M H(\cdot, t) \omega^n = 0 \quad \text{for all } t \in \mathbb{S}^1.$$

It turns out that $\overline{\text{spec}}(H_1) = \overline{\text{spec}}(H_2)$ for any normalized H_1, H_2 generating the same element $\phi \in \widehat{\text{Ham}}(M)$. Thus one can define $\text{spec}(\phi)$ for any $\phi \in \widehat{\text{Ham}}(M)$ as $\overline{\text{spec}}(H)$ for any normalized H generating ϕ .

Denote by $QH_*(M)$ the quantum homology ring of M (with coefficients in \mathbb{C}) and by $*$ the product in that ring. The fundamental class $[M]$ is the unit in the ring. To each non-zero quantum homology class $a \in QH_*(M)$ and each time-dependent Hamiltonian $H : M \times \mathbb{S}^1 \rightarrow \mathbb{R}$ one can associate a *spectral number* $\bar{c}(a, H)$. Spectral numbers have the following properties which are relevant for us:

Spectrality. $\bar{c}(a, H) \in \overline{\text{spec}}(H)$.

Shift property. $\bar{c}(a, H + \lambda(t)) = \bar{c}(a, H) + \int_0^1 \lambda(t) dt$ for any Hamiltonian H and function $\lambda : \mathbb{S}^1 \rightarrow \mathbb{R}$.

Monotonicity. If $H_1 \leq H_2$, then $\bar{c}(a, H_1) \leq \bar{c}(a, H_2)$.

²For the same reason the rationality assumption should be added to the results 2.5.3, 2.5.4, 2.6.1 and to part 4 of 2.4.2 in [19] which involve the spectral numbers.

Lipschitz property. The map $H \mapsto \bar{c}(a, H)$ is Lipschitz on the space of (time-dependent) Hamiltonians $H: M \times \mathbb{S}^1 \rightarrow \mathbb{R}$ with respect to the C^0 -norm.

Symplectic invariance. $\bar{c}(a, \phi^*H) = \bar{c}(a, H)$ for every $\phi \in \text{Symp}_0(M)$, $H \in C^\infty(M)$.

Normalization. $\bar{c}(a, 0) = 0$ for every even-dimensional singular homology class $a \in H_*(M, \mathbb{C})$.

Homotopy invariance. $\bar{c}(a, H_1) = \bar{c}(a, H_2)$ for any normalized H_1, H_2 generating the same $\phi \in \widetilde{\text{Ham}}(M)$. Thus one can define $c(a, \phi)$ for any $\phi \in \widetilde{\text{Ham}}(M)$ as $\bar{c}(a, H)$ for any normalized H generating ϕ . Note that $c(a, \phi) \in \text{spec}(\phi)$.

Triangle inequality. $c(a * b, \phi\psi) \leq c(a, \phi) + c(b, \psi)$.

6. From a Calabi quasi-morphism to a symplectic quasi-state

In this section we prove Theorem 3.1. Assume that M is spherically monotone. In this case the Novikov ring of M is a field of complex Laurent series in one variable. The even-degree part $QH_{\text{ev}}(M)$ of $QH_*(M)$ is a commutative algebra over this field. Assume that the algebra $QH_{\text{ev}}(M)$ is semi-simple in the sense of [20] – this holds, for instance, if M is one of the symplectic manifolds listed in the statement of Theorem 3.1: the standard \mathbb{CP}^n , a complex Grassmannian, $\mathbb{CP}^{n_1} \times \cdots \times \mathbb{CP}^{n_k}$ with a monotone product symplectic structure, the monotone symplectic blow-up of \mathbb{CP}^2 at one point. Denote by $\text{vol}(M^{2n}) := \int_M \omega^n$ the total symplectic volume of M . The main result of [20] states that for a suitable choice of an idempotent $a \in QH_{\text{ev}}(M)$, the function $\mu: \widetilde{\text{Ham}}(M) \rightarrow \mathbb{R}$ given by

$$\mu(\phi) := -\text{vol}(M) \cdot \lim_{k \rightarrow +\infty} c(a, \phi^k)/k \quad (3)$$

is a *homogeneous quasi-morphism* on the group $\widetilde{\text{Ham}}(M)$ with a number of additional properties. More precisely, the following holds:

Quasi-additivity. There exists $K > 0$, which depends only on μ , so that

$$|\mu(\phi\psi) - \mu(\phi) - \mu(\psi)| \leq K \quad \text{for all elements } \phi, \psi \in \widetilde{\text{Ham}}(M).$$

Homogeneity. $\mu(\phi^m) = m\mu(\phi)$ for each ϕ and each $m \in \mathbb{Z}$.

To proceed with properties of μ we need the following notations. For a (time-dependent) Hamiltonian H on M write ϕ_H for the element of $\widetilde{\text{Ham}}(M)$ represented the identity-based path in $\text{Ham}(M)$ given by the $[0, 1]$ -time Hamiltonian flow generated by H . For an open $U \subset M$ denote by $\widetilde{\text{Ham}}(U) \subset \widetilde{\text{Ham}}(M)$ the subgroup of elements

generated by Hamiltonians $H(x, t) = H_t(x)$ with $\text{supp } H_t \subset U$ for all $t \in \mathbb{S}^1$. Denote by $\text{Cal}: \widetilde{\text{Ham}}(U) \rightarrow \mathbb{R}$ the classical Calabi homomorphism: $\text{Cal}(\phi_H) := \int_0^1 \int_U H_t \omega^n dt$, where $\text{supp } H_t \subset U$ for all t .

Calabi property. If $U \subset M$ is open and displaceable, then the restriction of μ on $\widetilde{\text{Ham}}(U) \subseteq \widetilde{\text{Ham}}(M)$ is the Calabi homomorphism $\text{Cal}: \widetilde{\text{Ham}}(U) \rightarrow \mathbb{R}$.

Lipschitz property. $|\mu(\phi_F) - \mu(\phi_H)| \leq \text{vol}(M) \cdot \|F - H\|_{C^0}$.

Define now $\zeta: C^\infty(M) \rightarrow \mathbb{R}$ by

$$\zeta(F) = \frac{\int_M F \omega^n}{\text{vol}(M)} - \frac{\mu(\phi_F)}{\text{vol}(M)} = \lim_{k \rightarrow +\infty} \frac{\bar{c}(a, kF)}{k}. \quad (4)$$

Using the Lipschitz property of μ , we readily extend ζ to a functional on $C(M)$. Let us check that ζ satisfies the axioms of a symplectic quasi-state. Since $\mu(\mathbf{1}) = 0$ in view of homogeneity of μ , we get the normalization axiom. Invariance and monotonicity of spectral invariants yield the invariance and the monotonicity axioms respectively. The Calabi property of μ yields the vanishing axiom. To check the strong quasi-additivity axiom, note that if $\{F, G\} = 0$ the diffeomorphisms ϕ_F and ϕ_G commute and $\phi_{F+G} = \phi_F \phi_G$. The desired result follows from the following general fact (which is an easy exercise): restriction of a homogeneous quasi-morphism to any abelian subgroup is a homomorphism. This completes the proof of Theorem 3.1. \square

7. A partial symplectic quasi-state

Let M be a closed strongly semi-positive and rational symplectic manifold. For an element $\phi \in \widetilde{\text{Ham}}(M)$ write for brevity $c(\phi) = c([M], \phi)$ and, as above, define μ as a homogenization of $c([M], \cdot)$:

$$\mu(\phi) := -\text{vol}(M) \cdot \lim_{k \rightarrow +\infty} c(\phi^k)/k. \quad (5)$$

It is easy to see that μ is *not* a quasi-morphism already when M is the 2-torus – see the discussion following Question 8.7 in Section 8. Moreover, a similar argument actually shows that for any (strongly semi-positive, rational) symplectic direct product $M \times \mathbb{T}^{2n}$ the homogenization of *any* spectral number $c(a, \cdot)$ cannot be a quasi-morphism.

In spite of this, μ has a number of nice properties which will enable us to show that the functional ζ given by (4) is a partial symplectic quasi-state. We shall need the following definition. Given a displaceable open set $U \subset M$, each $\phi \in \widetilde{\text{Ham}}(M)$ can be represented as a product of elements of the form $\psi \theta \psi^{-1}$ with $\theta \in \widetilde{\text{Ham}}(U)$. This follows from Banyaga's fragmentation lemma [9]. Denote by $\|\phi\|_U$ the minimal number of factors in such a product.

Theorem 7.1. *Suppose M is strongly semi-positive and rational. The functional $\mu: \widetilde{\text{Ham}}(M) \rightarrow \mathbb{R}$, given by (5), is well defined and has the following properties:*

Controlled quasi-additivity. *Given a displaceable open subset U of M , there exists a constant K , depending only on U , so that*

$$|\mu(\phi\psi) - \mu(\phi) - \mu(\psi)| \leq K \min\{\|\phi\|_U, \|\psi\|_U\}$$

for any $\phi, \psi \in \widetilde{\text{Ham}}(M)$.

Semi-homogeneity. $\mu(\phi^m) = m\mu(\phi)$ for any ϕ and any $m \in \mathbb{Z}_{\geq 0}$.

In addition it has the Calabi and Lipschitz properties defined in the previous section.

Postponing the proof, we first prove Theorem 4.1.

Proof of Theorem 4.1 (assuming Theorem 7.1). Define a functional $\zeta: C^\infty(M) \rightarrow \mathbb{R}$ by formula (4). We claim that ζ is a partial symplectic quasi-state. Arguing exactly as in the end of the previous section, we check the monotonicity, vanishing, normalization and invariance axioms. Semi-homogeneity of μ yields that $\zeta(\lambda F) = \lambda\zeta(F)$ for $\lambda \in \mathbb{N}$ and all smooth F . As a logical consequence we get that the same holds for all positive rational λ . Using the Lipschitz property of μ , we pass to the limit and get this for all positive λ , thus establishing the semi-homogeneity axiom.

It remains to verify the partial additivity axiom. Assume that $\{F, H\} = 0$ and $\text{supp } H$ is contained in a displaceable open subset U . Note that $\|\phi_H^k\|_U = 1$ for all $k \in \mathbb{N}$. Since ϕ_F and ϕ_H commute we have (using controlled quasi-additivity of μ)

$$\mu(\phi_F \phi_H) = \frac{1}{k} \mu((\phi_F \phi_H)^k) = \frac{1}{k} (k\mu(\phi_F) + k\mu(\phi_H) + r_k),$$

where $|r_k| \leq K$. Taking the limit as $k \rightarrow +\infty$ we get that

$$\mu(\phi_F \phi_H) = \mu(\phi_F) + \mu(\phi_H) = \mu(\phi_F) + \int_M H \omega^n,$$

where the last equality follows from the Calabi property and the fact that $\text{supp } H$ is displaceable. Further, $\phi_{F+H} = \phi_F \phi_H$ since F and H commute. Substituting this into the definition of ζ , we get $\zeta(F + H) = \zeta(F)$, as required. This completes the proof. \square

Proof of Theorem 7.1. The proof is divided into a sequence of lemmas. In what follows we fix an open displaceable subset U of M and write for simplicity $\|\phi\| := \|\phi\|_U$.

Lemma 7.2. *There exists a constant $C \geq 0$ such that for any $\phi \in \widetilde{\text{Ham}}(U)$*

$$0 \leq c(\phi) + c(\phi^{-1}) \leq C.$$

Proof. Suppose $f \in \widetilde{\text{Ham}}(M)$ is a lift of a Hamiltonian diffeomorphism displacing U . Then the “shift of the spectrum” trick of Y. Ostrover [38] (cf. [19], [20]) yields that for a certain $E \in \mathbb{R}$, depending on ϕ ,

$$c(f\phi) = c(f) + E,$$

$$c(f\phi^{-1}) = c(f) - E.$$

Here we use the spectrality and the Lipschitz property of c . The signs in the formulae above comply with the sign convention as in [20]. Thus

$$c(f\phi) + c(f\phi^{-1}) = 2c(f).$$

In view of the triangle inequality³,

$$0 \leq c(\phi) + c(\phi^{-1}),$$

$$c(\phi) \leq c(f\phi) + c(f^{-1}),$$

$$c(\phi^{-1}) \leq c(f\phi^{-1}) + c(f^{-1}).$$

Hence

$$0 \leq c(\phi) + c(\phi^{-1}) \leq c(f\phi) + c(f\phi^{-1}) + 2c(f^{-1}) \leq 2c(f) + 2c(f^{-1}).$$

Set $C := 2c(f) + 2c(f^{-1})$. This is a non-negative number because of the triangle inequality. The lemma is proved. \square

Lemma 7.3. *For any $\phi \in \widetilde{\text{Ham}}(U)$ and any $\psi \in \widetilde{\text{Ham}}(M)$ one has*

$$c(\phi) + c(\psi) - C \leq c(\phi\psi) \leq c(\phi) + c(\psi),$$

where C is the constant from the previous lemma.

Proof. The second inequality is just the triangle inequality. To obtain the first one, observe that the triangle inequality yields

$$c(\psi) \leq c(\phi\psi) + c(\phi^{-1}).$$

³Note that, since $[M]$ is the unit in $QH_*(M)$, the triangle inequality for $c(\cdot) = c([M], \cdot)$ has the form $c(\phi\psi) \leq c(\phi) + c(\psi)$.

This, along with the previous lemma, implies

$$c(\phi\psi) \geq c(\psi) - c(\phi^{-1}) \geq c(\psi) + c(\phi) - C. \quad \square$$

Using a straightforward inductive argument one generalizes the lemma above as follows. Take any $\phi_1, \dots, \phi_m, \psi \in \widehat{\text{Ham}}(M)$ with $\|\phi_i\| = 1$ for all i . Then

$$|c(\phi_1 \dots \phi_m \psi) - \sum_{i=1}^m c(\phi_i) - c(\psi)| \leq mC. \quad (6)$$

This formula (with $\psi = \mathbf{1}$) yields

$$|c((\phi_1 \dots \phi_m)^l) - l \sum_{i=1}^m c(\phi_i)| \leq lmC. \quad (7)$$

Take any $\phi \in \text{Ham}(M)$ and represent it as $\phi = \phi_1 \dots \phi_m$ with $\|\phi_i\| = 1$ for all i . Formula (7) implies that for some large enough positive E (depending on ϕ) the sequence $\{c(\phi^l) + El\}_{l \in \mathbb{N}}$ is non-negative. On the other hand, because of the triangle inequality, this sequence is sub-additive. This yields the existence and finiteness of $\lim_{l \rightarrow +\infty} (c(\phi^l) + El)/l$ and, accordingly, of $\lim_{l \rightarrow +\infty} c(\phi^l)/l$. Therefore the function μ is well defined. The semi-homogeneity of μ follows immediately from its definition. The proof of the Lipschitz property of μ simply repeats the proof of a similar Proposition 3.5 in [20].

Now we are going to check controlled quasi-additivity of μ . Assume without loss of generality that the volume of M equals 1, so that

$$\mu(\phi) = - \lim_{k \rightarrow +\infty} c(\phi^k)/k.$$

We claim that for $\phi, \psi \neq \mathbf{1}$,

$$|\mu(\phi\psi) - \mu(\phi) - \mu(\psi)| \leq 2C \cdot \min(2\|\phi\| - 1, 2\|\psi\| - 1). \quad (8)$$

The controlled quasi-additivity follows immediately from (8) if one sets $K := 4C$. We prove the claim by induction on $m := \min(\|\phi\|, \|\psi\|)$.

Induction basis $m = 1$. Assume without loss of generality that $\|\phi\| = 1$. Note that

$$(\phi\psi)^k = \left(\prod_{i=0}^{k-1} \psi^i \phi \psi^{-i} \right) \cdot \psi^k.$$

Applying (6) and using the conjugation invariance of $c(\cdot)$ we get

$$|c((\phi\psi)^k) - kc(\phi) - c(\psi^k)| \leq Ck.$$

Combining this with inequality

$$|c(\phi^k) - kc(\phi)| \leq Ck,$$

which follows from (7), dividing by k and passing to the limit as $k \rightarrow +\infty$ we get the desired result.

Induction step $m \mapsto m + 1$. Assume without loss of generality that $\|\phi\| = m + 1$. Then ϕ can be decomposed as $\phi = \phi_m \phi_1$ where $\|\phi_m\| = m$ and $\|\phi_1\| = 1$. Using the induction assumption we have

$$|\mu(\phi_m \phi_1 \psi) - \mu(\phi_m) - \mu(\phi_1 \psi)| \leq 2C(2m - 1),$$

$$|\mu(\phi_1 \psi) - \mu(\phi_1) - \mu(\psi)| \leq 2C$$

and

$$|\mu(\phi_1) + \mu(\phi_m) - \mu(\phi_m \phi_1)| \leq 2C.$$

Adding up these inequalities we get that

$$|\mu(\phi \psi) - \mu(\phi) - \mu(\psi)| \leq 2C(2m + 1),$$

as desired. This completes the proof of the claim and of the controlled quasi-additivity.

Finally, the proof of the Calabi property of μ virtually repeats the proof of a similar Proposition 3.3 in [20]. The symplectic invariance of μ follows from the symplectic invariance of the spectral numbers. This finishes the proof of Theorem 7.1. \square

8. Symplectic quasi-states on surfaces

Symplectic quasi-measures. A quasi-measure on a symplectic manifold M is called *symplectic* if it is $\text{Symp}_0(M)$ -invariant and vanishes on displaceable closed subsets.

Here we discuss this notion in the case when M is a closed surface equipped with an area form. According to the general construction from [3], any quasi-measure τ gives rise to a quasi-state ζ_τ . Roughly speaking, the definition of ζ_τ is as follows. For a function $F \in C(M)$ define a measure σ_F on \mathbb{R} by its values on intervals

$$\sigma_F([a; b)) := \tau(\{F \geq a\}) - \tau(\{F \geq b\}),$$

and put $\zeta_\tau(F) := \int_{\mathbb{R}} s \cdot d\sigma_F(s)$. If τ is a symplectic quasi-measure, the quasi-state ζ_τ automatically satisfies all the axioms of a symplectic quasi-state except, possibly, strong quasi-additivity stating that ζ_τ is linear on the centralizer (with respect to the Poisson bracket) of any smooth function F .

Theorem 8.1. *On a closed surface, the strong quasi-additivity axiom follows from the usual quasi-linearity. In particular, any symplectic quasi-measure gives rise to a symplectic quasi-state.*

Proof. Let F, G be a pair of C^∞ -smooth functions on a closed surface M with $\{F, G\} = 0$. The Poisson-commutativity can be interpreted as follows: the differential of the map

$$\Phi: M \rightarrow \mathbb{R}^2, \quad x \mapsto (F(x), G(x)),$$

has rank ≤ 1 for at each point $x \in M$. Put $\Delta := \text{Image}(\Phi)$. Denote by d_c and d_h the covering dimension and the Hausdorff dimension of Δ respectively. It is a standard fact of dimension theory that $d_c \leq d_h$, see e.g. the proof of Theorem (6.2.10) in Edgar's book [17]. Further, $d_h \leq 1$. This follows from a result of Dubovickii [16] which is a partial case of a more general theorem of Sard [40]. Therefore $d_c \leq 1$.

Define a quasi-state η on $C(\Delta)$ by $\eta(H) := \zeta_\tau(\Phi^*H)$. The Wheeler–Shakhmatov Theorem [26], [44] implies that every quasi-state on a normal topological space (and hence on any metric space) of covering dimension ≤ 1 is linear. Hence η is linear. Applying this result to the restriction of the coordinate functions on \mathbb{R}^2 to Δ we get that

$$\zeta_\tau(F + G) = \zeta_\tau(F) + \zeta_\tau(G), \quad (9)$$

as required. \square

Note that in the proof above we used that the functions F and G are infinitely smooth in order to deduce inequality $d_h \leq 1$ from the Dubovickii–Sard theorem.

Problem 8.2. Extend identity (9) to Poisson-commuting functions of finite smoothness.

For instance, one can try to find a uniform approximation of the pair (F, G) by a Poisson-commuting pair of C^∞ -functions.

Remark 8.3. In contrast to the case of surfaces, the only known to us example of a symplectic quasi-measure on higher-dimensional manifolds comes from the “Floer-homological” symplectic quasi-state whose existence is established in Theorem 3.1.

Remark 8.4. In the case $\dim M > 2$ D. Grubb [25] constructed examples of quasi-states which are not strongly quasi-additive. The quasi-measures in the examples of Grubb do not necessarily vanish on displaceable sets (and hence are not symplectic) but may vanish on a stem. The push-forward of such a quasi-measure by a moment map of a finite-dimensional Poisson-commutative subspace of $C^\infty(M)$ is not necessarily a measure.

Now we address a question about existence and uniqueness of symplectic quasi-states and quasi-measures on surfaces.

The 2-sphere. The group $\text{Ham}(\mathbb{S}^2)$ admits a Calabi quasi-morphism [20], which in accordance with our discussion in Section 6 yields existence of a symplectic quasi-state and a symplectic quasi-measure on $C(\mathbb{S}^2)$. Theorem 5.2 in [20] shows that any two Calabi quasi-morphisms on $\text{Ham}(\mathbb{S}^2)$ coincide on the set of elements generated by time-independent Hamiltonians. The same argument proves that any two symplectic quasi-states coincide on the set of smooth Morse functions on \mathbb{S}^2 . Hence $C(\mathbb{S}^2)$ carries unique symplectic quasi-state and quasi-measure.

An explicit calculation presented in [20] shows that the restriction of this symplectic quasi-state, say ζ , to the subalgebra $\mathcal{A}_F \subset C(M)$ generated by a single Morse function $F \in C^\infty(\mathbb{S}^2)$ is multiplicative: $\zeta(GH) = \zeta(G)\zeta(H)$ for all $G, H \in \mathcal{A}_F$. Using this along with the continuity of ζ one can easily show that ζ is multiplicative on \mathcal{A}_F for any $F \in C(\mathbb{S}^2)$. Now a theorem of Aarnes [4] yields that *the corresponding quasi-measure is simple: it takes values 0 and 1 only*. It is unclear whether this phenomenon persists in higher dimensions, thus we pose the next question.

Question 8.5. Consider the “Floer-homological” symplectic quasi-state ζ on the complex projective space \mathbb{CP}^n constructed in Theorem 3.1. Is it multiplicative when $n \geq 2$? In particular, is it true that $\zeta(F^2) = \zeta(F)^2$ for all continuous functions F on \mathbb{CP}^n ?

For completeness, we present the formula for ζ on \mathcal{A}_F , where F is a Morse function, obtained in [20]. Assume that the total area of the sphere equals 1. One shows that there exists unique (may be, singular) connected component of a level set of F , say γ , so that the area of any connected component of $\mathbb{S}^2 \setminus \gamma$ is $\leq \frac{1}{2}$. Note that every $G \in \mathcal{A}_F$ is constant on connected components of level sets of F . It turns out that

$$\zeta(G) = G(\gamma).$$

A symplectic quasi-measure τ corresponding to ζ can be described as follows (we thank D. Grubb who pointed this out to us). A set $A \subset \mathbb{S}^2$ is called *solid* if both A and $\mathbb{S}^2 \setminus A$ are connected. According to the results of Aarnes [5] and Aarnes and Rustad [6], the quasi-measure τ is completely defined by the following condition: for a closed solid set $A \subset \mathbb{S}^2$ one has $\tau(A) = 1$ if the Lebesgue measure of A is greater or equal to $1/2$ and $\tau(A) = 0$ otherwise.

The 2-torus. Existence of a symplectic quasi-measure, say τ , in this case follows from a work of Grubb (see Theorem 32 of [24], where the auxiliary quasi-measures used in the definition of τ are taken to be the standard Lebesgue measure). The value of τ on any 2-dimensional smooth connected closed submanifold with boundary $W \subset \mathbb{T}^2$ can be calculated as follows (see Theorem 32 of [24]). If W is contractible

in \mathbb{T}^2 we have $\tau(W) = 0$. If W is non-contractible and ∂W has $k \geq 0$ contractible connected components that bound pair-wise disjoint discs D_1, \dots, D_k (in case $k = 0$ there are no discs), then

$$\tau(W) = \text{Area}(W) + \sum_{i=1}^k \text{Area}(D_i).$$

Remark 8.6. It would be interesting to describe all symplectic quasi-measures on the 2-torus; for more examples of such quasi-measures see a recent preprint [28] by Knudsen.

By Theorem 8.1 above, a symplectic quasi-measure on \mathbb{T}^2 gives rise to a symplectic quasi-state.

Question 8.7. Is Grubb's symplectic quasi-measure associated to a quasi-morphism on $\text{Ham}(\mathbb{T}^2)$?

Such a quasi-morphism, if exists, cannot come from spectral numbers described in Section 5. To see this denote by τ any symplectic quasi-measure on \mathbb{T}^2 . Introduce coordinates $(p, q) \bmod 1$ on \mathbb{T}^2 so that the symplectic form is given by $dp \wedge dq$. Let $\alpha = \{p = 0\}$ and $\beta = \{p = 1/2\}$ be two meridians dividing the torus into two open annuli $A = \{p \in (0; 1/2)\}$ and $B = \{p \in (1/2; 1)\}$ of equal area. Note that

$$\tau(A) + \tau(B) + \tau(\alpha) + \tau(\beta) = 1.$$

The Symp_0 -invariance of τ yields $\tau(A) = \tau(B)$ as well as $\tau(\alpha) = \tau(\beta) = 0$ (the torus contains an arbitrarily large number of pair-wise disjoint symplectic shifts of a meridian). Thus, putting $A' = A \cup \alpha \cup \beta$, we have $\tau(A') = 1/2$. On the other hand, choose a sequence of cut-off functions $F_i(p)$ approximating the characteristic function of A' so that the only critical values of F_i are 0 and 1. The key feature of the Hamiltonian flow generated by F_i is that its only *contractible* closed orbits are the critical points, hence the action spectrum $\overline{\text{spec}}(tF_i)$ equals $\{0; t\}$. Hence, using continuous dependence of spectral numbers on the Hamiltonian, we get that for every homology class $a \in H_*(\mathbb{T}^2)$, we have either $\bar{c}(a, tF_i) = 0$ or $\bar{c}(a, tF_i) = t$. Substituting this into the right term of formula (4), we get that $\zeta(F_i)$, if well defined, must be either 0 or 1 and hence $\tau(A') \neq 1/2$. This contradiction proves the claim.

Note that Hamiltonians F_i above have a wealth of non-contractible periodic orbits. In principle, the symplectic field theory [18], or, more precisely, its version called branched Floer homology (work in progress by V. Ginzburg and E. Kerman) which deals with Hamiltonian diffeomorphisms, may lead to a generalization of spectral numbers which takes into account non-contractible orbits as well. It would be interesting to understand whether this path leads to a symplectic quasi-measure.

9. Digging out a stem

Assume that one faces the problem of the following type: “Prove that a certain specific Lagrangian submanifold L of a symplectic manifold M is non-displaceable”. The mainstream approach to this problem is to show that the Lagrangian Floer homology of L is well defined and does not vanish. Our results above give rise to another potential approach (cf. [13]): show that L is a stem (see Definition 2.3) and deduce the non-displaceability of L from Theorem 2.1. Let us emphasize that this approach is not “soft”: unveiling the proof, one sees that we use an information about the asymptotic behaviour of Hamiltonian Floer homology for Hamiltonians concentrated near L . While in certain situations our method is simpler, it does not provide a lower bound on the number of intersections (assuming they are all transversal) between L and its image under a Hamiltonian isotopy – a bound which is usually given by the Lagrangian Floer homology approach whenever it works.

Below we illustrate our approach for the Lagrangian Clifford torus in \mathbb{CP}^n and for a similar torus in a monotone blow-up of \mathbb{CP}^2 at one point.

The Clifford torus in \mathbb{CP}^n . This example is taken from [13]. Let M be \mathbb{CP}^n with the Fubini–Study symplectic form. Consider the standard Hamiltonian \mathbb{T}^n -action on M whose moment polytope is a simplex in \mathbb{R}^n . Denote by L the Lagrangian torus which is the fiber of the moment map over the barycenter of the simplex – it is called the *Clifford torus* and can be described as

$$L := \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n \mid |z_0| = \cdots = |z_n|\}.$$

All the fibers of the moment map, other than L , are displaceable – this easily follows from the observation that permutations of homogeneous coordinates can be realized by Hamiltonian diffeomorphisms of \mathbb{CP}^n coming from the natural action of $\mathrm{PU}(n+1)$ on \mathbb{CP}^n . Thus L is a stem. Hence, according to Corollary 2.2, L is non-displaceable [13].

In fact, the non-displaceability of L can be also proved by means of the Lagrangian Floer homology. In this way Cho [14] showed that if an image of L under a Hamiltonian isotopy is transversal to L then the number of intersections between them must be at least 2^n (which is the sum of the Betti numbers of L).

The Clifford torus in the monotone blow-up of \mathbb{CP}^2 at one point. Our interest in this example is due to the fact that in this case the obstructions to displaceability of L coming from the Lagrangian Floer homology do vanish according to a result by Cho and Oh [15].

Here is the description of L . Consider a spherical shell W lying in the standard symplectic linear space \mathbb{C}^2 :

$$W = \{(u_1, u_2) \in \mathbb{C}^2 \mid \frac{1}{3} \leq \pi(|u_1|^2 + |u_2|^2) \leq 1\}.$$

Making a symplectic cut (i.e. collapsing the boundaries along the fibers of the characteristic foliation) we get a closed symplectic 4-manifold M which is one of the models of the blow up of \mathbb{CP}^2 at one point. The details of the construction can be extracted from the description of the symplectic structure on M given in [7], p. 61. The symplectic manifold M is spherically monotone. A Lagrangian torus

$$L := \{\pi|u_1|^2 = \pi|u_2|^2 = \tfrac{1}{3}\}$$

is called *the Clifford torus* of M – it can be viewed as the Clifford torus in \mathbb{CP}^2 which “survived” the blow-up.

Theorem 9.1. *The Clifford torus $L \subset M$ is a stem.*

Combining this with Corollary 2.2 we get that L is non-displaceable.

Proof. Consider a Hamiltonian action of the 2-torus on M , which in the spherical shell model is defined by its moment map

$$\Phi: W \rightarrow \mathbb{R}^2, \quad (u_1, u_2) \mapsto (\pi|u_1|^2, \pi|u_2|^2).$$

We shall show that L is the stem of a Poisson-commutative subspace generated by the coordinate functions of Φ . The image Δ of Φ is a trapezoid $ABCD$ in the plane with the vertices

$$A = (0, 1/3), \quad B = (1/3, 0), \quad C = (1, 0), \quad D = (0, 1).$$

The Clifford torus L is given by $\Phi^{-1}(Q)$, where $Q = (1/3, 1/3)$.

Claim. The fiber $\Phi^{-1}(X)$ is displaceable for every $X \neq Q$.

We use the following notation for lines and segments on the plane: PR stands for the line passing through points P and R , $[PR)$ denotes the segment with vertices P and R so that P is included and R is excluded and so on. We write $|PR|$ for the Euclidean length of $[PR]$.

Consider the points

$$P = (1/6, 1/6) \in [AB], \quad R = (1/2, 1/2) \in [CD].$$

Case I: $X \notin [PR]$. The unitary transformation $S: (u_1, u_2) \rightarrow (u_2, u_1)$ of W commutes with the \mathbb{T}^2 -action and induces the symmetry of Δ over the line PR which sends X to a point $X' \neq X$. Hence $S(\Phi^{-1}(X)) \cap \Phi^{-1}(X) = \emptyset$, which proves the claim in this case.

In order to proceed further, take the point $E = (2/3, 1/3) \in [CD]$. The segment $[AE]$ divides Δ into a triangle Δ' and a parallelogram Π . We assume that Δ' contains

segments $[DA)$ and $[DE)$ and does not contain $[AE]$, while Π contains $[BA)$ and $[BC)$ and does not contain the two other edges.

Case II: $X \in (QR]$. The set $\Phi^{-1}(\Delta')$ is \mathbb{T}^2 -equivariantly symplectomorphic to the standard symplectic ball $\{\pi(|w_1|^2 + |w_2|^2) < 2/3\}$ (the vertex D corresponds to the center of the ball). This follows from the local version of Delzant theorem – see [27]. The unitary transformation $(w_1, w_2) \rightarrow (w_2, w_1)$ of this ball commutes with the \mathbb{T}^2 -action and induces an affine involution of Δ' whose fixed point set coincides with $[DQ)$. This involution sends X to some point $X' \neq X$. We conclude that the torus $\Phi^{-1}(X)$ can be sent to $\Phi^{-1}(X')$ by a Hamiltonian isotopy, and therefore is displaceable.

Case III: $X \in [PQ)$. The set $\Phi^{-1}(\Pi)$ is \mathbb{T}^2 -equivariantly symplectomorphic to the standard symplectic polydisc $\mathcal{C} = \mathcal{D}_1 \times \mathcal{D}_2$ with

$$\mathcal{D}_1 = \{\pi|w_1|^2 < 2/3\}, \quad \mathcal{D}_2 = \{\pi|w_2|^2 < 1/3\}$$

(the vertex B corresponds to the center of the polydisc). This again follows from the local version of Delzant theorem [27]. The projection of \mathcal{C} to \mathcal{D}_1 sends the torus $\Phi^{-1}(X)$ to a circle $\Gamma := \{\pi|w_1|^2 = r\}$ which encloses a disc of area r . The area r corresponding to the point X can be calculated as follows. Let Y be the projection of X to BC along AB . Then

$$\frac{r}{\text{Area}(\mathcal{D}_1)} = \frac{|BY|}{|BC|} < \frac{1}{2}.$$

This inequality guarantees that Γ is displaceable in \mathcal{D}_1 by a Hamiltonian transformation of \mathcal{D}_1 . Lifting this transformation to \mathcal{C} we get that $\Phi^{-1}(X)$ is displaceable. This completes the proof of the claim, and hence of the theorem. \square

Sometimes even in seemingly simple situations it is hard to decide whether a given Lagrangian submanifold is a stem. For instance, we do not know an answer to the following question:

Question 9.2. Consider $\mathbb{RP}^2 \subset \mathbb{CP}^2$ or the anti-diagonal in the monotone $\mathbb{S}^2 \times \mathbb{S}^2$. Are they stems?

10. On the history and the physical meaning of quasi-states

The notion of quasi-state has an amusing history. To discuss it let us recall the mathematical model of quantum mechanics which goes back to von Neumann's famous book [43] published in 1932: Its basic ingredients are the real Lie algebra of observables \mathcal{A}_q (q for quantum) whose elements (in the simplest version of the theory)

are hermitian operators on a finite-dimensional complex Hilbert space H and the Lie bracket is given by

$$[A, B]_{\hbar} = \frac{i}{\hbar}(AB - BA),$$

where \hbar is the Planck constant. Observables represent physical quantities such as energy, position, momentum etc. The state of a quantum system is given by a functional $\zeta : \mathcal{A}_q \rightarrow \mathbb{R}$ which satisfies the following axioms:

Additivity. $\zeta(A + B) = \zeta(A) + \zeta(B)$ for all $A, B \in \mathcal{A}_q$.

Homogeneity. $\zeta(cA) = c\zeta(A)$ for all $c \in \mathbb{R}$ and $A \in \mathcal{A}_q$.

Positivity. $\zeta(A) \geq 0$ provided $A \geq 0$.

Normalization. $\zeta(\mathbf{1}) = 1$.

As a consequence of these axioms von Neumann proved that for every quantum state ζ there exists a non-negative Hermitian operator U_{ζ} with trace 1 such that $\zeta(A) = \text{tr}(U_{\zeta}A)$ for all $A \in \mathcal{A}_q$. An easy consequence of this formula is that for every state ζ there exists an observable A such that

$$\zeta(A^2) - \zeta(A)^2 > 0. \quad (10)$$

In his book von Neumann adopted a statistical interpretation of quantum mechanics according to which the value $\zeta(A)$ is considered as the expectation of a physical quantity represented by A in the state ζ . In this interpretation the equation (10) says that there are no dispersion-free states. This result led von Neumann to a conclusion which in the language of quantum mechanics can be formulated as the impossibility to introduce hidden variables into the quantum theory. This conclusion caused a (seemingly never ending) discussion among physicists which (citing Ballentine [8], p. 374) “was unfortunately clouded by emotionalism”. A number of prominent physicists, including Bohm and Bell, disagreed with the additivity axiom of a quantum state. Their reasoning was that the formula $\zeta(A + B) = \zeta(A) + \zeta(B)$ makes sense *a priori* only if observables A and B are simultaneously measurable, that is commute: $[A, B]_{\hbar} = 0$. We refer to Bell’s paper [10] for an account of this discussion.

In 1957 Gleason [23] proved a remarkable rigidity-type theorem which can be considered as an additional argument in favor of von Neumann’s additivity axiom. Recall that two hermitian operators on a finite-dimensional Hilbert space commute if and only if they can be written as polynomials of the same self-adjoint operator. Let us introduce a *quasi-state* on \mathcal{A}_q as a real-valued functional which satisfies the homogeneity, positivity and normalization axioms above, while the additivity axiom is replaced by one of the two *equivalent* axioms:

Quasi-additivity-I. $\zeta(A + B) = \zeta(A) + \zeta(B)$ provided that A and B commute: $[A, B]_{\hbar} = 0$.

Quasi-additivity-II. $\zeta(A + B) = \zeta(A) + \zeta(B)$ provided that A and B belong to a single-generated subalgebra of \mathcal{A}_q .

According to the Gleason theorem, every quasi-state on \mathcal{A}_q is linear (that is, a state) provided the complex dimension of the Hilbert space H is at least 3 (it is an easy exercise to show that in the two-dimensional case there are plenty of non-linear quasi-states).

Let us turn now to the mathematical model of classical mechanics. Here the algebra \mathcal{A}_c of observables (c for classical) is the space of continuous functions $C(M)$ on a symplectic manifold M . The Lie bracket is defined as the Poisson bracket on the dense subspace $C^\infty(M) \subset C(M)$. A natural question is whether the conclusion of the Gleason theorem remains valid in the classical context. We immediately face a dilemma: which of two definitions of quasi-additivity one should adopt as the starting point of such an extension. Adopting the second one, we arrive to the definition of a quasi-state given by Aarnes. It does not involve the symplectic structure and gives rise to the theory of quasi-states on general topological spaces. Adopting the first one, and taking into account the Correspondence Principle according to which the bracket $[\cdot, \cdot]_\hbar$ corresponds to the Poisson bracket $\{\cdot, \cdot\}$ in the classical limit $\hbar \rightarrow 0$, we get a definition which involves the strong quasi-additivity axiom: $\zeta(F + G) = \zeta(F) + \zeta(G)$ whenever $\{F, G\} = 0$, see Section 3. According to Theorem 8.1 above both definitions coincide in dimension 2. However, as it was mentioned in Remark 8.4, strong quasi-additivity is strictly stronger in higher dimensions. For the sake of brevity, we refer to non-linear strongly quasi-additive quasi-states on symplectic manifolds as to *strong quasi-states*.

In light of this discussion, Theorems 3.1 and 8.1 above which establish the existence of strong quasi-states on certain symplectic manifolds can be viewed as an “anti-Gleason phenomenon” in classical mechanics. This interpretation is far from being transparent. Let us indicate two points which require further clarification.

First, recall that the algebra \mathcal{A}_c of classical observables can be considered as a suitable limit of matrix algebras \mathcal{A}_q where the dimension N of the underlying Hilbert space H tends to ∞ and the Planck constant \hbar tends to 0. We refer the reader to Madore’s paper [33] dealing with the case where the classical phase space is the 2-dimensional sphere. For certain symplectic manifolds the algebra \mathcal{A}_c carries a strong quasi-state, say, ζ . At the same time the Gleason theorem rules out existence of a non-linear quasi-state on \mathcal{A}_q for every given values of N and \hbar . It would be interesting to understand what is a footprint of ζ in the quantum world. For instance, do the algebras \mathcal{A}_q carry a weaker object (a kind of “approximate quasi-state” still to be defined) which converges to ζ ?

Second, by the analogy with quantum mechanics, one can speculate that Poisson non-commuting functions F and G with $\zeta(F + G) \neq \zeta(F) + \zeta(G)$ are not simultaneously measurable. Does there exist an explanation of this phenomenon in terms of classical mechanics? An extra difficulty here is due to the fact that some strong quasi-states are dispersion-free. Therefore one cannot refer to uncertainty as to the

reason for the lack of simultaneous measurability.

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