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Embeddings of Danielewski surfaces in affine spaces

A. Dubouloz

Abstract. We construct explicit embeddings of Danielewski surfaces [4] in affine spaces. The equations defining these embeddings are obtained from the 2×2 minors of a matrix attached to a weighted rooted tree γ . We characterize those surfaces S_γ with a trivial Makar-Limanov invariant in terms of their associated trees. We prove that every Danielewski surface S with a nontrivial Makar-Limanov invariant admits a closed embedding in an affine space \mathbb{A}_k^n in such a way that every $\mathbb{G}_{a,k}$ -action on S extends to an action on \mathbb{A}^n defined by a triangular derivation. We show that a Danielewski surface S with a trivial Makar-Limanov invariant and non-isomorphic to a hypersurface with equation $xz - P(y) = 0$ in \mathbb{A}_k^3 admits nonconjugated algebraically independent $\mathbb{G}_{a,k}$ -actions.

Mathematics Subject Classification (2000). 14R20, 14R25.

Keywords. Danielewski surfaces, additive group actions, Makar-Limanov invariant.

Introduction

A *Danielewski surface* over a field k of characteristic zero is an integral affine surface S equipped with a morphism $\pi: S \rightarrow \mathbb{A}_k^1 = \text{Spec}(k[x])$ restricting to the trivial line bundle over $\mathbb{A}_k^1 \setminus \{0\}$ and such that the fiber $\pi^{-1}(0)$ is nonempty and reduced, consisting of a disjoint union of affine lines \mathbb{A}_k^1 . For instance, a surface $S_{P,n} \subset \text{Spec}(k[x, y, z])$ with equation $x^n z - P(y) = 0$, where P is a nonconstant polynomial with $\deg(P)$ simple roots, is a Danielewski surface $\text{pr}_x: S_{P,n} \rightarrow \text{Spec}(k[x])$. Danielewski surfaces appear naturally as locally trivial fiber bundles $\rho: S \rightarrow \tilde{X}$ over an affine line with a multiple origin (see e.g. [5]). More precisely, see [4], every such bundle ρ is a principal homogeneous bundle under the action of a line bundle $p: L \rightarrow \tilde{X}$. These principal L -bundles are uniquely determined by data consisting of an invertible sheaf \mathcal{L} on \tilde{X} and a Čech 1-cocycle g with values in the dual \mathcal{L}^\vee of \mathcal{L} for a suitable covering \mathcal{U} of \tilde{X} . In turn, the pair (\mathcal{L}, g) is encoded in a combinatorial datum consisting of a rooted tree with weighted edges, which we call a *weighted tree* (see [4, Example 1.6 and Theorem 3.2] and 2.2 below). Here we use weighted trees in a different way to construct embeddings of Danielewski surfaces into affine spaces.

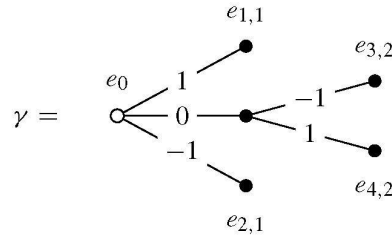
More precisely, starting with a suitable class of k -weighted trees γ , we construct explicit ideals of certain polynomial rings. In turn, these ideals define affine surfaces S_γ which are naturally Danielewski surfaces over the affine line \mathbb{A}_k^1 .

The paper is divided as follows. In Section 1 we recall basic facts on weighted trees. We associate to every *fine k -weighted tree* $\gamma = (\Gamma, w)$ (see Definition 1.3 below) a polynomial ring $k[\Gamma]$ and a collection of polynomials in $k[\Gamma]$ defined recursively through the *weight function* w .

In Section 2, we review the construction of Danielewski surfaces as locally trivial bundles over the affine line with an n -fold origin given in [4]. Then we associate to every fine k -weighted tree γ a closed affine subscheme $S_\gamma = \text{Spec}(B_\gamma)$ of $\mathbb{A}_k^1 \times \text{Spec}(k[\Gamma])$, and we prove the following result (Theorem 2.9).

Theorem. *For every fine k -weighted tree γ , the scheme S_γ is a Danielewski surface over \mathbb{A}_k^1 for the restriction of the projection $\text{pr}_1 : \mathbb{A}_k^1 \times \text{Spec}(k[\Gamma]) \rightarrow \mathbb{A}_k^1$.*

For instance, the surface corresponding to the following fine k -weighted tree



is the Bandman and Makar-Limanov surface [1] $S \subset k[x][y, z, u]$ with equations

$$xz - y(y^2 - 1) = 0, \quad yu - z(z^2 - 1) = 0, \quad xu - (y^2 - 1)(z^2 - 1) = 0.$$

It is a Danielewski surface over $X = \text{Spec}(k[x])$ via the projection morphism $\text{pr}_x : S \rightarrow X$.

Then we show that every embedded Danielewski surface S_γ as above comes canonically equipped with actions of the additive group $\mathbb{G}_{a,k}$ which are the restrictions to S_γ of certain $\mathbb{G}_{a,k}$ -actions on the ambient space $\mathbb{A}_k^1 \times \text{Spec}(k[\Gamma])$ defined by explicit locally nilpotent derivations $\tilde{\partial}_\gamma$ (see Proposition 2.15). In Section 3, we prove the following result (Corollary 3.8).

Theorem. *Every Danielewski surface $\pi : S \rightarrow X = \mathbb{A}_k^1$ is X -isomorphic to an embedded Danielewski surface $\pi_\gamma : S_\gamma = \text{Spec}(B_\gamma) \rightarrow X$ for an appropriate fine k -weighted tree γ .*

Moreover, we establish that every $\mathbb{G}_{a,X}$ -action on $\pi : S \rightarrow X$ is induced by a locally nilpotent derivation $\tilde{\partial}_\gamma$ as above. As a consequence of this description, we

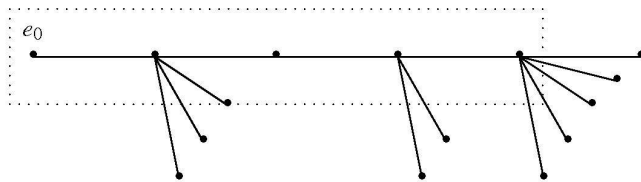
deduce that every Danielewski surface $\pi : S \rightarrow X = \mathbb{A}_k^1$ can be embedded in a relative affine space \mathbb{A}_X^d in such a way that every $\mathbb{G}_{a,X}$ -action on S extends to an action on \mathbb{A}_X^d (Corollary 3.11). This generalizes a result obtained by Makar-Limanov ([8], [9]) for the Danielewski hypersurfaces $S_{P,n}$ above.

The *Makar-Limanov invariant* [6] of an affine k -scheme $X = \text{Spec}(B)$ is defined as the sub-algebra $\text{ML}(X) \subset B$ consisting of regular functions which are invariant under all $\mathbb{G}_{a,k}$ -actions on X . If $\text{ML}(X) = k$, then we say that X has a trivial Makar-Limanov invariant. For Danielewski surfaces with a nontrivial Makar-Limanov invariant, we prove the following result.

Theorem. *Every Danielewski surface with a nontrivial Makar-Limanov invariant can be embedded in an affine space $\mathbb{A}_k^d = \text{Spec}(k[x_1, \dots, x_d])$ in such a way that every $\mathbb{G}_{a,k}$ -action on S extends to an action on \mathbb{A}_k^N . Furthermore, every such action is induced by a triangular locally nilpotent derivation of $k[x_1, \dots, x_d]$.*

In Section 4, we study Danielewski surfaces with a trivial Makar-Limanov invariant, that is, Danielewski surfaces S which admits two nontrivial $\mathbb{G}_{a,k}$ -actions with distinct general orbits. We obtain the following criterion which generalizes Theorem 5.4 in [4].

Theorem. *An embedded Danielewski surface $\pi : S_\gamma = \text{Spec}(B_\gamma) \rightarrow \mathbb{A}_k^1$ defined by a fine k -weighted tree γ has a trivial Makar-Limanov invariant if and only if γ is a comb, i.e. a tree such that every element has at most one non-terminal direct descendant (see Definition 4.1 below).*



A comb rooted in e_0 .

We obtain the following description (see 4.7 below). For every Danielewski surface S with a trivial Makar-Limanov invariant, there exists a collection of monic polynomials $P_0, \dots, P_{h-1} \in k[t]$ with simple roots $a_{i,j} \in k^*, i = 0, \dots, h-1, j = 1, \dots, \deg_t(P_i)$, such that S is isomorphic to the nonsingular surface $S_{P_0, \dots, P_{h-1}} \subset$

$\text{Spec}(k[x][y_{-1}, \dots, y_{h-2}][z])$ defined by the equations

$$\begin{aligned} xz - y_{h-2} \prod_{l=0}^{h-1} P_l(y_{l-1}) &= 0, \\ zy_{i-1} - y_i y_{h-2} \prod_{l=i+1}^{h-1} P_l(y_{l-1}) &= 0, \quad xy_i - y_{i-1} \prod_{l=0}^i P_l(y_{l-1}) = 0, \quad 0 \leq i \leq h-2, \\ y_{i-1} y_j - y_i y_{j-1} \prod_{l=i+1}^j P_l(y_{l-1}) &= 0, \quad 0 \leq i < j \leq h-2. \end{aligned}$$

On an affine surface $S = \text{Spec}(B)$, two $\mathbb{G}_{a,k}$ -actions μ_1 and μ_2 with associated quotient fibrations $\pi_1: S \rightarrow \mathbb{A}_k^1$ and $\pi_2: S \rightarrow \mathbb{A}_k^1$ respectively are said to be *algebraically independent* if the general fibers of π_1 and π_2 do not coincide. In this situation, we say that μ_1 and μ_2 are *conjugated* if there exists an automorphism ϕ of S sending the fibers of π_1 onto the fibers of π_2 . This means equivalently that there exists an automorphism ϕ^* of B such that $\text{Ker}(\partial_2) = \phi^*(\text{Ker}(\partial_1))$, where ∂_1 and ∂_2 denote the locally nilpotent derivations of B corresponding to the actions μ_1 and μ_2 respectively. Daigle [2] established that all the $\mathbb{G}_{a,k}$ -actions on a Danielewski surface $S_{P,1} = \{xz - P(y) = 0\}$ are conjugated. From the explicit description above, we obtain the following result (Theorem 4.12).

Theorem. *If a Danielewski surface S non isomorphic to a surface $S_{P,1}$ admits two independent $\mathbb{G}_{a,k}$ -actions, then it admits two algebraically independent nonconjugated $\mathbb{G}_{a,k}$ -actions.*

We also deduce the following characterization (Corollary 4.13) of the Danielewski surfaces $S_{P,1}$, which generalizes the ones previously obtained by Bandman and Makar-Limanov [1] and Daigle [2].

Theorem. *For a Danielewski surface $\pi: S \rightarrow X = \mathbb{A}_k^1$ with a trivial Makar-Limanov invariant, the following are equivalent.*

- 1) S admits a free $\mathbb{G}_{a,X}$ -action.
- 2) The canonical sheaf ω_S of S is trivial.
- 3) S is isomorphic to a surface $S_{P,1} \subset \mathbb{A}_k^3 = \text{Spec}(k[x, y, z])$ with the equation $xz - P(y) = 0$ for a certain nonconstant polynomial P with $\deg(P)$ simple roots.
- 4) All $\mathbb{G}_{a,k}$ -actions on S are conjugated.

1. Preliminaries

Weighted rooted trees. A *poset* is a nonempty finite partially ordered set $G = (G, \leq)$. A totally ordered subset $C \subset G$ is called a *chain of length* $l(C) = \text{Card}(C) - 1$. A chain which is maximal for the inclusion is called a *maximal chain*. For every $e \in G$, we let

$$(\uparrow e)_G = \{e' \in G, e \leq e'\} \quad \text{and} \quad (\downarrow e)_G = \{e' \in G, e' \leq e\}.$$

A subset $\overleftarrow{e'e}$ with two elements $e' < e$ such that $(\uparrow e')_G \cap (\downarrow e)_G = \{e' < e\}$ is called an *edge* of G . We denote the set of all edges in G by $E(G)$.

Definition 1.1. A (*rooted*) *tree* $\Gamma = (\Gamma, \leq)$ is a poset with a unique minimal element e_0 called the *root*, and such that $(\downarrow e)_\Gamma$ is a chain for every $e \in \Gamma$. A subposet $\Gamma' \subset \Gamma$ which is tree for the induced ordering is called a *subtree* of Γ . Given $e \in \Gamma$, the *maximal (rooted) subtree of Γ rooted in e* is the subtree $\Gamma(e) = (\uparrow e)_\Gamma$.

1.2. An element e such that $l(\downarrow e)_\Gamma = m$ is said to be *at level m* . The maximal elements $e_i = e_{i,m_i}$, where $m_i = l(\downarrow e_i)_\Gamma$, of a tree Γ are called the *leaves* of Γ . We denote the set of those elements by $L(\Gamma)$. The maximal chains of Γ are the chains

$$\Gamma_{e_i,m_i} = (\downarrow e_{i,m_i})_\Gamma = \{e_{i,0} = e_0 < e_{i,1} < \dots < e_{i,m_i}\}, \quad e_{i,m_i} \in L(\Gamma). \quad (1.1)$$

We say that Γ has *height* $h(\Gamma) = \max(m_i)$. An element of $\Gamma \setminus L(\Gamma)$ is called a *parent*, and we denote the set of those elements by $\mathbf{P}(\Gamma)$. Given $e \in \Gamma \setminus \{e_0\}$, an element of the chain $\text{Anc}(e) = (\downarrow e) \setminus \{e\}$ is called an *ancestor* of e . The *parent of e* is the maximal element $\text{Par}(e)$ of $\text{Anc}(e)$. More generally, the *n -th ancestor* of e is defined recursively by $\text{Par}^n(e) = \text{Par}(\text{Par}^{n-1}(e)) \in \text{Anc}(e)$. Given two different elements $e, e' \in \Gamma$, the *first common ancestor* of e and e' is the maximal element $\text{Anc}(e, e')$ of the chain $\text{Anc}(e) \cap \text{Anc}(e')$. If e is not a leaf of Γ , then the minimal elements of $(\uparrow e)_\Gamma \setminus \{e\}$ are called the *children* of e , and we denote the set of those elements by $\text{Ch}(e)$. The *degree* $\text{deg}(e)$ of an element e is the number of its children.

Definition 1.3. Let Γ be a tree. A *fine weight function* on Γ , with values in a field k , is a function $w: E(\Gamma) \rightarrow k$, which assigns an element $a_{e',e} = w(\overleftarrow{e'e}) \in k$ to every edge $\overleftarrow{e'e}$ of Γ , in such a way that $a_{e',e_1} \neq a_{e',e_2}$ whenever e_1 and e_2 share the same parent e' . A tree Γ equipped with such a function w is referred to as a *fine k -weighted tree* $\gamma = (\Gamma, w)$.

Definition 1.4. An *morphism of fine k -weighted trees* $\tau: \gamma' = (\Gamma', w') \rightarrow \gamma = (\Gamma, w)$ is an order-preserving map $\tau: \Gamma' \rightarrow \Gamma$ satisfying the following properties.

- a) The image of a maximal subchain of Γ' is a maximal subchain of Γ .

- b) For every $e' \in \Gamma'$, $\tau^{-1}(\tau(e'))$ is either e' itself or a maximal subtree of Γ' .
c) For every edge $\overleftarrow{e'e}$ of Γ' such that $\tau(e) \neq \tau(e')$, we have $w'(\overleftarrow{e'e}) = w(\overleftarrow{\tau(e')\tau(e)})$.

Remark 1.5. A morphism of fine k -weighted trees maps the root e'_0 of Γ' on the root e_0 of Γ and a leaf e'_{i,m'_i} of Γ' at level m'_i onto a leaf $e_{j(i),m_{j(i)}}$ of Γ at level $m_{j(i)} \leq m'_i$. Then b) guarantees that $\tau(e'_{i,k}) = e_{j,\min(m_{j(i)},k)}$ for every $k = 0, \dots, m'_i$, and so, condition c) above makes sense.

Genealogical matrix of a weighted tree. Here we associate to every fine k -weighted tree $\gamma = (\Gamma, w)$ a matrix with coefficients in a polynomial ring $k[\Gamma]$.

Definition 1.6. Given a tree Γ rooted in e_0 , we associate to every parent $e \in \mathbf{P}(\Gamma)$ a symbol X_e . If $e' \in \mathbf{P}(\Gamma)$ is the parent of a given $e \in \mathbf{P}(\Gamma)$, then we will sometimes denote $X_{e'}$ as $X_{\text{Par}(e)}$. We also extend this relationship between the X_e 's by introducing the symbol $X_{e_{-1}} = X_{\text{Par}(e_0)}$. We let $k[\Gamma] = k[(X_e)_{e \in \mathbf{P}(\Gamma) \cup \{e_{-1}\}}]$ be the corresponding polynomial ring in $d(\Gamma) = \text{Card}(\mathbf{P}(\Gamma)) + 1$ variables.

For every element $e \in \mathbf{P}(\Gamma)$ of a given fine k -weighted tree $\gamma = (\Gamma, w)$ rooted in e_0 , we introduce below three polynomials $F_e(\gamma), A_e(\gamma), G_e(\gamma) \in k[\Gamma]$, defined recursively through the weight function $w: E(\Gamma) \rightarrow k, \overleftarrow{e'e} \mapsto a_{e',e} = w(\overleftarrow{e'e})$.

Definition 1.7. For every $e' \in \mathbf{P}(\Gamma)$ and every subset $J \subset \text{Ch}(e')$ we let

$$F_{e'}^J = F_{e'}^J(\gamma) = \prod_{e \in (\text{Ch}(e') \setminus J)} (X_{\text{Par}(e')} - a_{e',e}) \in k[X_{\text{Par}(e')}] \subset k[\Gamma].$$

The polynomial $F_{e'} := F_{e'}^{\emptyset}$ is called *the fatherhood polynomial* of e' .

The *ancestral polynomial* $A_e = A_e(\gamma)$ of $e \in \Gamma$ is the polynomial defined recursively by

$$A_{e_0} = 1 \quad \text{and} \quad A_e = F_{\text{Par}(e)}^{[e]} A_{\text{Par}(e)} \in k[X_{e_{-1}}, (X_{e'})_{e' \in \text{Anc}(\text{Par}(e))}] \subset k[\Gamma].$$

The *genealogical polynomial* of $e \in \mathbf{P}(\Gamma)$ with respect to $e' \in \text{Anc}(e)$ is the polynomial

$$G_{e',e} = G_{e',e}(\gamma) = A_{e'}^{-1} A_e F_e \in k[X_{e_{-1}}, (X_{e''})_{e'' \in \text{Anc}(e) \setminus \text{Anc}(\text{Par}(e'))}] \subset k[\Gamma].$$

The polynomial $G_e = G_{e_0,e}$ is simply referred to as *the genealogical polynomial* of e .

Remark 1.8. Up to changing the variables, $G_{e',e}(\gamma)$ coincides with the genealogical polynomial $G_e(\gamma')$ of e as an element of the maximal weighted subtree $\gamma(e') = ((\uparrow e')_\Gamma, w|_{\Gamma(e')})$ of γ rooted in e' , considered as a fine k -weighted tree disregarding the inclusion $\gamma(e') \hookrightarrow \gamma$.

Definition 1.9. The *genealogical matrix* of a fine k -weighted tree $\gamma = (\Gamma, w)$ is the matrix $M(\gamma) \in \text{Mat}_{d(\Gamma)-1,2}(k[\Gamma])$ with the rows $M_e = (G_e, X_e) \in \text{Mat}_{1,2}(k[\Gamma])$, $e \in \mathbf{P}(\Gamma)$.

2. Danielewski surfaces defined by weighted trees

In [4], the author gives a method to construct a Danielewski surface $\pi : S^\gamma \rightarrow X$ over $X = \text{Spec}(k[x])$ from the data consisting of a fine k -weighted tree γ . Here we review briefly this construction. Then we introduce a new procedure to associate to every such tree γ a second Danielewski surface $\pi : S_\gamma \rightarrow X$, which comes embedded in a relative affine space $\mathbb{A}_X^d = X \times \mathbb{A}_k^d$.

Notation 2.1. Throughout this section, we fix a field k of characteristic zero. We let $A = k[x]$, $X = \text{Spec}(A) \simeq \mathbb{A}_k^1$, and we denote by $X_* \simeq \text{Spec}(A_x)$ the open complement in X of the origin $x_0 \in \mathbb{A}_k^1$. We consider Danielewski surfaces over the fixed base X . We denote by $\text{pr}_X : \mathbb{A}_X^1 = \text{Spec}(A[X_{e_{-1}}]) \rightarrow X$ the trivial line bundle over X . The additive group scheme with base X is denoted by $\mathbb{G}_{a,X} = \text{Spec}(A[T])$.

Abstract Danielewski surface defined by a fine k -weighted tree. Given a fine k -weighted tree $\gamma = (\Gamma, w)$ of height $h = h(\Gamma)$ with leaves $e_{1,m_1}, \dots, e_{n,m_n}$, we construct a Danielewski surface $\pi : S^\gamma \rightarrow X$ as follows. Using the maximal weighted subchains

$$\gamma_{e_{i,m_i}} = ((\downarrow e_{i,m_i}), w) = \{e_0 = e_{i,0} < e_{i,1} < \dots < e_{i,m_i-1} < e_{i,m_i}\}_w, \quad i = 1, \dots, n,$$

of γ , we define a collection of polynomials

$$\sigma = \{\sigma_i = \sum_{j=0}^{m_i-1} w(\overleftarrow{e_{i,j}e_{i,j+1}})x^j \in k[x]\}_{i=1,\dots,n}.$$

For every $i \neq j$, we let $g_{ij} = x^{-m_i}(\sigma_j - \sigma_i) \in A_x$. These *transition functions* g_{ij} satisfy the cocycle relation $g_{ik} = g_{ij} + x^{m_j-m_i}g_{jk}$ in A_x for every triple $i \neq j \neq k$.

2.2. We let $\pi : S^\gamma \rightarrow X$ be the X -scheme obtained by gluing n copies $S_i = \text{Spec}(A[T_i])$ of \mathbb{A}_X^1 over X_* by means of the A_x -algebra isomorphisms

$$\tau_{ij} : A_x[T_i] \rightarrow A_x[T_j], \quad T_i \mapsto g_{ij} + x^{m_j-m_i}T_j, \quad i \neq j, \quad i, j = 1, \dots, n.$$

Since γ is a fine k -weighted tree, it follows from 2.8 in [4] that S^γ is a Danielewski surface $\pi : S^\gamma \rightarrow X$. The irreducible components of $\pi^{-1}(x_0)$ are the curves $C_i = \pi^{-1}(x_0) \cap S_i \simeq \text{Spec}(k[T_i])$, $i = 1, \dots, n$. It comes equipped with a canonical birational X -morphism $\psi : S^\gamma \rightarrow \mathbb{A}_X^1 = \text{Spec}(A[X_{e_{-1}}])$ corresponding to the section $s_{e_{-1}} \in B^\gamma = \Gamma(S^\gamma, \mathcal{O}_{S^\gamma})$ with restrictions $s_{e_{-1}}|_{S_i} = \sigma_i + x^{m_i}T_i \in A[T_i]$,

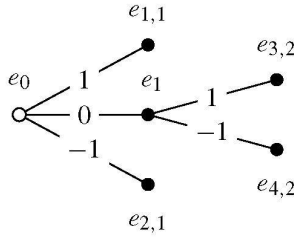
$i = 1, \dots, n$. By Theorem 3.2 in [4], every Danielewski surface $\pi: S \rightarrow X$ is X -isomorphic to an *abstract Danielewski surface* $\pi: S^\vee \rightarrow X$ obtained by this procedure.

2.3. A Danielewski surface $\pi: S \rightarrow X$ admits nontrivial actions of the additive group scheme $\mathbb{G}_{a,X}$. Indeed, since by definition $S|_{X_*}$ is isomorphic to the trivial line bundle $\mathbb{A}_{X_*}^1 = \text{Spec}(A_x[X_{e_{-1}}])$ over X_* , there exists $r \geq 0$ such that the A -derivation $x^m \partial_{X_{e_{-1}}}$ extends to a locally nilpotent A -derivation ∂ of $\Gamma(S, \mathcal{O}_S)$, corresponding to a nontrivial $\mathbb{G}_{a,X}$ -action on S . By Proposition 2.12 in [4], every nontrivial $\mathbb{G}_{a,X}$ -action on a Danielewski surface S^\vee is induced by the extension $\partial_{a,m}$ to B^\vee of a locally nilpotent A -derivation $ax^m \partial_{X_{e_{-1}}}$ of $B^\vee \otimes_A A_x \simeq A_x[X_{e_{-1}}]$, where $m \geq h(\Gamma)$ and $a \in A \setminus \{0\}$. We denote the corresponding $\mathbb{G}_{a,X}$ -actions on \mathbb{A}_X^1 and S^\vee by $t_{a,m}$ and $t_{a,m}^\vee$ respectively. On the open subsets $S_i = \text{Spec}(A[T_i])$, $t_{a,m}^\vee$ coincides with the *twisted translation* $t_{a,m-m_i}$ defined by the group co-action homomorphism

$$A[T_i] \rightarrow A[T_i, T] \simeq A[T_i] \otimes_A A[T], \quad T_i \mapsto T_i + ax^{m-m_i}T, \quad i = 1, \dots, n.$$

The canonical morphism $\psi: S^\vee \rightarrow \mathbb{A}_X^1$ is $\mathbb{G}_{a,X}$ -equivariant when S^\vee and \mathbb{A}_X^1 are equipped with the $\mathbb{G}_{a,X}$ -actions $t_{a,m}^\vee$ and $t_{a,m}$ respectively.

Example 2.4. The collection of polynomials σ corresponding to the following fine k -weighted tree $\gamma = (\Gamma, w)$ with leaves $e_{1,1}, e_{2,1}, e_{3,2}, e_{4,2}$



is $\sigma = \{1, -1, x, -x\}$. The associated transition functions $g = \{g_{ij}\}_{1 \leq i < j \leq 4}$ are

$$g_{12} = g_{34} = -2x^{-1}, \quad g_{13} = -g_{24} = x^{-1}(x-1), \\ g_{23} = -g_{14} = x^{-1}(x+1).$$

The gluing homomorphisms $\{\tau_{ij}\}_{1 \leq i < j \leq 4}$ are given by

$$\tau_{ij}: k[x, x^{-1}][T_i] \rightarrow k[x, x^{-1}][T_j], \\ T_i \mapsto \begin{cases} g_{ij} + T_j, & \text{if } (i, j) \in \{(1, 2), (3, 4)\}, \\ g_{ij} + xT_j, & \text{if } (i, j) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}. \end{cases}$$

The $\mathbb{G}_{a,X}$ -action $t_{1,2}^\gamma$ on $\pi: S^\gamma \rightarrow X$ is a non-free action which restricts on $S_i = \text{Spec}(A[T_i])$ to the action

$$T_i \mapsto \begin{cases} T_i + xT, & \text{if } i = 1, 2, \\ T_i + T, & \text{if } i = 3, 4. \end{cases}$$

Letting $P(t) = t^2 - 1 \in k[t]$, we will see in Example 3.2 below that S^γ is X -isomorphic to the Bandman and Makar-Limanov surface [1] $S \subset \text{Spec}(k[x][y, z, u])$ with equations

$$xu - yP(y) = 0, \quad yu - zP(z) = 0, \quad xu - P(y)P(z) = 0,$$

and that $t_{1,2}^\gamma$ coincides with the action on S induced by the triangular derivation

$$\partial_{1,2} = x^2\partial_y + x(3y^2 - 1)\partial_z + (2P(y)(3y^2 - 1)z + 2xyP(z))\partial_u \in \text{Der}_{k[x]}(k[x][y, z, u]).$$

Embedded Danielewski surface defined by a fine k -weighted tree. Given a fine k -weighted tree $\gamma = (\Gamma, w)$, we construct a Danielewski surface $\pi: S_\gamma \rightarrow X$ which comes embedded in a relative affine space $\mathbb{A}_X^{d(\Gamma)}$, where $d(\Gamma) = \text{Card}(\mathbf{P}(\Gamma)) + 1$. These surfaces are canonically equipped with the restrictions of certain actions of the additive group $\mathbb{G}_{a,X}$ on the ambient space $\mathbb{A}_X^{d(\Gamma)}$, defined by explicit locally nilpotent derivations.

2.5. Given a fine k -weighted tree $\gamma = (\Gamma, w)$, we let $A[\Gamma] = A \otimes_k k[\Gamma] \simeq A[X_{e_{-1}}, (X_e)_{e \in \mathbf{P}(\Gamma)}]$ (see Definition 1.6). We let $\bar{M}(\gamma) \in \text{Mat}_{d(\Gamma), 2}(A[\Gamma])$ be the matrix with the rows $M_{e_{-1}} = (x, 1)$ and $M_e = (G_e(\gamma), X_e)$, $e \in \mathbf{P}(\Gamma)$, i.e. $\bar{M}(\gamma) = (M_{e_{-1}}, M(\gamma))$, where $M(\gamma) \in \text{Mat}_{d(\Gamma)-1, 2}(k[\Gamma])$ denotes the genealogical matrix of γ (Definition 1.9).

Definition 2.6. Given a fine k -weighted tree $\gamma = (\Gamma, w)$, we let $I_\gamma \subset A[\Gamma]$ be the ideal generated by the *simplified genealogical minors* of $\bar{M}(\gamma)$

$$\Delta_{e',e} = \Delta_{e',e}(\gamma) = A_{e'}^{-1} \det(M_{\text{Par}(e')}, M_e) \in A[\Gamma], \quad (e, e') \in \mathbf{P}(\Gamma) \times (\downarrow e)_\Gamma. \quad (2.1)$$

We let $B_\gamma = A[\Gamma]/I_\gamma$, and we let $\pi: S_\gamma = \text{Spec}(B_\gamma) \rightarrow X$ be the corresponding closed sub- X -scheme of the relative affine space $\mathbb{A}_X^{d(\Gamma)} = \text{Spec}(A[\Gamma])$.

2.7. By construction, $\Delta_e := \Delta_{e_0,e} = xX_e - G_e \in A[(X_{e'})_{e' \in (\downarrow e)_\Gamma \cup \{e_{-1}\}}]$ for every $e \in \mathbf{P}(\Gamma)$, whereas $\Delta_{e',e} = (X_{\text{Par}^2(e')} - a_{\text{Par}(e'),e'})X_e - X_{\text{Par}(e')}G_{e',e}$ for every pair $(e, e') \in \mathbf{P}(\Gamma) \times ((\downarrow e)_\Gamma \setminus \{e_0\})$. As a consequence, for every triple $e_0 < e'' \leq e' \leq e$ in $\mathbf{P}(\Gamma)$, the following relations hold in $A[\Gamma]$:

$$\left. \begin{aligned} A_{e'}\Delta_{e',e} &= X_{\text{Par}(e')}\Delta_e - \Delta_{\text{Par}(e')}X_e, \\ x\Delta_{e',e} &= (X_{\text{Par}^2(e')} - a_{\text{Par}(e'),e'})\Delta_e - \Delta_{\text{Par}(e')}G_{e',e}, \\ (X_{\text{Par}^2(e'')} - a_{\text{Par}(e''),e''})\Delta_{e',e} &= (X_{\text{Par}^2(e')} - a_{\text{Par}(e'),e'})\Delta_{e'',e} - \Delta_{e'',e'}G_{e',e}. \end{aligned} \right\} \quad (2.2)$$

2.8. If $\gamma = (\Gamma, w)$ is the trivial tree with just one element e_0 , then the first projection $\pi: S_\gamma = \text{Spec}(k[x][X_{e_{-1}}]) \rightarrow X$ is a Danielewski surface. Similarly, if Γ has height 1, then $G_{e_0} \in k[X_{e_{-1}}]$ is a monic polynomial with simple roots $a_{e_0, e} = w(\overleftarrow{e_0 e}) \in k$, $e \in \text{Ch}(e_0)$. Therefore,

$$\pi: S_\gamma = \text{Spec}(A[\Gamma]/I_\gamma) = \text{Spec}(k[x][X_{e_{-1}}, X_{e_0}]/xX_{e_0} - G_{e_0}(X_{e_{-1}})) \rightarrow X$$

is a Danielewski surface, and the irreducible components of $\pi^{-1}(x_0)$ are the curves $C_e \simeq \text{Spec}(k[X_{e_0}])$ with defining ideals $I_{\gamma, e} = (I_\gamma, X_{e_{-1}} - a_{e, e_0}) \subset A[\Gamma]$, $e \in \text{Ch}(e_0)$. More generally, we have the following result.

Theorem 2.9. *For every fine k -weighted tree $\gamma = (\Gamma, w)$ with leaves $e_{1, m_1}, \dots, e_{n, m_n}$, $\pi: S_\gamma \rightarrow X$ is a Danielewski surface. Furthermore, the fiber $\pi^{-1}(x_0)$ is the disjoint union of the curves $C_{e_i, m_i} \simeq \text{Spec}(k[X_{e_i, m_i-1}])$ with defining ideals*

$$I_{\gamma, e_i, m_i} = (I_\gamma, x, (X_{e_i, j-1} - a_{e_i, j, e_i, j+1})_{0 \leq j \leq m_i-1}) \subset A[\Gamma], \quad i = 1, \dots, n.$$

The proof is divided as follows. In 2.10, Lemmas 2.11 and 2.12 below, we show that S_γ is an integral scheme. Then, in Lemma 2.13, we describe explicitly the irreducible components of $\pi^{-1}(x_0)$.

2.10. We first observe that S_γ restricts to the trivial line bundle $\mathbb{A}_{X_x}^1 = \text{Spec}(A_x[X_{e_{-1}}])$ over X_* . Indeed, the second relation of (2.2) guarantees that the ideal $I_\gamma A_x[\Gamma]$ of $A_x[\Gamma] \simeq A[\Gamma] \otimes_A A_x$ is generated by the polynomials $x^{-1}\Delta_e = X_e - x^{-1}G_e$, $e \in \mathbf{P}(\Gamma)$. Since G_e only involves the variables $X_{e'}$, where $e' \in \text{Anc}(e)$, we recursively arrive at an A_x -algebra isomorphism $A_x[\Gamma]/I_\gamma A_x[\Gamma] \simeq A_x[X_{e_{-1}}]$. Thus S_γ is a Danielewski surface with base $(k[x], x)$ provided that x is not a zero divisor in B_γ and that B_γ/xB_γ is isomorphic to a nonempty direct product of polynomial rings in one variable over k . Indeed, the first condition guarantees that the canonical map $B_\gamma \rightarrow B_\gamma \otimes_A A_x \simeq A_x[X_{e_{-1}}]$ is injective. In turn, this implies that B_γ is a sub-domain of $A_x[X_{e_{-1}}]$. The second one means equivalently that the fiber $\pi^{-1}(x_0)$ decomposes as a nonempty disjoint union of affine lines \mathbb{A}_k^1 .

To show that x is not a zero divisor in B_γ , it suffices to find a covering of S_γ by principal affine open subsets $Y_i = \text{Spec}(B_i)$ such that x is not a zero divisor in B_i for every $i = 1, \dots, n$.

Lemma 2.11. *If $\gamma = (\Gamma, w)$ is a fine k -weighted tree with the leaves e_1, \dots, e_n , then S_γ is covered by the principal open subsets $Y_i = \text{Spec}(A[\Gamma][T]/(I_\gamma, A_{e_i}T - 1))$, $i = 1, \dots, n$.*

Proof. For every $e \in \mathbf{P}(\Gamma)$ the polynomials $F_e^{(e')}$ in $A[X_{\text{Par}(e)}]$, $e' \in \text{Ch}(e)$ generate the unit ideal of $A[X_{\text{Par}(e)}]$ as γ is a fine k -weighted tree. Therefore, there exist

polynomials $\Lambda_{e'} \in A[\Gamma]$, $e' \in \text{Ch}(e)$, such that

$$A_e = A_e \sum_{e' \in \text{Ch}(e)} \Lambda_{e'} F_e^{\{e'\}} = \sum_{e' \in \text{Ch}(e)} \Lambda_{e'} A_{e'}.$$

It follows by induction that the image of $A_{e_0} = 1$ in B_γ belongs to the ideal generated by the images $a_i \in B_\gamma$ of the ancestral polynomials A_{e_i} of the leaves of Γ . This means equivalently that the open subsets $\text{Spec}((B_\gamma)_{a_i}) \simeq \text{Spec}(A[\Gamma][T]/(I_\gamma, A_{e_i}T - 1))$ cover S_γ . \square

Lemma 2.12. *For every $i = 1, \dots, n$, Y_i is an integral scheme.*

Proof. Let us denote by $e_j = e_{i,j}$, $j = 0, \dots, m = m_i$, the elements of the maximal subchain $(\downarrow e_{i,m_i})_\Gamma$ of Γ associated with the leaf e_{i,m_i} . For every $i = 1, \dots, m - 2$, the polynomial $A_{e_{i+1}}$ divides A_{e_m} . Similarly, for every $e \in \mathbf{P}(\Gamma) \setminus (\downarrow e_m)$, the first common ancestor of e and e_m is an element e_i , $i \leq m - 1$, such that $e' = \text{Ch}(e_i) \cap (\downarrow e) \neq e_{i+1}$, and so $(X_{e_{i-1}} - a_{e_i,e'})$ divides A_{e_m} . Therefore, these polynomials become invertible in $A[\Gamma]_{A_{e_m}}$. We claim that the ideal $I_\gamma A[\Gamma]_{A_{e_m}}$ is generated by the polynomials

$$\begin{aligned} \delta_{e_i} &= A_{e_{i+1}}^{-1} \Delta_{e_i} = -(X_{e_{i-1}} - a_{e_i,e_{i+1}}) + A_{e_{i+1}}^{-1} x X_{e_i}, \quad i = 1, \dots, m - 2, \\ \delta_{e_i,e} &= (X_{e_{i-1}} - a_{e_i,e'})^{-1} \Delta_{e_i,e} \\ &= X_e - (X_{e_{i-1}} - a_{e_i,e'})^{-1} X_{e_i} G_{e_i,e}, \end{aligned} \quad \begin{cases} e \in \mathbf{P}(\Gamma) \setminus (\downarrow e_m), \\ \text{Anc}(e, e_m) = e_i, \\ e' = \text{Ch}(e_i) \cap (\downarrow e)_\Gamma. \end{cases}$$

Indeed, the second relation of (2.2) guarantees that the polynomials Δ_e , where $e \in \mathbf{P}(\Gamma) \setminus (\downarrow e_m)_\Gamma$, can be expressed in $A[\Gamma]_{A_{e_m}}$ in terms of the δ_{e_i} 's and $\delta_{e_i,e}$'s. In turn, we deduce from the first and the third ones that all the polynomials $\Delta_{e',e}$, $(e, e') \in \mathbf{P}(\Gamma) \times ((\downarrow e)_\Gamma \setminus \{e_0\})$ belong to the ideal of $A[\Gamma]_{A_{e_m}}$ generated by the δ_{e_i} 's and the $\delta_{e_i,e}$'s. Since the polynomials A_{e_i} and $G_{e_i,e}$ above only involve the variables corresponding to the elements in $(\downarrow e_{i-2})_\Gamma$ and $(\uparrow e')_\Gamma \cap (\downarrow \text{Anc}(e))_\Gamma$ respectively, we conclude by induction that there exists a nonconstant polynomial $P \in A[X_{e_{m-1}}]$ such that $A[\Gamma]_{A_{e_m}}/I_\gamma A[\Gamma]_{A_{e_m}} \simeq A[X_{e_{m-1}}]_P$. Since A is a domain and P is nonconstant, it follows that $(B_\gamma)_{a_i} \simeq A[X_{e_{m-1}}]_P$ is a nonzero domain too. \square

Summing up, we have established that for every fine k -weighted tree γ , $\pi : S_\gamma \rightarrow X$ is an integral affine scheme restricting to the trivial bundle $\mathbb{A}_{X_*}^1$ over X_* . The following result completes the proof of Theorem 2.9.

Lemma 2.13. *For every fine k -weighted tree $\gamma = (\Gamma, w)$ with leaves $e_{1,m_1}, \dots, e_{n,m_n}$, the fiber $\pi^{-1}(x_0)$ of $\pi : S_\gamma \rightarrow X$ is the disjoint union of the curves $C_{e_i,m_i} \simeq \text{Spec}(k[X_{e_i,m_i-1}])$ with defining ideals*

$$I_{\gamma, e_i, m_i} = (I_\gamma, x, (X_{e_{i,j-1}} - a_{e_{i,j}, e_{i,j+1}})_{0 \leq j \leq m_i-1}) \subset A[\Gamma], \quad i = 1, \dots, n.$$

Proof. We proceed by induction on the height h of Γ . If $h = 0$ then $S_\gamma = \text{Spec}(A[X_{e_{-1}}])$ and $\pi^{-1}(x_0) \simeq \text{Spec}(k[X_{e_{-1}}])$. Otherwise, if $\text{Ch}(e_0) \neq \emptyset$ then, since γ is a fine k -weighted tree, it follows that the polynomials $X_{e_{-1}} - a_{e_0, e}$, $e \in \text{Ch}(e_0)$ are pairwise relatively prime. Therefore $\pi^{-1}(x_0) = \text{Spec}(A[\Gamma]/(x, I_\gamma))$ decomposes as the disjoint union of curves $D_e = \text{Spec}(A[\Gamma]/(x, X_{e_{-1}} - a_{e_0, e}, I_\gamma))$, $e \in \text{Ch}(e_0)$. We let $\gamma(e) = (\Gamma(e), w|_{\Gamma(e)})$ be the maximal fine k -weighted subtree of γ rooted in e . Clearly, the ideal $(x, X_{e_{-1}} - a_{e_0, e}, I_\gamma)$ coincides with the ideal $I_e \subset A[\Gamma]$ generated by x , $X_{e_{-1}} - a_{e_0, e}$ and the polynomials

$$\begin{aligned} G_{e, e'}(\gamma), & \quad e' \in \mathbf{P}(\Gamma(e)), \\ \Delta_{e'', e'}(\gamma), & \quad (e', e'') \in \mathbf{P}(\Gamma(e)) \times (\text{Anc}_{\Gamma(e)}(e')), \\ \delta_{e, e'} = (a_{e_0, e} - a_{e_0, e''})X_{e'} - X_{e_0}G_{e, e'}(\gamma), & \quad \begin{cases} e' \in \mathbf{P}(\Gamma) \setminus (\{e_0\} \cup \mathbf{P}(\Gamma(e))), \\ e'' = \text{Ch}(e_0) \cap (\downarrow e') \neq e. \end{cases} \end{aligned}$$

By definition, we have $A[\Gamma(e)] = A[X_{e_{-1}}, (X_{e'})_{e' \in \mathbf{P}(\Gamma)}] \simeq A[X_{e_0}, (X_{e'})_{e' \in \mathbf{P}(\Gamma)}]$ as $e_0 \notin \Gamma(e)$. This choice of coordinates yields the identities

$$\begin{aligned} G_{e'}(\gamma(e)) &= G_{e, e'}(\gamma), \quad e' \in \mathbf{P}(\Gamma(e)), \\ G_{e'', e'}(\gamma(e)) &= G_{e'', e'}(\gamma), \quad (e', e'') \in \mathbf{P}(\Gamma(e)) \times \text{Anc}_{\Gamma(e)}(e'), \end{aligned}$$

and we conclude that $A[\Gamma]/(x, X_{e_{-1}} - a_{e_0, e}, I_\gamma) \simeq A[\Gamma]/I_e \simeq A[\Gamma(e)]/(x, I_{\gamma(e)})$. This means equivalently that $\pi^{-1}(x_0)$ is isomorphic to the disjoint union of the fibers $\pi_{\gamma(e)}^{-1}(x_0)$ of the corresponding surfaces $\pi_{\gamma(e)}: S_{\gamma(e)} \rightarrow X$, $e \in \text{Ch}(e_0)$. Since the fine k -weighted tree $\gamma(e)$ has height $h - 1$, it follows from our induction hypothesis that these fibers are nonempty and reduced, consisting of disjoint unions of affine lines \mathbb{A}_k^1 . So the same holds for $\pi^{-1}(x_0)$. Finally, the precise description of the irreducible components of $\pi^{-1}(x_0)$ follows easily by induction again. \square

Remark 2.14. A Danielewski surface $\pi: S_\gamma \rightarrow X = \mathbb{A}_k^1$ is a flat (or rather a smooth) X -scheme. In general, the scheme $\tilde{\pi}: \tilde{S}_\gamma \rightarrow X$ with defining ideal \tilde{I}_γ generated only by the polynomials Δ_e , $e \in \mathbf{P}(\Gamma)$, is not flat over X . The above discussion together with the second relation of (2.2) imply that S_γ coincides with the flat limit over X of the trivial family of affine lines $\tilde{S}_\gamma|_{X_*} \simeq \mathbb{A}_{X_*}^1$ defined by the equations $\Delta_e = 0$, $e \in \mathbf{P}(\Gamma)$, in $\mathbb{A}_{X_*}^{d(\Gamma)} = \text{Spec}(A_x[\Gamma])$. This explains why the polynomials $\Delta_{e', e}$, $(e, e') \in \mathbf{P}(\Gamma) \times ((\downarrow e)_\Gamma \setminus \{e_0\})$, should be added to the obvious ones Δ_e , $e \in \mathbf{P}(\Gamma)$, to define the surface S_γ .

The following result shows that the *embedded Danielewski surface* $\pi: S_\gamma \rightarrow X$ defined by a fine k -weighted tree $\gamma = (\Gamma, w)$ admits nontrivial actions of the additive group $\mathbb{G}_{a, X}$, which come as the restrictions of certain $\mathbb{G}_{a, X}$ -actions on the ambient space $\mathbb{A}_X^{d(\Gamma)}$.

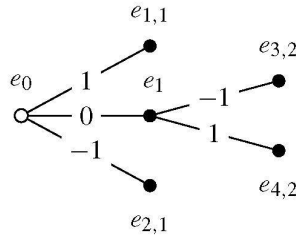
Proposition 2.15. *Let $\gamma = (\Gamma, w)$ be a fine k -weighted tree of height $h \geq 0$. Then, for every $m \geq h$ and every $a \in A \setminus \{0\}$, the derivation $\tilde{\partial}_{\gamma,a,m} \in \text{Der}_A(A[\Gamma], A_x[\Gamma])$ defined recursively by*

$$\tilde{\partial}_{\gamma,a,m} = ax^m \partial_{X_{e_{-1}}} + x^{-1} \sum_{e \in \mathbf{P}(\Gamma)} \tilde{\partial}_{\gamma,a,m}(G_e(\gamma)) \partial_{X_e}$$

is a triangular derivation of $A[\Gamma]$ inducing a locally nilpotent A -derivation $\partial_{\gamma,a,m}$ of B_γ .

Proof. It suffices to prove the assertion for the derivation $\tilde{\partial} = \partial_{\gamma,1,h}$ as $\tilde{\partial}_{\gamma,a,m} = ax^{m-h} \tilde{\partial}$. For every $e \in \mathbf{P}(\Gamma)$ at level $i < h$, the polynomial G_e only involves the variables X_0 and $X_{e'}$, $e' \in \text{Anc}(e)$. So we conclude recursively that $\tilde{\partial}(X_e) \in x^{h-i-1} A[X_{e_{-1}}, (X_{e'})_{e' \in \text{Anc}(e)}]$. Thus $\tilde{\partial}$ restricts to a triangular A -derivation of $A[\Gamma]$. By construction, $\tilde{\partial}$ annihilates Δ_e for every $e \in \mathbf{P}(\Gamma)$. Moreover, $x\tilde{\partial}(\Delta_{e',e}) = \tilde{\partial}(x\Delta_{e',e}) \in I_\gamma$ for every pair $(e, e') \in (\mathbf{P}(\Gamma) \setminus \{e_0\}) \times ((\downarrow e)_\Gamma \setminus \{e_0\})$ by virtue of (2.2). Thus $\tilde{\partial}(\Delta_{e',e}) \in I_\gamma$ as I_γ is a prime ideal which does not contain x . Hence $\tilde{\partial}(I_\gamma) \subset I_\gamma$ and so, $\tilde{\partial}$ induces a locally nilpotent A -derivation ∂ of B_γ . \square

Example 2.16. We consider the following fine k -weighted tree $\tilde{\gamma} = (\Gamma, \tilde{w})$ with the leaves $e_{1,1}, e_{2,1}, e_{3,2}, e_{4,2}$.



We have $A[\Gamma] = k[x][X_{e_{-1}}, X_{e_0}, X_{e_1}]$ and

$${}^t \bar{M}(\tilde{\gamma}) = \begin{pmatrix} x & X_{e_{-1}} P(X_{e_{-1}}) & P(X_{e_{-1}}) P(X_{e_0}) \\ 1 & X_{e_0} & X_{e_1} \end{pmatrix},$$

where $P(t) = t^2 - 1 \in k[t]$. Therefore $\pi : S_{\tilde{\gamma}} \rightarrow X$ is the surface with equations

$$\begin{aligned} xX_{e_0} - X_{e_{-1}} P(X_{e_{-1}}) &= 0, & X_{e_{-1}} X_{e_1} - X_{e_0} P(X_{e_0}) &= 0, \\ xX_{e_1} - P(X_{e_{-1}}) P(X_{e_0}) &= 0. \end{aligned}$$

Letting $y = X_{e_{-1}}, z = X_{e_0}$ and $u = X_{e_1}$, the locally nilpotent derivation $\tilde{\partial}_{\tilde{\gamma},1,2} \in \text{Der}_A(A[\Gamma])$ is simply the derivation $\partial_{1,2} \in \text{Der}_{k[x]}(k[x][y, z, u])$ of Example 2.4.

3. Embeddings of Danielewski surfaces in affine spaces

In this section, we compare the two constructions of Danielewski surfaces by means of fine k -weighted trees. We describe a certain class of morphisms of Danielewski surfaces as the restrictions of suitable linear projections.

From abstract to embedded Danielewski surfaces. Here we prove the following result.

Theorem 3.1. *For every abstract Danielewski surface $\pi : S^\gamma \rightarrow X$ defined by a fine k -weighted tree $\gamma = (\Gamma, w)$, there exists another fine weight function $\tilde{w} : E(\Gamma) \rightarrow k$ on the tree Γ , and a closed embedding $\zeta : S^\gamma \hookrightarrow \mathbb{A}_X^{d(\Gamma)}$ inducing an isomorphism between S^γ and the embedded Danielewski surface $S_{\tilde{\gamma}}$ defined by the fine k -weighted tree $\tilde{\gamma} = (\Gamma, \tilde{w})$. Moreover, ζ is equivariant when we equip S^γ and $\mathbb{A}_X^{d(\Gamma)}$ with the $\mathbb{G}_{a,X}$ -actions corresponding to the locally nilpotent A -derivations $\partial_{a,m} \in \text{Der}_A(B^\gamma)$ (see 2.3) and $\tilde{\partial}_{\tilde{\gamma},a,m} \in \text{Der}_A(A[\Gamma])$ (Proposition 2.15) respectively.*

Example 3.2. We consider the abstract Danielewski surface $\pi : S^\gamma \rightarrow X$ defined by the fine k -weighted tree of Example 2.4. The canonical morphism $\psi : S^\gamma \rightarrow \mathbb{A}_X^1 = \text{Spec}(k[x][X_{e-1}])$ is given by the section $s_{e-1} \in B^\gamma$ whose restrictions on the canonical open subsets $S_i = \text{Spec}(k[x][T_i])$ are given by

$$s_{e-1}|_{S_i} = \begin{cases} (-1)^{i+1} + xT_i, & \text{if } i = 1, 2, \\ (-1)^{i+1}x + x^2T_i, & \text{if } i = 3, 4. \end{cases}$$

Letting $C_i = \pi^{-1}(x_0) \cap S_i$, $i = 1, \dots, 4$, be the irreducible components of $\pi^{-1}(x_0)$, we see that s_{e-1} restricts to a coordinate function on every fiber $\pi^{-1}(y)$, $y \in X_*$, and is locally constant on $\pi^{-1}(x_0)$ with the values 1, -1 and 0 on C_1 , C_2 and $C_3 \cup C_4$ respectively. Therefore, letting $P(t) = (t^2 - 1) \in k[t]$, the section $x^{-1}s_{e-1}P(s_{e-1}) \in B^\gamma \otimes_{k[x]} k[x, x^{-1}]$ extends to a section $s_{e_0} \in B^\gamma$ whose restrictions on the S_i 's are given by

$$s_{e_0}|_{S_i} = \begin{cases} 2T_1 + 3xT_1^2 + x^2T_1^3, & \text{if } i = 1, \\ 2T_2 - 3xT_2^2 + x^2T_2^3, & \text{if } i = 2, \\ -1 - xT_3 + x^2\xi_3(x, T_3), & \text{if } i = 3, \\ 1 - xT_4 + x^2\xi_4(x, T_4), & \text{if } i = 4, \end{cases}$$

for certain polynomials $\xi_3(x, t), \xi_4(x, t) \in k[x, t]$. Thus s_{e_0} restricts to a coordinate function on C_1 and C_2 , and is constant on C_3 and C_4 with the values -1 and 1 respectively. Again, $x^{-1}P(s_{e_0}) \in B^\gamma \otimes_{k[x]} k[x, x^{-1}]$ extends to a regular function on $S_3 \cup S_4 \subset S^\gamma$ which restricts to a coordinate function on C_3 and C_4 . Clearly, $x^{-1}P(s_{e-1})P(s_{e_0})$ extends to a section $s_{e_1} \in B^\gamma$ with the same property as

$P(s_{e_{-1}})|_{C_i} = -1, i = 3, 4$. The A -algebra homomorphism $A[X_{e_{-1}}, X_{e_0}, X_{e_1}] \rightarrow B^\nu, X_e \mapsto s_e$ defines a closed embedding $\zeta: S^\nu \rightarrow \mathbb{A}_X^3$, inducing an X -isomorphism between S^ν and the embedded Danielewski surface $S_{\tilde{\gamma}}$ defined by the fine k -weighted tree $\tilde{\gamma} = (\Gamma, \tilde{w})$ of Example 2.16.

3.3. To prove Theorem 3.1, we proceed in a similar way as in the previous example. More precisely, given an abstract Danielewski surface S^ν defined by a fine k -weighted tree $\gamma = (\Gamma, w)$, we construct in 3.4 and Lemmas 3.5–3.7 below a fine weight function $\tilde{w}: E(\Gamma) \rightarrow k$ on Γ and a collection of sections $s_e \in B^\nu, e \in \mathbf{P}(\Gamma) \cup \{e_{-1}\}$, which define a closed embedding $\zeta: S^\nu \hookrightarrow \mathbb{A}_X^{d(\Gamma)}$ inducing an X -isomorphism $\phi: S^\nu \xrightarrow{\sim} S_{\tilde{\gamma}}$ between S^ν and the embedded Danielewski surface defined by the tree $\tilde{\gamma} = (\Gamma, \tilde{w})$.

3.4. Given a fine k -weighted tree $\gamma = (\Gamma, w)$ with the leaves $e_{1,m_1}, \dots, e_{n,m_n}$, we denote by $\tau_i: B^\nu = \Gamma(S^\nu, \mathcal{O}_{S^\nu}) \rightarrow A[T_i]$ the localization homomorphisms corresponding to the canonical open covering of the abstract Danielewski surface S^ν by the open subsets $S_i = \text{Spec}(A[T_i]), i = 1, \dots, n$. The canonical X -morphism $\psi: S^\nu \rightarrow \mathbb{A}_X^1 = \text{Spec}(A[X_{e_{-1}}])$ (2.2) corresponds to the section $s_{e_{-1}} \in B^\nu$ such that

$$\tau_i(s_{e_{-1}}) = \sum_{j=0}^{m_i} w_{i,j} x^j \in A[T_i],$$

where

$$w_{i,j} = \begin{cases} w(\overleftarrow{e_{i,j}e_{i,j+1}}), & \text{if } 0 \leq j \leq m_i - 1, \\ T_i, & \text{if } j = m_i. \end{cases}$$

For every $e \in \Gamma$, we let

$$C_e = \bigsqcup_{\{e_i, m_i \in L((\uparrow e)\Gamma)\}} (\pi^{-1}(x_0) \cap S_i) \simeq \text{Spec} \left(\prod_{\{e_i, m_i \in L((\uparrow e)\Gamma)\}} \text{Spec}(k[T_i]) \right).$$

If γ has height $h = 0$ then Γ is the trivial tree with one element $\{e_0\}$ and $\psi: S^\nu \rightarrow \mathbb{A}_X^1$ is an isomorphism. Otherwise, if $h \geq 1$, then we have the following result.

Lemma 3.5. *If $h \geq 1$ then there exists a fine weight function $\tilde{w}: E(\Gamma) \rightarrow k, \overleftarrow{e'}e \mapsto \tilde{a}_{e',e}$ defining a fine k -weighted tree $\tilde{\gamma} = (\Gamma, \tilde{w})$, and a collection of sections $(s_e)_{e \in \mathbf{P}(\Gamma) \cup \{e_{-1}\}} \in B^\nu$ with the following properties.*

- a) For every $e_{i,j} \in \mathbf{P}(\Gamma), s_{e_{i,j}} = x^{-1}G_{e_{i,j}}(\tilde{\gamma})(s_{e_{-1}}, s_{e_0}, s_{e_{i,1}}, \dots, s_{e_{i,j-1}})$.
- b) If $\text{Ch}(e_{i,j}) = \{e_{i_1,j+1}, \dots, e_{i_r,j+1}\}$, then $s_{e_{i,j-1}}$ is constant on $C_{e_{i_l,j+1}} \subset \pi^{-1}(x_0)$ with the value $\tilde{a}_{e_{i,j},e_{i_l,j+1}} \in k, l = 1, \dots, r$.
- c) For every leaf e_{i,m_i} of $\Gamma, s_{e_{i,m_i-1}}$ induces an coordinate function on $C_{e_{i,m_i}} \simeq \mathbb{A}_k^1$.

Proof. We construct the weight function \tilde{w} and the sections s_e by induction as follows. For every $m = 0, \dots, h$, we denote by Γ_m the subtree of Γ with the elements $e \in \Gamma$ at levels $l \leq m$. At step m , we suppose that the weight function $\tilde{w}_m: E(\Gamma_m) \rightarrow k$ is constructed on Γ_m , as well as the sections s_e for every $e \in \Gamma_{m-2}$, and we define the sections $s_e, e \in \Gamma_{m-1} \setminus \Gamma_{m-2}$. Then we extend \tilde{w}_m to a weight function $\tilde{w}_{m+1}: E(\Gamma_{m+1}) \rightarrow k$.

Step 0. We let $s_{e_{-1}} \in B^\gamma$ be the section corresponding to the canonical morphism $\psi: S^\gamma \rightarrow \mathbb{A}_X^1$. By definition, $\tau_i(s_{e_{-1}}) = w_{i,0} + x\xi_i$ for a certain $\xi_i \in A[T_i]$ for every $i = 1, \dots, n$. Thus b) is satisfied provided that we define the weight function \tilde{w}_1 on $\Gamma_1 \setminus \{e_0\}$ by

$$\tilde{a}_{e_0, e_{i,1}} = \tilde{w}_1(\overleftarrow{e_0 e_{i,1}}) = s_{e_{-1}}|_{C_{e_{i,1}}} = w_{i,0} \in k$$

for every $e_{i,1} \in \text{Ch}(e_0)$. Note that if $e_{j,1} = e_{i,1}$, then $w_{i,0} = w_{j,0}$ as $w_{i,0} \neq w_{j,0}$ if and only if e_0 is the first common ancestor of the leaves e_{i,m_i} and e_{j,m_j} . Thus $\tilde{\gamma}_1 = (\Gamma_1, \tilde{w}_1)$ is a fine k -weighted tree and we are done with Step 0.

Step 1. By construction, the rational section $x^{-1}G_{e_0}(\tilde{\gamma}_1)(s_{\gamma, e_{-1}}) \in B^\gamma \otimes_A A_x$ extends to a section s_{e_0} of B^γ satisfying a). Since γ is a fine k -weighted tree, we deduce from Taylor's Formula that for every $i = 1, \dots, n$, there exists a pair $(\alpha_{i,1} = F_{e_0}^{(e_{i,1})}(w_{i,0}), \beta_{i,1}) \in k^* \times k$ depending only of the subchain $(\downarrow e_{i,1})_\Gamma$, and a polynomial $\xi_{i,1} \in A[T_i]$ such that

$$\tau_i(s_{e_0}) = \alpha_{i,1}w_{i,1} + \beta_{i,1} + x\xi_{i,1} \in A[T_i].$$

Thus, if $e_{i,1}$ is a leaf of Γ then $w_{i,1} = T_i$ and so c) is satisfied. Otherwise, if $e_{j,2}$ and $e_{j',2}$ are children of $e_{i,1}$ then $\alpha_{j,1} = \alpha_{j',1} = \alpha_{i,1}$ and $\beta_{j,1} = \beta_{j',1} = \beta_{i,1}$ as $e_{j,1} = e_{j',1} = e_{i,1}$, whereas $w_{j,1} \neq w_{j',1}$ as γ is a fine k -weighted tree. Thus $\tilde{\gamma}_2 = (\Gamma_2, \tilde{w}_2)$ is a fine k -weighted tree for the weight function $\tilde{w}_2: E(\Gamma_2) \rightarrow k$ restricting to \tilde{w}_1 on $\Gamma_1 \subset \Gamma_2$ and such that

$$\tilde{a}_{e_{i,1}, e_{i,2}} = \tilde{w}_2(\overleftarrow{e_{i,1} e_{i,2}}) = s_{e_0}|_{C_{e_{i,2}}} = (\alpha_{i,0}w_{i,1} + \beta_{i,1}) \in k, \quad i = 1, \dots, n.$$

By construction, b) is also satisfied. This completes Step 1.

Step $m, m \geq 2$. By induction hypothesis, $\tilde{\gamma}_m = (\Gamma_m, \tilde{w}_m)$ is a fine k -weighted tree, and the sections $s_e \in B^\gamma, e \in \Gamma_{m-2}$, satisfying the hypothesis of Lemma 3.5 have been defined. So the formula

$$s_{e_{i,m-1}} = x^{-1}G_{e_{i,m-1}}(\tilde{\gamma}_m)(s_{e_{-1}}, s_{e_0}, s_{e_{i,1}}, \dots, s_{e_{i,m-2}})$$

makes sense and defines an element of $B^\gamma \otimes_A A_x$. Similarly as in Step 1, we deduce from Taylor's Formula that for every $j = 0, \dots, m-1$ there exists a pair $(\tilde{\alpha}_{i,j}, \tilde{\beta}_{i,j}) \in k^* \times k$ depending only on the subchain $(\downarrow e_{i,j})_\Gamma$, and a polynomial $\tilde{\xi}_{i,j} \in A[T_i]$ such that

$$\tau_i(s_{e_{i,j-1}}) = a_{e_{i,j+1}e_{i,j}} + x(\tilde{\alpha}_{i,j}w_{i,j+1} + \tilde{\beta}_{i,j}) + x^2\tilde{\xi}_{i,j} \in A[T_i].$$

By applying Taylor’s Formula again, we conclude that there exists a pair $(\alpha_{i,m}, \beta_{i,m}) \in k^* \times k$ depending only on the subchain $(\downarrow e_{i,m})_\Gamma$ and a polynomial $\xi_{i,m} \in A[T_i]$ such that

$$\tau_i(s_{e_{i,m-1}}) = \alpha_{i,m} w_{i,m} + \beta_{i,m} + x \xi_{i,m} \in A[T_i].$$

Thus, if $e_{i,m-1} \in (\downarrow e_{j,m_j})$ then $e_{i,m-1} = e_{j,m-1}$ and so $\tau_j(s_{e_{i,m-1}}) \in A[T_j]$. Otherwise, for every index j such that $e_{i,m-1} \notin (\downarrow e_{j,m_j})_\Gamma$, the first common ancestor of $e_{i,m-1}$ and e_{j,m_j} is an element $e_{i,l} = e_{j,l}$ at level $l \leq \min(m-2, m_j-1)$. Thus $(X_{e_{j,l-1}} - \tilde{a}_{e_{j,l}, e_{j,l+1}})$ divides the genealogical polynomial $G_{e_{i,m-1}}(\tilde{\gamma}_m)$ of $e_{i,m-1}$. Since $\tau_j(s_{e_{j,l-1}} - \tilde{a}_{e_{j,l}, e_{j,l+1}}) \in xA[T_j]$, we conclude that

$$x\tau_j(s_{e_{i,m-1}}) = G_{e_{i,m-1}}(\tilde{\gamma}_m)(\tau_j(s_{e_{-1}}), \tau_j(s_{e_{i,0}}), \tau_j(s_{e_{i,1}}), \dots, \tau_j(s_{e_{i,m-2}})) \in xA[T_j].$$

Thus $\tau_j(s_{e_{i,m-1}}) \in A[T_j]$ for every $j = 1, \dots, n$, and hence, $s_{\gamma, e_{i,m-1}} \in B^\gamma$. If $e_{i,m}$ is a leaf of Γ then $w_{i,m} = w_{i,m_i} = T_i$ by definition. Thus $s_{e_{i,m-1}}$ satisfies a) and c). Finally, the same argument as in Step 1 shows that $\tilde{\gamma}_{m+1} = (\Gamma_{m+1}, \tilde{w}_{m+1})$ is a fine k -weighted tree for the weight function $\tilde{w}_{m+1}: E(\Gamma_{m+1}) \rightarrow k$ restricting to \tilde{w}_m on $\Gamma_m \subset \Gamma_{m+1}$ and such that

$$\tilde{a}_{e_{i,m}, e_{i,m+1}} = \tilde{w}_{m+1}(\overleftarrow{e_{i,m} e_{i,m+1}}) = s_{e_{i,m-1}}|_{C(e_{i,m+1})} = (\alpha_{i,m} w_{i,m} + \beta_{i,m}) \in k,$$

whenever $e_{i,m}$ is not a leaf of Γ . This completes Step m as b) is satisfied by construction.

After $h = h(\Gamma)$ steps, the above procedure stops, and we obtain a fine k -weighted tree $\tilde{\gamma} = \tilde{\gamma}_h = (\Gamma, \tilde{w}_h)$ and a collection of sections $(s_e)_{e \in \mathbf{P}(\Gamma) \cup \{e_{-1}\}} \in B^\gamma$ satisfying conditions a), b) and c). This completes the proof. \square

The following lemma implies the first assertion of Theorem 3.1.

Lemma 3.6. *The X -morphism $\zeta: S^\gamma \rightarrow \mathbb{A}_X^{d(\Gamma)}$ induced by the A -algebra homomorphism $\zeta^*: A[\Gamma] \rightarrow B^\gamma, X_e \mapsto s_e, e \in \mathbf{P}(\Gamma) \cup \{e_{-1}\}$, is a closed embedding inducing an X -isomorphism $\phi: S^\gamma \xrightarrow{\sim} S_{\tilde{\gamma}}$.*

Proof. By construction, $s_{e_{-1}}$ corresponds to the canonical birational morphism $\psi: S^\gamma \rightarrow \mathbb{A}_X^1$, whence induces a X_* -isomorphism $S^\gamma|_{X_*} \xrightarrow{\sim} \mathbb{A}_{X_*}^1$. By b) of Lemma 3.5, for every pair e_{i,m_i}, e_{j,m_j} of leaves of Γ with first common ancestor $e \in \Gamma$, the section $s_{\text{Par}(e)}$ takes distinct constant values on $C_{e_{i,m_i}}$ and $C_{e_{j,m_j}}$. Thus ζ distinguishes the irreducible components of the fiber $\pi^{-1}(x_0)$. Finally, c) of Lemma 3.5 implies that for every $i = 1, \dots, n, s_{e_{i,m_i-1}}$ induces a coordinate function on $C_{e_{i,m_i}} \simeq \mathbb{A}_k^1$. This proves that $\zeta: S^\gamma \rightarrow \mathbb{A}_X^{d(\Gamma)}$ is an embedding. By construction, $\zeta^*(\Delta_e(\tilde{\gamma})) = 0$ in B^γ for every $e \in \mathbf{P}(\Gamma)$. Thus $x\zeta^*(\Delta_{e',e}(\tilde{\gamma})) = \zeta^*(x\Delta_{e',e}(\tilde{\gamma})) = 0$ for every $(e, e') \in (\mathbf{P}(\Gamma) \setminus \{e_0\}) \times ((\downarrow e)_\Gamma \setminus \{e_0\})$ by virtue of (2.2), and so, $\zeta^*(\Delta_{e',e}(\tilde{\gamma})) = 0$

as B^γ is an integral A -algebra. This proves that the image of ζ is contained in the embedded Danielewski surface $S_{\tilde{\gamma}}$. It is clear by construction that the induced X -morphism $\phi: S^\gamma \rightarrow S_{\tilde{\gamma}}$ restricts to a bijection between the sets of closed points of S^γ and $S_{\tilde{\gamma}}$ respectively. So the result follows from Zariski's Main Theorem as $S_{\tilde{\gamma}}$ is smooth over k , whence, in particular, normal. \square

The following result completes the proof of Theorem 3.1.

Lemma 3.7. *For every nontrivial $\mathbb{G}_{a,X}$ -action $t_{\gamma,a,m}$ (2.3) on an abstract Danielewski surface $\pi: S^\gamma \rightarrow X$ defined by a fine k -weighted tree $\gamma = (\Gamma, w)$, the closed embedding $\zeta: S^\gamma \hookrightarrow \mathbb{A}_X^{d(\Gamma)}$ in Lemma 3.6 is equivariant when we equip $\mathbb{A}_X^{d(\Gamma)}$ with the $\mathbb{G}_{a,X}$ -action induced by the locally nilpotent A -derivation $\tilde{\partial}_{\tilde{\gamma},a,m} \in \text{Der}_A(A[\Gamma])$ (Proposition 2.15).*

Proof. By definition (see 2.3), the twisted translation $t_{\gamma,a,m}$ on S^γ is induced by the extension $\partial_{a,m}$ to B^γ of the locally nilpotent derivation $\delta_{a,m} = ax^m \partial_{X_{e_{-1}}}$ of $B^\gamma \otimes_A A_x \simeq A_x[X_{e_{-1}}]$, where $m \geq h(\Gamma)$ and $a \in A \setminus \{0\}$. By construction, for every $e \in \mathbf{P}(\Gamma)$, we have $s_e = x^{-1} G_e(\tilde{\gamma})(s_{e_{-1}}, s_{e_0}, \dots, s_{\text{Par}(e)}) \in B^\gamma \subset A_x[X_{e_{-1}}]$ and so,

$$\partial_{a,m}(s_e) = x^{-1} \sum_{e' \in \text{Anc}(e) \cup \{e_{-1}\}} \partial_{X_{e'}} G_e(\tilde{\gamma})(s_{e_{-1}}, s_{e_0}, \dots, s_{\text{Par}(e)}) \partial_{a,m}(s_{e'}) \in B^\gamma \otimes_A A_x.$$

In view of the definition of $\tilde{\partial}_{\tilde{\gamma},a,m} \in \text{Der}_A(A[\Gamma])$ (see Proposition 2.15), this means precisely that the embedding $\zeta: S^\gamma \hookrightarrow \mathbb{A}_X^{d(\Gamma)}$ is equivariant when we equip S^γ and $\mathbb{A}_X^{d(\Gamma)}$ with the actions corresponding to the locally nilpotent derivation $\partial_{a,m}$ and $\tilde{\partial}_{\tilde{\gamma},a,m}$. \square

Corollary 3.8. *Every Danielewski surface $\pi: S \rightarrow X$ equipped with a nontrivial $\mathbb{G}_{a,X}$ -action is equivariantly X -isomorphic to an embedded Danielewski surface S_γ defined by a fine k -weighted tree $\gamma = (\Gamma, w)$, equipped with the $\mathbb{G}_{a,X}$ -action corresponding to a suitable locally nilpotent derivation $\partial_{\gamma,a,m} \in \text{Der}_A(B_\gamma)$, where $m \geq h(\Gamma)$ and $a \in A \setminus \{0\}$.*

Proof. By Theorem 3.2 in [4], every Danielewski surface S is isomorphic to an abstract Danielewski surface S^γ defined by a fine k -weighted tree γ . Moreover, by Proposition 2.12 in *loc. cit.*, every nontrivial $\mathbb{G}_{a,X}$ -action on S^γ coincides with a twisted translation $t_{\gamma,a,m}$ for a suitable pair ($m \geq h(\Gamma)$, $a \in A \setminus \{0\}$). So the result follows from Theorem 3.1. \square

Corollary 3.9. *Every $\mathbb{G}_{a,X}$ -action on an embedded Danielewski surface S_γ defined by a fine k -weighted tree $\gamma = (\Gamma, w)$ is induced by a locally nilpotent derivation $\partial_{\gamma,a,m} \in \text{Der}_A(B_\gamma)$.*

Since the locally nilpotent derivations $\partial_{\gamma,a,m} \in \text{Der}_A(B_\gamma)$ are induced by locally nilpotent derivations $\tilde{\partial}_{\gamma,a,m} \in \text{Der}_A(A[\Gamma])$, we obtain the following result.

Corollary 3.10. *Every Danielewski surface $\pi : S \rightarrow X$ admits a closed embedding $\zeta : S \hookrightarrow \mathbb{A}_X^d$ into a relative affine space \mathbb{A}_X^d , where $d \geq 1$, such that every $\mathbb{G}_{a,X}$ -action on S extends to an action on \mathbb{A}_X^d .*

In particular, if the Makar-Limanov invariant of S is nontrivial, then $\pi : S \rightarrow X$ is a unique \mathbb{A}^1 -fibration on S up to automorphisms of X . Therefore, the general orbits of a $\mathbb{G}_{a,k}$ -action on S coincide with the general fibers of π . This leads to the following result.

Corollary 3.11. *Every Danielewski surface S with a nontrivial Makar-Limanov invariant admits a closed embedding into an affine space \mathbb{A}_k^d in such a way that every $\mathbb{G}_{a,k}$ -action on S extends to an action on \mathbb{A}_k^d .*

Morphisms of Danielewski surfaces as linear projections. A morphism of Danielewski surfaces is a birational X -morphism $\beta : S' \rightarrow S$, restricting to an isomorphism over X_* . In other words, β is an affine modification [7] restricting to an isomorphism over the complement of the support of the principal divisor $\pi^{-1}(x_0) = \text{div}(x) \subset S$. Thus, letting $S = \text{Spec}(B)$, there exists an ideal $I \subset B$ containing a power x^m of x such that S' is isomorphic to the open subset $\text{Spec}(B[It]/(1 - x^m t))$ of the spectrum of the Rees algebra $B[It]$. In turn, this implies that $S' \simeq \text{Spec}(B[t_1, \dots, t_r]/J)$ for a certain ideal J . In these coordinates, the morphism $\beta : S' \rightarrow S$ coincides with the restriction to S' of the projection $\text{pr}_S : \mathbb{A}_S^{r+1} = \text{Spec}(B[t_1, \dots, t_r]) \rightarrow S$. Here we give a more precise description of this situation.

3.12. To every morphism $\tau : \gamma' = (\Gamma', w') \rightarrow \gamma = (\Gamma, w)$ of fine k -weighted tree (see Definition 1.4), we associate a morphism $\beta_\tau : S^{\gamma'} \rightarrow S^\gamma$ between the associated abstract Danielewski surfaces in the following manner. We let $\sigma' = \{\sigma'_i \in A\}_{i=1, \dots, n'}$ and $\sigma = \{\sigma_j \in A\}_{j=1, \dots, n}$ be the collection of polynomials associated with γ' and γ , and we let $g' = \{g'_{ij} \in A_x\}$ and $g = \{g_{ij} \in A_x\}$ be the corresponding transition functions. We denote by $S'_i = \text{Spec}(A[T'_i])$, $i = 1, \dots, n'$, and $S_j = \text{Spec}(A[T_j])$, $j = 1, \dots, n$, the open subsets of the canonical coverings of $S^{\gamma'}$ and S^γ respectively. By Remark 1.5, the image of a leaf e'_{i,m'_i} of Γ' by τ is a leaf $e_{j(i),m_{j(i)}}$ of Γ such that $m'_i \geq m_{j(i)}$ and $\tau(e'_{i,k}) = e_{j(i),\min(k,m_{j(i)})}$ for every $k = 0, \dots, m'_i$. Since $w(\overleftarrow{\tau(e'_{i,k})\tau(e'_{i,k+1})}) = w'(\overleftarrow{e'_{i,k}e'_{i,k+1}})$ whenever $\tau(e'_{i,k}) \neq \tau(e'_{i,k+1})$, we conclude that there exists a collection $\sigma'' = \{\sigma''_i \in A\}_{i=1, \dots, n'}$ such that $\sigma'_i = \sigma_{j(i)} + x^{m_{j(i)}}\sigma''_i \in A$ for every $i = 1, \dots, n'$. Then for every $i = 1, \dots, n'$, the A -algebra homomorphism

$$A[T_{j(i)}] \longrightarrow A[T'_i], \quad T_{j(i)} \mapsto \sigma''_i + x^{m'_i - m_{j(i)}} T'_i$$

defines a birational X -morphism $\beta_\tau^{(i)}: S'_i \rightarrow S_{j(i)}$ restricting to an isomorphism over X_* . Since the transition functions satisfy the relation $x^{m_{i'}-m_{j(i)}} g'_{il} = g_{j(i)j(l)} + x^{m_{j(i)}-m_{j(i)}} \sigma'_i + \sigma''_i$ for every $i, l = 1, \dots, n'$, it follows that these local morphisms $\beta_\tau^{(i)}$ glue to a morphism of Danielewski surfaces $\beta_\tau: S^{\nu'} \rightarrow S^\nu$. By Proposition 3.8 and Corollary 3.9 in [4], for every morphism of Danielewski surfaces $\beta: S' \rightarrow S$, there exists X -isomorphisms $\phi': S' \xrightarrow{\sim} S^{\nu'}$ and $\phi: S \xrightarrow{\sim} S^\nu$ for suitable fine k -weighted trees γ' and γ such that $\phi \circ \beta \circ (\phi')^{-1}$ is the morphism β_τ induced by a morphism of fine k -weighted tree $\tau: \gamma' \rightarrow \gamma$.

3.13. Every morphism of fine k -weighted tree $\tau: \gamma' \rightarrow \gamma$ factors through a surjective morphism $\tau': \gamma' \rightarrow \tau(\gamma')$ followed by an injection $\tau(\gamma') \hookrightarrow \gamma$. As a consequence, every morphism of Danielewski surfaces factors through a *quasi-surjective morphism* $\beta': S^{\nu'} \rightarrow S^{\tau(\gamma')}$, i.e. a morphism of Danielewski surfaces such that $\beta'^{-1}(C) \neq \emptyset$ for every irreducible component C of the fiber $\pi_{\tau(\gamma')}^{-1}(x_0) \subset S^{\tau(\gamma')}$ followed by the open immersion of $S^{\tau(\gamma')}$ in S^ν as the complement of irreducible components of $\pi_\nu^{-1}(x_0) \subset S^\nu$ corresponding to the leaves of Γ which are not in the image of τ .

3.14. Given a fine k -weighted tree $\gamma = (\Gamma, w)$, we consider the tree $\tilde{\gamma} = (\Gamma, \tilde{w})$ constructed in Lemma 3.5. For every edge $\overleftarrow{e'e}$ of Γ , the weight $\tilde{w}(\overleftarrow{e'e}) \in k$ is uniquely determined by the weights w of the edges of the subtree of Γ with elements $(\downarrow e)_\Gamma \cup \bigcup_{e' \in (\downarrow e)_\Gamma} \text{Ch}(e')$. Therefore, every *surjective* morphism of fine k -weighted trees $\tau: \gamma' = (\Gamma', w') \rightarrow \gamma$ gives rise to a surjective morphism of fine k -weighted trees $\tilde{\tau}: \tilde{\gamma}' = (\Gamma', \tilde{w}') \rightarrow \tilde{\gamma}$ which restricts to the same morphism as τ between the underlying trees Γ' and Γ of $\tilde{\gamma}'$ and $\tilde{\gamma}$ respectively¹. Since the subset $\Gamma'' = \{e' \in \Gamma', \tau^{-1}(\tau(e')) = \{e'\}\} \subset \Gamma'$ is a subtree of Γ' isomorphic to Γ , we obtain that

$$A[\Gamma'] = A[\Gamma''] \otimes_A A[(X_{e'})_{e' \in \mathbf{P}(\Gamma') \cap (\Gamma' \setminus \mathbf{P}(\Gamma''))}] \simeq A[\Gamma] \otimes_A A[(X_{e'})_{e' \in \mathbf{P}(\Gamma') \cap (\Gamma' \setminus \mathbf{P}(\Gamma''))}].$$

Moreover, for every $e' \in \mathbf{P}(\Gamma'')$, the genealogical polynomial $G_{e'}(\tilde{\gamma}')$ of e' is an element of $A[\Gamma''] \subset A[\Gamma']$ which coincides with the genealogical polynomial $G_{\tau(e')}(\tilde{\gamma}) \in A[\Gamma]$ of $\tau(e')$ via the isomorphism above. In turn, this implies that the genealogical matrix (see Definition 1.9) $M(\tilde{\gamma})$ of $\tilde{\gamma}$ is obtained from $M(\tilde{\gamma}')$ by deleting the rows corresponding to the elements in $\mathbf{P}(\Gamma') \setminus \mathbf{P}(\Gamma'')$. By construction of the embedding of S^ν into $\mathbb{A}_X^{d(\Gamma)}$ as the Danielewski surface $S_{\tilde{\gamma}}$, we obtain the following result.

Theorem 3.15. *Let $\tau: \gamma' = (\Gamma', w') \rightarrow \gamma = (\Gamma, w)$ be a surjective morphism of fine k -weighted trees and let $\tilde{\tau}: \tilde{\gamma}' \rightarrow \tilde{\gamma}$ be the morphism obtained above. Let*

¹Actually, the functor $\gamma \mapsto \tilde{\gamma}, \tau \mapsto \tilde{\tau}$ is an automorphism of the category $\mathcal{T}_{w,k}^s$ of fine k -weighted trees equipped with surjective morphisms.

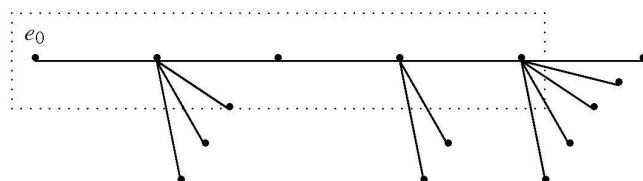
$\zeta': S^{\gamma'} \hookrightarrow \mathbb{A}_X^{d(\Gamma')}$ and $\zeta: S^\gamma \hookrightarrow \mathbb{A}_X^{d(\Gamma)}$ are the embeddings from Lemma 3.6 of $S^{\gamma'}$ and S^γ as the Danielewski surfaces $S_{\tilde{\gamma}'}$ and $S_{\tilde{\gamma}}$ respectively. Then $\zeta \circ \beta = p_{\Gamma'/\Gamma} \circ \zeta'$, where $p_{\Gamma'/\Gamma}: \mathbb{A}_X^{d(\Gamma')} \rightarrow \mathbb{A}_X^{d(\Gamma)}$ is the projection induced by the inclusion $A[\Gamma] \simeq A[\Gamma'] \subset A[\Gamma']$.

4. Danielewski surfaces with a trivial Makar-Limanov invariant

The Makar-Limanov [6] invariant of an affine variety $V = \text{Spec}(B)$ over a field k of characteristic zero is the sub-algebra $\text{ML}(V) \subset B$ of regular functions on V which are invariant under every $\mathbb{G}_{a,k}$ -action on V . A surface S has a trivial Makar-Limanov invariant $\text{ML}(S) = k$ if and only if it admits two nontrivial $\mathbb{G}_{a,k}$ -actions with distinct general orbits. In view of the correspondence between nontrivial $\mathbb{G}_{a,k}$ -actions $\mathbb{G}_{a,k} \times S \rightarrow S$ on S and quotient \mathbb{A}^1 -fibrations $\pi: S \rightarrow X = S/\mathbb{G}_{a,k}$, this means in turn that S has a trivial Makar-Limanov invariant if and only if it admits two \mathbb{A}^1 -fibrations with distinct general fibers. In this section, we characterize among Danielewski surfaces the ones with a trivial Makar-Limanov invariant.

Danielewski surfaces defined by weighted combs

Definition 4.1. A nontrivial (oriented) comb of height $h \geq 1$ is a tree Γ such that for every $e \in \mathbf{P}(\Gamma)$ of degree $\text{deg}_\Gamma(e) \geq 1$, all but possibly one of the children of e are leaves of Γ . This means equivalently that the subtree $C_\Gamma = \mathbf{P}(\Gamma) = \{e_0 < \dots < e_{h-1}\}$ of Γ is a nonempty chain of length $h - 1$, called the dorsal chain of Γ .



A comb rooted in e_0 .

4.2. By Theorem 5.4 in [4], a Danielewski surface S defined over an algebraically closed field $k = \bar{k}$ of characteristic zero has a trivial Makar-Limanov invariant if and only if it is isomorphic to an abstract Danielewski surface S^γ defined by a fine k -weighted comb. This result is based on a characterization of normal affine surfaces S with a trivial Makar-Limanov invariant in terms on the boundary divisors of certain

minimal completions \bar{S} of S (see [3]). Unfortunately, no such criterion exists for a normal affine surface defined over an arbitrary field k of characteristic zero. However, the following result shows that the combinatorial characterization of Danielewski surfaces with a trivial Makar-Limanov invariant remains valid in this more general setting.

Theorem 4.3. *A Danielewski surface $S \not\cong \mathbb{A}_X^1$, defined over a field k of characteristic zero, has a trivial Makar-Limanov invariant if and only if it is isomorphic to an abstract Danielewski surface S^γ defined by a fine k -weighted comb. If this is the case, then there exist an integer $h \geq 1$ and a collection of monic polynomials $P_0, \dots, P_{h-1} \in k[t]$ with simple roots $a_{i,j} \in k^*$, $i = 0, \dots, h-1$, $j = 1, \dots, \deg_t(P_i)$, such that S is isomorphic to the surface $S_{P_0, \dots, P_{h-1}} \subset \text{Spec}(k[x][y_{-1}, \dots, y_{h-2}][z])$ defined by the equations*

$$\begin{aligned} xz - y_{h-2} \prod_{l=0}^{h-1} P_l(y_{l-1}) &= 0, \\ zy_{i-1} - y_i y_{h-2} \prod_{l=i+1}^{h-1} P_l(y_{l-1}) &= 0, \quad xy_i - y_{i-1} \prod_{l=0}^i P_l(y_{l-1}) = 0, \quad 0 \leq i \leq h-2, \\ y_{i-1} y_j - y_i y_{j-1} \prod_{l=i+1}^j P_l(y_{l-1}) &= 0, \quad 0 \leq i < j \leq h-2. \end{aligned}$$

4.4. The proof is given in 4.5–4.7 below. We first observe that the condition is necessary. Indeed, suppose that the Makar-Limanov invariant of S is trivial. We let $\gamma = (\Gamma, w)$ be a fine k -weighted tree such that $S \simeq S^\gamma$, and we let $i: k \hookrightarrow \bar{k}$ be the injection of k in an algebraic closure \bar{k} . Then the Danielewski surface $S_{\bar{k}} = S \times_{\text{Spec}(k)} \text{Spec}(\bar{k}) \rightarrow X_{\bar{k}} = X \times_{\text{Spec}(k)} \text{Spec}(\bar{k})$ is $X_{\bar{k}}$ -isomorphic to the abstract Danielewski surface $S^\gamma \times_{\text{Spec}(k)} \text{Spec}(\bar{k})$ defined by the tree γ considered a fine \bar{k} -weighted tree via the weight function $i \circ w: E(\Gamma) \rightarrow \bar{k}$. Since every nontrivial $\mathbb{G}_{a,k}$ -action on S lifts to a nontrivial action of $\mathbb{G}_{a,\bar{k}} = \mathbb{G}_{a,k} \times_{\text{Spec}(k)} \text{Spec}(\bar{k})$ on $S_{\bar{k}}$, we conclude that $S_{\bar{k}}$ has a trivial Makar-Limanov invariant too. Thus the tree γ is a comb by virtue of Theorem 5.4 in [4].

4.5. Conversely, the same argument shows that if S is isomorphic to an abstract Danielewski surface S^γ defined by a fine k -weighted comb γ , then $S_{\bar{k}}$ has a trivial Makar-Limanov invariant. Unfortunately, in general, there is no guarantee that a given $\mathbb{G}_{a,\bar{k}}$ -action on $S_{\bar{k}}$ appears as the lifting of an action of $\mathbb{G}_{a,k}$ on S . Therefore, to show that the condition is sufficient, we must proceed in a different way. We will exploit the fact that S is isomorphic to an embedded surface S_γ defined by a fine k -weighted comb γ to construct two explicit \mathbb{A}^1 -fibrations on S with distinct general fibers.

4.6. By construction, a Danielewski surface S is isomorphic to \mathbb{A}_X^1 if and only if it is isomorphic to an abstract surface S^γ defined by a fine k -weighted chain γ . In this case it is also isomorphic to the surface $S_{\{e_0\}}$ defined by the trivial tree with one element $\{e_0\}$. More generally, it follows from Theorem 3.10 in [4] that every Danielewski surface $S \not\cong \mathbb{A}_X^1$ isomorphic to an abstract Danielewski surface S^γ defined by a fine k -weighted comb γ is also isomorphic to a surface S^{γ_0} defined by a fine k -weighted comb $\gamma_0 = (\Gamma, w_0)$ of height $h \geq 1$, with dorsal chain $C_\Gamma = \{e_0 < e_1 < \dots < e_{h-1}\}$, satisfying the following properties:

- a) The root e_0 of Γ as at least two children.
- b) For every $i = 0, \dots, h - 2$, $w_0(\overleftarrow{e_i e_{i+1}}) = 0 \in k$.
- c) There exists $e_h \in \text{Ch}(e_{h-1})$ such that $w_0(\overleftarrow{e_{h-1} e_h}) = 0 \in k$.

By definition, the restriction of the canonical morphism $\psi: S^{\gamma_0} \rightarrow \mathbb{A}_X^1$ to an open subset $S_i = \text{Spec}(A[T_i])$ corresponding to a leaf e_{i,m_i} of Γ at level $m_i \geq 1$ is induced by the section $w_0(\overleftarrow{e_{m_i-1} e_{i,m_i}})x^{m_i-1} + x^{m_i}T_i$. Thus, by applying the procedure used in the proof of Lemma 3.5 to this comb γ_0 , we obtain a fine k -weighted comb $\tilde{\gamma}_0 = (\Gamma, \tilde{w}_0)$ with the same underlying comb Γ as γ_0 such that $\tilde{w}_0(\overleftarrow{e_i e_{i+1}}) = 0 \in k$ for every $i = 0, \dots, h - 1$.

4.7. By construction of the tree $\tilde{\gamma}_0$, there exists monic polynomials $P_0, \dots, P_{h-1} \in k[t]$, of degrees $\deg(P_i) = \deg_\Gamma(e_i) - 1$, with simple roots $\tilde{a}_{e,e_i} \in k^*$, $e \in \text{Ch}(e_i) \setminus \{e_{i+1}\}$ respectively, such that $F_{e_i}(\tilde{\gamma}_0) = X_{e_{i-1}}P_i(X_{e_{i-1}})$ for every $i = 0, \dots, h - 1$. Letting $y_{-1} = X_{e_{-1}}$, $y_0 = X_{e_0}, \dots, y_{h-2} = X_{e_{h-2}}, z = X_{e_{h-1}}$, we conclude that the embedded Danielewski surface $S_{\tilde{\gamma}_0}$ is X -isomorphic to the surface $S_{P_0, \dots, P_{h-1}}$ of Theorem 4.3. This shows that every abstract Danielewski surface $S^\gamma \not\cong \mathbb{A}_X^1$ defined by a fine k -weighted comb γ is X -isomorphic to a surface $S_{P_0, \dots, P_{h-1}} \subset \mathbb{A}_X^{h+1}$. Thus, to complete the proof of Theorem 4.3, it suffices to show that a surface $S = S_{P_0, \dots, P_{h-1}} = \text{Spec}(B)$ has a trivial Makar-Limanov invariant. A similar argument as in 2.10 shows that $B \otimes_{k[z]} k[z, z^{-1}] \simeq k[z, z^{-1}][y_{h-2}]$. This means equivalently that the projection $\pi_2 = \text{pr}_z|_S: S \rightarrow Z = \text{Spec}(k[z])$ in an \mathbb{A}^1 -fibration restricting to the trivial line bundle $\mathbb{A}_{Z^*}^1 = \text{Spec}(k[z, z^{-1}][y_{h-2}])$ over Z^* . Since the general fibers of the two projections $\pi_1 = \text{pr}_x|_S: S \rightarrow X = \text{Spec}(k[x])$ and $\pi_2: S \rightarrow Z$ do not coincide, we conclude that S has a trivial Makar-Limanov invariant. This completes the proof of Theorem 4.3.

Remark 4.8. The same argument as in the proof of Proposition 2.15 applied to the fibration π_2 shows that the locally nilpotent derivation $z^h \partial_{y_{h-2}}$ of $B \otimes_{k[z]} k[z, z^{-1}] \simeq k[z, z^{-1}][y_{h-2}]$ extends to a locally nilpotent derivation of B , induced by a triangular $k[z]$ -derivation of $k[z][y_{h-2}, \dots, y_{-1}, x]$. This proves that every Danielewski surface S with a trivial Makar-Limanov invariant can be embedded in an affine space \mathbb{A}_k^d in such a way that at least two algebraically independent $\mathbb{G}_{a,k}$ -actions on S extend to $\mathbb{G}_{a,k}$ -actions on \mathbb{A}_k^d .

Nonconjugated \mathbb{G}_a -actions on a Danielewski surface. By a result of Daigle [2], all the $\mathbb{G}_{a,k}$ -actions on a Danielewski surface $S_{P,1} = \{xz - P(y)\}$ are conjugated under the action of the automorphism group $\text{Aut}(S_{P,1})$ of $S_{P,1}$.

4.9. This means that for every pair of nontrivial locally nilpotent derivations ∂_1 and ∂_2 of $B = \Gamma(S_{P,1}, \mathcal{O}_{S_{P,1}})$, there exists a k -algebra automorphism ϕ of B such that $\phi(\text{Ker}(\partial_1)) = \text{Ker}(\partial_2)$. This implies in particular that the fibers of corresponding quotient \mathbb{A}^1 -fibrations $\pi_1: S_{P,1} \rightarrow \mathbb{A}_k^1$ and $\pi_2: S_{P,1} \rightarrow \mathbb{A}_k^1$ have the same scheme-theoretic structures. By 4.7 above, a Danielewski surface $S = S_{P_0, \dots, P_{h-1}} = \text{Spec}(B)$ admits two \mathbb{A}^1 -fibrations $\pi_1: S \rightarrow X = \text{Spec}(k[x])$ and $\pi_2: S \rightarrow Z = \text{Spec}(k[z])$. Moreover π_2 restricts to the trivial line bundle over $Z_* = \text{Spec}(k[z, z^{-1}])$, and a similar argument as in Lemma 2.13 shows that the fiber $(\pi_2^{-1}(0))_{\text{red}}$ decomposes as a disjoint union of curves isomorphic to the affine line \mathbb{A}_k^1 . However, we have the following result.

Lemma 4.10. *If $h \geq 2$, then $\pi_2: S = S_{P_0, \dots, P_{h-1}} \rightarrow Z$ is not a Danielewski surface over Z .*

Proof. It suffices to show that the intersection of the fiber $\pi_2^{-1}(0)$ with the complement of the fiber $\pi_1^{-1}(0)$ is a nonreduced scheme. By (2.2), the defining ideal I_* of $S \setminus \pi_1^{-1}(0) \simeq \mathbb{A}_{X_*}^1$ in $k[x, x^{-1}][y_{-1}, \dots, y_{h-2}][z]$ is generated by the polynomials $c_i = y_i - x^{-1}y_{i-1} \prod_{l=0}^i P_l(y_{l-1})$, $i = 0, \dots, h-2$ and $d = z - x^{-1}y_{h-2} \prod_{l=0}^{h-1} P_l(y_{l-1})$. We conclude recursively that there exists a polynomial $R \in k[x, x^{-1}][y_{-1}]$ such that

$$d \equiv z - x^{-h}y_{-1}(P_0(y_{-1}))^h R(y_{-1})$$

modulo c_0, \dots, c_{h-2} . Since the polynomial P_0 is nonconstant (see 4.6),

$$\begin{aligned} (S \setminus \pi_1^{-1}(0)) \cap \pi_2^{-1}(0) &\simeq \text{Spec}(k[x, x^{-1}][y_{-1}, \dots, y_{h-2}, z]/(I_*, z)) \\ &\simeq \text{Spec}(k[x, x^{-1}][y_{-1}]/(x^{-h}y_{-1}(P_0(y_{-1}))^h R(y_{-1}))) \end{aligned}$$

is clearly nonreduced whenever $h \geq 2$. This completes the proof. \square

4.11. The above result implies that if $h \geq 2$, then the degenerate fibers of π_1 and π_2 have different scheme-theoretic structures. Therefore two $\mathbb{G}_{a,k}$ -actions on $S_{P_0, \dots, P_{h-1}}$ with associated quotient \mathbb{A}^1 -fibrations $\pi_1: S \rightarrow X$ and $\pi_2: S \rightarrow Z$ respectively can not be conjugated in the sense of (4.9) above. This leads to the following result.

Theorem 4.12. *A Danielewski surface $S \not\cong S_{P,1}$ with a trivial Makar-Limanov invariant admits two algebraically independent nonconjugated $\mathbb{G}_{a,k}$ -actions.*

As a consequence of this description, we obtain the following characterization of ordinary Danielewski surfaces $S_{P,1}$.

Corollary 4.13. *Let $\pi: S \rightarrow X = \text{Spec}(k[x])$, where k is an arbitrary field of characteristic zero, be a Danielewski surface with a trivial Makar-Limanov invariant. Then the following are equivalent.*

- a) *S admits a free $\mathbb{G}_{a,X}$ -action.*
- b) *S is isomorphic to a surface $S_{P,1} = \{xz - P(y) = 0\}$ in $\mathbb{A}_k^3 = \text{Spec}(k[x, y, z])$, where P is a nonconstant polynomial with $\deg P$ simple roots.*
- c) *The canonical sheaf ω_S is trivial.*
- d) *All $\mathbb{G}_{a,k}$ -actions on S are conjugated.*

Proof. The equivalence b) \Leftrightarrow d) follows from [2] and the above discussion. All the other equivalences can be obtained in the same way as in Corollary 5.7 in [4]. \square

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