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**Arithmetic properties of  $\varphi(n)/\lambda(n)$  and the structure of the multiplicative group modulo  $n$** 

William D. Banks, Florian Luca and Igor E. Shparlinski

**Abstract.** For a positive integer  $n$ , we let  $\varphi(n)$  and  $\lambda(n)$  denote the Euler function and the Carmichael function, respectively. We define  $\xi(n)$  as the ratio  $\varphi(n)/\lambda(n)$  and study various arithmetic properties of  $\xi(n)$ .

**Mathematics Subject Classification (2000).** 11A25.

**Keywords.** Euler function, Carmichael function.

**1. Introduction and notation**

Let  $\varphi(n)$  denote the *Euler function*, which is defined as usual by

$$\varphi(n) = \#(\mathbb{Z}/n\mathbb{Z})^\times = \prod_{p^v \parallel n} p^{v-1}(p-1), \quad n \geq 1.$$

The *Carmichael function*  $\lambda(n)$  is defined for all  $n \geq 1$  as the largest order of any element in the multiplicative group  $(\mathbb{Z}/n\mathbb{Z})^\times$ . More explicitly, for any prime power  $p^v$ , one has

$$\lambda(p^v) = \begin{cases} p^{v-1}(p-1) & \text{if } p \geq 3 \text{ or } v \leq 2, \\ 2^{v-2} & \text{if } p = 2 \text{ and } v \geq 3, \end{cases}$$

and for an arbitrary integer  $n \geq 2$ ,

$$\lambda(n) = \text{lcm}(\lambda(p_1^{v_1}), \dots, \lambda(p_k^{v_k})),$$

where  $n = p_1^{v_1} \dots p_k^{v_k}$  is the prime factorization of  $n$ . Clearly,  $\lambda(1) = 1$ .

Despite their many similarities, the functions  $\varphi(n)$  and  $\lambda(n)$  often exhibit remarkable differences in their arithmetic behavior, and a vast number of results about the growth rate and various arithmetical properties of  $\varphi(n)$  and  $\lambda(n)$  have been obtained; see for example [4], [5], [7], [8], [9], [11], [15]. In this paper, we consider the

arithmetical function defined by

$$\xi(n) = \frac{\varphi(n)}{\lambda(n)}, \quad n \geq 1,$$

and we study some of its arithmetic properties.

In particular, letting  $P(k)$  denote the largest prime factor of a positive integer  $k$  (with the convention that  $P(1) = 1$ ), we study the behavior of  $P(\xi(n))$ . Our results imply that typically  $\xi(n)$  is much “smoother” than a random integer  $k$  of the same size. To make this comparison, it is useful to recall that Theorem 2 of [9] implies that the estimate

$$\xi(n) = \exp(\log_2 n \log_3 n + C \log_2 n + o(\log_2 n)) \quad (1)$$

holds on a set of positive integers  $n$  of asymptotic density 1 with some absolute constant  $C > 0$ . Here, and in the sequel, for a real number  $z > 0$  and a natural number  $\ell$ , we write  $\log_\ell z$  for the recursively defined function given by  $\log_1 z = \max\{\log z, 1\}$ , where  $\log z$  denotes the natural logarithm of  $z$ , and  $\log_\ell z = \max\{\log(\log_{\ell-1} z), 1\}$  for  $\ell > 1$ . When  $\ell = 1$ , we omit the subscript (however, we still assume that all the logarithms that appear below are at least 1). Of course, when  $z$  is sufficiently large, then  $\log_\ell z$  is nothing more than the  $\ell$ -fold composition of the natural logarithm evaluated at  $z$ .

We also use  $\Omega(n)$  and  $\omega(n)$  with their usual meanings:  $\Omega(n)$  denotes the total number of prime divisors of  $n > 1$  counted with multiplicity, while  $\omega(n)$  is the number of distinct prime factors of  $n > 1$ ; as usual, we put  $\Omega(1) = \omega(1) = 0$ . In this paper, we also study the functions  $\Omega(\xi(n))$  and  $\omega(\xi(n))$ .

Observe that a prime  $p$  divides  $\xi(n)$  if and only if the  $p$ -Sylow subgroup of the group  $(\mathbb{Z}/n\mathbb{Z})^\times$  is not cyclic. Thus,  $P(\xi(n))$  and  $\omega(\xi(n))$  can be viewed as measures of “non-cyclicity” of this group. In particular,  $\omega(\xi(n))$  is the number of non-cyclic Sylow subgroups of  $(\mathbb{Z}/n\mathbb{Z})^\times$ .

We also remark that any prime  $p \mid \xi(n)$  has that property that  $p^2 \mid \varphi(n)$ . Thus, while studying the prime factors of  $\xi(n)$ , one is naturally lead to an associated question concerning the difference  $\Omega(\varphi(n)) - \omega(\varphi(n))$ , a question that we address here as well.

As usual, for a large number  $x$ ,  $\pi(x)$  denotes the number of primes  $p \leq x$ , and for positive integers  $a, k$  with  $\gcd(a, k) = 1$ ,  $\pi(x; k, a)$  denotes the number of primes  $p \leq x$  with  $p \equiv a \pmod{k}$ .

We use the Vinogradov symbols  $\gg, \ll, \asymp$  as well as the Landau symbols  $O$  and  $o$  with their usual meanings. The implied constants in the symbols  $O, \gg, \ll$  and  $\asymp$  are always absolute unless indicated otherwise.

Finally, we say that a certain property holds for “almost all”  $n$  if it holds for all  $n \leq x$  with at most  $o(x)$  exceptions, as  $x \rightarrow \infty$ .

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## 2. Distribution of $P(\xi(n))$ , $\omega(\xi(n))$ and $\Omega(\xi(n))$

In what follows, let us call a real-valued function  $\varepsilon(x)$  *admissible* if

- $\varepsilon(x)$  is a decreasing function, with limit 0 as  $x \rightarrow \infty$ ;
- $\varepsilon(x) \log_2 x$  is an increasing function, tending to  $\infty$  as  $x \rightarrow \infty$ .

We begin with the following statement, which may be of independent interest.

**Lemma 1.** *For any admissible function  $\varepsilon(x)$  and any prime  $q \leq \varepsilon(x) \log_2 x$ , every positive integer  $n \leq x$  has at least  $(\log_2 n)/2q$  distinct prime factors  $p \equiv 1 \pmod{q}$ , with at most  $o(x)$  exceptions.*

*Proof.* Let  $\omega(n, q)$  denote the number of distinct prime factors  $p$  of  $n$  such that  $p \equiv 1 \pmod{q}$ . For any real number  $y \geq 1$  and integer  $a \geq 1$ , put

$$S(y, a) = \sum_{\substack{p \leq y \\ p \equiv 1 \pmod{a}}} \frac{1}{p}. \quad (2)$$

It is known (see Theorem 1 in [18] or Lemma 6.3 in [17]) that

$$S(y, a) = \frac{\log_2 y}{\varphi(a)} + O(1). \quad (3)$$

In particular, the estimate

$$S(n, q) = \frac{\log_2 n}{q-1} + O(1) \gg \varepsilon(x)^{-1}$$

holds for all  $q$  in the stated range and all  $n > x^{1/2}$ , once  $x$  is sufficiently large. By the classical result of Turán [20], we also have that the estimate

$$\omega(n, q) = S(n, q) + O(S(n, q)^{2/3})$$

holds for all  $n$  in the interval  $x^{1/2} < n \leq x$ , with at most

$$O(xS(n, q)^{-1/6}) = O(x\varepsilon(x)^{1/6}) = o(x)$$

possible exceptions, and the result now follows.  $\square$

**Lemma 2.** For real numbers  $x \geq y > 1$  let

$$\Xi(x, y) = \#\{n \leq x : P(\xi(n)) > y\}.$$

Then,

$$\Xi(x, y) \ll \frac{x(\log_2 x)^2}{y \log y}.$$

*Proof.* If a prime  $q$  divides  $\xi(n)$ , then clearly  $q^2 \mid \varphi(n)$ . The upper bound

$$\#\{n \leq x : \varphi(n) \equiv 0 \pmod{q^2}\} \ll \frac{x(\log_2 x)^2}{q^2}$$

is a special partial case of Lemma 2 of [5] (see also the proof of Theorem 7.1 in [4]). In particular,

$$\#\{n \leq x : P(\xi(n)) = q\} \ll \frac{x(\log_2 x)^2}{q^2}. \quad (4)$$

It now follows that

$$\Xi(x, y) = \sum_{y < q \leq x} \sum_{\substack{n \leq x \\ P(\xi(n))=q}} 1 \ll \sum_{y < q \leq x} \frac{x(\log_2 x)^2}{q^2}.$$

Using Abel summation, we estimate

$$\sum_{y < q \leq x} \frac{1}{q^2} = \frac{\pi(x)}{x^2} - \frac{\pi(y)}{y^2} + 2 \int_y^x \frac{\pi(t)}{t^3} dt \ll \frac{1}{x \log x} + \int_y^x \frac{1}{t^2 \log t} dt \ll \frac{1}{y \log y},$$

and the lemma follows.  $\square$

**Theorem 1.** If  $\varepsilon(x)$  is any admissible function, then the inequalities

$$\varepsilon(n) \log_2 n \leq P(\xi(n)) \leq \frac{(\log_2 n)^2}{\varepsilon(n) \log_3 n}$$

hold for almost all positive integers  $n$ .

*Proof.* By the Prime Number Theorem, for all sufficiently large real numbers  $x$  there exists a prime  $q$  in the interval:

$$\varepsilon(x) \log_2 x < q \leq 2 \varepsilon(x) \log_2 x.$$

If  $n$  is an integer with two prime factors  $p_1 \equiv p_2 \equiv 1 \pmod{q}$ , then  $q \mid \xi(n)$ . By Lemma 1, we derive that

$$\sum_{\substack{x^{1/2} < n \leq x \\ P(\xi(n)) \geq \varepsilon(n) \log_2 n}} 1 \geq \sum_{\substack{x^{1/2} < n \leq x \\ P(\xi(n)) \geq q}} 1 \geq \sum_{\substack{x^{1/2} < n \leq x \\ \omega(n, q) \geq 2}} 1 = x + o(x).$$

This proves the lower bound. The upper bound is a direct application of Lemma 2.  $\square$

We remark that the upper bound of Theorem 1 improves the corollary to Theorem 2 in [9].

**Theorem 2.** *As  $x \rightarrow \infty$ , we have*

$$(1 + o(1)) x \log_3 x \leq \sum_{n \leq x} \log P(\xi(n)) \leq (2 + o(1)) x \log_3 x.$$

*Proof.* The above lower bound follows from the lower bound from Theorem 1. For the upper bound above, we write

$$\sum_{n \leq x} \log P(\xi(n)) = \sum_{q \leq x} \log q \sum_{\substack{n \leq x \\ P(\xi(n))=q}} 1.$$

For  $q \leq y$ , we trivially have

$$\sum_{q \leq y} \log q \sum_{\substack{n \leq x \\ P(\xi(n))=q}} 1 \leq \log y \sum_{q \leq y} \sum_{\substack{n \leq x \\ P(\xi(n))=q}} 1 \leq \log y \sum_{n \leq x} 1 \leq x \log y,$$

while for  $q > y$ , we have, by (4):

$$\sum_{y < q \leq x} \log q \sum_{\substack{n \leq x \\ P(\xi(n))=q}} 1 \ll x (\log_2 x)^2 \sum_{y < q \leq x} \frac{\log q}{q^2} \ll xy^{-1} (\log_2 x)^2,$$

where we have used Abel summation to estimate

$$\begin{aligned} \sum_{y < q \leq x} \frac{\log q}{q^2} &= \pi(x) \frac{\log x}{x^2} - \pi(y) \frac{\log y}{y^2} - \int_y^x \pi(t) \left( \frac{1}{t^3} - \frac{2 \log t}{t^3} \right) dt \\ &\ll x^{-1} + \int_y^x t^{-2} dt \ll y^{-1}. \end{aligned}$$

Setting  $y = (\log_2 x)^2$ , we obtain the desired upper bound.  $\square$

**Theorem 3.** *As  $x \rightarrow \infty$ , we have*

$$\sum_{n \leq x} P(\xi(n)) \asymp x (\log_2 x)^3.$$

*Proof.* Let  $y = (\log_2 x)^3$ ,  $z = \exp((\log x)^{1/2})$  and  $w = \exp((\log x)^{2/3})$ . We also put  $v = z^6$ . In what follows,  $x$  is taken to be arbitrarily large.

Taking  $A = 5/2$ ,  $\varepsilon = 1/2$ , and  $\delta = 1/15$  in the statement of Theorem 2.1 of [1], we see that there exists an absolute constant  $D \geq 0$  and a set  $\mathcal{D}$  of cardinality  $\#\mathcal{D} \leq D$ , with  $\min\{m : m \in \mathcal{D}\} \geq \log v = 6(\log x)^{1/2}$ , such that the inequality

$$\pi(t; d, 1) \geq \frac{\pi(t)}{2\varphi(d)} \quad (5)$$

holds for all positive reals  $t$  provided that  $1 \leq d \leq \min\{tv^{-2/3}, z^2\}$  and that  $d$  is not divisible by any element of  $\mathcal{D}$ . Note that if  $x$  is sufficiently large and  $t \geq w$ , then  $tv^{-2/3} \geq wv^{-2/3} \geq z^2$ .

Letting  $\mathcal{Q}$  denote the set of primes  $q \in [y, z] \setminus \mathcal{D}$ , we therefore see that the lower bound (5) holds for all  $t \in [w, x]$  and all integers  $d \in [1, z^2]$  whose prime factors all lie in  $\mathcal{Q}$ . Together with the Brun–Titchmarsh theorem (see for example Theorem 3.7 in Chapter 3 of [12]), we conclude that

$$\pi(t; d, 1) \asymp \frac{\pi(t)}{\varphi(d)}$$

holds uniformly for all  $t \in [w, x]$  and all integers  $d$  of the form  $d = q$  or  $d = q_1 q_2$  composed of one or two (not necessarily distinct) primes from  $\mathcal{Q}$ . Moreover, for any sufficiently large constant  $\gamma > 1$ , we also have

$$\pi(t; d, 1) - \pi(t/\gamma; d, 1) \asymp \frac{\pi(t)}{\varphi(d)} \quad (6)$$

under the same conditions.

We now let

$$k = \left\lceil \frac{\log w}{\log \gamma} \right\rceil \quad \text{and} \quad K = \left\lfloor \frac{\log x}{2 \log \gamma} \right\rfloor - 1.$$

For any prime  $q \in \mathcal{Q}$ , we have, by (6):

$$\sum_{\substack{w < p \leq x^{1/2} \\ p \equiv 1 \pmod{q}}} \frac{1}{p} \geq \sum_{j=k}^K \frac{\pi(\gamma^{j+1}; d, 1) - \pi(\gamma^j; d, 1)}{\gamma^{j+1}} \gg \frac{1}{q} \sum_{j=k}^K \frac{1}{j} \gg \frac{\log_2 x}{q}.$$

On the other hand, the upper bound (3.1) in [7] (see also Lemma 1 of [5]) provides an upper bound of the same size as the above lower bound. Consequently,

$$\sum_{\substack{w < p \leq x^{1/2} \\ p \equiv 1 \pmod{q}}} \frac{1}{p} \asymp \frac{\log_2 x}{q}. \quad (7)$$

We now fix a prime number  $q$  in  $\mathcal{Q}$ . We denote by  $N(x, q)$  the number of integers  $n \leq x$  for which there exists a unique representation of the form  $n = p_1 p_2 m$  for some integer  $m$  and two primes  $w < p_1 < p_2 \leq x^{1/2}$  with  $p_1 \equiv p_2 \equiv 1 \pmod{q}$  and such that  $q$  is the only prime in  $\mathcal{Q}$  dividing  $\gcd(p_1 - 1, p_2 - 1)$ . We then have

$$N(x, q) \geq T_0(x, q) - T_1(x, q) - T_2(x, q) - T_3(x, q),$$

where

- $T_0(x, q)$  is the total number of ordered triples  $(p_1, p_2, m)$  with primes  $w < p_1 < p_2 \leq x^{1/2}$ ,  $p_1 \equiv p_2 \equiv 1 \pmod{q}$ , and an integer  $m \leq x/p_1 p_2$ . Therefore, using (7), we obtain that

$$\begin{aligned} T_0(x, q) &\gg x \sum_{\substack{w < p_1 < p_2 \leq x^{1/2} \\ p_1 \equiv p_2 \equiv 1 \pmod{q}}} \frac{1}{p_1 p_2} \\ &= \frac{x}{2} \left( \sum_{\substack{w < p \leq x^{1/2} \\ p \equiv 1 \pmod{q}}} \frac{1}{p} \right)^2 - \frac{x}{2} \sum_{\substack{w < p \leq x^{1/2} \\ p \equiv 1 \pmod{q}}} \frac{1}{p^2} \\ &\gg \frac{x}{2} \left( \frac{\log_2 x}{q} \right)^2 - \frac{x}{2q} \sum_{\substack{w < p \leq x^{1/2} \\ p \equiv 1 \pmod{q}}} \frac{1}{p} \\ &= \frac{x(\log_2 x)^2}{2q^2} + O\left(\frac{x \log_2 x}{q^2}\right) \gg \frac{x(\log_2 x)^2}{q^2}. \end{aligned}$$

- $T_1(x, q)$  is the number of triples  $(p_1, p_2, m)$  as above for which there exists another prime  $\ell \in \mathcal{Q}$ ,  $\ell \neq q$ , such that  $p_1 \equiv p_2 \equiv 1 \pmod{\ell}$ . Then, by (7), we have that

$$\begin{aligned} T_1(x, q) &\ll x \sum_{\substack{\ell \in \mathcal{Q} \\ \ell \neq q}} \sum_{\substack{w < p_1 < p_2 \leq x^{1/2} \\ p_1 \equiv p_2 \equiv 1 \pmod{\ell}}} \frac{1}{p_1 p_2} \leq x \sum_{\ell \in \mathcal{Q}} \left( \sum_{\substack{w < p \leq x^{1/2} \\ p \equiv 1 \pmod{\ell}}} \frac{1}{p} \right)^2 \\ &\ll x \sum_{\ell \in \mathcal{Q}} \frac{(\log_2 x)^2}{q^2 \ell^2} \ll \frac{x(\log_2 x)^2}{q^2} \sum_{\ell > y} \frac{1}{\ell^2} \\ &\ll \frac{x(\log_2 x)^2}{q^2 y \log y} = o\left(\frac{x(\log_2 x)^2}{q^2}\right). \end{aligned}$$

- $T_2(x, q)$  is the number of triples  $(p_1, p_2, m)$  as above for which there exists another prime  $p_3$ ,  $w < p_3 \leq x^{1/2}$ , which divides  $m$ , and for some prime  $\ell \in \mathcal{Q}$



(possibly  $\ell = q$ ) one has  $p_3 \equiv 1 \pmod{\ell}$ , and either  $p_1 \equiv 1 \pmod{\ell}$ , or  $p_2 \equiv 1 \pmod{\ell}$ . Therefore, by (7), we see that

$$\begin{aligned}
 T_2(x, q) &\ll x \sum_{\ell \in \mathcal{Q}} \sum_{\substack{w < p_1, p_2 \leq x^{1/2} \\ w < p_3 \leq x^{1/2} \\ p_1 \equiv p_2 \equiv 1 \pmod{q} \\ p_3 \equiv 1 \pmod{\ell}}} \frac{1}{p_1 p_2 p_3} \\
 &\ll x \sum_{\ell \in \mathcal{Q}} \sum_{\substack{w < p_1 \leq x^{1/2} \\ p_1 \equiv 1 \pmod{q}}} \frac{1}{p_1} \sum_{\substack{w < p_2 \leq x^{1/2} \\ p_2 \equiv 1 \pmod{q\ell}}} \frac{1}{p_2} \sum_{\substack{w < p_3 \leq x^{1/2} \\ p_3 \equiv 1 \pmod{\ell}}} \frac{1}{p_3} \\
 &\ll x (\log_2 x)^3 \sum_{y \leq \ell \leq z} \frac{1}{q^2 \ell^2} \ll \frac{x (\log_2 x)^3}{q^2 y \log y} = o\left(\frac{x (\log_2 x)^2}{q^2}\right).
 \end{aligned}$$

- $T_3(x, q)$  is the number of triples  $(p_1, p_2, m)$  as above for which there exists another triple  $(r_1, r_2, k)$  with primes  $w \leq r_1 < r_2 \leq x^{1/2}$  such that  $r_1 \equiv r_2 \equiv 1 \pmod{\ell}$  for some  $\ell \in \mathcal{Q}$ , and  $p_1 p_2 m = r_1 r_2 k$ . Applying (7) once again, we obtain that

$$\begin{aligned}
 T_3(x, q) &\ll x \sum_{\ell \in \mathcal{Q}} \sum_{\substack{w < p_1 < p_2 \leq x^{1/2} \\ p_1 \equiv p_2 \equiv 1 \pmod{q}}} \frac{1}{p_1 p_2} \sum_{\substack{w < r_1 < r_2 \leq x^{1/2} \\ r_1 \equiv r_2 \equiv 1 \pmod{\ell}}} \frac{1}{r_1 r_2} \\
 &\ll x (\log_2 x)^4 \sum_{y \leq \ell \leq z} \frac{1}{q^2 \ell^2} \ll \frac{x (\log_2 x)^4}{q^2 y \log y} = o\left(\frac{x (\log_2 x)^2}{q^2}\right).
 \end{aligned}$$

Consequently, we have

$$N(x, q) \geq T_0(x, q) - T_1(x, q) - T_2(x, q) - T_3(x, q) \gg \frac{x (\log_2 x)^2}{q^2}.$$

We note that  $P(\xi(n)) \geq q$  for all  $n \in N(x, q)$  and that the sets  $N(x, q)$  are disjoint for different choices of  $q \in \mathcal{Q}$ . Thus,

$$\begin{aligned}
 \sum_{n \leq x} P(\xi(n)) &\gg \sum_{q \in \mathcal{Q}} q \#N(x, q) \gg x (\log_2 x)^2 \sum_{q \in \mathcal{Q}} \frac{1}{q} \\
 &\geq x (\log_2 x)^2 \left( \sum_{y \leq q \leq z} \frac{1}{q} - \frac{D}{6 (\log x)^{1/2}} \right) \\
 &\gg x (\log_2 x)^2 (\log_2 z - \log_2 y + o(1)) \gg x (\log_2 x)^3.
 \end{aligned}$$

To prove the upper bound, we simply use (4) to derive that

$$\sum_{n \leq x} P(\xi(n)) \leq \sum_{q \leq x} q \sum_{\substack{n \leq x \\ P(\xi(n))=q}} 1 \ll x(\log_2 x)^2 \sum_{q \leq x} \frac{1}{q} \ll x(\log_2 x)^3.$$

This completes the proof.  $\square$

Concerning the minimal order of  $P(\xi(n))$ , little need be said; clearly  $P(\xi(n)) \geq 1$  for all  $n \geq 1$ , and equality holds if and only if  $n = 2, 4, p^\nu$  or  $2p^\nu$  for some odd prime  $p$  and  $\nu \geq 1$ . As for the maximal order, we have the following:

**Theorem 4.** *The inequality*

$$P(\xi(n)) \leq \frac{(3n+1)^{1/2} - 2}{6}$$

*holds for all  $n \geq 276$ , and the inequality*

$$P(\xi(n)) \gg n^{0.3335}$$

*holds for infinitely many  $n$ .*

*Proof.* For  $n$  in the range  $276 \leq n \leq 579$ , the upper bound can be verified case by case; hence, we assume that  $n \geq 580$  in what follows. Without loss of generality, we may further assume that  $q = P(\xi(n)) > 3$ , since

$$3 \leq \frac{(3n+1)^{1/2} - 2}{6} \quad \text{holds for all } n \geq 133.$$

If  $P(\xi(n)) = q$ , then either  $n$  has a prime divisor  $p \equiv 1 \pmod{q}$  and  $q^2 p \mid n$ , or  $n$  has two distinct prime divisors  $p_1 \equiv p_2 \equiv 1 \pmod{q}$ . In the first case, we see that

$$q < (q^2 p/2)^{1/3} \leq (n/2)^{1/3} \leq \frac{(3n+1)^{1/2} - 2}{6},$$

the last inequality being valid for all  $n \geq 580$ . In the second case, suppose  $p_1 = aq + 1$  and  $p_2 = bq + 1$ , where  $a < b$  are distinct even integers. Now if  $2q + 1$  is prime, then  $4q + 1$  is divisible by 3; thus, we must have  $a \geq 2, b \geq 6$ . Then

$$(2q+1)(6q+1) \leq (aq+1)(bq+1) = p_1 p_2 \leq n,$$

and we obtain the stated upper bound.

To establish the lower bound, we recall the result of Fouvry [10], which asserts that for all large  $x$ , the set  $\mathcal{Q}$  of primes  $p$  in the interval  $x^{1/2} \leq p \leq x$  and satisfying  $P(p-1) \gg p^{0.667}$  is of cardinality  $\#\mathcal{Q} \gg x/\log x$ . We also recall that, by Brun's

method (see Theorem 2.2 in [12]), for any integer  $m$ , the number of primes of the form  $p = mq + 1 \leq x$  for some other prime  $q$  is

$$O\left(\frac{x}{\varphi(m)(\log(x/m))^2}\right) = O\left(\frac{x}{\varphi(m)(\log x)^2}\right)$$

provided that  $m < x^{1/2}$ . Summing up the above inequalities over all positive integers  $m \leq \log_2 x$ , we see that

$$\#\{p \leq x : P(p-1) \geq x/\log_2 x\} \ll \frac{x}{\log^2 x} \sum_{m < \log x} \frac{1}{\varphi(m)} \ll \frac{x \log_2 x}{\log^2 x} = o(\mathcal{Q}).$$

Thus, most of the primes  $p$  in  $\mathcal{Q}$  in the interval have  $q = P(p-1) < x/\log_2 x$ , and therefore there exist two primes  $p_1, p_2 \in \mathcal{Q}$  with the same value of  $P(p_1-1) = P(p_2-1) = q$ . With  $n = p_1 p_2$ , we see that  $P(\xi(n)) \geq q \gg \max\{p_1^{0.667}, p_2^{0.667}\} \gg n^{0.3335}$ .  $\square$

As is clear from the proof, the upper bound of Theorem 4 is tight under the prime  $k$ -tuple conjecture of Hardy and Littlewood (see, for example, [3]). We also remark that the trivial upper bound  $P(\xi(n)) \leq n^{1/2}$  holds for all  $n \geq 1$ .

Unfortunately, our method of proof for the lower bound of Theorem 4 can not be combined with the more recent results of [2], since the set of primes considered there is too thin.

**Theorem 5.** *The inequalities*

$$\Omega(\xi(n)) = (1 + o(1)) \log_2 n \log_4 n \quad \text{and} \quad \frac{\log_2 n}{(\log_3 n)^2} \ll \omega(\xi(n)) \ll \log_2 n$$

*hold for almost all positive integers  $n$ .*

*Proof.* We start with  $\Omega(\xi(n))$  and first turn our attention to the upper bound. Let  $x$  be a large positive real number, and let  $\mathcal{A}_1$  be the set of all positive integers  $n$  in the interval  $[x/\log x, x]$ . Clearly,  $\mathcal{A}_1$  contains all but  $o(x)$  positive integers  $n \leq x$ . Let  $\mathcal{A}_2$  be the set of those integers  $n \in \mathcal{A}_1$  for which  $P(\xi(n)) \leq (\log_2 x)^2$ ; by Theorem 1,  $\mathcal{A}_2$  contains all but  $o(x)$  positive integers  $n \leq x$ . Let  $y = (\log_2 x)^2$ . For any positive integer  $m$ , we write

$$\omega_y(m) = \sum_{\substack{p < y \\ p \mid m}} 1 \quad \text{and} \quad \Omega_y(m) = \sum_{\substack{p < y \\ p^v \parallel m}} v.$$

Thus, the inequality  $\Omega(\xi(n)) \leq \Omega_y(\varphi(n))$  holds for all  $n \in \mathcal{A}_2$ . The argument on page 349 in [8] shows that

$$\sum_{n \leq x} |\Omega_y(\varphi(n)) - \log_2 x \log_2 y|^2 \ll x \log_2 x (\log_2 y)^2. \quad (8)$$

Now let  $\varepsilon_1(x) = (\log_2 x)^{-1/3}$ , and let  $\mathcal{B}$  be the set of those  $n \leq x$  such that

$$\Omega_y(\varphi(n)) > (1 + \varepsilon_1(x)) \log_2 x \log_2 y.$$

Using (8), it follows that

$$\#\mathcal{B} \ll \frac{x}{\varepsilon_1(x)^2 \log_2 x} = o(x).$$

The set  $\mathcal{A}_3 = \mathcal{A}_2 \setminus \mathcal{B}$  contains all but  $o(x)$  positive integers  $n \leq x$ , and for each  $n \in \mathcal{A}_3$  we have

$$\Omega(\xi(n)) \leq \Omega_y(\varphi(n)) \leq (1 + \varepsilon_1(x)) \log_2 x \log_2 y = (1 + o(1)) \log_2 x \log_4 x. \quad (9)$$

Since  $n \geq x/\log x$  for all  $n \in \mathcal{A}_3$ , this shows that

$$\Omega(\xi(n)) \leq (1 + o(1)) \log_2 n \log_4 n$$

for almost all positive integers  $n$ .

Next we turn to the lower bound for  $\Omega(\xi(n))$ . As before, let  $x$  be a large real number, and put  $\varepsilon_2(x) = (\log_3 x)^{-1/3}$  and  $Q = (\log_2 x)^{1/2}$ . For natural numbers  $n$  and  $q$ , we again write  $\omega(n, q)$  for the number of prime factors  $p$  of  $n$  that are congruent to 1 modulo  $q$ . For a prime  $q \leq Q$  we define the sets

$$\mathcal{C}_q = \left\{ n \leq x : \omega(n, q) \leq (1 - \varepsilon_2(x)) \frac{\log_2 x}{\varphi(q)} \right\},$$

and

$$\mathcal{C} = \bigcup_{q \leq Q} \mathcal{C}_q.$$

We claim that  $\#\mathcal{C} = o(x)$  as  $x \rightarrow \infty$ . Indeed, for a fixed prime  $q \leq Q$ , by a result of Turán [20] (see also (1.2) of [17]), we have

$$\#\mathcal{C}_q \ll \frac{xq}{\varepsilon_2^2(x) \log_2 x} \ll \frac{x(\log_3 x)^{2/3}}{\log_2 x} q.$$

Therefore,

$$\#\mathcal{C} \leq \sum_{q \leq Q} \#\mathcal{C}_q \ll \frac{x(\log_3 x)^{2/3}}{\log_2 x} \sum_{q \leq (\log_2 x)^{1/2}} q \ll \frac{x}{(\log_3 x)^{1/3}} = o(x).$$

Now let  $\mathcal{D}$  be the set of those positive integers  $n \leq x$  not lying in  $\mathcal{C}$ . Then for each

$n \in \mathcal{D}$ , one has

$$\begin{aligned}
\Omega(\xi(n)) &\geq \sum_{q \leq Q} (\omega(n, q) - 1) = \sum_{q \leq Q} \omega(n, q) - \pi(Q) \\
&\geq (1 - \varepsilon_2(x)) \log_2 x \sum_{q \leq Q} \frac{1}{\varphi(q)} - \pi(Q) \\
&\geq (1 - \varepsilon_2(x)) \log_2 x \sum_{q \leq Q} \frac{1}{q} - \pi(Q) \\
&\geq (1 + o(1)) \log_2 x \log_4 x \geq (1 + o(1)) \log_2 n \log_4 n.
\end{aligned}$$

This completes the proof of the normal order of  $\Omega(\xi(n))$ .

We now turn our attention to  $\omega(\xi(n))$  and start with the lower bound. Again, let  $x$  be a large positive real number, and let  $\varepsilon_3(x)$  be any admissible function. Let  $q$  be a prime number and let  $v_q(m)$  denote the largest power of  $q$  dividing a natural number  $m$ . It suffices to show that there exists a constant  $c_1$  such that for all but  $o(x)$  positive integers  $n \leq x$ , the estimate

$$v_q(\xi(n)) \geq \varepsilon_3(x) \log_2 x, \quad (10)$$

holds simultaneously for all primes  $q \leq c_1 \log_2 x / \log_3 x$ .

Let us define

$$\mathcal{W}_q = \left\{ n \leq x : \omega(n, q) < \frac{\log_2 x}{2\varphi(q)} \right\}.$$

By the result of Turán mentioned above, we have  $\#\mathcal{W}_q \ll xq / \log_2 x$ ; summing up these estimates for all  $q \leq (\log_3 x)^{1/2}$ , we see that

$$\sum_{q \leq (\log_3 x)^{1/2}} \#\mathcal{W}_q \ll \frac{x}{\log_2 x} \sum_{q \leq (\log_3 x)^{1/2}} q \ll \frac{x \log_3 x}{\log_2 x \log_4 x} = o(x).$$

We also note that for  $q \leq (\log_3 x)^{1/2}$ , we have

$$\frac{\log_2 x}{2\varphi(q)} \gg \frac{\log_2 x}{(\log_3 x)^{1/2}}$$

which establishes (10) for  $q$  in this small range if  $\varepsilon_3(x) \leq (\log_3 x)^{-1/2}$ , which we now assume.

Next we consider the case in which  $q > (\log_3 x)^{1/2}$ .

Let us denote by  $\omega_y(n)$  the number of prime factors  $p$  of  $n$  with  $p \leq y$ . Let  $\mathcal{N}$  be the set of integers  $x^{1/2} \leq n \leq x$  for which

$$\omega_y(n) = \log_2 y + O((\log_2 y)^{2/3})$$

holds simultaneously for  $y = \exp((\log x)^{1/2})$  and for  $y = x$ . By [20], we have that  $\#\mathcal{N} = x + o(x)$ .

Let  $\mathcal{E}_q$  be the set of  $n \in \mathcal{N}$  such that  $p^2 \mid n$  for some  $p \equiv 1 \pmod{q}$  and let  $\mathcal{E}$  be the union of all  $\mathcal{E}_q$  for  $q > (\log_3 x)^{1/2}$ . Clearly,

$$\#\mathcal{E}_q \ll \sum_{p \equiv 1 \pmod{q}} \frac{x}{p^2} \leq \frac{x}{q^2} \sum_{t \geq 1} \frac{1}{t^2} \ll \frac{x}{q^2},$$

and therefore

$$\#\mathcal{E} \leq \sum_{q > (\log_3 x)^{1/2}} \#\mathcal{E}_q \ll x \sum_{q > (\log_3 x)^{1/2}} \frac{1}{q^2} = o\left(\frac{x}{(\log_3 x)^{1/2}}\right) = o(x).$$

For a fixed positive integer  $k$  and primes  $p_1 \equiv \dots \equiv p_k \equiv 1 \pmod{q}$ , let  $\mathcal{N}_{k,q}(p_1, \dots, p_k)$  be the set of integers  $n \in \mathcal{N} \setminus \mathcal{E}$  such that  $n = p_1 \dots p_k m$  holds with some integer  $m$  with  $\omega(m, q) = 0$ .

We first show that if  $k \leq 0.5 \log_2 x$ , then  $\mathcal{N}_{k,q}(p_1, \dots, p_k)$  is empty unless

$$\frac{x}{p_1 \dots p_k} \geq z, \quad (11)$$

where  $z = \exp((\log x)^{1/2})$ . Indeed, in the opposite case, we see that for  $n \in \mathcal{N}_{k,q}(p_1, \dots, p_k)$ ,

$$\omega(n) \leq k + \omega(m) \leq k + \omega_z(n) \leq 0.5 \log_2 x + O((\log_2 x)^{1/2}),$$

which is impossible because  $\omega(n) \sim \log_2 n \sim \log_2 x$  for  $n \in \mathcal{N}$ .

We now have

$$\#\mathcal{N}_{k,q}(p_1, \dots, p_k) \leq \sum_{\substack{m \leq x/(p_1 \dots p_k) \\ q \nmid \varphi(m)}} 1. \quad (12)$$

It has been shown in the proof of Theorem 4.1 of [7] that there exists an absolute constant  $c_2 > 0$  such that the upper bound

$$\sum_{\substack{m \leq t \\ q \nmid \varphi(m)}} 1 \ll t \exp(-c_2 S(t, q))$$

holds uniformly when  $\log t > q$ , where  $S(t, q)$  is given by (2). By Theorem 3.4 of [7], we know that the lower bound

$$S(t, q) \gg \frac{\log_2 t}{q}$$

holds provided that  $q < \log t$ . Thus, assuming (11), and remarking that  $\log z = (\log x)^{1/2} > q$ , we derive from (12) that the estimate

$$\#\mathcal{N}_{k,q}(p_1, \dots, p_k) \ll \frac{x}{p_1 \dots p_k} \exp\left(-c_3 \frac{\log_2 x}{q}\right)$$

holds with some absolute constant  $c_3 > 0$ .

Therefore, the set  $\mathcal{N}_{k,q}$  consisting of all integers  $n$  in  $\mathcal{N} \setminus \mathcal{E}$  that belong to at least one of the sets  $\mathcal{N}_{k,q}(p_1, \dots, p_k)$ , for fixed  $k$  and  $q$ , has cardinality at most

$$\begin{aligned} \#\mathcal{N}_{k,q} &= \frac{1}{k!} \sum_{\substack{p_1 < x \\ p_1 \equiv 1 \pmod{q}}} \dots \sum_{\substack{p_k < x \\ p_k \equiv 1 \pmod{q}}} \#\mathcal{N}_{k,q}(p_1, \dots, p_k) \\ &\leq \frac{1}{k!} \sum_{\substack{p_1 < x \\ p_1 \equiv 1 \pmod{q}}} \dots \sum_{\substack{p_k < x \\ p_k \equiv 1 \pmod{q}}} \frac{x}{p_1 \dots p_k} \exp\left(-c_3 \frac{\log_2 x}{q}\right) \\ &\leq \frac{x}{k!} \exp\left(-c_3 \frac{\log_2 x}{q}\right) S(x, q)^k. \end{aligned}$$

Put  $K_q = \varepsilon_3(x)(\log_2 x)/q$ . Recalling the bound (3) and using the Stirling formula, we obtain

$$\begin{aligned} \sum_{k \leq K_q} \#\mathcal{N}_{k,q} &\ll x \exp\left(-c_3 \frac{\log_2 x}{q}\right) \sum_{k \leq K_q} \frac{(2 \log_2 x)^k}{q^k k!} \\ &\ll x \exp\left(-c_3 \frac{\log_2 x}{q}\right) \sum_{k \leq K_q} \left(\frac{6 \log_2 x}{qk}\right)^k. \end{aligned}$$

Furthermore, we derive

$$\begin{aligned} \sum_{k \leq K_q} \left(\frac{6 \log_2 x}{qk}\right)^k &\ll \sum_{0 \leq i \leq \log K_q} \sum_{K_q e^{-i-1} \leq k \leq K_q e^{-i}} \left(\frac{6e^{i+1} \log_2 x}{qK_q}\right)^k \\ &= \sum_{0 \leq i \leq \log K_q} \sum_{K_q e^{-i-1} \leq k \leq K_q e^{-i}} \left(6\varepsilon_3^{-1}(x)e^{i+1}\right)^k \\ &\ll \sum_{0 \leq i \leq \log K_q} \left(6\varepsilon_3^{-1}(x)e^{i+1}\right)^{K_q e^{-i}} \\ &\ll \exp\left(c_4 K_q \log(\varepsilon_3^{-1}(x))\right) \end{aligned}$$

for some constant  $c_4$ . Therefore, for an appropriate constant  $c_1$ ,

$$\begin{aligned} & \sum_{q \leq c_1 \log_2 x / \log_3 x} \sum_{k \leq K_q} \# \mathcal{N}_{k,q} \\ & \ll x \sum_{q \leq c_1 \log_2 x / \log_3 x} \exp \left( -c_3 \frac{\log_2 x}{q} + c_4 K_q \log (\varepsilon_3^{-1}(x)) \right) \\ & \ll x \sum_{q \leq c_1 \log_2 x / \log_3 x} \exp \left( -0.5c_3 \frac{\log_2 x}{q} \right) = o(x) \end{aligned}$$

provided that  $x$  is large enough. Clearly, the inequality (10) implies the desired lower bound on  $\omega(\xi(n))$ .

We now prove the upper bound on  $\omega(\xi(n))$ . By (1), we know that the inequality

$$\log(\xi(n)) \ll \log_2 n \log_3 n \quad (13)$$

holds on a set of positive integers  $n$  of asymptotic density 1. The upper bound on  $\omega(\xi(n))$  claimed by our Theorem 5 follows now from inequality (13) above combined with the classical estimate

$$\omega(\xi(n)) \ll \frac{\log \xi(n)}{\log_2 \xi(n)},$$

which concludes the proof.  $\square$

It is easy to see that Theorem 5 implies that for some constant  $c_5 > 0$ , the bound

$$\tau(\xi(n)) \geq 2^{\omega(\xi(n))} \gg \exp \left( c_5 \frac{\log_2 n}{(\log_3 n)^2} \right)$$

holds for almost all positive integers  $n$ , where, as usual,  $\tau(k)$  denotes the number of divisors of an integer  $k \geq 1$ .

It is also clear that for any positive integer  $n$

$$\omega(\xi(n)) \leq \omega(\varphi(n)) \ll \frac{\log \varphi(n)}{\log_2 \varphi(n)} \ll \frac{\log n}{\log_2 n}$$

and

$$\Omega(\xi(n)) \ll \Omega(\varphi(n)) \ll \log \varphi(n) \ll \log n.$$

**Theorem 6.** *The inequalities*

$$\Omega(\xi(n)) \gg \log n \quad \text{and} \quad \omega(\xi(n)) \gg \frac{\log n}{\log_2 n}$$

*hold for infinitely many positive integers  $n$ .*



*Proof.* Let  $k$  be a sufficiently large integer, and then let  $p_1$  and  $p_2$  be the first two primes in the arithmetic progression  $1 \pmod{2^k}$ . By Linnik's Theorem, in the form given by Heath-Brown [13], we know that  $\max\{p_1, p_2\} \ll 2^{11k/2}$ . With  $n = p_1 p_2$ , we have that  $2^k \mid \xi(n)$ ; therefore  $\Omega(\xi(n)) \geq k \gg \log n$ . Finally, let  $y$  be large and let  $M = \prod_{p < y} p$ . By the Prime Number Theorem, we have  $\log M = (1 + o(1))y$ . Let  $p_1$  and  $p_2$  be the first two primes in the arithmetic progression  $1 \pmod{M}$ . We again have that  $\max\{p_1, p_2\} \ll M^{11/2}$ , and with  $n = p_1 p_2$  we have that  $M \mid \xi(n)$ . Thus,

$$\omega(\xi(n)) \gg \omega(M) = \pi(y) \gg \frac{\log M}{\log_2 M} \gg \frac{\log n}{\log_2 n},$$

which finishes the proof.  $\square$

### 3. Average $q$ -adic norm and order of $\varphi(n)$

Let  $q$  be a prime, and let  $|m|_q$  be the  $q$ -adic norm of  $m$ , that is,  $|m|_q = q^{-v_q(m)}$  where, as before,  $v_q(m)$  is the largest power of  $q$  dividing  $m$ . In this section, we address the average value of  $|\varphi(n)|_q$  and  $v_q(\varphi(n))$ .

Recall that an arithmetic function  $f(n)$  is said to be *multiplicative* if  $f(nm) = f(n)f(m)$  for any integers  $n$  and  $m$  with  $\gcd(n, m) = 1$ . Accordingly, if  $f(nm) = f(n) + f(m)$  for any integers  $n$  and  $m$  with  $\gcd(n, m) = 1$  then  $f(n)$  is called *additive*.

In particular,  $v_q(\varphi(n))$  is an additive function. Thus,  $|\varphi(n)|_q$  is a bounded multiplicative function, and therefore it is natural that our principal tool is the following theorem of Wirsing [21].

**Lemma 3.** *Assume that a real-valued multiplicative function  $f(n)$  satisfies the following conditions:*

- $f(n) \geq 0$ ,  $n = 1, 2, \dots$ ;
- $f(p^v) \leq ab^v$ ,  $v = 2, 3, \dots$ , for some constants  $a, b > 0$  with  $b < 2$ ;
- there exists a constant  $\tau > 0$  such that

$$\sum_{p \leq x} f(p) = (\tau + o(1)) \frac{x}{\log x}.$$

Then, for any  $x \geq 0$ ,

$$\sum_{n \leq x} f(n) = \left( \frac{1}{e^{\gamma\tau} \Gamma(\tau)} + o(1) \right) \frac{x}{\log x} \prod_{p \leq x} \left( \sum_{v=0}^{\infty} \frac{f(p^v)}{p^v} \right),$$

where  $\gamma$  is the Euler constant, and

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

is the  $\Gamma$ -function.

**Lemma 4.** For any fixed prime  $q$ ,

$$\prod_{p \leq x} \left( 1 + \frac{|p-1|_q}{p-1} \right) = (\eta_q + o(1)) (\log_2 x)^{\alpha_q},$$

where  $\alpha_q = (q^2 - q - 1)/(q^2 - 1)$ , and  $\eta_q$  is a constant depending only on  $q$ .

*Proof.* We have

$$\log \left( 1 + \frac{|p-1|_q}{p-1} \right) = \frac{|p-1|_q}{p} + O \left( \frac{|p-1|_q}{p^2} \right),$$

therefore the series

$$\zeta_q = \sum_p \left| \log \left( 1 + \frac{|p-1|_q}{p-1} \right) - \frac{|p-1|_q}{p} \right|$$

converges absolutely. Hence, it is enough to show that

$$\sum_{p \leq x} \frac{|p-1|_q}{p} = \alpha_q \log_2 x + \beta_q + o(1) \quad (14)$$

holds with some constant  $\beta_q$ .

We have:

$$\begin{aligned} \sum_{p \leq x} \frac{|p-1|_q}{p} &= \sum_{k=0}^{\infty} \left( \sum_{\substack{p \leq x \\ p \equiv 1 \pmod{q^k}}} \frac{q^{-k}}{p} - \sum_{\substack{p \leq x \\ p \equiv 1 \pmod{q^{k+1}}}} \frac{q^{-k}}{p} \right) \\ &= S(x, 1) - (q-1) \sum_{k=1}^{\infty} q^{-k} S(x, q^k), \end{aligned} \quad (15)$$

where, as before,  $S(x, q^k)$  is given by (2).

We write  $K$  for the largest positive integer such that  $q^K \leq \log_2 x$ ; thus,  $K \asymp \log_3 x$ . Using the classical *Page bound* (see Chapter 20 of [6]) and partial summation (see a remark in Chapter 22 of [6]), we have

$$\pi(t; q^k, 1) = \frac{t}{(q-1)q^{k-1} \log t} + O \left( \frac{t}{q^k (\log t)^2} \right) \quad (16)$$

for all positive integers  $k \leq K$  and real  $t \geq e^K$ .

Therefore, using the same partial summation arguments as in the proof of Theorem 1 of [18] (see also Lemma 6.3 of [17]), and using (16) in the appropriate place (starting with the value of  $t \geq e^K$ ), we derive that for every  $k \leq K$ ,

$$S(x, q^k) = \frac{\log_2 x}{(q-1)q^{k-1}} + A_{k,q} + O\left(\frac{1}{(\log x)^{1/2}}\right), \quad (17)$$

for some constants  $A_{k,q}$  depending only on  $k$  and  $q$ . Moreover, by Theorem 1 of [18] or Lemma 6.3 of [17],  $A_{k,q} = O(1)$  uniformly for  $q$  and  $k = 0, 1, \dots$  (see (3)).

For  $k \geq K$ , we use the fact that

$$S(x, q^k) \ll \frac{\log_2 x}{(q-1)q^{k-1}} \quad (18)$$

(see the bound (3.1) in [7] and also Lemma 1 of [5]). Define

$$\beta_q = A_{k,0} - (q-1) \sum_{k \geq 1} \frac{A_{k,q}}{q^k}.$$

Using (17) and (18) in (15), and taking into account that

$$1 - (q-1) \sum_{k \geq 1} \frac{1}{(q-1)q^{2k-1}} = \frac{q^2 - q - 1}{q^2 - 1} = \alpha_q,$$

we get (14) and thus finish the proof.  $\square$

**Theorem 7.** *For any prime  $q$ ,*

$$\sum_{n \leq x} |\varphi(n)|_q = (\gamma_q + o(1)) x (\log x)^{-q/(q^2-1)},$$

where  $\gamma_q$  is a constant depending only on  $q$ .

*Proof.* For  $p \neq q$ , we have

$$\sum_{v=0}^{\infty} \frac{|\varphi(p^v)|_q}{p^v} = 1 + \sum_{v=1}^{\infty} \frac{|p-1|_q}{p^v} = \frac{|p-1|_q}{p-1},$$

and certainly

$$\sum_{v=0}^{\infty} \frac{|\varphi(q^v)|_q}{q^v} = 1 + \sum_{v=1}^{\infty} \frac{1}{q^{2v-1}} = 1 + \frac{q}{q^2-1} = \frac{q^2+q-1}{q^2-1}.$$

Combining Lemma 3 and Lemma 4, we obtain the desired result.  $\square$

We now show that the classical *Turán–Kubilius* inequality can be used to study the normal order of  $v_q(\varphi(n))$ .

**Theorem 8.** *For any prime  $q$ , the estimate*

$$v_q(\varphi(n)) = \left( \frac{q}{(q-1)^2} + o(1) \right) \log_2 n$$

*holds for almost all positive integers  $n$ .*

*Proof.* Because  $v_q(\varphi(n))$  is an additive function, by the Turán–Kubilius inequality (see [14], [19]), we have

$$\frac{1}{x} \sum_{n \leq x} |v_q(\varphi(n)) - A_q(x)|^2 \ll D_q(x)$$

where

$$A_q(x) = \sum_{p^r \leq x} \frac{v_q(\varphi(p^r))}{p^r} \quad \text{and} \quad D_q(x) = \sum_{p^r \leq x} \frac{v_q^2(\varphi(p^r))}{p^r},$$

and in both sums the summation is extended over all prime powers  $p^r \leq x$ . Thus, it is enough to show that

$$A_q(x) = \left( \frac{q}{(q-1)^2} + o(1) \right) \log_2 x \quad \text{and} \quad D_q(x) = o((\log_2 x)^2). \quad (19)$$

Because  $v_q(\varphi(p)) \ll \log p$ , using the Prime Number Theorem, we derive that

$$\sum_{\substack{p^r \leq x \\ r \geq 2}} \frac{v_q(\varphi(p))}{p^r} \ll \sum_{r=2}^x \sum_{k=2}^{\infty} \frac{\log k}{(0.5k \log k)^r} \ll \sum_{r=2}^x \sum_{k=2}^{\infty} \frac{1}{k^r} \ll \sum_{r=2}^x 2^{-r} \ll 1.$$

Thus

$$A_q(x) = \sum_{p \leq x} \frac{v_q(\varphi(p))}{p} + O(1) = \sum_{\substack{p \leq x \\ p \neq q}} \frac{v_q(\varphi(p))}{p} + O(1).$$

Furthermore, as in the proof of Lemma 4, we derive that

$$\begin{aligned} \sum_{\substack{p \leq x \\ p \neq q}} \frac{v_q(\varphi(p))}{p} &= \sum_{k=1}^{\infty} \left( \sum_{\substack{p \leq x \\ p \equiv 1 \pmod{q^k}}} \frac{k}{p} - \sum_{\substack{p \leq x \\ p \equiv 1 \pmod{q^{k+1}}}} \frac{k}{p} \right) \\ &= \sum_{k=1}^{\infty} S(x, q^k) = \left( \frac{q}{(q-1)^2} + o(1) \right) \log_2 x. \end{aligned}$$

Similar arguments show that  $D_q(x) = O(\log_2 x)$  (in fact, our arguments give an asymptotic formula for  $D_q(x)$ ). Therefore, we obtain (19), which finishes the proof.  $\square$

#### 4. Distribution of $\Omega(\varphi(n)) - \omega(\varphi(n))$

It has been shown in [8] that for almost all positive integers  $n$ , both  $\Omega(\varphi(n))$  and  $\omega(\varphi(n))$  are close to  $0.5(\log_2 n)^2$ . Here, we study the behavior of the difference  $\Omega(\varphi(n)) - \omega(\varphi(n))$ .

**Theorem 9.** *The estimate*

$$\Omega(\varphi(n)) - \omega(\varphi(n)) = (1 + o(1)) \log_2 n \log_4 n$$

holds for almost all positive integers  $n$ .

*Proof.* By Theorem 5, we know that

$$\Omega(\xi(n)) = (1 + o(1)) \log_2 n \log_4 n$$

holds for almost all positive integers  $n$ . Since

$$\Omega(\varphi(n)) - \omega(\varphi(n)) = \Omega(\varphi(n)) - \omega(\lambda(n)) \geq \Omega(\varphi(n)) - \Omega(\lambda(n)) \geq \Omega(\xi(n)),$$

we see that

$$\Omega(\varphi(n)) - \omega(\varphi(n)) \geq (1 + o(1)) \log_2 n \log_4 n$$

holds for almost all positive integers  $n$ .

To obtain the upper bound, let  $x$  be a large positive real number, and let  $y = (\log_2 x)^2$ . The argument on page 404 of [16] shows that the set of all positive integers  $n \leq x$  such that  $\varphi(n)$  is not divisible by the square of any prime  $q > y$  has cardinality  $x + o(x)$  (see the bound on  $\#\mathcal{E}_2$  in Theorem 9 of [16]). Thus, for all but  $o(x)$  positive integers  $n \leq x$ , we have that

$$\Omega(\varphi(n)) - \omega(\varphi(n)) = \Omega_y(\varphi(n)) - \omega_y(\varphi(n)) \leq \Omega_y(\varphi(n)).$$

Now using (9) (which is established with the same value of  $y$ ), we finish the proof.  $\square$

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