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## Embeddings of Danielewski surfaces in affine spaces

A. Dubouloz

**Abstract.** We construct explicit embeddings of Danielewski surfaces [4] in affine spaces. The equations defining these embeddings are obtained from the  $2 \times 2$  minors of a matrix attached to a weighted rooted tree  $\gamma$ . We characterize those surfaces  $S_\gamma$  with a trivial Makar-Limanov invariant in terms of their associated trees. We prove that every Danielewski surface  $S$  with a nontrivial Makar-Limanov invariant admits a closed embedding in an affine space  $\mathbb{A}_k^n$  in such a way that every  $\mathbb{G}_{a,k}$ -action on  $S$  extends to an action on  $\mathbb{A}^n$  defined by a triangular derivation. We show that a Danielewski surface  $S$  with a trivial Makar-Limanov invariant and non-isomorphic to a hypersurface with equation  $xz - P(y) = 0$  in  $\mathbb{A}_k^3$  admits nonconjugated algebraically independent  $\mathbb{G}_{a,k}$ -actions.

**Mathematics Subject Classification (2000).** 14R20, 14R25.

**Keywords.** Danielewski surfaces, additive group actions, Makar-Limanov invariant.

### Introduction

A *Danielewski surface* over a field  $k$  of characteristic zero is an integral affine surface  $S$  equipped with a morphism  $\pi: S \rightarrow \mathbb{A}_k^1 = \operatorname{Spec}(k[x])$  restricting to the trivial line bundle over  $\mathbb{A}_k^1 \setminus \{0\}$  and such that the fiber  $\pi^{-1}(0)$  is nonempty and reduced, consisting of a disjoint union of affine lines  $\mathbb{A}_k^1$ . For instance, a surface  $S_{P,n} \subset \operatorname{Spec}(k[x, y, z])$  with equation  $x^n z - P(y) = 0$ , where  $P$  is a nonconstant polynomial with  $\deg(P)$  simple roots, is a Danielewski surface  $\operatorname{pr}_x: S_{P,n} \rightarrow \operatorname{Spec}(k[x])$ . Danielewski surfaces appear naturally as locally trivial fiber bundles  $\rho: S \rightarrow \tilde{X}$  over an affine line with a multiple origin (see e.g. [5]). More precisely, see [4], every such bundle  $\rho$  is a principal homogeneous bundle under the action of a line bundle  $p: L \rightarrow \tilde{X}$ . These principal  $L$ -bundles are uniquely determined by data consisting of an invertible sheaf  $\mathcal{L}$  on  $\tilde{X}$  and a Čech 1-cocycle  $g$  with values in the dual  $\mathcal{L}^\vee$  of  $\mathcal{L}$  for a suitable covering  $\mathcal{U}$  of  $\tilde{X}$ . In turn, the pair  $(\mathcal{L}, g)$  is encoded in a combinatorial datum consisting of a rooted tree with weighted edges, which we call a *weighted tree* (see [4, Example 1.6 and Theorem 3.2] and 2.2 below). Here we use weighted trees in a different way to construct embeddings of Danielewski surfaces into affine spaces.

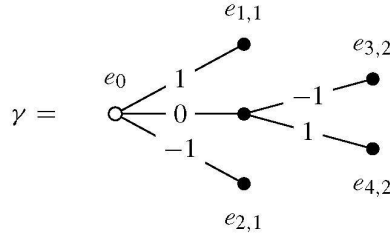
More precisely, starting with a suitable class of  $k$ -weighted trees  $\gamma$ , we construct explicit ideals of certain polynomial rings. In turn, these ideals define affine surfaces  $S_\gamma$  which are naturally Danielewski surfaces over the affine line  $\mathbb{A}_k^1$ .

The paper is divided as follows. In Section 1 we recall basic facts on weighted trees. We associate to every *fine  $k$ -weighted tree*  $\gamma = (\Gamma, w)$  (see Definition 1.3 below) a polynomial ring  $k[\Gamma]$  and a collection of polynomials in  $k[\Gamma]$  defined recursively through the *weight function*  $w$ .

In Section 2, we review the construction of Danielewski surfaces as locally trivial bundles over the affine line with an  $n$ -fold origin given in [4]. Then we associate to every fine  $k$ -weighted tree  $\gamma$  a closed affine subscheme  $S_\gamma = \text{Spec}(B_\gamma)$  of  $\mathbb{A}_k^1 \times \text{Spec}(k[\Gamma])$ , and we prove the following result (Theorem 2.9).

**Theorem.** *For every fine  $k$ -weighted tree  $\gamma$ , the scheme  $S_\gamma$  is a Danielewski surface over  $\mathbb{A}_k^1$  for the restriction of the projection  $\text{pr}_1 : \mathbb{A}_k^1 \times \text{Spec}(k[\Gamma]) \rightarrow \mathbb{A}_k^1$ .*

For instance, the surface corresponding to the following fine  $k$ -weighted tree



is the Bandman and Makar-Limanov surface [1]  $S \subset k[x][y, z, u]$  with equations

$$xz - y(y^2 - 1) = 0, \quad yu - z(z^2 - 1) = 0, \quad xu - (y^2 - 1)(z^2 - 1) = 0.$$

It is a Danielewski surface over  $X = \text{Spec}(k[x])$  via the projection morphism  $\text{pr}_x : S \rightarrow X$ .

Then we show that every embedded Danielewski surface  $S_\gamma$  as above comes canonically equipped with actions of the additive group  $\mathbb{G}_{a,k}$  which are the restrictions to  $S_\gamma$  of certain  $\mathbb{G}_{a,k}$ -actions on the ambient space  $\mathbb{A}_k^1 \times \text{Spec}(k[\Gamma])$  defined by explicit locally nilpotent derivations  $\tilde{\partial}_\gamma$  (see Proposition 2.15). In Section 3, we prove the following result (Corollary 3.8).

**Theorem.** *Every Danielewski surface  $\pi : S \rightarrow X = \mathbb{A}_k^1$  is  $X$ -isomorphic to an embedded Danielewski surface  $\pi_\gamma : S_\gamma = \text{Spec}(B_\gamma) \rightarrow X$  for an appropriate fine  $k$ -weighted tree  $\gamma$ .*

Moreover, we establish that every  $\mathbb{G}_{a,X}$ -action on  $\pi : S_\gamma \rightarrow X$  is induced by a locally nilpotent derivation  $\tilde{\partial}_\gamma$  as above. As a consequence of this description, we

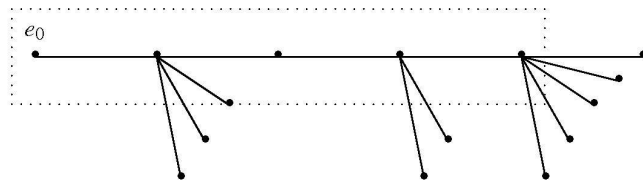
deduce that every Danielewski surface  $\pi: S \rightarrow X = \mathbb{A}_k^1$  can be embedded in a relative affine space  $\mathbb{A}_X^d$  in such a way that every  $\mathbb{G}_{a,X}$ -action on  $S$  extends to an action on  $\mathbb{A}_X^d$  (Corollary 3.11). This generalizes a result obtained by Makar-Limanov ([8], [9]) for the Danielewski hypersurfaces  $S_{P,n}$  above.

The *Makar-Limanov invariant* [6] of an affine  $k$ -scheme  $X = \operatorname{Spec}(B)$  is defined as the sub-algebra  $\operatorname{ML}(X) \subset B$  consisting of regular functions which are invariant under all  $\mathbb{G}_{a,k}$ -actions on  $X$ . If  $\operatorname{ML}(X) = k$ , then we say that  $X$  has a trivial Makar-Limanov invariant. For Danielewski surfaces with a nontrivial Makar-Limanov invariant, we prove the following result.

**Theorem.** *Every Danielewski surface with a nontrivial Makar-Limanov invariant can be embedded in an affine space  $\mathbb{A}_k^d = \operatorname{Spec}(k[x_1, \dots, x_d])$  in such a way that every  $\mathbb{G}_{a,k}$ -action on  $S$  extends to an action on  $\mathbb{A}_k^N$ . Furthermore, every such action is induced by a triangular locally nilpotent derivation of  $k[x_1, \dots, x_d]$ .*

In Section 4, we study Danielewski surfaces with a trivial Makar-Limanov invariant, that is, Danielewski surfaces  $S$  which admits two nontrivial  $\mathbb{G}_{a,k}$ -actions with distinct general orbits. We obtain the following criterion which generalizes Theorem 5.4 in [4].

**Theorem.** *An embedded Danielewski surface  $\pi: S_\gamma = \operatorname{Spec}(B_\gamma) \rightarrow \mathbb{A}_k^1$  defined by a fine  $k$ -weighted tree  $\gamma$  has a trivial Makar-Limanov invariant if and only if  $\gamma$  is a comb, i.e. a tree such that every element has at most one non-terminal direct descendant (see Definition 4.1 below).*



A comb rooted in  $e_0$ .

We obtain the following description (see 4.7 below). For every Danielewski surface  $S$  with a trivial Makar-Limanov invariant, there exists a collection of monic polynomials  $P_0, \dots, P_{h-1} \in k[t]$  with simple roots  $a_{i,j} \in k^*$ ,  $i = 0, \dots, h-1$ ,  $j = 1, \dots, \deg_t(P_i)$ , such that  $S$  is isomorphic to the nonsingular surface  $S_{P_0, \dots, P_{h-1}} \subset$



$\text{Spec}(k[x][y_{-1}, \dots, y_{h-2}][z])$  defined by the equations

$$\begin{aligned} xz - y_{h-2} \prod_{l=0}^{h-1} P_l(y_{l-1}) &= 0, \\ zy_{i-1} - y_i y_{h-2} \prod_{l=i+1}^{h-1} P_l(y_{l-1}) &= 0, \quad xy_i - y_{i-1} \prod_{l=0}^i P_l(y_{l-1}) = 0, \quad 0 \leq i \leq h-2, \\ y_{i-1} y_j - y_i y_{j-1} \prod_{l=i+1}^j P_l(y_{l-1}) &= 0, \quad 0 \leq i < j \leq h-2. \end{aligned}$$

On an affine surface  $S = \text{Spec}(B)$ , two  $\mathbb{G}_{a,k}$ -actions  $\mu_1$  and  $\mu_2$  with associated quotient fibrations  $\pi_1: S \rightarrow \mathbb{A}_k^1$  and  $\pi_2: S \rightarrow \mathbb{A}_k^1$  respectively are said to be *algebraically independent* if the general fibers of  $\pi_1$  and  $\pi_2$  do not coincide. In this situation, we say that  $\mu_1$  and  $\mu_2$  are *conjugated* if there exists an automorphism  $\phi$  of  $S$  sending the fibers of  $\pi_1$  onto the fibers of  $\pi_2$ . This means equivalently that there exists an automorphism  $\phi^*$  of  $B$  such that  $\text{Ker}(\partial_2) = \phi^*(\text{Ker}(\partial_1))$ , where  $\partial_1$  and  $\partial_2$  denote the locally nilpotent derivations of  $B$  corresponding to the actions  $\mu_1$  and  $\mu_2$  respectively. Daigle [2] established that all the  $\mathbb{G}_{a,k}$ -actions on a Danielewski surface  $S_{P,1} = \{xz - P(y) = 0\}$  are conjugated. From the explicit description above, we obtain the following result (Theorem 4.12).

**Theorem.** *If a Danielewski surface  $S$  non isomorphic to a surface  $S_{P,1}$  admits two independent  $\mathbb{G}_{a,k}$ -actions, then it admits two algebraically independent nonconjugated  $\mathbb{G}_{a,k}$ -actions.*

We also deduce the following characterization (Corollary 4.13) of the Danielewski surfaces  $S_{P,1}$ , which generalizes the ones previously obtained by Bandman and Makar-Limanov [1] and Daigle [2].

**Theorem.** *For a Danielewski surface  $\pi: S \rightarrow X = \mathbb{A}_k^1$  with a trivial Makar-Limanov invariant, the following are equivalent.*

- 1)  $S$  admits a free  $\mathbb{G}_{a,X}$ -action.
- 2) The canonical sheaf  $\omega_S$  of  $S$  is trivial.
- 3)  $S$  is isomorphic to a surface  $S_{P,1} \subset \mathbb{A}_k^3 = \text{Spec}(k[x, y, z])$  with the equation  $xz - P(y) = 0$  for a certain nonconstant polynomial  $P$  with  $\deg(P)$  simple roots.
- 4) All  $\mathbb{G}_{a,k}$ -actions on  $S$  are conjugated.

## 1. Preliminaries

**Weighted rooted trees.** A *poset* is a nonempty finite partially ordered set  $G = (G, \leq)$ . A totally ordered subset  $C \subset G$  is called a *chain of length*  $l(C) = \text{Card}(C) - 1$ . A chain which is maximal for the inclusion is called a *maximal chain*. For every  $e \in G$ , we let

$$(\uparrow e)_G = \{e' \in G, e \leq e'\} \quad \text{and} \quad (\downarrow e)_G = \{e' \in G, e' \leq e\}.$$

A subset  $\overleftarrow{e'e}$  with two elements  $e' < e$  such that  $(\uparrow e')_G \cap (\downarrow e)_G = \{e' < e\}$  is called an *edge* of  $G$ . We denote the set of all edges in  $G$  by  $E(G)$ .

**Definition 1.1.** A (*rooted*) *tree*  $\Gamma = (\Gamma, \leq)$  is a poset with a unique minimal element  $e_0$  called the *root*, and such that  $(\downarrow e)_\Gamma$  is a chain for every  $e \in \Gamma$ . A subposet  $\Gamma' \subset \Gamma$  which is tree for the induced ordering is called a *subtree* of  $\Gamma$ . Given  $e \in \Gamma$ , the *maximal (rooted) subtree of  $\Gamma$  rooted in  $e$*  is the subtree  $\Gamma(e) = (\uparrow e)_\Gamma$ .

**1.2.** An element  $e$  such that  $l(\downarrow e)_\Gamma = m$  is said to be *at level  $m$* . The maximal elements  $e_i = e_{i, m_i}$ , where  $m_i = l(\downarrow e_i)_\Gamma$ , of a tree  $\Gamma$  are called the *leaves* of  $\Gamma$ . We denote the set of those elements by  $L(\Gamma)$ . The maximal chains of  $\Gamma$  are the chains

$$\Gamma_{e_i, m_i} = (\downarrow e_{i, m_i})_\Gamma = \{e_{i, 0} = e_0 < e_{i, 1} < \cdots < e_{i, m_i}\}, \quad e_{i, m_i} \in L(\Gamma). \quad (1.1)$$

We say that  $\Gamma$  has *height*  $h(\Gamma) = \max(m_i)$ . An element of  $\Gamma \setminus L(\Gamma)$  is called a *parent*, and we denote the set of those elements by  $\mathbf{P}(\Gamma)$ . Given  $e \in \Gamma \setminus \{e_0\}$ , an element of the chain  $\text{Anc}(e) = (\downarrow e) \setminus \{e\}$  is called an *ancestor* of  $e$ . The *parent of  $e$*  is the maximal element  $\text{Par}(e)$  of  $\text{Anc}(e)$ . More generally, the  $n$ -th *ancestor* of  $e$  is defined recursively by  $\text{Par}^n(e) = \text{Par}(\text{Par}^{n-1}(e)) \in \text{Anc}(e)$ . Given two different elements  $e, e' \in \Gamma$ , the *first common ancestor of  $e$  and  $e'$*  is the maximal element  $\text{Anc}(e, e')$  of the chain  $\text{Anc}(e) \cap \text{Anc}(e')$ . If  $e$  is not a leaf of  $\Gamma$ , then the minimal elements of  $(\uparrow e)_\Gamma \setminus \{e\}$  are called the *children* of  $e$ , and we denote the set of those elements by  $\text{Ch}(e)$ . The *degree*  $\deg(e)$  of an element  $e$  is the number of its children.

**Definition 1.3.** Let  $\Gamma$  be a tree. A *fine weight function on  $\Gamma$ , with values in a field  $k$* , is a function  $w: E(\Gamma) \rightarrow k$ , which assigns an element  $a_{e', e} = w(\overleftarrow{e'e}) \in k$  to every edge  $\overleftarrow{e'e}$  of  $\Gamma$ , in such a way that  $a_{e', e_1} \neq a_{e', e_2}$  whenever  $e_1$  and  $e_2$  share the same parent  $e'$ . A tree  $\Gamma$  equipped with such a function  $w$  is referred to as a *fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$* .

**Definition 1.4.** An morphism of fine  $k$ -weighted trees  $\tau: \gamma' = (\Gamma', w') \rightarrow \gamma = (\Gamma, w)$  is an order-preserving map  $\tau: \Gamma' \rightarrow \Gamma$  satisfying the following properties.

- a) The image of a maximal subchain of  $\Gamma'$  is a maximal subchain of  $\Gamma$ .

- b) For every  $e' \in \Gamma'$ ,  $\tau^{-1}(\tau(e'))$  is either  $e'$  itself or a maximal subtree of  $\Gamma'$ .
- c) For every edge  $\overleftarrow{e'e}$  of  $\Gamma'$  such that  $\tau(e) \neq \tau(e')$ , we have  $w'(\overleftarrow{e'e}) = w(\overleftarrow{\tau(e')\tau(e)})$ .

**Remark 1.5.** A morphism of fine  $k$ -weighted trees maps the root  $e'_0$  of  $\Gamma'$  on the root  $e_0$  of  $\Gamma$  and a leaf  $e'_{i,m'_i}$  of  $\Gamma'$  at level  $m'_i$  onto a leaf  $e_{j(i),m_{j(i)}}$  of  $\Gamma$  at level  $m_{j(i)} \leq m'_i$ . Then b) guarantees that  $\tau(e'_{i,k}) = e_{j,\min(m_{j(i)},k)}$  for every  $k = 0, \dots, m'_i$ , and so, condition c) above makes sense.

**Genealogical matrix of a weighted tree.** Here we associate to every fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$  a matrix with coefficients in a polynomial ring  $k[\Gamma]$ .

**Definition 1.6.** Given a tree  $\Gamma$  rooted in  $e_0$ , we associate to every parent  $e \in \mathbf{P}(\Gamma)$  a symbol  $X_e$ . If  $e' \in \mathbf{P}(\Gamma)$  is the parent of a given  $e \in \mathbf{P}(\Gamma)$ , then we will sometimes denote  $X_{e'}$  as  $X_{\text{Par}(e)}$ . We also extend this relationship between the  $X_e$ 's by introducing the symbol  $X_{e_{-1}} = X_{\text{Par}(e_0)}$ . We let  $k[\Gamma] = k[(X_e)_{e \in \mathbf{P}(\Gamma) \cup \{e_{-1}\}}]$  be the corresponding polynomial ring in  $d(\Gamma) = \text{Card}(\mathbf{P}(\Gamma)) + 1$  variables.

For every element  $e \in \mathbf{P}(\Gamma)$  of a given fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$  rooted in  $e_0$ , we introduce below three polynomials  $F_e(\gamma), A_e(\gamma), G_e(\gamma) \in k[\Gamma]$ , defined recursively through the weight function  $w: E(\Gamma) \rightarrow k, \overleftarrow{e'e} \mapsto a_{e',e} = w(\overleftarrow{e'e})$ .

**Definition 1.7.** For every  $e' \in \mathbf{P}(\Gamma)$  and every subset  $J \subset \text{Ch}(e')$  we let

$$F_{e'}^J = F_{e'}^J(\gamma) = \prod_{e \in (\text{Ch}(e') \setminus J)} (X_{\text{Par}(e')} - a_{e',e}) \in k[X_{\text{Par}(e')}] \subset k[\Gamma].$$

The polynomial  $F_{e'} := F_{e'}^\emptyset$  is called *the fatherhood polynomial* of  $e'$ .

The *ancestral polynomial*  $A_e = A_e(\gamma)$  of  $e \in \Gamma$  is the polynomial defined recursively by

$$A_{e_0} = 1 \quad \text{and} \quad A_e = F_{\text{Par}(e)}^{[e]} A_{\text{Par}(e)} \in k[X_{e_{-1}}, (X_{e'})_{e' \in \text{Anc}(\text{Par}(e))}] \subset k[\Gamma].$$

The *genealogical polynomial* of  $e \in \mathbf{P}(\Gamma)$  with respect to  $e' \in \text{Anc}(e)$  is the polynomial

$$G_{e',e} = G_{e',e}(\gamma) = A_{e'}^{-1} A_e F_e \in k[X_{e_{-1}}, (X_{e''})_{e'' \in \text{Anc}(e) \setminus \text{Anc}(\text{Par}(e'))}] \subset k[\Gamma].$$

The polynomial  $G_e = G_{e_0,e}$  is simply referred to as *the genealogical polynomial* of  $e$ .

**Remark 1.8.** Up to changing the variables,  $G_{e',e}(\gamma)$  coincides with the genealogical polynomial  $G_e(\gamma')$  of  $e$  as an element of the maximal weighted subtree  $\gamma(e') = ((\uparrow e')_\Gamma, w|_{\Gamma(e')})$  of  $\gamma$  rooted in  $e'$ , considered as a fine  $k$ -weighted tree disregarding the inclusion  $\gamma(e') \hookrightarrow \gamma$ .

**Definition 1.9.** The *genealogical matrix* of a fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$  is the matrix  $M(\gamma) \in \text{Mat}_{d(\Gamma)-1,2}(k[\Gamma])$  with the rows  $M_e = (G_e, X_e) \in \text{Mat}_{1,2}(k[\Gamma])$ ,  $e \in \mathbf{P}(\Gamma)$ .

## 2. Danielewski surfaces defined by weighted trees

In [4], the author gives a method to construct a Danielewski surface  $\pi: S^\gamma \rightarrow X$  over  $X = \text{Spec}(k[x])$  from the data consisting of a fine  $k$ -weighted tree  $\gamma$ . Here we review briefly this construction. Then we introduce a new procedure to associate to every such tree  $\gamma$  a second Danielewski surface  $\pi: S_\gamma \rightarrow X$ , which comes embedded in a relative affine space  $\mathbb{A}_X^d = X \times \mathbb{A}_k^d$ .

**Notation 2.1.** Throughout this section, we fix a field  $k$  of characteristic zero. We let  $A = k[x]$ ,  $X = \text{Spec}(A) \simeq \mathbb{A}_k^1$ , and we denote by  $X_* \simeq \text{Spec}(A_x)$  the open complement in  $X$  of the origin  $x_0 \in \mathbb{A}_k^1$ . We consider Danielewski surfaces over the fixed base  $X$ . We denote by  $\text{pr}_X: \mathbb{A}_X^1 = \text{Spec}(A[X_{e-1}]) \rightarrow X$  the trivial line bundle over  $X$ . The additive group scheme with base  $X$  is denoted by  $\mathbb{G}_{a,X} = \text{Spec}(A[T])$ .

**Abstract Danielewski surface defined by a fine  $k$ -weighted tree.** Given a fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$  of height  $h = h(\Gamma)$  with leaves  $e_{1,m_1}, \dots, e_{n,m_n}$ , we construct a Danielewski surface  $\pi: S^\gamma \rightarrow X$  as follows. Using the maximal weighted subchains

$$\gamma_{e_{i,m_i}} = ((\downarrow e_{i,m_i}), w) = \{e_0 = e_{i,0} < e_{i,1} < \dots < e_{i,m_i-1} < e_{i,m_i}\}_w, \quad i = 1, \dots, n,$$

of  $\gamma$ , we define a collection of polynomials

$$\sigma = \{\sigma_i = \sum_{j=0}^{m_i-1} w(\overleftarrow{e_{i,j}e_{i,j+1}})x^j \in k[x]\}_{i=1,\dots,n}.$$

For every  $i \neq j$ , we let  $g_{ij} = x^{-m_i}(\sigma_j - \sigma_i) \in A_x$ . These *transition functions*  $g_{ij}$  satisfy the cocycle relation  $g_{ik} = g_{ij} + x^{m_j-m_i}g_{jk}$  in  $A_x$  for every triple  $i \neq j \neq k$ .

**2.2.** We let  $\pi: S^\gamma \rightarrow X$  be the  $X$ -scheme obtained by gluing  $n$  copies  $S_i = \text{Spec}(A[T_i])$  of  $\mathbb{A}_X^1$  over  $X_*$  by means of the  $A_x$ -algebra isomorphisms

$$\tau_{ij}: A_x[T_i] \rightarrow A_x[T_j], \quad T_i \mapsto g_{ij} + x^{m_j-m_i}T_j, \quad i \neq j, \quad i, j = 1, \dots, n.$$

Since  $\gamma$  is a fine  $k$ -weighted tree, it follows from 2.8 in [4] that  $S^\gamma$  is a Danielewski surface  $\pi: S^\gamma \rightarrow X$ . The irreducible components of  $\pi^{-1}(x_0)$  are the curves  $C_i = \pi^{-1}(x_0) \cap S_i \simeq \text{Spec}(k[T_i])$ ,  $i = 1, \dots, n$ . It comes equipped with a canonical birational  $X$ -morphism  $\psi: S^\gamma \rightarrow \mathbb{A}_X^1 = \text{Spec}(A[X_{e-1}])$  corresponding to the section  $s_{e-1} \in B^\gamma = \Gamma(S^\gamma, \mathcal{O}_{S^\gamma})$  with restrictions  $s_{e-1}|_{S_i} = \sigma_i + x^{m_i}T_i \in A[T_i]$ ,

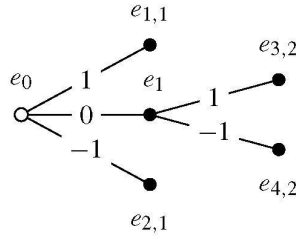
$i = 1, \dots, n$ . By Theorem 3.2 in [4], every Danielewski surface  $\pi: S \rightarrow X$  is  $X$ -isomorphic to an *abstract Danielewski surface*  $\pi: S^\vee \rightarrow X$  obtained by this procedure.

**2.3.** A Danielewski surface  $\pi: S \rightarrow X$  admits nontrivial actions of the additive group scheme  $\mathbb{G}_{a,X}$ . Indeed, since by definition  $S|_{X_*}$  is isomorphic to the trivial line bundle  $\mathbb{A}_{X_*}^1 = \text{Spec}(A_x[X_{e_{-1}}])$  over  $X_*$ , there exists  $r \geq 0$  such that the  $A$ -derivation  $x^m \partial_{X_{e_{-1}}}$  extends to a locally nilpotent  $A$ -derivation  $\partial$  of  $\Gamma(S, \mathcal{O}_S)$ , corresponding to a nontrivial  $\mathbb{G}_{a,X}$ -action on  $S$ . By Proposition 2.12 in [4], every nontrivial  $\mathbb{G}_{a,X}$ -action on a Danielewski surface  $S^\vee$  is induced by the extension  $\partial_{a,m}$  to  $B^\vee$  of a locally nilpotent  $A$ -derivation  $ax^m \partial_{X_{e_{-1}}}$  of  $B^\vee \otimes_A A_x \simeq A_x[X_{e_{-1}}]$ , where  $m \geq h(\Gamma)$  and  $a \in A \setminus \{0\}$ . We denote the corresponding  $\mathbb{G}_{a,X}$ -actions on  $\mathbb{A}_X^1$  and  $S^\vee$  by  $t_{a,m}$  and  $t_{a,m}^\vee$  respectively. On the open subsets  $S_i = \text{Spec}(A[T_i])$ ,  $t_{a,m}^\vee$  coincides with the *twisted translation*  $t_{a,m-m_i}$  defined by the group co-action homomorphism

$$A[T_i] \rightarrow A[T_i, T] \simeq A[T_i] \otimes_A A[T], \quad T_i \mapsto T_i + ax^{m-m_i}T, \quad i = 1, \dots, n.$$

The canonical morphism  $\psi: S^\vee \rightarrow \mathbb{A}_X^1$  is  $\mathbb{G}_{a,X}$ -equivariant when  $S^\vee$  and  $\mathbb{A}_X^1$  are equipped with the  $\mathbb{G}_{a,X}$ -actions  $t_{a,m}^\vee$  and  $t_{a,m}$  respectively.

**Example 2.4.** The collection of polynomials  $\sigma$  corresponding to the following fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$  with leaves  $e_{1,1}, e_{2,1}, e_{3,2}, e_{4,2}$



is  $\sigma = \{1, -1, x, -x\}$ . The associated transition functions  $g = \{g_{ij}\}_{1 \leq i < j \leq 4}$  are

$$\begin{aligned} g_{12} = g_{34} &= -2x^{-1}, & g_{13} &= -g_{24} = x^{-1}(x-1), \\ g_{23} &= -g_{14} = x^{-1}(x+1). \end{aligned}$$

The gluing homomorphisms  $\{\tau_{ij}\}_{1 \leq i < j \leq 4}$  are given by

$$\begin{aligned} \tau_{ij}: k[x, x^{-1}][T_i] &\rightarrow k[x, x^{-1}][T_j], \\ T_i &\mapsto \begin{cases} g_{ij} + T_j, & \text{if } (i, j) \in \{(1, 2), (3, 4)\}, \\ g_{ij} + xT_j, & \text{if } (i, j) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}. \end{cases} \end{aligned}$$

The  $\mathbb{G}_{a,X}$ -action  $t_{1,2}^\gamma$  on  $\pi: S^\gamma \rightarrow X$  is a non-free action which restricts on  $S_i = \text{Spec}(A[T_i])$  to the action

$$T_i \mapsto \begin{cases} T_i + xT, & \text{if } i = 1, 2, \\ T_i + T, & \text{if } i = 3, 4. \end{cases}$$

Letting  $P(t) = t^2 - 1 \in k[t]$ , we will see in Example 3.2 below that  $S^\gamma$  is  $X$ -isomorphic to the Bandman and Makar-Limanov surface [1]  $S \subset \text{Spec}(k[x][y, z, u])$  with equations

$$xz - yP(y) = 0, \quad yu - zP(z) = 0, \quad xu - P(y)P(z) = 0,$$

and that  $t_{1,2}^\gamma$  coincides with the action on  $S$  induced by the triangular derivation

$$\partial_{1,2} = x^2 \partial_y + x(3y^2 - 1) \partial_z + (2P(y)(3y^2 - 1)z + 2xyP(z)) \partial_u \in \text{Der}_{k[x]}(k[x][y, z, u]).$$

**Embedded Danielewski surface defined by a fine  $k$ -weighted tree.** Given a fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$ , we construct a Danielewski surface  $\pi: S_\gamma \rightarrow X$  which comes embedded in a relative affine space  $\mathbb{A}_X^{d(\Gamma)}$ , where  $d(\Gamma) = \text{Card}(\mathbf{P}(\Gamma)) + 1$ . These surfaces are canonically equipped with the restrictions of certain actions of the additive group  $\mathbb{G}_{a,X}$  on the ambient space  $\mathbb{A}_X^{d(\Gamma)}$ , defined by explicit locally nilpotent derivations.

**2.5.** Given a fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$ , we let  $A[\Gamma] = A \otimes_k k[\Gamma] \simeq A[X_{e_{-1}}, (X_e)_{e \in \mathbf{P}(\Gamma)}]$  (see Definition 1.6). We let  $\bar{M}(\gamma) \in \text{Mat}_{d(\Gamma), 2}(A[\Gamma])$  be the matrix with the rows  $M_{e_{-1}} = (x, 1)$  and  $M_e = (G_e(\gamma), X_e)$ ,  $e \in \mathbf{P}(\Gamma)$ , i.e.  $\bar{M}(\gamma) = (M_{e_{-1}}, M(\gamma))$ , where  $M(\gamma) \in \text{Mat}_{d(\Gamma)-1, 2}(k[\Gamma])$  denotes the genealogical matrix of  $\gamma$  (Definition 1.9).

**Definition 2.6.** Given a fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$ , we let  $I_\gamma \subset A[\Gamma]$  be the ideal generated by the *simplified genealogical minors* of  $\bar{M}(\gamma)$

$$\Delta_{e', e} = \Delta_{e', e}(\gamma) = A_{e'}^{-1} \det(M_{\text{Par}(e')}, M_e) \in A[\Gamma], \quad (e, e') \in \mathbf{P}(\Gamma) \times (\downarrow e)_\Gamma. \quad (2.1)$$

We let  $B_\gamma = A[\Gamma]/I_\gamma$ , and we let  $\pi: S_\gamma = \text{Spec}(B_\gamma) \rightarrow X$  be the corresponding closed sub- $X$ -scheme of the relative affine space  $\mathbb{A}_X^{d(\Gamma)} = \text{Spec}(A[\Gamma])$ .

**2.7.** By construction,  $\Delta_e := \Delta_{e_0, e} = xX_e - G_e \in A[(X_{e'})_{e' \in (\downarrow e)_\Gamma \cup \{e_{-1}\}}]$  for every  $e \in \mathbf{P}(\Gamma)$ , whereas  $\Delta_{e', e} = (X_{\text{Par}^2(e')} - a_{\text{Par}(e'), e'})X_e - X_{\text{Par}(e')}G_{e', e}$  for every pair  $(e, e') \in \mathbf{P}(\Gamma) \times ((\downarrow e)_\Gamma \setminus \{e_0\})$ . As a consequence, for every triple  $e_0 < e'' \leq e' \leq e$  in  $\mathbf{P}(\Gamma)$ , the following relations hold in  $A[\Gamma]$ :

$$\left. \begin{aligned} A_{e'} \Delta_{e', e} &= X_{\text{Par}(e')} \Delta_e - \Delta_{\text{Par}(e')} X_e, \\ x \Delta_{e', e} &= (X_{\text{Par}^2(e')} - a_{\text{Par}(e'), e'}) \Delta_e - \Delta_{\text{Par}(e')} G_{e', e}, \\ (X_{\text{Par}^2(e'')} - a_{\text{Par}(e''), e''}) \Delta_{e', e} &= (X_{\text{Par}^2(e')} - a_{\text{Par}(e'), e'}) \Delta_{e'', e} - \Delta_{e'', e'} G_{e', e}. \end{aligned} \right\} \quad (2.2)$$

**2.8.** If  $\gamma = (\Gamma, w)$  is the trivial tree with just one element  $e_0$ , then the first projection  $\pi : S_\gamma = \text{Spec}(k[x][X_{e_{-1}}]) \rightarrow X$  is a Danielewski surface. Similarly, if  $\Gamma$  has height 1, then  $G_{e_0} \in k[X_{e_{-1}}]$  is a monic polynomial with simple roots  $a_{e_0, e} = w(\overleftarrow{e_0 e}) \in k$ ,  $e \in \text{Ch}(e_0)$ . Therefore,

$$\pi : S_\gamma = \text{Spec}(A[\Gamma]/I_\gamma) = \text{Spec}(k[x][X_{e_{-1}}, X_{e_0}]/xX_{e_0} - G_{e_0}(X_{e_{-1}})) \rightarrow X$$

is a Danielewski surface, and the irreducible components of  $\pi^{-1}(x_0)$  are the curves  $C_e \simeq \text{Spec}(k[X_{e_0}])$  with defining ideals  $I_{\gamma, e} = (I_\gamma, X_{e_{-1}} - a_{e, e_0}) \subset A[\Gamma]$ ,  $e \in \text{Ch}(e_0)$ . More generally, we have the following result.

**Theorem 2.9.** *For every fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$  with leaves  $e_{1, m_1}, \dots, e_{n, m_n}$ ,  $\pi : S_\gamma \rightarrow X$  is a Danielewski surface. Furthermore, the fiber  $\pi^{-1}(x_0)$  is the disjoint union of the curves  $C_{e_{i, m_i}} \simeq \text{Spec}(k[X_{e_{i, m_i-1}}])$  with defining ideals*

$$I_{\gamma, e_{i, m_i}} = (I_\gamma, x, (X_{e_{i, j-1}} - a_{e_{i, j}, e_{i, j+1}})_{0 \leq j \leq m_i-1}) \subset A[\Gamma], \quad i = 1, \dots, n.$$

The proof is divided as follows. In 2.10, Lemmas 2.11 and 2.12 below, we show that  $S_\gamma$  is an integral scheme. Then, in Lemma 2.13, we describe explicitly the irreducible components of  $\pi^{-1}(x_0)$ .

**2.10.** We first observe that  $S_\gamma$  restricts to the trivial line bundle  $\mathbb{A}_{X_*}^1 = \text{Spec}(A_x[X_{e_{-1}}])$  over  $X_*$ . Indeed, the second relation of (2.2) guarantees that the ideal  $I_\gamma A_x[\Gamma]$  of  $A_x[\Gamma] \simeq A[\Gamma] \otimes_A A_x$  is generated by the polynomials  $x^{-1}\Delta_e = X_e - x^{-1}G_e$ ,  $e \in \mathbf{P}(\Gamma)$ . Since  $G_e$  only involves the variables  $X_{e'}$ , where  $e' \in \text{Anc}(e)$ , we recursively arrive at an  $A_x$ -algebra isomorphism  $A_x[\Gamma]/I_\gamma A_x[\Gamma] \simeq A_x[X_{e_{-1}}]$ . Thus  $S_\gamma$  is a Danielewski surface with base  $(k[x], x)$  provided that  $x$  is not a zero divisor in  $B_\gamma$  and that  $B_\gamma/xB_\gamma$  is isomorphic to a nonempty direct product of polynomial rings in one variable over  $k$ . Indeed, the first condition guarantees that the canonical map  $B_\gamma \rightarrow B_\gamma \otimes_A A_x \simeq A_x[X_{e_{-1}}]$  is injective. In turn, this implies that  $B_\gamma$  is a sub-domain of  $A_x[X_{e_{-1}}]$ . The second one means equivalently that the fiber  $\pi^{-1}(x_0)$  decomposes as a nonempty disjoint union of affine lines  $\mathbb{A}_k^1$ .

To show that  $x$  is not a zero divisor in  $B_\gamma$ , it suffices to find a covering of  $S_\gamma$  by principal affine open subsets  $Y_i = \text{Spec}(B_i)$  such that  $x$  is not a zero divisor in  $B_i$  for every  $i = 1, \dots, n$ .

**Lemma 2.11.** *If  $\gamma = (\Gamma, w)$  is a fine  $k$ -weighted tree with the leaves  $e_1, \dots, e_n$ , then  $S_\gamma$  is covered by the principal open subsets  $Y_i = \text{Spec}(A[\Gamma][T]/(I_\gamma, A_{e_i}T - 1))$ ,  $i = 1, \dots, n$ .*

*Proof.* For every  $e \in \mathbf{P}(\Gamma)$  the polynomials  $F_e^{(e')} \in A[X_{\text{Par}(e)}]$ ,  $e' \in \text{Ch}(e)$  generate the unit ideal of  $A[X_{\text{Par}(e)}]$  as  $\gamma$  is a fine  $k$ -weighted tree. Therefore, there exist

polynomials  $\Lambda_{e'} \in A[\Gamma]$ ,  $e' \in \text{Ch}(e)$ , such that

$$A_e = A_e \sum_{e' \in \text{Ch}(e)} \Lambda_{e'} F_e^{\{e'\}} = \sum_{e' \in \text{Ch}(e)} \Lambda_{e'} A_{e'}.$$

It follows by induction that the image of  $A_{e_0} = 1$  in  $B_\gamma$  belongs to the ideal generated by the images  $a_i \in B_\gamma$  of the ancestral polynomials  $A_{e_i}$  of the leaves of  $\Gamma$ . This means equivalently that the open subsets  $\text{Spec}((B_\gamma)_{a_i}) \simeq \text{Spec}(A[\Gamma][T]/(I_\gamma, A_{e_i}T - 1))$  cover  $S_\gamma$ .  $\square$

**Lemma 2.12.** *For every  $i = 1, \dots, n$ ,  $Y_i$  is an integral scheme.*

*Proof.* Let us denote by  $e_j = e_{i,j}$ ,  $j = 0, \dots, m = m_i$ , the elements of the maximal subchain  $(\downarrow e_{i,m_i})_\Gamma$  of  $\Gamma$  associated with the leaf  $e_{i,m_i}$ . For every  $i = 1, \dots, m - 2$ , the polynomial  $A_{e_{i+1}}$  divides  $A_{e_m}$ . Similarly, for every  $e \in \mathbf{P}(\Gamma) \setminus (\downarrow e_m)$ , the first common ancestor of  $e$  and  $e_m$  is an element  $e_i$ ,  $i \leq m - 1$ , such that  $e' = \text{Ch}(e_i) \cap (\downarrow e) \neq e_{i+1}$ , and so  $(X_{e_{i-1}} - a_{e_i,e'})$  divides  $A_{e_m}$ . Therefore, these polynomials become invertible in  $A[\Gamma]_{A_{e_m}}$ . We claim that the ideal  $I_\gamma A[\Gamma]_{A_{e_m}}$  is generated by the polynomials

$$\begin{aligned} \delta_{e_i} &= A_{e_{i+1}}^{-1} \Delta_{e_i} = -(X_{e_{i-1}} - a_{e_i,e_{i+1}}) + A_{e_{i+1}}^{-1} x X_{e_i}, \quad i = 1, \dots, m - 2, \\ \delta_{e_i,e} &= (X_{e_{i-1}} - a_{e_i,e'})^{-1} \Delta_{e_i,e} \\ &= X_e - (X_{e_{i-1}} - a_{e_i,e'})^{-1} X_{e_i} G_{e_i,e}, \end{aligned} \quad \begin{cases} e \in \mathbf{P}(\Gamma) \setminus (\downarrow e_m), \\ \text{Anc}(e, e_m) = e_i, \\ e' = \text{Ch}(e_i) \cap (\downarrow e)_\Gamma. \end{cases}$$

Indeed, the second relation of (2.2) guarantees that the polynomials  $\Delta_e$ , where  $e \in \mathbf{P}(\Gamma) \setminus (\downarrow e_m)_\Gamma$ , can be expressed in  $A[\Gamma]_{A_{e_m}}$  in terms of the  $\delta_{e_i}$ 's and  $\delta_{e_i,e}$ 's. In turn, we deduce from the first and the third ones that all the polynomials  $\Delta_{e',e}$ ,  $(e, e') \in \mathbf{P}(\Gamma) \times ((\downarrow e)_\Gamma \setminus \{e_0\})$  belong to the ideal of  $A[\Gamma]_{A_{e_m}}$  generated by the  $\delta_{e_i}$ 's and the  $\delta_{e_i,e}$ 's. Since the polynomials  $A_{e_i}$  and  $G_{e_i,e}$  above only involve the variables corresponding to the elements in  $(\downarrow e_{i-2})_\Gamma$  and  $(\uparrow e')_\Gamma \cap (\downarrow \text{Anc}(e))_\Gamma$  respectively, we conclude by induction that there exists a nonconstant polynomial  $P \in A[X_{e_{m-1}}]$  such that  $A[\Gamma]_{A_{e_m}}/I_\gamma A[\Gamma]_{A_{e_m}} \simeq A[X_{e_{m-1}}]_P$ . Since  $A$  is a domain and  $P$  is nonconstant, it follows that  $(B_\gamma)_{a_i} \simeq A[X_{e_{m-1}}]_P$  is a nonzero domain too.  $\square$

Summing up, we have established that for every fine  $k$ -weighted tree  $\gamma$ ,  $\pi: S_\gamma \rightarrow X$  is an integral affine scheme restricting to the trivial bundle  $\mathbb{A}_{X_*}^1$  over  $X_*$ . The following result completes the proof of Theorem 2.9.

**Lemma 2.13.** *For every fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$  with leaves  $e_{1,m_1}, \dots, e_{n,m_n}$ , the fiber  $\pi^{-1}(x_0)$  of  $\pi: S_\gamma \rightarrow X$  is the disjoint union of the curves  $C_{e_i,m_i} \simeq \text{Spec}(k[X_{e_i,m_i-1}])$  with defining ideals*

$$I_{\gamma,e_i,m_i} = (I_\gamma, x, (X_{e_{i,j-1}} - a_{e_i,j,e_{i,j+1}})_{0 \leq j \leq m_i-1}) \subset A[\Gamma], \quad i = 1, \dots, n.$$



*Proof.* We proceed by induction on the height  $h$  of  $\Gamma$ . If  $h = 0$  then  $S_\gamma = \text{Spec}(A[X_{e_{-1}}])$  and  $\pi^{-1}(x_0) \simeq \text{Spec}(k[X_{e_{-1}}])$ . Otherwise, if  $\text{Ch}(e_0) \neq \emptyset$  then, since  $\gamma$  is a fine  $k$ -weighted tree, it follows that the polynomials  $X_{e_{-1}} - a_{e_0, e}$ ,  $e \in \text{Ch}(e_0)$  are pairwise relatively prime. Therefore  $\pi^{-1}(x_0) = \text{Spec}(A[\Gamma]/(x, I_\gamma))$  decomposes as the disjoint union of curves  $D_e = \text{Spec}(A[\Gamma]/(x, X_{e_{-1}} - a_{e_0, e}, I_\gamma))$ ,  $e \in \text{Ch}(e_0)$ . We let  $\gamma(e) = (\Gamma(e), w|_{\Gamma(e)})$  be the maximal fine  $k$ -weighted subtree of  $\gamma$  rooted in  $e$ . Clearly, the ideal  $(x, X_{e_{-1}} - a_{e_0, e}, I_\gamma)$  coincides with the ideal  $I_e \subset A[\Gamma]$  generated by  $x$ ,  $X_{e_{-1}} - a_{e_0, e}$  and the polynomials

$$\begin{aligned} G_{e, e'}(\gamma), & \quad e' \in \mathbf{P}(\Gamma(e)), \\ \Delta_{e'', e'}(\gamma), & \quad (e', e'') \in \mathbf{P}(\Gamma(e)) \times (\text{Anc}_{\Gamma(e)}(e')), \\ \delta_{e, e'} = (a_{e_0, e} - a_{e_0, e''})X_{e'} - X_{e_0}G_{e, e'}(\gamma), & \quad \begin{cases} e' \in \mathbf{P}(\Gamma) \setminus (\{e_0\} \cup \mathbf{P}(\Gamma(e))), \\ e'' = \text{Ch}(e_0) \cap (\downarrow e') \neq e. \end{cases} \end{aligned}$$

By definition, we have  $A[\Gamma(e)] = A[X_{e_{-1}}, (X_{e'})_{e' \in \mathbf{P}(\Gamma)}] \simeq A[X_{e_0}, (X_{e'})_{e' \in \mathbf{P}(\Gamma)}]$  as  $e_0 \notin \Gamma(e)$ . This choice of coordinates yields the identities

$$\begin{aligned} G_{e'}(\gamma(e)) &= G_{e, e'}(\gamma), \quad e' \in \mathbf{P}(\Gamma(e)), \\ G_{e'', e'}(\gamma(e)) &= G_{e'', e'}(\gamma), \quad (e', e'') \in \mathbf{P}(\Gamma(e)) \times \text{Anc}_{\Gamma(e)}(e'), \end{aligned}$$

and we conclude that  $A[\Gamma]/(x, X_{e_{-1}} - a_{e_0, e}, I_\gamma) \simeq A[\Gamma]/I_e \simeq A[\Gamma(e)]/(x, I_{\gamma(e)})$ . This means equivalently that  $\pi^{-1}(x_0)$  is isomorphic to the disjoint union of the fibers  $\pi_{\gamma(e)}^{-1}(x_0)$  of the corresponding surfaces  $\pi_{\gamma(e)}: S_{\gamma(e)} \rightarrow X$ ,  $e \in \text{Ch}(e_0)$ . Since the fine  $k$ -weighted tree  $\gamma(e)$  has height  $h - 1$ , it follows from our induction hypothesis that these fibers are nonempty and reduced, consisting of disjoint unions of affine lines  $\mathbb{A}_k^1$ . So the same holds for  $\pi^{-1}(x_0)$ . Finally, the precise description of the irreducible components of  $\pi^{-1}(x_0)$  follows easily by induction again.  $\square$

**Remark 2.14.** A Danielewski surface  $\pi: S_\gamma \rightarrow X = \mathbb{A}_k^1$  is a flat (or rather a smooth)  $X$ -scheme. In general, the scheme  $\tilde{\pi}: \tilde{S}_\gamma \rightarrow X$  with defining ideal  $\tilde{I}_\gamma$  generated only by the polynomials  $\Delta_e$ ,  $e \in \mathbf{P}(\Gamma)$ , is not flat over  $X$ . The above discussion together with the second relation of (2.2) imply that  $S_\gamma$  coincides with the flat limit over  $X$  of the trivial family of affine lines  $\tilde{S}_\gamma|_{X_*} \simeq \mathbb{A}_{X_*}^1$  defined by the equations  $\Delta_e = 0$ ,  $e \in \mathbf{P}(\Gamma)$ , in  $\mathbb{A}_{X_*}^{d(\Gamma)} = \text{Spec}(A_X[\Gamma])$ . This explains why the polynomials  $\Delta_{e', e}$ ,  $(e, e') \in \mathbf{P}(\Gamma) \times ((\downarrow e)_\Gamma \setminus \{e_0\})$ , should be added to the obvious ones  $\Delta_e$ ,  $e \in \mathbf{P}(\Gamma)$ , to define the surface  $S_\gamma$ .

The following result shows that the *embedded Danielewski surface*  $\pi: S_\gamma \rightarrow X$  defined by a fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$  admits nontrivial actions of the additive group  $\mathbb{G}_{a, X}$ , which come as the restrictions of certain  $\mathbb{G}_{a, X}$ -actions on the ambient space  $\mathbb{A}_X^{d(\Gamma)}$ .

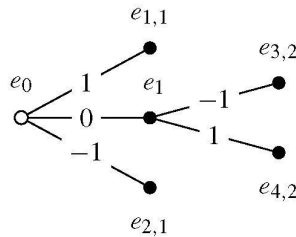
**Proposition 2.15.** *Let  $\gamma = (\Gamma, w)$  be a fine  $k$ -weighted tree of height  $h \geq 0$ . Then, for every  $m \geq h$  and every  $a \in A \setminus \{0\}$ , the derivation  $\tilde{\partial}_{\gamma,a,m} \in \text{Der}_A(A[\Gamma], A_x[\Gamma])$  defined recursively by*

$$\tilde{\partial}_{\gamma,a,m} = ax^m \partial_{X_{e_{-1}}} + x^{-1} \sum_{e \in \mathbf{P}(\Gamma)} \tilde{\partial}_{\gamma,a,m}(G_e(\gamma)) \partial_{X_e}$$

*is a triangular derivation of  $A[\Gamma]$  inducing a locally nilpotent  $A$ -derivation  $\partial_{\gamma,a,m}$  of  $B_\gamma$ .*

*Proof.* It suffices to prove the assertion for the derivation  $\tilde{\partial} = \partial_{\gamma,1,h}$  as  $\tilde{\partial}_{\gamma,a,m} = ax^{m-h} \tilde{\partial}$ . For every  $e \in \mathbf{P}(\Gamma)$  at level  $i < h$ , the polynomial  $G_e$  only involves the variables  $X_0$  and  $X_{e'}$ ,  $e' \in \text{Anc}(e)$ . So we conclude recursively that  $\tilde{\partial}(X_e) \in x^{h-i-1} A[X_{e_{-1}}, (X_{e'})_{e' \in \text{Anc}(e)}]$ . Thus  $\tilde{\partial}$  restricts to a triangular  $A$ -derivation of  $A[\Gamma]$ . By construction,  $\tilde{\partial}$  annihilates  $\Delta_e$  for every  $e \in \mathbf{P}(\Gamma)$ . Moreover,  $x\tilde{\partial}(\Delta_{e',e}) = \tilde{\partial}(x\Delta_{e',e}) \in I_\gamma$  for every pair  $(e, e') \in (\mathbf{P}(\Gamma) \setminus \{e_0\}) \times ((\downarrow e)_\Gamma \setminus \{e_0\})$  by virtue of (2.2). Thus  $\tilde{\partial}(\Delta_{e',e}) \in I_\gamma$  as  $I_\gamma$  is a prime ideal which does not contain  $x$ . Hence  $\tilde{\partial}(I_\gamma) \subset I_\gamma$  and so,  $\tilde{\partial}$  induces a locally nilpotent  $A$ -derivation  $\partial$  of  $B_\gamma$ .  $\square$

**Example 2.16.** We consider the following fine  $k$ -weighted tree  $\tilde{\gamma} = (\Gamma, \tilde{w})$  with the leaves  $e_{1,1}, e_{2,1}, e_{3,2}, e_{4,2}$ .



We have  $A[\Gamma] = k[x][X_{e_{-1}}, X_{e_0}, X_{e_1}]$  and

$${}^t \bar{M}(\tilde{\gamma}) = \begin{pmatrix} x & X_{e_{-1}} P(X_{e_{-1}}) & P(X_{e_{-1}}) P(X_{e_0}) \\ 1 & X_{e_0} & X_{e_1} \end{pmatrix},$$

where  $P(t) = t^2 - 1 \in k[t]$ . Therefore  $\pi: S_{\tilde{\gamma}} \rightarrow X$  is the surface with equations

$$\begin{aligned} xX_{e_0} - X_{e_{-1}}P(X_{e_{-1}}) &= 0, & X_{e_{-1}}X_{e_1} - X_{e_0}P(X_{e_0}) &= 0, \\ xX_{e_1} - P(X_{e_{-1}})P(X_{e_0}) &= 0. \end{aligned}$$

Letting  $y = X_{e_{-1}}, z = X_{e_0}$  and  $u = X_{e_1}$ , the locally nilpotent derivation  $\tilde{\partial}_{\tilde{\gamma},1,2} \in \text{Der}_A(A[\Gamma])$  is simply the derivation  $\partial_{1,2} \in \text{Der}_{k[x]}(k[x][y, z, u])$  of Example 2.4.

### 3. Embeddings of Danielewski surfaces in affine spaces

In this section, we compare the two constructions of Danielewski surfaces by means of fine  $k$ -weighted trees. We describe a certain class of morphisms of Danielewski surfaces as the restrictions of suitable linear projections.

**From abstract to embedded Danielewski surfaces.** Here we prove the following result.

**Theorem 3.1.** *For every abstract Danielewski surface  $\pi : S^\vee \rightarrow X$  defined by a fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$ , there exists another fine weight function  $\tilde{w} : E(\Gamma) \rightarrow k$  on the tree  $\Gamma$ , and a closed embedding  $\zeta : S^\vee \hookrightarrow \mathbb{A}_X^{d(\Gamma)}$  inducing an isomorphism between  $S^\vee$  and the embedded Danielewski surface  $S_{\tilde{\gamma}}$  defined by the fine  $k$ -weighted tree  $\tilde{\gamma} = (\Gamma, \tilde{w})$ . Moreover,  $\zeta$  is equivariant when we equip  $S^\vee$  and  $\mathbb{A}_X^{d(\Gamma)}$  with the  $\mathbb{G}_{a,X}$ -actions corresponding to the locally nilpotent  $A$ -derivations  $\partial_{a,m} \in \text{Der}_A(B^\vee)$  (see 2.3) and  $\tilde{\partial}_{\tilde{\gamma},a,m} \in \text{Der}_A(A[\Gamma])$  (Proposition 2.15) respectively.*

**Example 3.2.** We consider the abstract Danielewski surface  $\pi : S^\vee \rightarrow X$  defined by the fine  $k$ -weighted tree of Example 2.4. The canonical morphism  $\psi : S^\vee \rightarrow \mathbb{A}_X^1 = \text{Spec}(k[x][X_{e-1}])$  is given by the section  $s_{e-1} \in B^\vee$  whose restrictions on the canonical open subsets  $S_i = \text{Spec}(k[x][T_i])$  are given by

$$s_{e-1}|_{S_i} = \begin{cases} (-1)^{i+1} + xT_i, & \text{if } i = 1, 2, \\ (-1)^{i+1}x + x^2T_i, & \text{if } i = 3, 4. \end{cases}$$

Letting  $C_i = \pi^{-1}(x_0) \cap S_i$ ,  $i = 1, \dots, 4$ , be the irreducible components of  $\pi^{-1}(x_0)$ , we see that  $s_{e-1}$  restricts to a coordinate function on every fiber  $\pi^{-1}(y)$ ,  $y \in X_*$ , and is locally constant on  $\pi^{-1}(x_0)$  with the values 1,  $-1$  and 0 on  $C_1$ ,  $C_2$  and  $C_3 \cup C_4$  respectively. Therefore, letting  $P(t) = (t^2 - 1) \in k[t]$ , the section  $x^{-1}s_{e-1}P(s_{e-1}) \in B^\vee \otimes_{k[x]} k[x, x^{-1}]$  extends to a section  $s_{e_0} \in B^\vee$  whose restrictions on the  $S_i$ 's are given by

$$s_{e_0}|_{S_i} = \begin{cases} 2T_1 + 3xT_1^2 + x^2T_1^3, & \text{if } i = 1, \\ 2T_2 - 3xT_2^2 + x^2T_2^3, & \text{if } i = 2, \\ -1 - xT_3 + x^2\xi_3(x, T_3), & \text{if } i = 3, \\ 1 - xT_4 + x^2\xi_4(x, T_4), & \text{if } i = 4, \end{cases}$$

for certain polynomials  $\xi_3(x, t), \xi_4(x, t) \in k[x, t]$ . Thus  $s_{e_0}$  restricts to a coordinate function on  $C_1$  and  $C_2$ , and is constant on  $C_3$  and  $C_4$  with the values  $-1$  and 1 respectively. Again,  $x^{-1}P(s_{e_0}) \in B^\vee \otimes_{k[x]} k[x, x^{-1}]$  extends to a regular function on  $S_3 \cup S_4 \subset S^\vee$  which restricts to a coordinate function on  $C_3$  and  $C_4$ . Clearly,  $x^{-1}P(s_{e-1})P(s_{e_0})$  extends to a section  $s_{e_1} \in B^\vee$  with the same property as

$P(s_{e_{-1}})|_{C_i} = -1$ ,  $i = 3, 4$ . The  $A$ -algebra homomorphism  $A[X_{e_{-1}}, X_{e_0}, X_{e_1}] \rightarrow B^\gamma$ ,  $X_e \mapsto s_e$  defines a closed embedding  $\zeta: S^\gamma \rightarrow \mathbb{A}_X^3$ , inducing an  $X$ -isomorphism between  $S^\gamma$  and the embedded Danielewski surface  $S_{\tilde{\gamma}}$  defined by the fine  $k$ -weighted tree  $\tilde{\gamma} = (\Gamma, \tilde{w})$  of Example 2.16.

**3.3.** To prove Theorem 3.1, we proceed in a similar way as in the previous example. More precisely, given an abstract Danielewski surface  $S^\gamma$  defined by a fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$ , we construct in 3.4 and Lemmas 3.5–3.7 below a fine weight function  $\tilde{w}: E(\Gamma) \rightarrow k$  on  $\Gamma$  and a collection of sections  $s_e \in B^\gamma$ ,  $e \in \mathbf{P}(\Gamma) \cup \{e_{-1}\}$ , which define a closed embedding  $\zeta: S^\gamma \hookrightarrow \mathbb{A}_X^{d(\Gamma)}$  inducing an  $X$ -isomorphism  $\phi: S^\gamma \xrightarrow{\sim} S_{\tilde{\gamma}}$  between  $S^\gamma$  and the embedded Danielewski surface defined by the tree  $\tilde{\gamma} = (\Gamma, \tilde{w})$ .

**3.4.** Given a fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$  with the leaves  $e_{1,m_1}, \dots, e_{n,m_n}$ , we denote by  $\tau_i: B^\gamma = \Gamma(S^\gamma, \mathcal{O}_{S^\gamma}) \rightarrow A[T_i]$  the localization homomorphisms corresponding to the canonical open covering of the abstract Danielewski surface  $S^\gamma$  by the open subsets  $S_i = \text{Spec}(A[T_i])$ ,  $i = 1, \dots, n$ . The canonical  $X$ -morphism  $\psi: S^\gamma \rightarrow \mathbb{A}_X^1 = \text{Spec}(A[X_{e_{-1}}])$  (2.2) corresponds to the section  $s_{e_{-1}} \in B^\gamma$  such that

$$\tau_i(s_{e_{-1}}) = \sum_{j=0}^{m_i} w_{i,j} x^j \in A[T_i],$$

where

$$w_{i,j} = \begin{cases} w(\overleftarrow{e_{i,j}e_{i,j+1}}), & \text{if } 0 \leq j \leq m_i - 1, \\ T_i, & \text{if } j = m_i. \end{cases}$$

For every  $e \in \Gamma$ , we let

$$C_e = \bigsqcup_{\{e_{i,m_i} \in L((\uparrow e)\Gamma)\}} (\pi^{-1}(x_0) \cap S_i) \simeq \text{Spec} \left( \prod_{\{e_{i,m_i} \in L((\uparrow e)\Gamma)\}} \text{Spec}(k[T_i]) \right).$$

If  $\gamma$  has height  $h = 0$  then  $\Gamma$  is the trivial tree with one element  $\{e_0\}$  and  $\psi: S^\gamma \rightarrow \mathbb{A}_X^1$  is an isomorphism. Otherwise, if  $h \geq 1$ , then we have the following result.

**Lemma 3.5.** *If  $h \geq 1$  then there exists a fine weight function  $\tilde{w}: E(\Gamma) \rightarrow k$ ,  $\overleftarrow{e'}e \mapsto \tilde{a}_{e',e}$  defining a fine  $k$ -weighted tree  $\tilde{\gamma} = (\Gamma, \tilde{w})$ , and a collection of sections  $(s_e)_{e \in \mathbf{P}(\Gamma) \cup \{e_{-1}\}} \in B^\gamma$  with the following properties.*

- For every  $e_{i,j} \in \mathbf{P}(\Gamma)$ ,  $s_{e_{i,j}} = x^{-1} G_{e_{i,j}}(\tilde{\gamma})(s_{e_{-1}}, s_{e_0}, s_{e_{i,1}}, \dots, s_{e_{i,j-1}})$ .
- If  $\text{Ch}(e_{i,j}) = \{e_{i,j+1}, \dots, e_{i,r,j+1}\}$ , then  $s_{e_{i,j-1}}$  is constant on  $C_{e_{i,j+1}} \subset \pi^{-1}(x_0)$  with the value  $\tilde{a}_{e_{i,j}, e_{i,j+1}} \in k$ ,  $l = 1, \dots, r$ .
- For every leaf  $e_{i,m_i}$  of  $\Gamma$ ,  $s_{e_{i,m_i-1}}$  induces an coordinate function on  $C_{e_{i,m_i}} \simeq \mathbb{A}_k^1$ .

*Proof.* We construct the weight function  $\tilde{w}$  and the sections  $s_e$  by induction as follows. For every  $m = 0, \dots, h$ , we denote by  $\Gamma_m$  the subtree of  $\Gamma$  with the elements  $e \in \Gamma$  at levels  $l \leq m$ . At step  $m$ , we suppose that the weight function  $\tilde{w}_m: E(\Gamma_m) \rightarrow k$  is constructed on  $\Gamma_m$ , as well as the sections  $s_e$  for every  $e \in \Gamma_{m-2}$ , and we define the sections  $s_e, e \in \Gamma_{m-1} \setminus \Gamma_{m-2}$ . Then we extend  $\tilde{w}_m$  to a weight function  $\tilde{w}_{m+1}: E(\Gamma_{m+1}) \rightarrow k$ .

*Step 0.* We let  $s_{e_{-1}} \in B^\vee$  be the section corresponding to the canonical morphism  $\psi: S^\vee \rightarrow \mathbb{A}_X^1$ . By definition,  $\tau_i(s_{e_{-1}}) = w_{i,0} + x\xi_i$  for a certain  $\xi_i \in A[T_i]$  for every  $i = 1, \dots, n$ . Thus b) is satisfied provided that we define the weight function  $\tilde{w}_1$  on  $\Gamma_1 \setminus \{e_0\}$  by

$$\tilde{w}_{e_0, e_{i,1}} = \tilde{w}_1(\overleftarrow{e_0 e_{i,1}}) = s_{e_{-1}}|_{C_{e_{i,1}}} = w_{i,0} \in k$$

for every  $e_{i,1} \in \text{Ch}(e_0)$ . Note that if  $e_{j,1} = e_{i,1}$ , then  $w_{i,0} = w_{j,0}$  as  $w_{i,0} \neq w_{j,0}$  if and only if  $e_0$  is the first common ancestor of the leaves  $e_{i,m_i}$  and  $e_{j,m_j}$ . Thus  $\tilde{\gamma}_1 = (\Gamma_1, \tilde{w}_1)$  is a fine  $k$ -weighted tree and we are done with Step 0.

*Step 1.* By construction, the rational section  $x^{-1}G_{e_0}(\tilde{\gamma}_1)(s_{\gamma, e_{-1}}) \in B^\vee \otimes_A A_x$  extends to a section  $s_{e_0}$  of  $B^\vee$  satisfying a). Since  $\gamma$  is a fine  $k$ -weighted tree, we deduce from Taylor's Formula that for every  $i = 1, \dots, n$ , there exists a pair  $(\alpha_{i,1} = F_{e_0}^{(e_{i,1})}(w_{i,0}), \beta_{i,1}) \in k^* \times k$  depending only of the subchain  $(\downarrow e_{i,1})_\Gamma$ , and a polynomial  $\xi_{i,1} \in A[T_i]$  such that

$$\tau_i(s_{e_0}) = \alpha_{i,1}w_{i,1} + \beta_{i,1} + x\xi_{i,1} \in A[T_i].$$

Thus, if  $e_{i,1}$  is a leaf of  $\Gamma$  then  $w_{i,1} = T_i$  and so c) is satisfied. Otherwise, if  $e_{j,2}$  and  $e_{j',2}$  are children of  $e_{i,1}$  then  $\alpha_{j,1} = \alpha_{j',1} = \alpha_{i,1}$  and  $\beta_{j,1} = \beta_{j',1} = \beta_{i,1}$  as  $e_{j,1} = e_{j',1} = e_{i,1}$ , whereas  $w_{j,1} \neq w_{j',1}$  as  $\gamma$  is a fine  $k$ -weighted tree. Thus  $\tilde{\gamma}_2 = (\Gamma_2, \tilde{w}_2)$  is a fine  $k$ -weighted tree for the weight function  $\tilde{w}_2: E(\Gamma_2) \rightarrow k$  restricting to  $\tilde{w}_1$  on  $\Gamma_1 \subset \Gamma_2$  and such that

$$\tilde{w}_{e_{i,1}, e_{i,2}} = \tilde{w}_2(\overleftarrow{e_{i,1} e_{i,2}}) = s_{e_0}|_{C_{e_{i,2}}} = (\alpha_{i,0}w_{i,1} + \beta_{i,1}) \in k, \quad i = 1, \dots, n.$$

By construction, b) is also satisfied. This completes Step 1.

*Step  $m, m \geq 2$ .* By induction hypothesis,  $\tilde{\gamma}_m = (\Gamma_m, \tilde{w}_m)$  is a fine  $k$ -weighted tree, and the sections  $s_e \in B^\vee, e \in \Gamma_{m-2}$ , satisfying the hypothesis of Lemma 3.5 have been defined. So the formula

$$s_{e_{i,m-1}} = x^{-1}G_{e_{i,m-1}}(\tilde{\gamma}_m)(s_{e_{-1}}, s_{e_0}, s_{e_{i,1}}, \dots, s_{e_{i,m-2}})$$

makes sense and defines an element of  $B^\vee \otimes_A A_x$ . Similarly as in Step 1, we deduce from Taylor's Formula that for every  $j = 0, \dots, m-1$  there exists a pair  $(\tilde{\alpha}_{i,j}, \tilde{\beta}_{i,j}) \in k^* \times k$  depending only on the subchain  $(\downarrow e_{i,j})_\Gamma$ , and a polynomial  $\tilde{\xi}_{i,j} \in A[T_i]$  such that

$$\tau_i(s_{e_{i,j-1}}) = a_{e_{i,j+1}e_{i,j}} + x(\tilde{\alpha}_{i,j}w_{i,j+1} + \tilde{\beta}_{i,j}) + x^2\tilde{\xi}_{i,j} \in A[T_i].$$

By applying Taylor's Formula again, we conclude that there exists a pair  $(\alpha_{i,m}, \beta_{i,m}) \in k^* \times k$  depending only on the subchain  $(\downarrow e_{i,m})_\Gamma$  and a polynomial  $\xi_{i,m} \in A[T_i]$  such that

$$\tau_i(s_{e_{i,m-1}}) = \alpha_{i,m} w_{i,m} + \beta_{i,m} + x \xi_{i,m} \in A[T_i].$$

Thus, if  $e_{i,m-1} \in (\downarrow e_{j,m_j})$  then  $e_{i,m-1} = e_{j,m-1}$  and so  $\tau_j(s_{e_{i,m-1}}) \in A[T_j]$ . Otherwise, for every index  $j$  such that  $e_{i,m-1} \notin (\downarrow e_{j,m_j})_\Gamma$ , the first common ancestor of  $e_{i,m-1}$  and  $e_{j,m_j}$  is an element  $e_{i,l} = e_{j,l}$  at level  $l \leq \min(m-2, m_j-1)$ . Thus  $(X_{e_{j,l-1}} - \tilde{a}_{e_{j,l}, e_{j,l+1}})$  divides the genealogical polynomial  $G_{e_{i,m-1}}(\tilde{\gamma}_m)$  of  $e_{i,m-1}$ . Since  $\tau_j(s_{e_{j,l-1}} - \tilde{a}_{e_{j,l}, e_{j,l+1}}) \in xA[T_j]$ , we conclude that

$$x\tau_j(s_{e_{i,m-1}}) = G_{e_{i,m-1}}(\tilde{\gamma}_m)(\tau_j(s_{e_{-1}}), \tau_j(s_{e_{i,0}}), \tau_j(s_{e_{i,1}}), \dots, \tau_j(s_{e_{i,m-2}})) \in xA[T_j].$$

Thus  $\tau_j(s_{e_{i,m-1}}) \in A[T_j]$  for every  $j = 1, \dots, n$ , and hence,  $s_{\gamma, e_{i,m-1}} \in B^\gamma$ . If  $e_{i,m}$  is a leaf of  $\Gamma$  then  $w_{i,m} = w_{i,m_i} = T_i$  by definition. Thus  $s_{e_{i,m-1}}$  satisfies a) and c). Finally, the same argument as in Step 1 shows that  $\tilde{\gamma}_{m+1} = (\Gamma_{m+1}, \tilde{w}_{m+1})$  is a fine  $k$ -weighted tree for the weight function  $\tilde{w}_{m+1}: E(\Gamma_{m+1}) \rightarrow k$  restricting to  $\tilde{w}_m$  on  $\Gamma_m \subset \Gamma_{m+1}$  and such that

$$\tilde{a}_{e_{i,m}, e_{i,m+1}} = \tilde{w}_{m+1}(\overleftarrow{e_{i,m} e_{i,m+1}}) = s_{e_{i,m-1}}|_{C(e_{i,m+1})} = (\alpha_{i,m} w_{i,m} + \beta_{i,m}) \in k,$$

whenever  $e_{i,m}$  is not a leaf of  $\Gamma$ . This completes Step  $m$  as b) is satisfied by construction.

After  $h = h(\Gamma)$  steps, the above procedure stops, and we obtain a fine  $k$ -weighted tree  $\tilde{\gamma} = \tilde{\gamma}_h = (\Gamma, \tilde{w}_h)$  and a collection of sections  $(s_e)_{e \in \mathbf{P}(\Gamma) \cup \{e_{-1}\}} \in B^\gamma$  satisfying conditions a), b) and c). This completes the proof.  $\square$

The following lemma implies the first assertion of Theorem 3.1.

**Lemma 3.6.** *The  $X$ -morphism  $\zeta: S^\gamma \rightarrow \mathbb{A}_X^{d(\Gamma)}$  induced by the  $A$ -algebra homomorphism  $\zeta^*: A[\Gamma] \rightarrow B^\gamma, X_e \mapsto s_e, e \in \mathbf{P}(\Gamma) \cup \{e_{-1}\}$ , is a closed embedding inducing an  $X$ -isomorphism  $\phi: S^\gamma \xrightarrow{\sim} S_{\tilde{\gamma}}$ .*

*Proof.* By construction,  $s_{e_{-1}}$  corresponds to the canonical birational morphism  $\psi: S^\gamma \rightarrow \mathbb{A}_X^1$ , whence induces a  $X_*$ -isomorphism  $S^\gamma|_{X_*} \xrightarrow{\sim} \mathbb{A}_{X_*}^1$ . By b) of Lemma 3.5, for every pair  $e_{i,m_i}, e_{j,m_j}$  of leaves of  $\Gamma$  with first common ancestor  $e \in \Gamma$ , the section  $s_{\text{Par}(e)}$  takes distinct constant values on  $C_{e_{i,m_i}}$  and  $C_{e_{j,m_j}}$ . Thus  $\zeta$  distinguishes the irreducible components of the fiber  $\pi^{-1}(x_0)$ . Finally, c) of Lemma 3.5 implies that for every  $i = 1, \dots, n$ ,  $s_{e_{i,m_i-1}}$  induces a coordinate function on  $C_{e_{i,m_i}} \simeq \mathbb{A}_k^1$ . This proves that  $\zeta: S^\gamma \rightarrow \mathbb{A}_X^{d(\Gamma)}$  is an embedding. By construction,  $\zeta^*(\Delta_e(\tilde{\gamma})) = 0$  in  $B^\gamma$  for every  $e \in \mathbf{P}(\Gamma)$ . Thus  $x\zeta^*(\Delta_{e',e}(\tilde{\gamma})) = \zeta^*(x\Delta_{e',e}(\tilde{\gamma})) = 0$  for every  $(e, e') \in (\mathbf{P}(\Gamma) \setminus \{e_0\}) \times ((\downarrow e)_\Gamma \setminus \{e_0\})$  by virtue of (2.2), and so,  $\zeta^*(\Delta_{e',e}(\tilde{\gamma})) = 0$

as  $B^\vee$  is an integral  $A$ -algebra. This proves that the image of  $\zeta$  is contained in the embedded Danielewski surface  $S_{\tilde{\gamma}}$ . It is clear by construction that the induced  $X$ -morphism  $\phi: S^\vee \rightarrow S_{\tilde{\gamma}}$  restricts to a bijection between the sets of closed points of  $S^\vee$  and  $S_{\tilde{\gamma}}$  respectively. So the result follows from Zariski's Main Theorem as  $S_{\tilde{\gamma}}$  is smooth over  $k$ , whence, in particular, normal.  $\square$

The following result completes the proof of Theorem 3.1.

**Lemma 3.7.** *For every nontrivial  $\mathbb{G}_{a,X}$ -action  $t_{\gamma,a,m}$  (2.3) on an abstract Danielewski surface  $\pi: S^\vee \rightarrow X$  defined by a fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$ , the closed embedding  $\zeta: S^\vee \hookrightarrow \mathbb{A}_X^{d(\Gamma)}$  in Lemma 3.6 is equivariant when we equip  $\mathbb{A}_X^{d(\Gamma)}$  with the  $\mathbb{G}_{a,X}$ -action induced by the locally nilpotent  $A$ -derivation  $\tilde{\partial}_{\tilde{\gamma},a,m} \in \text{Der}_A(A[\Gamma])$  (Proposition 2.15).*

*Proof.* By definition (see 2.3), the twisted translation  $t_{\gamma,a,m}$  on  $S^\vee$  is induced by the extension  $\partial_{a,m}$  to  $B^\vee$  of the locally nilpotent derivation  $\delta_{a,m} = ax^m \partial_{X_{e_{-1}}}$  of  $B^\vee \otimes_A A_x \simeq A_x[X_{e_{-1}}]$ , where  $m \geq h(\Gamma)$  and  $a \in A \setminus \{0\}$ . By construction, for every  $e \in \mathbf{P}(\Gamma)$ , we have  $s_e = x^{-1} G_e(\tilde{\gamma})(s_{e_{-1}}, s_{e_0}, \dots, s_{\text{Par}(e)}) \in B^\vee \subset A_x[X_{e_{-1}}]$  and so,

$$\partial_{a,m}(s_e) = x^{-1} \sum_{e' \in \text{Anc}(e) \cup \{e_{-1}\}} \partial_{X_{e'}} G_e(\tilde{\gamma})(s_{e_{-1}}, s_{e_0}, \dots, s_{\text{Par}(e)}) \partial_{a,m}(s_{e'}) \in B^\vee \otimes_A A_x.$$

In view of the definition of  $\tilde{\partial}_{\tilde{\gamma},a,m} \in \text{Der}_A(A[\Gamma])$  (see Proposition 2.15), this means precisely that the embedding  $\zeta: S^\vee \hookrightarrow \mathbb{A}_X^{d(\Gamma)}$  is equivariant when we equip  $S^\vee$  and  $\mathbb{A}_X^{d(\Gamma)}$  with the actions corresponding to the locally nilpotent derivation  $\partial_{a,m}$  and  $\tilde{\partial}_{\tilde{\gamma},a,m}$ .  $\square$

**Corollary 3.8.** *Every Danielewski surface  $\pi: S \rightarrow X$  equipped with a nontrivial  $\mathbb{G}_{a,X}$ -action is equivariantly  $X$ -isomorphic to an embedded Danielewski surface  $S_\gamma$  defined by a fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$ , equipped with the  $\mathbb{G}_{a,X}$ -action corresponding to a suitable locally nilpotent derivation  $\partial_{\gamma,a,m} \in \text{Der}_A(B_\gamma)$ , where  $m \geq h(\Gamma)$  and  $a \in A \setminus \{0\}$ .*

*Proof.* By Theorem 3.2 in [4], every Danielewski surface  $S$  is isomorphic to an abstract Danielewski surface  $S^\vee$  defined by a fine  $k$ -weighted tree  $\gamma$ . Moreover, by Proposition 2.12 in *loc. cit.*, every nontrivial  $\mathbb{G}_{a,X}$ -action on  $S^\vee$  coincides with a twisted translation  $t_{\gamma,a,m}$  for a suitable pair  $(m \geq h(\Gamma), a \in A \setminus \{0\})$ . So the result follows from Theorem 3.1.  $\square$

**Corollary 3.9.** *Every  $\mathbb{G}_{a,X}$ -action on an embedded Danielewski surface  $S_\gamma$  defined by a fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$  is induced by a locally nilpotent derivation  $\partial_{\gamma,a,m} \in \text{Der}_A(B_\gamma)$ .*

Since the locally nilpotent derivations  $\partial_{\gamma,a,m} \in \text{Der}_A(B_\gamma)$  are induced by locally nilpotent derivations  $\tilde{\partial}_{\gamma,a,m} \in \text{Der}_A(A[\Gamma])$ , we obtain the following result.

**Corollary 3.10.** *Every Danielewski surface  $\pi : S \rightarrow X$  admits a closed embedding  $\zeta : S \hookrightarrow \mathbb{A}_X^d$  into a relative affine space  $\mathbb{A}_X^d$ , where  $d \geq 1$ , such that every  $\mathbb{G}_{a,X}$ -action on  $S$  extends to an action on  $\mathbb{A}_X^d$ .*

In particular, if the Makar-Limanov invariant of  $S$  is nontrivial, then  $\pi : S \rightarrow X$  is a unique  $\mathbb{A}^1$ -fibration on  $S$  up to automorphisms of  $X$ . Therefore, the general orbits of a  $\mathbb{G}_{a,k}$ -action on  $S$  coincide with the general fibers of  $\pi$ . This leads to the following result.

**Corollary 3.11.** *Every Danielewski surface  $S$  with a nontrivial Makar-Limanov invariant admits a closed embedding into an affine space  $\mathbb{A}_k^d$  in such a way that every  $\mathbb{G}_{a,k}$ -action on  $S$  extends to an action on  $\mathbb{A}_k^d$ .*

**Morphisms of Danielewski surfaces as linear projections.** A morphism of Danielewski surfaces is a birational  $X$ -morphism  $\beta : S' \rightarrow S$ , restricting to an isomorphism over  $X_*$ . In other words,  $\beta$  is an affine modification [7] restricting to an isomorphism over the complement of the support of the principal divisor  $\pi^{-1}(x_0) = \text{div}(x) \subset S$ . Thus, letting  $S = \text{Spec}(B)$ , there exists an ideal  $I \subset B$  containing a power  $x^m$  of  $x$  such that  $S'$  is isomorphic to the open subset  $\text{Spec}(B[It]/(1 - x^m t))$  of the spectrum of the Rees algebra  $B[It]$ . In turn, this implies that  $S' \simeq \text{Spec}(B[t_1, \dots, t_r]/J)$  for a certain ideal  $J$ . In these coordinates, the morphism  $\beta : S' \rightarrow S$  coincides with the restriction to  $S'$  of the projection  $\text{pr}_S : \mathbb{A}_S^{r+1} = \text{Spec}(B[t_1, \dots, t_r]) \rightarrow S$ . Here we give a more precise description of this situation.

**3.12.** To every morphism  $\tau : \gamma' = (\Gamma', w') \rightarrow \gamma = (\Gamma, w)$  of fine  $k$ -weighted tree (see Definition 1.4), we associate a morphism  $\beta_\tau : S^{\gamma'} \rightarrow S^\gamma$  between the associated abstract Danielewski surfaces in the following manner. We let  $\sigma' = \{\sigma'_i \in A\}_{i=1, \dots, n'}$  and  $\sigma = \{\sigma_j \in A\}_{j=1, \dots, n}$  be the collection of polynomials associated with  $\gamma'$  and  $\gamma$ , and we let  $g' = \{g'_{ij} \in A_x\}$  and  $g = \{g_{ij} \in A_x\}$  be the corresponding transition functions. We denote by  $S'_i = \text{Spec}(A[T'_i])$ ,  $i = 1, \dots, n'$ , and  $S_j = \text{Spec}(A[T_j])$ ,  $j = 1, \dots, n$ , the open subsets of the canonical coverings of  $S^{\gamma'}$  and  $S^\gamma$  respectively. By Remark 1.5, the image of a leaf  $e'_{i,m'_i}$  of  $\Gamma'$  by  $\tau$  is a leaf  $e_{j(i),m_{j(i)}}$  of  $\Gamma$  such that  $m'_i \geq m_{j(i)}$  and  $\tau(e'_{i,k}) = e_{j(i),\min(k,m_{j(i)})}$  for every  $k = 0, \dots, m'_i$ . Since  $w(\tau(e'_{i,k})\tau(e'_{i,k+1})) = w'(e'_{i,k}e'_{i,k+1})$  whenever  $\tau(e'_{i,k}) \neq \tau(e'_{i,k+1})$ , we conclude that there exists a collection  $\sigma'' = \{\sigma''_i \in A\}_{i=1, \dots, n'}$  such that  $\sigma'_i = \sigma_{j(i)} + x^{m_{j(i)}}\sigma''_i \in A$  for every  $i = 1, \dots, n'$ . Then for every  $i = 1, \dots, n'$ , the  $A$ -algebra homomorphism

$$A[T_{j(i)}] \longrightarrow A[T'_i], \quad T_{j(i)} \mapsto \sigma''_i + x^{m'_i - m_{j(i)}} T'_i$$



defines a birational  $X$ -morphism  $\beta_\tau^{(i)}: S'_i \rightarrow S_{j(i)}$  restricting to an isomorphism over  $X_*$ . Since the transition functions satisfy the relation  $x^{m_{i'}-m_{j(i)}} g'_{il} = g_{j(i)j(l)} + x^{m_{j(l)}-m_{j(i)}} \sigma''_i + \sigma''_i$  for every  $i, l = 1, \dots, n'$ , it follows that these local morphisms  $\beta_\tau^{(i)}$  glue to a morphism of Danielewski surfaces  $\beta_\tau: S^{\gamma'} \rightarrow S^\gamma$ . By Proposition 3.8 and Corollary 3.9 in [4], for every morphism of Danielewski surfaces  $\beta: S' \rightarrow S$ , there exists  $X$ -isomorphisms  $\phi': S' \xrightarrow{\sim} S^{\gamma'}$  and  $\phi: S \xrightarrow{\sim} S^\gamma$  for suitable fine  $k$ -weighted trees  $\gamma'$  and  $\gamma$  such that  $\phi \circ \beta \circ (\phi')^{-1}$  is the morphism  $\beta_\tau$  induced by a morphism of fine  $k$ -weighted tree  $\tau: \gamma' \rightarrow \gamma$ .

**3.13.** Every morphism of fine  $k$ -weighted tree  $\tau: \gamma' \rightarrow \gamma$  factors through a surjective morphism  $\tau': \gamma' \rightarrow \tau(\gamma')$  followed by an injection  $\tau(\gamma') \hookrightarrow \gamma$ . As a consequence, every morphism of Danielewski surfaces factors through a *quasi-surjective morphism*  $\beta': S^{\gamma'} \rightarrow S^{\tau(\gamma')}$ , i.e. a morphism of Danielewski surfaces such that  $\beta'^{-1}(C) \neq \emptyset$  for every irreducible component  $C$  of the fiber  $\pi_{\tau(\gamma')}^{-1}(x_0) \subset S^{\tau(\gamma')}$  followed by the open immersion of  $S^{\tau(\gamma')}$  in  $S^\gamma$  as the complement of irreducible components of  $\pi_\gamma^{-1}(x_0) \subset S^\gamma$  corresponding to the leaves of  $\Gamma$  which are not in the image of  $\tau$ .

**3.14.** Given a fine  $k$ -weighted tree  $\gamma = (\Gamma, w)$ , we consider the tree  $\tilde{\gamma} = (\Gamma, \tilde{w})$  constructed in Lemma 3.5. For every edge  $\overleftarrow{e'e}$  of  $\Gamma$ , the weight  $\tilde{w}(\overleftarrow{e'e}) \in k$  is uniquely determined by the weights  $w$  of the edges of the subtree of  $\Gamma$  with elements  $(\downarrow e)_\Gamma \cup \bigcup_{e' \in (\downarrow e)_\Gamma} \text{Ch}(e')$ . Therefore, every *surjective* morphism of fine  $k$ -weighted trees  $\tau: \gamma' = (\Gamma', w') \rightarrow \gamma$  gives rise to a surjective morphism of fine  $k$ -weighted trees  $\tilde{\tau}: \tilde{\gamma}' = (\Gamma', \tilde{w}') \rightarrow \tilde{\gamma}$  which restricts to the same morphism as  $\tau$  between the underlying trees  $\Gamma'$  and  $\Gamma$  of  $\tilde{\gamma}'$  and  $\tilde{\gamma}$  respectively<sup>1</sup>. Since the subset  $\Gamma'' = \{e' \in \Gamma', \tau^{-1}(\tau(e')) = \{e'\}\} \subset \Gamma'$  is a subtree of  $\Gamma'$  isomorphic to  $\Gamma$ , we obtain that

$$A[\Gamma'] = A[\Gamma''] \otimes_A A[(X_{e'})_{e' \in \mathbf{P}(\Gamma') \cap (\Gamma' \setminus \mathbf{P}(\Gamma''))}] \simeq A[\Gamma] \otimes_A A[(X_{e'})_{e' \in \mathbf{P}(\Gamma') \cap (\Gamma' \setminus \mathbf{P}(\Gamma''))}].$$

Moreover, for every  $e' \in \mathbf{P}(\Gamma'')$ , the genealogical polynomial  $G_{e'}(\tilde{\gamma}')$  of  $e'$  is an element of  $A[\Gamma''] \subset A[\Gamma']$  which coincides with the genealogical polynomial  $G_{\tau(e')}(\tilde{\gamma}) \in A[\Gamma]$  of  $\tau(e')$  via the isomorphism above. In turn, this implies that the genealogical matrix (see Definition 1.9)  $M(\tilde{\gamma})$  of  $\tilde{\gamma}$  is obtained from  $M(\tilde{\gamma}')$  by deleting the rows corresponding to the elements in  $\mathbf{P}(\Gamma') \setminus \mathbf{P}(\Gamma'')$ . By construction of the embedding of  $S^\gamma$  into  $\mathbb{A}_X^{d(\Gamma)}$  as the Danielewski surface  $S_{\tilde{\gamma}}$ , we obtain the following result.

**Theorem 3.15.** *Let  $\tau: \gamma' = (\Gamma', w') \rightarrow \gamma = (\Gamma, w)$  be a surjective morphism of fine  $k$ -weighted trees and let  $\tilde{\tau}: \tilde{\gamma}' \rightarrow \tilde{\gamma}$  be the morphism obtained above. Let*

<sup>1</sup>Actually, the functor  $\gamma \mapsto \tilde{\gamma}, \tau \mapsto \tilde{\tau}$  is an automorphism of the category  $\mathcal{T}_{w,k}^s$  of fine  $k$ -weighted trees equipped with surjective morphisms.

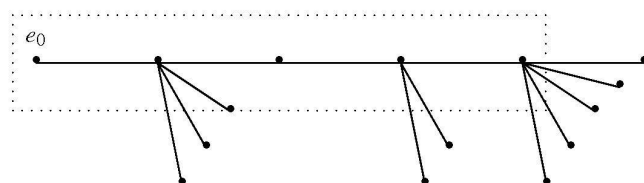
$\zeta': S^{\vee'} \hookrightarrow \mathbb{A}_X^{d(\Gamma')}$  and  $\zeta: S^\vee \hookrightarrow \mathbb{A}_X^{d(\Gamma)}$  are the embeddings from Lemma 3.6 of  $S^{\vee'}$  and  $S^\vee$  as the Danielewski surfaces  $S_{\tilde{\gamma}'}$  and  $S_{\tilde{\gamma}}$  respectively. Then  $\zeta \circ \beta = p_{\Gamma'/\Gamma} \circ \zeta'$ , where  $p_{\Gamma'/\Gamma}: \mathbb{A}_X^{d(\Gamma')} \rightarrow \mathbb{A}_X^{d(\Gamma)}$  is the projection induced by the inclusion  $A[\Gamma] \cong A[\Gamma''] \subset A[\Gamma']$ .

#### 4. Danielewski surfaces with a trivial Makar-Limanov invariant

The Makar-Limanov [6] invariant of an affine variety  $V = \text{Spec}(B)$  over a field  $k$  of characteristic zero is the sub-algebra  $\text{ML}(V) \subset B$  of regular functions on  $V$  which are invariant under every  $\mathbb{G}_{a,k}$ -action on  $V$ . A surface  $S$  has a trivial Makar-Limanov invariant  $\text{ML}(S) = k$  if and only if it admits two nontrivial  $\mathbb{G}_{a,k}$ -actions with distinct general orbits. In view of the correspondence between nontrivial  $\mathbb{G}_{a,k}$ -actions  $\mathbb{G}_{a,k} \times S \rightarrow S$  on  $S$  and quotient  $\mathbb{A}^1$ -fibrations  $\pi: S \rightarrow X = S/\mathbb{G}_{a,k}$ , this means in turn that  $S$  has a trivial Makar-Limanov invariant if and only if it admits two  $\mathbb{A}^1$ -fibrations with distinct general fibers. In this section, we characterize among Danielewski surfaces the ones with a trivial Makar-Limanov invariant.

##### Danielewski surfaces defined by weighted combs

**Definition 4.1.** A nontrivial (oriented) comb of height  $h \geq 1$  is a tree  $\Gamma$  such that for every  $e \in \mathbf{P}(\Gamma)$  of degree  $\deg_\Gamma(e) \geq 1$ , all but possibly one of the children of  $e$  are leaves of  $\Gamma$ . This means equivalently that the subtree  $C_\Gamma = \mathbf{P}(\Gamma) = \{e_0 < \dots < e_{h-1}\}$  of  $\Gamma$  is a nonempty chain of length  $h - 1$ , called the *dorsal chain* of  $\Gamma$ .



A comb rooted in  $e_0$ .

**4.2.** By Theorem 5.4 in [4], a Danielewski surface  $S$  defined over an algebraically closed field  $k = \bar{k}$  of characteristic zero has a trivial Makar-Limanov invariant if and only if it is isomorphic to an abstract Danielewski surface  $S^\vee$  defined by a fine  $k$ -weighted comb. This result is based on a characterization of normal affine surfaces  $S$  with a trivial Makar-Limanov invariant in terms on the boundary divisors of certain

minimal completions  $\bar{S}$  of  $S$  (see [3]). Unfortunately, no such criterion exists for a normal affine surface defined over an arbitrary field  $k$  of characteristic zero. However, the following result shows that the combinatorial characterization of Danielewski surfaces with a trivial Makar-Limanov invariant remains valid in this more general setting.

**Theorem 4.3.** *A Danielewski surface  $S \not\cong \mathbb{A}_X^1$ , defined over a field  $k$  of characteristic zero, has a trivial Makar-Limanov invariant if and only if it is isomorphic to an abstract Danielewski surface  $S^\gamma$  defined by a fine  $k$ -weighted comb. If this is the case, then there exist an integer  $h \geq 1$  and a collection of monic polynomials  $P_0, \dots, P_{h-1} \in k[t]$  with simple roots  $a_{i,j} \in k^*$ ,  $i = 0, \dots, h-1$ ,  $j = 1, \dots, \deg_t(P_i)$ , such that  $S$  is isomorphic to the surface  $S_{P_0, \dots, P_{h-1}} \subset \text{Spec}(k[x][y_{-1}, \dots, y_{h-2}][z])$  defined by the equations*

$$\begin{aligned} xz - y_{h-2} \prod_{l=0}^{h-1} P_l(y_{l-1}) &= 0, \\ zy_{i-1} - y_i y_{h-2} \prod_{l=i+1}^{h-1} P_l(y_{l-1}) &= 0, \quad xy_i - y_{i-1} \prod_{l=0}^i P_l(y_{l-1}) = 0, \quad 0 \leq i \leq h-2, \\ y_{i-1} y_j - y_i y_{j-1} \prod_{l=i+1}^j P_l(y_{l-1}) &= 0, \quad 0 \leq i < j \leq h-2. \end{aligned}$$

**4.4.** The proof is given in 4.5–4.7 below. We first observe that the condition is necessary. Indeed, suppose that the Makar-Limanov invariant of  $S$  is trivial. We let  $\gamma = (\Gamma, w)$  be a fine  $k$ -weighted tree such that  $S \simeq S^\gamma$ , and we let  $i: k \hookrightarrow \bar{k}$  be the injection of  $k$  in an algebraic closure  $\bar{k}$ . Then the Danielewski surface  $S_{\bar{k}} = S \times_{\text{Spec}(k)} \text{Spec}(\bar{k}) \rightarrow X_{\bar{k}} = X \times_{\text{Spec}(k)} \text{Spec}(\bar{k})$  is  $X_{\bar{k}}$ -isomorphic to the abstract Danielewski surface  $S^\gamma \times_{\text{Spec}(k)} \text{Spec}(\bar{k})$  defined by the tree  $\gamma$  considered a fine  $\bar{k}$ -weighted tree via the weight function  $i \circ w: E(\Gamma) \rightarrow \bar{k}$ . Since every nontrivial  $\mathbb{G}_{a,k}$ -action on  $S$  lifts to a nontrivial action of  $\mathbb{G}_{a,\bar{k}} = \mathbb{G}_{a,k} \times_{\text{Spec}(k)} \text{Spec}(\bar{k})$  on  $S_{\bar{k}}$ , we conclude that  $S_{\bar{k}}$  has a trivial Makar-Limanov invariant too. Thus the tree  $\gamma$  is a comb by virtue of Theorem 5.4 in [4].

**4.5.** Conversely, the same argument shows that if  $S$  is isomorphic to an abstract Danielewski surface  $S^\gamma$  defined by a fine  $k$ -weighted comb  $\gamma$ , then  $S_{\bar{k}}$  has a trivial Makar-Limanov invariant. Unfortunately, in general, there is no guarantee that a given  $\mathbb{G}_{a,\bar{k}}$ -action on  $S_{\bar{k}}$  appears as the lifting of an action of  $\mathbb{G}_{a,k}$  on  $S$ . Therefore, to show that the condition is sufficient, we must proceed in a different way. We will exploit the fact that  $S$  is isomorphic to an embedded surface  $S_\gamma$  defined by a fine  $k$ -weighted comb  $\gamma$  to construct two explicit  $\mathbb{A}^1$ -fibrations on  $S$  with distinct general fibers.

**4.6.** By construction, a Danielewski surface  $S$  is isomorphic to  $\mathbb{A}_X^1$  if and only if it is isomorphic to an abstract surface  $S^\gamma$  defined by a fine  $k$ -weighted chain  $\gamma$ . In this case it is also isomorphic to the surface  $S_{\{e_0\}}$  defined by the trivial tree with one element  $\{e_0\}$ . More generally, it follows from Theorem 3.10 in [4] that every Danielewski surface  $S \not\cong \mathbb{A}_X^1$  is isomorphic to an abstract Danielewski surface  $S^\gamma$  defined by a fine  $k$ -weighted comb  $\gamma$  is also isomorphic to a surface  $S^{\gamma_0}$  defined by a fine  $k$ -weighted comb  $\gamma_0 = (\Gamma, w_0)$  of height  $h \geq 1$ , with dorsal chain  $C_\Gamma = \{e_0 < e_1 < \dots < e_{h-1}\}$ , satisfying the following properties:

- a) The root  $e_0$  of  $\Gamma$  has at least two children.
- b) For every  $i = 0, \dots, h-2$ ,  $w_0(\overleftarrow{e_i e_{i+1}}) = 0 \in k$ .
- c) There exists  $e_h \in \text{Ch}(e_{h-1})$  such that  $w_0(\overleftarrow{e_{h-1} e_h}) = 0 \in k$ .

By definition, the restriction of the canonical morphism  $\psi: S^{\gamma_0} \rightarrow \mathbb{A}_X^1$  to an open subset  $S_i = \text{Spec}(A[T_i])$  corresponding to a leaf  $e_{i,m_i}$  of  $\Gamma$  at level  $m_i \geq 1$  is induced by the section  $w_0(\overleftarrow{e_{m_i-1} e_{i,m_i}})x^{m_i-1} + x^{m_i}T_i$ . Thus, by applying the procedure used in the proof of Lemma 3.5 to this comb  $\gamma_0$ , we obtain a fine  $k$ -weighted comb  $\tilde{\gamma}_0 = (\Gamma, \tilde{w}_0)$  with the same underlying comb  $\Gamma$  as  $\gamma_0$  such that  $\tilde{w}_0(\overleftarrow{e_i e_{i+1}}) = 0 \in k$  for every  $i = 0, \dots, h-1$ .

**4.7.** By construction of the tree  $\tilde{\gamma}_0$ , there exists monic polynomials  $P_0, \dots, P_{h-1} \in k[t]$ , of degrees  $\deg(P_i) = \deg_\Gamma(e_i) - 1$ , with simple roots  $\tilde{a}_{e,e_i} \in k^*$ ,  $e \in \text{Ch}(e_i) \setminus \{e_{i+1}\}$  respectively, such that  $F_{e_i}(\tilde{\gamma}_0) = X_{e_{i-1}}P_i(X_{e_{i-1}})$  for every  $i = 0, \dots, h-1$ . Letting  $y_{-1} = X_{e_{-1}}$ ,  $y_0 = X_{e_0}$ ,  $\dots$ ,  $y_{h-2} = X_{e_{h-2}}$ ,  $z = X_{e_{h-1}}$ , we conclude that the embedded Danielewski surface  $S_{\tilde{\gamma}_0}$  is  $X$ -isomorphic to the surface  $S_{P_0, \dots, P_{h-1}}$  of Theorem 4.3. This shows that every abstract Danielewski surface  $S^\gamma \not\cong \mathbb{A}_X^1$  defined by a fine  $k$ -weighted comb  $\gamma$  is  $X$ -isomorphic to a surface  $S_{P_0, \dots, P_{h-1}} \subset \mathbb{A}_X^{h+1}$ . Thus, to complete the proof of Theorem 4.3, it suffices to show that a surface  $S = S_{P_0, \dots, P_{h-1}} = \text{Spec}(B)$  has a trivial Makar-Limanov invariant. A similar argument as in 2.10 shows that  $B \otimes_{k[z]} k[z, z^{-1}] \simeq k[z, z^{-1}][y_{h-2}]$ . This means equivalently that the projection  $\pi_2 = \text{pr}_z|_S: S \rightarrow Z = \text{Spec}(k[z])$  in an  $\mathbb{A}^1$ -fibration restricting to the trivial line bundle  $\mathbb{A}_{Z^*}^1 = \text{Spec}(k[z, z^{-1}][y_{h-2}])$  over  $Z^*$ . Since the general fibers of the two projections  $\pi_1 = \text{pr}_x|_S: S \rightarrow X = \text{Spec}(k[x])$  and  $\pi_2: S \rightarrow Z$  do not coincide, we conclude that  $S$  has a trivial Makar-Limanov invariant. This completes the proof of Theorem 4.3.

**Remark 4.8.** The same argument as in the proof of Proposition 2.15 applied to the fibration  $\pi_2$  shows that the locally nilpotent derivation  $z^h \partial_{y_{h-2}}$  of  $B \otimes_{k[z]} k[z, z^{-1}] \simeq k[z, z^{-1}][y_{h-2}]$  extends to a locally nilpotent derivation of  $B$ , induced by a triangular  $k[z]$ -derivation of  $k[z][y_{h-2}, \dots, y_{-1}, x]$ . This proves that every Danielewski surface  $S$  with a trivial Makar-Limanov invariant can be embedded in an affine space  $\mathbb{A}_k^d$  in such a way that at least two algebraically independent  $\mathbb{G}_{a,k}$ -actions on  $S$  extend to  $\mathbb{G}_{a,k}$ -actions on  $\mathbb{A}_k^d$ .

**Nonconjugated  $\mathbb{G}_a$ -actions on a Danielewski surface.** By a result of Daigle [2], all the  $\mathbb{G}_{a,k}$ -actions on a Danielewski surface  $S_{P,1} = \{xz - P(y)\}$  are conjugated under the action of the automorphism group  $\text{Aut}(S_{P,1})$  of  $S_{P,1}$ .

**4.9.** This means that for every pair of nontrivial locally nilpotent derivations  $\partial_1$  and  $\partial_2$  of  $B = \Gamma(S_{P,1}, \mathcal{O}_{S_{P,1}})$ , there exists a  $k$ -algebra automorphism  $\phi$  of  $B$  such that  $\phi(\text{Ker}(\partial_1)) = \text{Ker}(\partial_2)$ . This implies in particular that the fibers of corresponding quotient  $\mathbb{A}^1$ -fibrations  $\pi_1: S_{P,1} \rightarrow \mathbb{A}_k^1$  and  $\pi_2: S_{P,1} \rightarrow \mathbb{A}_k^1$  have the same scheme-theoretic structures. By 4.7 above, a Danielewski surface  $S = S_{P_0, \dots, P_{h-1}} = \text{Spec}(B)$  admits two  $\mathbb{A}^1$ -fibrations  $\pi_1: S \rightarrow X = \text{Spec}(k[x])$  and  $\pi_2: S \rightarrow Z = \text{Spec}(k[z])$ . Moreover  $\pi_2$  restricts to the trivial line bundle over  $Z_* = \text{Spec}(k[z, z^{-1}])$ , and a similar argument as in Lemma 2.13 shows that the fiber  $(\pi_2^{-1}(0))_{\text{red}}$  decomposes as a disjoint union of curves isomorphic to the affine line  $\mathbb{A}_k^1$ . However, we have the following result.

**Lemma 4.10.** *If  $h \geq 2$ , then  $\pi_2: S = S_{P_0, \dots, P_{h-1}} \rightarrow Z$  is not a Danielewski surface over  $Z$ .*

*Proof.* It suffices to show that the intersection of the fiber  $\pi_2^{-1}(0)$  with the complement of the fiber  $\pi_1^{-1}(0)$  is a nonreduced scheme. By (2.2), the defining ideal  $I_*$  of  $S \setminus \pi_1^{-1}(0) \simeq \mathbb{A}_{X_*}^1$  in  $k[x, x^{-1}][y_{-1}, \dots, y_{h-2}][z]$  is generated by the polynomials  $c_i = y_i - x^{-1}y_{i-1} \prod_{l=0}^i P_l(y_{l-1})$ ,  $i = 0, \dots, h-2$  and  $d = z - x^{-1}y_{h-2} \prod_{l=0}^{h-1} P_l(y_{l-1})$ . We conclude recursively that there exists a polynomial  $R \in k[x, x^{-1}][y_{-1}]$  such that

$$d \equiv z - x^{-h}y_{-1}(P_0(y_{-1}))^h R(y_{-1})$$

modulo  $c_0, \dots, c_{h-2}$ . Since the polynomial  $P_0$  is nonconstant (see 4.6),

$$\begin{aligned} (S \setminus \pi_1^{-1}(0)) \cap \pi_2^{-1}(0) &\simeq \text{Spec}(k[x, x^{-1}][y_{-1}, \dots, y_{h-2}, z]/(I_*, z)) \\ &\simeq \text{Spec}(k[x, x^{-1}][y_{-1}]/(x^{-h}y_{-1}(P_0(y_{-1}))^h R(y_{-1}))) \end{aligned}$$

is clearly nonreduced whenever  $h \geq 2$ . This completes the proof.  $\square$

**4.11.** The above result implies that if  $h \geq 2$ , then the degenerate fibers of  $\pi_1$  and  $\pi_2$  have different scheme-theoretic structures. Therefore two  $\mathbb{G}_{a,k}$ -actions on  $S_{P_0, \dots, P_{h-1}}$  with associated quotient  $\mathbb{A}^1$ -fibrations  $\pi_1: S \rightarrow X$  and  $\pi_2: S \rightarrow Z$  respectively can not be conjugated in the sense of (4.9) above. This leads to the following result.

**Theorem 4.12.** *A Danielewski surface  $S \not\cong S_{P,1}$  with a trivial Makar-Limanov invariant admits two algebraically independent nonconjugated  $\mathbb{G}_{a,k}$ -actions.*

As a consequence of this description, we obtain the following characterization of ordinary Danielewski surfaces  $S_{P,1}$ .

**Corollary 4.13.** *Let  $\pi: S \rightarrow X = \operatorname{Spec}(k[x])$ , where  $k$  is an arbitrary field of characteristic zero, be a Danielewski surface with a trivial Makar-Limanov invariant. Then the following are equivalent.*

- a)  *$S$  admits a free  $\mathbb{G}_{a,X}$ -action.*
- b)  *$S$  is isomorphic to a surface  $S_{P,1} = \{xz - P(y) = 0\}$  in  $\mathbb{A}_k^3 = \operatorname{Spec}(k[x, y, z])$ , where  $P$  is a nonconstant polynomial with  $\deg P$  simple roots.*
- c) *The canonical sheaf  $\omega_S$  is trivial.*
- d) *All  $\mathbb{G}_{a,k}$ -actions on  $S$  are conjugated.*

*Proof.* The equivalence b) $\Leftrightarrow$ d) follows from [2] and the above discussion. All the other equivalences can be obtained in the same way as in Corollary 5.7 in [4].  $\square$

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