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Complete proper minimal surfaces in convex bodies of $\mathbb{R}^3$, II. The behavior of the limit set

Francisco Martín* and Santiago Morales*

Abstract. Let D be a regular, strictly convex bounded domain of $\mathbb{R}^3$, and consider a Jordan curve $\Gamma \subset \partial D$. Then, for each $\varepsilon > 0$, we obtain the existence of a complete proper minimal immersion $\psi_\varepsilon \colon \mathbb{D} \to D$ satisfying that the Hausdorff distance $\delta^H(\psi_\varepsilon(\partial\mathbb{D}), \Gamma) < \varepsilon$, where $\psi_\varepsilon(\partial\mathbb{D})$ represents the limit set of the minimal disk $\psi_\varepsilon(\mathbb{D})$.

This result has some interesting consequences. Among other things, we can prove that any bounded regular domain R in $\mathbb{R}^3$ admits a complete proper minimal immersion $\psi : \mathbb{D} \to R$.

Mathematics Subject Classification (2000). Primary 53A10; Secondary 49Q05, 49Q10, 53C42.

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1. Introduction and background

Last few years have seen an important progress on many long-standing problems in global theory of complete minimal surfaces in $\mathbb{R}^3$. One of these has been the Calabi–Yau problem, which dates back to the 1960s. Calabi asked whether or not it is possible for a complete minimal surface in $\mathbb{R}^3$ to be contained in the ball $\mathbb{B} = \{x \in \mathbb{R}^3 \mid \|x\| < 1\}$. Much work has been done on it over the past four decades. The most important result in this line was obtained by N. Nadirashvili in [13] where he constructed a complete minimal surface in $\mathbb{B}$. After Nadirashvili's negative solution to Calabi's question, the conjecture was revisited by S.-T. Yau in [15], where he stated new questions related to the embeddedness and properness of surfaces of this type.

Regarding the existence of complete embedded minimal surfaces in a ball, T. Colding and W. Minicozzi [1] have proved that a complete embedded minimal surface with finite topology in $\mathbb{R}^3$ must be properly embedded in $\mathbb{R}^3$. In particular it cannot be contained in a ball. Very recently, Colding–Minicozzi result has been generalized in two different directions. On one hand W. H. Meeks III, J. Pérez and A. Ros [9] have

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proved that if M is a complete embedded minimal surface in $\mathbb{R}^3$ with finite genus and a countable number of ends, then M is properly embedded in $\mathbb{R}^3$. On the other hand, Meeks and Rosenberg [12] have obtained that if a complete embedded minimal surface M has injectivity radius $I_M > 0$, then M is proper in space.

Concerning properness, it is important to note that Nadirashvili's technique did not guarantee the immersion was proper. In [5], F. Martín and S. Morales introduced an additional ingredient into Nadirashvili's machinery in order to produce a complete minimal disk which is properly immersed in a ball of $\mathbb{R}^3$. Recently [6], they improved on their original techniques and were able to show that every convex domain (not necessarily bounded or smooth) admits a complete properly immersed minimal disk.

The present paper can be considered as a continuation of the above mentioned work developed by the authors about the construction of complete proper minimal surfaces in (open) convex bodies of Euclidean space. After the discovering of those examples a natural question arose: What is the asymptotic behavior of such a surface? If we consider a proper minimal immersion $\psi : \mathbb{D} \to C$, where C is an open convex body, then we define the limit set as $\psi(\partial\mathbb{D}) \stackrel{\text{def}}{=} \overline{\psi(\mathbb{D})} - \psi(\mathbb{D})$. It is obvious that ψ is proper if, and only if, $\psi(\partial\mathbb{D}) \subset \partial C$. Furthermore, one can easily check that $\psi(\mathbb{D})$ is closed and connected. In this paper we show:

Theorem A. *Let C be a regular strictly convex*[1] *bounded domain of $\mathbb{R}^3$, and consider a regular Jordan curve $\Gamma \subset \partial C$. Then, for each $\varepsilon > 0$, we obtain the existence of a complete proper minimal immersion $\psi_{(\Gamma,\varepsilon)} : \mathbb{D} \to C$ satisfying that the Hausdorff distance $\delta^H(\psi_{(\Gamma,\varepsilon)}(\partial\mathbb{D}), \Gamma) < \varepsilon$, where $\psi_{(\Gamma,\varepsilon)}(\partial\mathbb{D})$ represents the limit set of the minimal disk $\psi_{(\Gamma,\varepsilon)}(\mathbb{D})$.*

The main obstacle in the study of the asymptotic behavior of a complete proper minimal surface in a convex domain is that all the methods of construction up until today are implicit. So, it was almost impossible to obtain any control about the behavior of the known examples near their ends. From this point of view, the new methods of construction introduced in this paper are significant. The main of these tools is Theorem 1 whose proof is based on a Meeks' idea that appeared first in [6] but that has been entirely developed and exploited in this article.

Theorem 1 represents by itself an interesting density result. It asserts that any minimal disk with boundary D can be approximated (in terms of uniform convergence) by a complete minimal disk $\widetilde{D}$. Moreover, it is possible to find a thin tube around ∂D such that the part of $\widetilde{D}$ which lies in the exterior of this tube is compact. In other words, the part of $\widetilde{D}$ where the intrinsic metric exploits is contained in the interior of this thin tube (see Figure 1).

[1] *Strictly convex* means that the principal curvatures of ∂C associated to the inward pointing unit normal are positive everywhere.

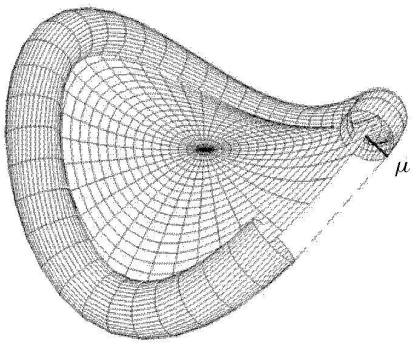

Figure 1. The minimal disk $X(P)$ and the tube $T(\Gamma, \mu)$.

The other important ingredient in the proof of the main theorem is Lemma 2. This approximation lemma is crucial to obtain the properness in Theorem A and it was proved in [6]. It essentially asserts that a minimal disk with boundary can be perturbed outside a compact set in such a way that the boundary of the resulting surface achieves the boundary of a prescribed convex domain. The hypothesis of C being strictly convex is crucial at this point, otherwise we could not obtain an upper bound for the distances between the boundary of a minimal disk and the boundary of the deformed one.

It is natural to ask what is the limit of the complete proper minimal surfaces given by Theorem A as ε tends to 0. We would like to point out that the limit as ε goes to zero of $\psi_{(\Gamma,\varepsilon)}$ exists, but it is not complete. Actually, this limit coincides with the minimal disk (with boundary) spanned by the curve Γ.

Theorem A has some interesting consequences. We would like to point out two of them.

Theorem B. *Let C be a regular strictly convex bounded domain, and consider a* connected compact set $K \subset \partial C$. *Then, for each $\varepsilon > 0$, there exists a complete proper minimal immersion $\varphi_{(K,\varepsilon)} : \mathbb{D} \to C$ satisfying that the Hausdorff distance $\delta^H(\varphi_{(K,\varepsilon)}(\partial\mathbb{D}), K) < \varepsilon$.*

The above theorem follows from the fact that Jordan curves are dense in the space of compact sets of ∂C with the Hausdorff metric. Among other things, Theorem B tells us that the limit set of a complete proper minimal surface can be very small. This means that we can work with small pieces of the boundary of a given domain in order to prove the following result.

Theorem C. *Every bounded domain with regular boundary admits a complete properly immersed minimal disk.*

In contrast to these existence results for complete properly immersed minimal disks in bounded domains, Meeks, Nadirashvili and the first author [4] have constructed domains of $\mathbb{R}^3$ which do not contain any complete proper minimal surface with finite topology. It is our belief that these open domains are in fact *universal* according to the following definition: A connected region of space which is open or the closure of an open set is *universal for minimal surfaces*, if every complete properly immersed minimal surface in the region is recurrent for Brownian motions. In particular, a bounded domain is universal if and only if it contains no complete properly immersed minimal surfaces.

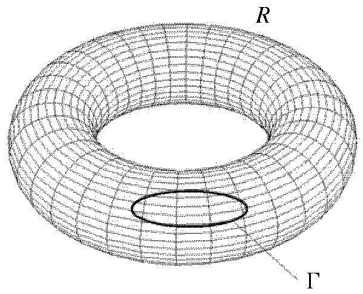

Figure 2. Theorem C states that every bounded domain with regular boundary admits a complete properly immersed minimal surface. Besides, the boundary limit set is close to a small simple closed curve Γ on the positively curved part of the boundary of the domain.

As we mentioned before, Colding and Minicozzi proved that any complete embedded minimal surface in $\mathbb{R}^3$ with finite topology is properly embedded in $\mathbb{R}^3$. By results of Meeks and Rosenberg, [10], [11], any properly embedded minimal surface of finite topology in $\mathbb{R}^3$ is recurrent for Brownian motion. Hence, every domain in $\mathbb{R}^3$ is universal for embedded minimal surfaces of finite topology. Finally, we remark that Collin, Kusner, Meeks and Rosenberg [2] proved that any properly immersed minimal surface with boundary in a closed convex domain in $\mathbb{R}^3$ has full harmonic measure on its boundary.

The paper is displayed as follows. Section 2 is dedicated to state and demonstrate the preliminary results that we need to obtain the main theorems. These main theorems are proved in Section 3 where we also establish some results which show that Theorem A is sharp. Thus, at the end of Section 3 we show that Theorem A is false if the boundary of the domain contains an open set where the mean curvature is non positive.

1.1. Minimal surface background. Given $X = (X_1, X_2, X_3)\colon M \to \mathbb{R}^3$ a conformal minimal immersion we denote by $g\colon M \to \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ its stereographically

projected Gauss map that is a meromorphic function and by ϕ_3 the holomorphic differential defined as $\phi_3 = dX_3 + \star \mathrm{i}\, dX_3$, where $\star$ denotes the Hodge operator on M. The pair (g, ϕ_3) is usually referred to as the Weierstrass data of the minimal surface, and the minimal immersion X can be expressed, up to translations, solely in terms of these data as

$$X = \mathrm{Re} \int^z (\phi_1, \phi_2, \phi_3) = \mathrm{Re} \int^z \left(\frac{1}{2}\left(\frac{1}{g} - g\right), \frac{\mathrm{i}}{2}\left(\frac{1}{g} + g\right), 1 \right) \phi_3, \tag{1}$$

where Re stands for real part and z is a conformal parameter on M. The pair (g, ϕ_3) satisfies certain compatibility conditions:

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 0, \tag{2}$$

$$\|\phi_1\|^2 + \|\phi_2\|^2 + \|\phi_3\|^2 \neq 0, \tag{3}$$

and all periods of the ϕ_j are purely imaginary; $j = 1, 2, 3$.

Conversely, if M is a Riemann surface, $g\colon M \to \overline{\mathbb{C}}$ is a meromorphic function and ϕ_3 is a holomorphic one-form on M fulfilling the conditions (2) and (3) then the map $X\colon M \to \mathbb{R}^3$ given by (1) is a conformal minimal immersion with Weierstrass data (g, ϕ_3).

The conditions about the periods stated above deals with the independence of (1) on the integration path, and it is usually called the period problem. In this article, all the minimal immersions are defined on simply connected domains of $\mathbb{C}$. Then, the Weierstrass 1-forms have no periods, and so the only requirements are (2) and (3). In this case, the differential $\eta \overset{\text{def}}{=} \phi_3/g$ can be written as $\eta = f(z)\, dz$. The metric of X can be expressed as

$$\mathcal{S}_X{}^2 = \tfrac{1}{2}\|\phi\|^2 = \left(\tfrac{1}{2}\left(1 + |g|^2\right) |f|\, |dz|\right)^2. \tag{4}$$

Throughout the paper, we will use several orthonormal bases of $\mathbb{R}^3$. Given S an orthonormal basis and $v \in \mathbb{R}^3$, let $v_{(k,S)}$ denote the k-th coordinate of v in S. The first two coordinates of v in this basis will be represented by $v_{(*,S)} = (v_{(1,S)}, v_{(2,S)})$.

Given a curve α in M, by $\ell(\alpha, X)$ we mean the length of α with respect to the metric $\mathcal{S}_X$. Given a subset $W \subset M$, we define:

- $\mathrm{dist}_{(W,X)}(p, q) = \inf\{\ell(\alpha, \mathcal{S}_X) \mid \alpha : [0, 1] \to W,\ \alpha(0) = p, \alpha(1) = q\}$, for any $p, q \in W$;
- $\mathrm{dist}_{(W,X)}(T_1, T_2) = \inf\{\mathrm{dist}_{(W,\mathcal{S}_X)}(p, q) \mid p \in T_1,\ q \in T_2\}$, for any $T_1, T_2 \subset W$.

The Euclidean metric on $\mathbb{C}$ will be denoted as $\langle \cdot, \cdot \rangle$. Note that $\mathcal{S}_X{}^2 = \lambda_X^2 \langle \cdot, \cdot \rangle$, where the conformal coefficient λ_X is given by (4).

Given a domain $D \subset \mathbb{C}$, we will say that a function, or a 1-form, is harmonic, holomorphic, meromophic,...on $\overline{D}$, if it is harmonic, holomorphic, meromorphic,...on a domain containing $\overline{D}$.

Let P be a simple closed polygonal curve in $\mathbb{C}$. We let Int P denote the bounded connected component of $\mathbb{C} - P$. We will assume that the origin is in the interior region of all the polygons that appears in the paper. Given $\xi > 0$, small enough, we define P^{ξ} to be the parallel polygonal curve in Int P, satisfying the property that the distance between parallel sides is equal to ξ. Whenever we write P^{ξ} in the paper we are assuming that ξ is small enough to define the polygon properly.

1.2. Background on convex bodies and Hausdorff distance. Given E a bounded regular convex domain of $\mathbb{R}^3$ and $p \in \partial E$, we will let $\kappa_2(p) \geq \kappa_1(p) \geq 0$ denote the principal curvatures of ∂E at p (associated to the inward pointing unit normal). Moreover, we write

$$\kappa_1(\partial E) \stackrel{\text{def}}{=} \min\{\kappa_1(p) \mid p \in \partial E\}, \quad \kappa_2(\partial E) \stackrel{\text{def}}{=} \max\{\kappa_2(p) \mid p \in \partial E\}.$$

Remark 1. By regular we mean that ∂E is C^{∞}. Nevertheless, one can see that the results we have proved only need that E has a $C^{2,\alpha}$ boundary.

If we consider $\mathcal{N}\colon \partial E \to \mathbb{S}^2$ the outward pointing unit normal or Gauss map of ∂E, then there exists a constant $a > 0$ (depending on E) such that $\partial E_t = \{p + t \cdot \mathcal{N}(p) \mid p \in \partial E\}$ is a regular (convex) surface for all $t \in [-a, +\infty[$. Let E_t denote the convex domain bounded by ∂E_t. The normal projection to E is represented as

$$\begin{aligned} \mathcal{P}_E\colon \mathbb{R}^3 \setminus E_{-a} &\to \partial E, \\ p + t \cdot \mathcal{N}(p) &\mapsto p. \end{aligned}$$

For a curve Υ in $\mathbb{R}^3$ and a real $r > 0$, we define the tube of radius r along Υ in the following way:

$$T(\Upsilon, r) = \Upsilon + \mathbb{B}(0, r),$$

where $\mathbb{B}(0, r) = \{p \in \mathbb{R}^3 \mid \|p\| < r\}$.

The set $\mathcal{C}^n$ of convex bodies of $\mathbb{R}^n$ can be made into a metric space in several geometrically reasonable ways. The Hausdorff metric is particularly convenient and applicable. The natural domain for this metric is the set $\mathcal{K}^n$ of the nonempty compact subsets of $\mathbb{R}^n$.

For $C, D \in \mathcal{K}^n$ the *Hausdorff distance* is defined by:

$$\delta^H(C, D) = \max\Big\{\sup_{x \in C} \inf_{y \in D} \|x - y\|, \sup_{y \in D} \inf_{x \in C} \|x - y\|\Big\}$$

or, equivalently, by

$$\delta^H(C, D) = \min\big\{\lambda \geq 0 \mid C \subset D + \lambda\mathbb{B}^n,\ D \subset C + \lambda\mathbb{B}^n\big\},$$

where $\mathbb{B}^n = \{p \in \mathbb{R}^n \mid \|p\| < 1\}$. Then δ^H is a metric on $\mathcal{K}^n$, the *Hausdorff metric.* For more details we refer to [14].

2. Preliminary lemmas

As we indicated in the introduction, the proofs of the main theorems of the paper require the technical results of this section. To be more precise, the two principal tools are Theorem 1 and Lemma 2. Lemma 3 is a suitable combination of the two previous results that allows us to prove Theorem 2. Regarding Lemma 1, it is a necessary instrument in the demonstration of Theorem 1.

Lemma 1 (Completeness Lemma). *Consider Γ a closed analytic curve. Let P be a polygon, $X : \overline{\operatorname{Int} P} \to \mathbb{R}^3$ a conformal minimal immersion, and r, ε positive constants satisfying:*

1. $X(\overline{\operatorname{Int} P - \operatorname{Int} P^\varepsilon}) \subset T(\Gamma, r)$;
2. *$X(P^\varepsilon)$ contains a cycle which is homologically equivalent to Γ in $T(\Gamma, r)$.*

Then, for all $s > 0$ there exist a polygon $\widetilde{P}$ and a conformal minimal immersion $\widetilde{X} : \overline{\operatorname{Int} \widetilde{P}} \to \mathbb{R}^3$ verifying:

(a.1) $\operatorname{Int} P^\varepsilon \subset \operatorname{Int} \widetilde{P} \subset \overline{\operatorname{Int} \widetilde{P}} \subset \operatorname{Int} P$;

(a.2) $\|X(z) - \widetilde{X}(z)\| < \varepsilon$ *for all* $z \in \operatorname{Int} P^\varepsilon$;

(a.3) $\operatorname{dist}_{(\overline{\operatorname{Int} \widetilde{P}}, \widetilde{X})}(z, P^\varepsilon) > s$ *for all* $z \in \widetilde{P}$;

(a.4) $\widetilde{X}\big(\overline{\operatorname{Int} \widetilde{P} - \operatorname{Int} P^\varepsilon}\big) \subset T(\Gamma, R)$ *where* $R = \sqrt{(2s)^2 + r^2} + \varepsilon$;

(a.5) *$X(P^\varepsilon)$ and $\widetilde{X}(P^\varepsilon)$ are homologically equivalent in $T(\Gamma, R)$;*

(a.6) *$\widetilde{X}(\widetilde{P})$ contains a cycle with the same homology than Γ in the tube $T(\Gamma, R)$.*

As we mentioned before, Lemma 1 is merely a tool in the proof of the next theorem, that we have called Meeks' trick because it is based on an idea that W. H. Meeks III suggested to us in 2004. Roughly speaking this theorem asserts that complete minimal disks are "dense" in the space of minimal disks with boundary.

Theorem 1 (Meeks' trick). *Let $U \subset \mathbb{C}$ be a bounded domain, and $P \subset U$ a polygon. Consider $X : U \to \mathbb{R}^3$ a conformal minimal immersion, with $X(0) = 0$. Then, for every $\mu > 0$ there exists a simply connected domain $\Sigma \subset \mathbb{C}$ and a complete minimal immersion $\widehat{X} : \Sigma \to \mathbb{R}^3$, with $\widehat{X}(0) = 0$ such that:*

(b.1) $\overline{\operatorname{Int} P} \subset \Sigma \subset \overline{\Sigma} \subset U$;

(b.2) $\|X(z) - \widehat{X}(z)\| < \mu$ *for all* $z \in \operatorname{Int} P$;

(b.3) $\widehat{X}(\Sigma - \operatorname{Int} P) \subset T(X(P), \mu)$;

(b.4) *$\widehat{X}(P)$ is homologous to $X(P)$ in the open neighborhood $T(X(P), \mu)$.*

The next lemma was obtained in [6] (see Lemma 1 and Remark 3). We have stated it here just to make this paper self-contained. We would like to point out that the assumption of C being strictly convex is essential in this lemma, otherwise Item (c.5) has no sense.

Lemma 2 (Properness Lemma, [6]). *Let E and E' be two regular bounded strictly convex domains in $\mathbb{R}^3$, with $0 \in E \subset \overline{E} \subset E'$. Let $X\colon O \to \mathbb{R}^3$ be a conformal minimal immersion defined on a simply connected domain O, $0 \in O$, with $X(0) = 0$. Consider a polygon P with $P \subset O$, satisfying*

$$X(O \setminus \operatorname{Int} P) \subset E' - \overline{E}. \tag{5}$$

Then, for any $b_1, b_2 > 0$, such that E'_{-b_2} and E_{-2b_2} exist, there exist a polygon Q and a conformal minimal immersion $Y\colon \overline{\operatorname{Int} Q} \to \mathbb{R}^3$, with $Y(0) = 0$, such that:

(c.1) $P \subset \operatorname{Int} Q \subset \overline{\operatorname{Int} Q} \subset O$;

(c.2) $\|Y(z) - X(z)\| < b_1$ *for all* $z \in \overline{\operatorname{Int} P}$;

(c.3) $Y(Q) \subset E' - E'_{-b_2}$;

(c.4) $Y(\operatorname{Int} Q \setminus \operatorname{Int} P) \subset \mathbb{R}^3 \setminus E_{-2b_2}$;

(c.5) *for any $z \in \operatorname{Int} Q \setminus \operatorname{Int} P$, one has $\|X(z) - Y(z)\| < M(b_1, b_2, E, E')$, where*

$$M(b_1, b_2, E, E') = \left[\left(2b_2 + \frac{1}{\kappa_1(\partial E)} + \delta^H(E, E')\right)^2 - \left(2b_2 + \frac{1}{\kappa_1(\partial E)}\right)^2 + (2b_2)^2\right]^{\frac{1}{2}} + b_1;$$

(c.6) *$X(P)$ and $Y(P)$ are homologous cycles in $T(X(P), M(b_1, b_2, E, E'))$.*

The last lemma of this section is a combination of Theorem 1 and Lemma 2. The proof of our main result (Theorem 2) consists of constructing a sequence of minimal disks with boundary whose limit is the immersion which proves the theorem. The sequence is defined in a recursive way and the tool to obtain an element of that sequence from the previous one in Lemma 3 below.

Lemma 3. *Let Γ be a smooth Jordan curve of $\mathbb{R}^3$ and let $\rho > 0$ be a sufficiently small constant so that the tube $T(\Gamma, \rho)$ is homeomorphic to a solid torus. Let E and E' be two regular bounded strictly convex domains in $\mathbb{R}^3$, with $0 \in E \subset \overline{E} \subset E'$. Consider P a polygon, $X\colon \overline{\operatorname{Int} P} \to \mathbb{R}^3$ be a conformal minimal immersion, with $X(0) = 0$, and ε, a, b and c positive constants, such that:*

(1) $X(\overline{\operatorname{Int} P \setminus \operatorname{Int} P^{\varepsilon}}) \subset E \setminus E_{-a}$;

(2) $\delta^H(\Gamma, X(\overline{\operatorname{Int} P - \operatorname{Int} P^{\varepsilon}}) \leq c$;

(3) *$X(P)$ is homologous to Γ in $T(\Gamma, c)$;*

(4) *$c + M(a, b, \varepsilon, E, E') < \rho$, where*

$$M(a, b, \varepsilon, E, E') = \left[\left(2(2a+b) + \frac{1}{\kappa_1(\partial E)} + \delta^H(E, E')\right)^2 - \left(2(a+b) + \frac{1}{\kappa_1(\partial E)}\right)^2 + (2b)^2\right]^{\frac{1}{2}} + \varepsilon.$$

Then, there exist a polygon Q and a conformal minimal immersion $Y : \overline{\operatorname{Int} Q} \to \mathbb{R}^3$, with $Y(0) = 0$, and verifying:

(d.1) $\overline{\operatorname{Int} P^{\varepsilon}} \subset \operatorname{Int} Q \subset \overline{\operatorname{Int} Q} \subset \operatorname{Int} P$;

(d.2) $\frac{1}{\varepsilon} < \operatorname{dist}_{(\overline{\operatorname{Int} Q}, \delta_Y)}(z, P^{\varepsilon})$ *for all* $z \in Q$;

(d.3) $Y(Q) \subset E' - E'_{-b}$;

(d.4) $Y(\operatorname{Int} Q - \operatorname{Int} P^{\varepsilon}) \subset \mathbb{R}^3 - E_{-2(a+b)}$;

(d.5) $\|Y(z) - X(z)\| < \varepsilon$ *for all* $z \in \overline{\operatorname{Int} P^{\varepsilon}}$;

(d.6) *the Hausdorff distance* $\delta^H(\Gamma, Y(\overline{\operatorname{Int} Q - \operatorname{Int} P^{\varepsilon}})) < c + M(a, b, \varepsilon, E, E')$;

(d.7) *the cycle $Y(Q)$ is homologous to Γ in the open tube $T(\Gamma, c+M(a, b, \varepsilon, E, E'))$.*

2.1. Proof of Lemma 1. Consider P, the polygon given in the statement of the lemma. As usual in constructions of this kind, our first step will consist of describing a labyrinth on Int P, which depends on P and a positive integer N.

Let ℓ be the number of sides of P. From now on, N will be a positive multiple of ℓ.

Remark 2. Throughout the proof of the lemma a set of real positive constants depending on X, P, r, ε and s will appear. These constants will be represented by the symbol const. Notice that the choice of these constants does not depend on the integer N.

Let $\zeta_0 > 0$ small enough so that P^{ζ_0} is well defined and $\overline{\operatorname{Int}(P^{\varepsilon})} \subset \operatorname{Int}(P^{\zeta_0})$. From now on, we will only consider $N \in \mathbb{N}$ such that $2/N < \zeta_0$. Let $v_1, \ldots, v_{2N}$ be a set of points in the polygon P (containing the vertices of P) that divide each side of P into $\frac{2N}{\ell}$ equal parts. We can transfer this partition to the polygon $P^{2/N}$: $v'_1, \ldots, v'_{2N}$. We define the following sets:

- L_i = the segment that joins v_i and v_i', $i = 1, \dots, 2N$;
- $\mathcal{P}_i = P^{i/N^3}$, $i = 0, \dots, 2N^2$;
- $\mathcal{E} = \bigcup_{i=0}^{N^2-1} \overline{\mathrm{Int}(\mathcal{P}_{2i}) - \mathrm{Int}(\mathcal{P}_{2i+1})}$ and $\widetilde{\mathcal{E}} = \bigcup_{i=1}^{N^2} \overline{\mathrm{Int}(\mathcal{P}_{2i-1}) - \mathrm{Int}(\mathcal{P}_{2i})}$;
- $\mathcal{R} = \bigcup_{i=0}^{2N^2} \mathcal{P}_i$;
- $\mathcal{B} = \bigcup_{i=1}^{N} L_{2i}$ and $\widetilde{\mathcal{B}} = \bigcup_{i=0}^{N-1} L_{2i+1}$;
- $L = \mathcal{B} \cap \mathcal{E}$, $\widetilde{L} = \widetilde{\mathcal{B}} \cap \widetilde{\mathcal{E}}$, and $H = \mathcal{R} \cup L \cup \widetilde{L}$;
- $\Xi_N = \{z \in \mathrm{Int}(\mathcal{P}_0) - \mathrm{Int}(\mathcal{P}_{2N^2}) \mid \mathrm{dist}_{ds_0,\mathbb{C}}(z, H) \geq \frac{1}{4N^3}\}$, where ds_0 is the Euclidean metric on $\mathbb{C}$.

We define ω_i as the union of the segment L_i and those connected components of Ξ_N that have nonempty intersection with L_i for $i = 1, \dots, 2N$. Finally, we label $\varpi_i = \{z \in \mathbb{C} \mid \mathrm{dist}_{ds_0,\mathbb{C}}(z, \omega_i) < \delta(N)\}$, where $\delta(N) > 0$ is chosen in such a way that the sets $\overline{\varpi_i}$ $(i = 1, \dots, 2N)$ are pairwise disjoint.

The shape of the labyrinth formed by the sets ω_i guarantee the following claims if N is large enough:

Claim 2.1. *The Euclidean diameter of ϖ_i is less than $\frac{\mathrm{const}}{N}$.*

Claim 2.2. *If $\lambda^2 \langle \cdot, \cdot \rangle$ is a conformal metric on $\overline{\mathrm{Int}\, P}$ and verifies*

$$\lambda \geq \begin{cases} c & \text{in } \mathrm{Int}\, P, \\ cN^4 & \text{in } \Xi_N, \end{cases}$$

for $c \in \mathbb{R}^+$, and if α is a curve in $\overline{\mathrm{Int}\, P}$ connecting P^ε and P, then

$$\ell(\alpha, \lambda \langle \cdot, \cdot \rangle) > \frac{c \cdot \mathrm{const} \cdot N}{2}.$$

Claim 2.2 is a consequence of the fact that a curve α, that does not go through the connected components of Ξ_N, must have a large Euclidean length.

Provided that N is large enough, we can assume that $X(\overline{\varpi_i})$ has a sufficiently small diameter in $\mathbb{R}^3$ so that

$$X(\overline{\varpi_i}) \subset \mathbb{B}(p_i, r), \quad \text{where } p_i \in \Gamma, i = 1, \dots 2N. \tag{6}$$

For each point $p \in \mathbb{B}(p_i, r) - \{p_i\}$ we define

$$\mathcal{N}_i(p) = \frac{p - p_i}{\|p - p_i\|}.$$

We pursue the construction of a finite sequence of minimal immersions (with boundary); $F_0 = X$, $F_1, \dots, F_{2N}$, satisfying:

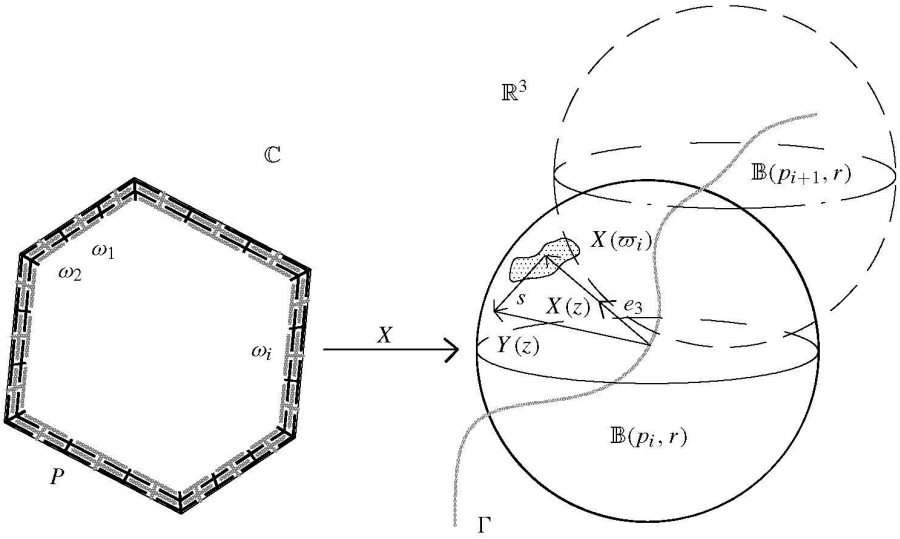

Figure 3. The set ϖ_i and its image in $\mathbb{R}^3$.

(A.1$_i$) $\|\phi^i(z) - \phi^{i-1}(z)\| \leq \frac{1}{N^2}$ for all $z \in \overline{\operatorname{Int} P} - \varpi_i$;

(A.2$_i$) $\|\phi^i(z)\| \geq N^{7/2}$ for all $z \in \omega_i$;

(A.3$_i$) $\|\phi^i(z)\| \geq \frac{\text{const}}{\sqrt{N}}$ for all $z \in \varpi_i$;

(A.4$_i$) $\operatorname{dist}_{(\mathbb{S}^2, ds_0)}(G_i(z), G_{i-1}(z)) < \frac{1}{N^2}$ for all $z \in \overline{\operatorname{Int} P} - \varpi_i$;

(A.5$_i$) there exists an orthonormal basis of $\mathbb{R}^3$, $S_i = \{e_1^i, e_2^i, e_3^i\}$, so that:

(A.5.1$_i$) If $z \in \varpi_i$ and $\|X(z) - p_i\| \geq \frac{1}{\sqrt{N}}$, then $\| (X(z) - p_i)_{(*, S_i)} \| < \frac{\text{const}}{\sqrt{N}}$,

(A.5.2$_i$) $(F_i(z))_{(3, S_i)} = (F_{i-1}(z))_{(3, S_i)}$ for all $z \in \overline{\operatorname{Int} P}$;

(A.6$_i$) $\|F_i(z) - F_{i-1}(z)\| \leq \frac{\text{const}}{N^2}$ for all $z \in \overline{\operatorname{Int} P} - \varpi_i$;

(A.7$_i$) $F_i(P^\varepsilon)$ and Γ are homologous in $T(\Gamma, R)$.

Here $\phi^i(z)\,dz$ is the Weierstrass representation of F_i and G_i represents its spherical Gauss map.

In order to obtain the sequence $F_0, \ldots, F_{2N}$ we follow an inductive method. Assume we have constructed $F_0, \ldots, F_{i-1}$ verifying Properties (A.1$_k$), ..., (A.7$_k$) for $k = 1, \ldots, i-1$. We define F_i as follows.

First, observe that (for a large enough N) one has:

(B.1) There are positive constants so that $\text{const}_1 \leq \|\phi^{i-1}(z)\| \leq \text{const}_2$, for all $z \in \operatorname{Int} P - \bigcup_{k=1}^{i-1} \varpi_k$; To obtain this property, it suffices to apply $(\text{A}2_j)$ for $j = 1, \ldots, i-1$.

(B.2) The diameter in $\mathbb{R}^3$ of $F_{i-1}(\varpi_i)$ is less than $\frac{1}{\sqrt{N}}$. This is a consequence of (B.1), Claim 2.1, and (4).

(B.3) The diameter in $\mathbb{S}^2$ of $G_{i-1}(\varpi_i)$ is less than $\frac{1}{\sqrt{N}}$. In particular, the set $G_{i-1}(\varpi_i)$ can be included in a cone $\operatorname{Cone}\big(g, \frac{1}{\sqrt{N}}\big)$, for a suitable $g \in G_{i-1}(\varpi_i)$.
From Claim 2.1, the diameter of $G_0(\varpi_i)$ is bounded. Then (B.3) holds after successive applications of $(\text{A}.5_j)$, $j = 1, \ldots, i-1$.

(B.4) There exists an orthogonal frame $S_i = \{e_1, e_2, e_3\}$ in $\mathbb{R}^3$, where:

(B.4.1) If $z \in \varpi_i$ and $\|X(z) - p_i\| \geq \frac{1}{\sqrt{N}}$, then $\angle\,(e_3, X(z) - p_i) \leq \frac{\text{const}}{\sqrt{N}}$;

(B.4.2) $\angle(\pm e_3, G_{i-1}(z)) \geq \frac{\text{const}}{\sqrt{N}}$ for all $z \in \varpi_i$.

The proof of (B.4) is slightly more complicated. Let $C = \operatorname{Cone}\big(g, \frac{2}{\sqrt{N}}\big)$ where g is given by Property (B.3). To obtain (B.4.2) it suffices to take e_3 in $\mathbb{S}^2 - H$, where $H = C \cup (-C)$. On the other hand, in order to verify (B.4.1), the vector e_3 must be chosen as follows:

- If $(\mathbb{S}^2 \setminus H) \cap \mathcal{N}_i(X(\varpi_i) - \{p_i\})) \neq \emptyset$, then we take $e_3 \in (\mathbb{S}^2 \setminus H) \cap \mathcal{N}_i(X(\varpi_i) - \{p_i\})$.
- If $(\mathbb{S}^2 \setminus H) \cap \mathcal{N}_i(X(\varpi_i) - \{p_i\}) = \emptyset$, then we take $e_3 \in \mathbb{S}^2 - H$ satisfying $\angle(e_3, q') < \frac{2}{\sqrt{N}}$ for some $q' \in \mathcal{N}_i(X(\varpi_i) - \{p_i\})$.

It is straightforward to check that this choice of e_3 guarantees (B.4).

At this point we are able to construct the i-th element of our sequence F_i. Let (g^{i-1}, ϕ_3^{i-1}) be the Weierstrass data of F_{i-1} in the orthonormal basis S_i. Applying Runge's theorem, we can construct a family of holomorphic functions, $h_\alpha : \mathbb{C} \to \mathbb{C}^*$ verifying the following:

(a) $|h_\alpha(z) - 1| < \frac{1}{\alpha}$ for all $z \in \overline{\operatorname{Int} P} - \varpi_i$,

(b) $|h_\alpha(z) - \alpha| < \frac{1}{\alpha}$ for all $z \in \omega_i$,

where $\alpha \in \mathbb{R}_+$. It is important to note that the family $\{h_\alpha|_{\overline{\operatorname{Int} P} - \varpi_i} \mid \alpha \in \mathbb{R}_+\}$ is continuous in the parameter α.

Using h_α as a López–Ros function, we define this new Weierstrass data

$$g^{(i,\alpha)} = \frac{g^{i-1}}{h_\alpha}, \quad \phi_3^i = \phi_3^{i-1},$$

and the associated conformal minimal immersion

$$F_{(i,\alpha)}(z) = \frac{1}{2}\,\mathrm{Re}\int^z \left(\frac{1}{g^{(i,\alpha)}} - g^{(i,\alpha)}, \mathrm{i}\left(\frac{1}{g^{(i,\alpha)}} + g^{(i,\alpha)}\right), 2\right)\phi_3^i.$$

Now, we have to check that there exists a real $\alpha_0 > 0$ such that $F_i \stackrel{\text{def}}{=} F_{(i,\alpha_0)}$ satisfies Properties (A.1_i), ..., (A.7_i).

Since $h_\alpha \to 1$, uniformly on $\overline{\mathrm{Int}\,P} - \varpi_i$, and $h_\alpha \to \infty$, uniformly on ω_i, as $\alpha \to \infty$, then (A.1_i), (A.2_i), (A.4_i) and (A.6_i) trivially hold for any $\alpha > \alpha_0$, provided α_0 is large enough in terms of N.

As we mentioned before, the family $\{h_\alpha|_{\overline{\mathrm{Int}\,P}-\varpi_i},\ \alpha \in \mathbb{R}_+\}$ depends continuously on α, and $h_\alpha \to 1$ as $\alpha \to +\infty$, uniformly on $\overline{\mathrm{Int}\,P} - \varpi_i$. As P^ε is contained in $\overline{\mathrm{Int}\,P} - \varpi_i$, then we can see $F_{(i,\alpha)}(P^\varepsilon)$, $\alpha \in [\alpha_0, +\infty]$, as a continuous deformation between $F_{(i,\alpha_0)}(P^\varepsilon)$ and $F_{i-1}(P^\varepsilon)$. Furthermore, if N is large enough, Property (A.6_i) implies that $F_{(i,\alpha)}(P^\varepsilon)$ lies in the interior of $T(\Gamma, R)$, for all $\alpha \in [\alpha_0, +\infty]$. So, Property (A.$7_i$) is a consequence of these facts and Property (A.7_{i-1}).

In order to check (A.3_i) we have to use (B.4.2). This property gives us

$$\frac{\sin\left(\frac{\text{const}}{\sqrt{N}}\right)}{1+\cos\left(\frac{\text{const}}{\sqrt{N}}\right)} \le |g^{i-1}| \le \frac{\sin\left(\frac{\text{const}}{\sqrt{N}}\right)}{1-\cos\left(\frac{\text{const}}{\sqrt{N}}\right)} \quad \text{in } \varpi_i,$$

and so, taking (B.1) into account one has (if N is large enough):

$$\|\phi^i\| \ge |\phi_3^i| = |\phi_3^{i-1}| \ge \sqrt{2}\|\phi^{i-1}\|\frac{|g^{i-1}|}{1+|g^{i-1}|^2} \ge \text{const}\cdot\sin\left(\frac{\text{const}}{\sqrt{N}}\right) \ge \frac{\text{const}}{\sqrt{N}} \quad \text{in } \varpi_i.$$

Using (B.4.1), we get (A.5.1_i). Finally, to obtain (A.5.2_i), we use that $\phi_3^{i-1} = \phi_3^i$ in the frame S_i.

Hence, we have constructed the immersions $F_0, F_1, \ldots, F_{2N}$ verifying claims (A.1_i),...,(A.7_i) for $i = 1, \ldots, 2N$. The following proposition states all the properties of F_{2N} we will need.

Proposition 1. *If N is sufficiently large, then we have:*

(I) $\mathrm{dist}_{(\overline{\mathrm{Int}\,P},F_{2N})}(P, P^\varepsilon) > 2s$.

(II) $\|F_{2N}(z) - X(z)\| < \mathrm{const}/N$ *for all* $z \in \overline{\mathrm{Int}\,P} - \bigcup_{i=1}^{2N} \varpi_i$.

(III) *There exists a polygon $\widetilde{P}$ in $\mathbb{C}$ such that:*

(III. a) $\overline{\mathrm{Int}\,P^\varepsilon} \subset \mathrm{Int}\,\widetilde{P} \subset \overline{\mathrm{Int}\,\widetilde{P}} \subset \mathrm{Int}\,P$,

(III. b) $s < \mathrm{dist}_{(\overline{\mathrm{Int}\,P},F_{2N})}(z, P^\varepsilon) < 2s$ *for all* $z \in \widetilde{P}$,

(III. c) $F_{2N}\left(\overline{\mathrm{Int}\,\widetilde{P} - \mathrm{Int}\,P^\varepsilon}\right) \subset T(\Gamma, R)$.

(IV) *$F_{2N}(P^\varepsilon)$ and Γ are homologous in $T(\Gamma, R)$.*

Proof. Item (I) is an standard consequence of Claim 2.2, making use of (A.1_i), (A.2_i) and (A.3_i), $i = 1, \dots, 2N$. Similarly, a successive application of (A.6_i) implies Item (II).

The demonstration of Item (III) is a bit more delicate. First we need to construct the polygon $\widetilde{P}$ in (III). To do this, we consider the set $\mathcal{A} = \{z \in \operatorname{Int} P - \operatorname{Int} P^\varepsilon \mid s < \operatorname{dist}_{(\overline{\operatorname{Int} P}, F_{2N})}(z, P^\varepsilon) < 2s\}$. Note that $\mathcal{A}$ is nonempty and that P and P^ε are in different connected components of $\mathbb{C} - \mathcal{A}$. Then, the existence of the polygon $\widetilde{P}$ satisfying Items (III.a) and (III.b) is obvious.

As a next step, we are going to check that F_{2N} verifies Item (III.c). We consider $\eta \in \overline{\operatorname{Int} \widetilde{P} - \operatorname{Int} P^\varepsilon}$. We will assume that $F_{2N}(\eta) \notin \Gamma$, otherwise we have nothing to prove. Hence, we distinguish two cases.

Case 1. The point η belongs to $\operatorname{Int} P - \bigcup_{i=1}^{2N} \varpi_i$.

In this case we know that $\|F_{2N}(\eta) - X(\eta)\| \le \frac{\text{const}}{N}$. Since $X(\eta)$ lies in the interior of $T(\Gamma, r)$, then we can choose N large enough so that $F_{2N}(\eta) \in T(\Gamma, r) \subset T(\Gamma, R)$.

Case 2. There exists $i \in \{1, \dots, 2N\}$ such that $\eta \in \varpi_i$.

Consider now a curve $\beta\colon [0, 1] \to \operatorname{Int} P$ so that $\beta(0) \in P^\varepsilon$, $\beta(1) = \eta$ and $\ell(\beta, F_{2N}) \le 2s$. Note that the existence of such a curve is guaranteed by (III.b). Let us define $\bar{t} = \operatorname{Supremum}\{t \in [0, 1] \mid \beta(t) \in \partial\varpi_i\}$ and $\bar{\eta} = \beta(\bar{t})$. It is important to note that $\bar{t}$ exists because $\varpi_i \subset \operatorname{Int} P - \operatorname{Int} P^\varepsilon$.

For our purposes, we need to prove first the following inequality:

$$\|F_i(\bar{\eta}) - F_i(\eta)\| \le \frac{\text{const}}{N} + 2s. \tag{7}$$

Indeed, by using Properties (A.6_k), for $k = 1, \dots, 2N$, we obtain

$$\begin{aligned}\|F_i(\bar{\eta}) - F_i(\eta)\| &\le \|F_i(\bar{\eta}) - F_{2N}(\bar{\eta})\| + \|F_{2N}(\bar{\eta}) - F_{2N}(\eta)\| + \|F_{2N}(\eta) - F_i(\eta)\| \\ &\le \frac{\text{const}}{N} + \ell(\beta, F_{2N}) + \frac{\text{const}}{N} \le \frac{\text{const}}{N} + 2s.\end{aligned}$$

Let p_i be the point given by condition (6). Then we have

$$\|F_{2N}(\eta) - p_i\| \le \|F_i(\eta) - p_i\| + \frac{\text{const}}{N}. \tag{8}$$

We again distinguish two cases.

Case 2.1. If $\|X(\eta) - p_i\| \le 1/\sqrt{N}$, then one has

$$\begin{aligned}\|F_i(\eta) - p_i\| &\le \|F_i(\eta) - F_i(\bar{\eta})\| + \|F_i(\bar{\eta}) - F_{i-1}(\bar{\eta})\| + \|F_{i-1}(\bar{\eta}) - F_{i-1}(\eta)\| \\ &\quad + \|F_{i-1}(\eta) - X(\eta)\| + \|X(\eta) - p_i\| \\ &\le \frac{\text{const}}{N} + 2s + \frac{\text{const}}{N^2} + \frac{\text{const}}{\sqrt{N}} + \frac{\text{const}}{N} + \frac{1}{\sqrt{N}} \le R,\end{aligned}$$

where the last inequality occurs if N is sufficiently large.

Case 2.2. If $\|X(\eta) - p_i\| > 1/\sqrt{N}$, then we use (A.5.2$_i$) to get a bound for the third coordinate of $F_i(\eta) - p_i$ in the orthonormal basis S_i. We proceed as follows:

$$\begin{aligned} |(F_i(\eta) - p_i)_{(3,S_i)}| &= |(F_{i-1}(\eta) - p_i)_{(3,S_i)}| \\ &\le |(F_{i-1}(\eta) - X(\eta))_{(3,S_i)}| + |(X(\eta) - p_i)_{(3,S_i)}| \\ &\le \frac{\text{const}}{N} + r. \end{aligned} \tag{9}$$

On the other hand, we can apply Property (A.5.1$_i$) to find an upper bound for the first two coordinates of $F_i(\eta) - p_i$,

$$\begin{aligned} \|(F_i(\eta) - p_i)_{(*,S_i)}\| &\le \|(F_i(\eta) - F_i(\bar{\eta}))_{(*,S_i)}\| + \|(F_i(\bar{\eta}) - F_{i-1}(\bar{\eta}))_{(*,S_i)} \\ &\quad + \|(F_{i-1}(\bar{\eta}) - F_i(\eta))_{(*,S_i)}\| \\ &\quad + \|(F_{i-1}(\eta) - X(\eta))_{(*,S_i)}\| \\ &\quad + \|(X(\eta) - p_i)_{(*,S_i)}\| \\ &\le \frac{\text{const}}{N} + 2s + \frac{\text{const}}{N^2} + \frac{1}{\sqrt{N}} + \frac{\text{const}}{\sqrt{N}} \le 2s + \frac{\text{const}}{\sqrt{N}}. \end{aligned} \tag{10}$$

By Pythagoras' theorem and taking into account (9) and (10), we easily infer that

$$\|F_i(\eta) - p_i\| < \sqrt{\left(2s + \frac{\text{const}}{\sqrt{N}}\right)^2 + \left(r + \frac{\text{const}}{N}\right)^2}.$$

Using this upper bound, inequality (8) becomes

$$\|F_{2N}(\eta) - p_i\| \le \sqrt{\left(2s + \frac{\text{const}}{\sqrt{N}}\right)^2 + \left(r + \frac{\text{const}}{N}\right)^2} + \frac{\text{const}}{N}.$$

So, for a large enough N, it is clear that $F_{2N}(\eta) \in T(\Gamma, R)$ where $R = \sqrt{r^2 + (2s)^2} + \varepsilon$, for all $\eta \in \operatorname{Int} \widetilde{P} - \operatorname{Int} P^{\varepsilon}$. This completes the proof of Item (III.c).

Finally, Item (IV) in the proposition is a direct consequence of property (A.7$_{2N}$). □

Obviously, the immersion $F_{2N}\colon \operatorname{Int} \widetilde{P} \to \mathbb{R}^3$ is the immersion $\widetilde{X}$ we are looking for. Proposition 1 tells us that $\widetilde{X}$ so defined verifies Items (a.1), ..., (a.6) in the lemma.

2.2. Proof of Theorem 1. Let $c_0 > 0$, $r_1 > 0$ and $\rho_1 > 0$ to be specified later, and define

$$r_n = \sqrt{r_{n-1}^2 + \left(\frac{2c_0}{n}\right)^2} + \frac{c_0}{n^2},$$

and $\rho_n = \rho_1 + \sum_{i=2}^{n} c_0/i$, $n \geq 2$. The constants r_1 and c_0 are chosen in such a way that

$$\lim_{n\to\infty} r_n < \frac{\mu}{4} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{c_0}{n^2} < \frac{\mu}{4}. \tag{11}$$

Consider also a regular curve Γ in $\mathbb{R}^3$, so that $\delta^H(X(P), \Gamma) < r_1$ and both curves are homologous in $T(\Gamma, r_1)$. Our strategy consists of using Lemma 3 to define a sequence

$$\chi_n = (X_n \colon \overline{\operatorname{Int} P_n} \to \mathbb{R}^3,\ P_n, \varepsilon_n, \xi_n),$$

where X_n is a conformal minimal immersion, P_n is a polygon, $\{\varepsilon_n\}$, $\{\xi_n\}$ are decreasing sequences of non vanishing terms satisfying $\varepsilon_n, \xi_n < c_0/n^2$, and:

(A_n) $\overline{\operatorname{Int} P_{n-1}^{\xi_{n-1}}} \subset \operatorname{Int} P_{n-1}^{\varepsilon_n} \subset \overline{\operatorname{Int} P_{n-1}^{\varepsilon_n}} \subset \operatorname{Int} P_n^{\xi_n} \subset \overline{\operatorname{Int} P_n^{\xi_n}} \subset \operatorname{Int} P_n \subset \overline{\operatorname{Int} P_n} \subset \operatorname{Int} P_{n-1}$,

(B_n) $\rho_n < \operatorname{dist}_{(X_n, \overline{\operatorname{Int} P_n^{\xi_n}})}(0, P_n^{\xi_n})$,

(C_n) $\|X_n - X_{n-1}\| < \varepsilon_n$ in $\operatorname{Int} P_{n-1}^{\varepsilon_n}$,

(D_n) $\lambda_{X_n} \geq \alpha_n \lambda_{X_{n-1}}$ in $\operatorname{Int} P_{n-1}^{\xi_{n-1}}$, where $\{\alpha_i\}_{i\in\mathbb{N}}$ is a sequence of real numbers such that $0 < \alpha_i < 1$ and $\left\{\prod_{i=1}^{n} \alpha_i\right\}_n$ converges to $1/2$,

(E_n) $X_n(P_{n-1}^{\varepsilon_n})$ and Γ are homologous in $T(\Gamma, r_n)$,

(F_n) $X_n\left(\overline{\operatorname{Int} P_n - \operatorname{Int} P_{n-1}^{\varepsilon_n}}\right) \subset T(\Gamma, r_n)$.

The choice of the first element of the sequence is not difficult. First, we take $X_1 = X$. Let P_1 be a polygon parallel to P and satisfying $P_1^{\xi_1} = P$. The constant and the polygon must be chosen in such a way that $\delta^H(X_1(P_1 - P_1^{\xi_1}), \Gamma) < r_1$. Finally, we choose ρ_1 and ε_1 satisfying

$$\rho_1 < \operatorname{dist}_{(X_1, \overline{\operatorname{Int} P_1})}(0, P_1) \quad \text{and} \quad \varepsilon_1 < \min\{c_0, r_1\}.$$

Suppose that we have defined $\chi_1, \ldots, \chi_n$. Then, we will construct the $(n+1)$-th term in the following way.

Take a sequence $\{\widehat{\varepsilon}_m\} \searrow 0$, with $\widehat{\varepsilon}_m < \frac{c_0}{(n+1)^2}$ for all m. For each m, we consider $\widetilde{P}_m$ and $Y_m \colon \operatorname{Int} \widetilde{P}_m \to \mathbb{R}^3$ given by Lemma 1, for the data

$$X = X_n, \quad P = P_n, \quad r = r_n, \quad s = \frac{c_0}{n+1}, \quad \nu = \widehat{\varepsilon}_m.$$

If m is large enough, Items (a.1) and (a.2) in Lemma 1 tell us that $\overline{\operatorname{Int} P_n^{\xi_n}} \subset \operatorname{Int} \widetilde{P}_m$ and the sequence $\{Y_m\}$ converges to X_n uniformly in $\overline{\operatorname{Int} P_n^{\xi_n}}$. In particular, $\{\lambda_{Y_m}\}$

converges uniformly to λ_{X_n} in $\overline{\operatorname{Int} P_n^{\xi_n}}$. Therefore there is an $m_0 \in \mathbb{N}$ such that

$$\overline{\operatorname{Int} P_n^{\xi_n}} \subset \operatorname{Int} P_n^{\hat{\varepsilon}_{m_0}} \subset \operatorname{Int} \widetilde{P}_{m_0}, \tag{12}$$

$$\rho_n < \operatorname{dist}_{(Y_{m_0}, \overline{\operatorname{Int} P_n^{\xi_n}})}(0, P_n^{\xi_n}), \tag{13}$$

$$\lambda_{Y_{m_0}} \geq \alpha_{n+1} \lambda_{X_n} \qquad \text{in } \operatorname{Int} P_n^{\xi_n}. \tag{14}$$

We define $X_{n+1} = Y_{m_0}$, $P_{n+1} = \widetilde{P}_{m_0}$, and $\varepsilon_{n+1} = \widehat{\varepsilon}_{m_0}$. From (12), (13) and Item (a.3) in Lemma 1, it is not hard to see that $\rho_{n+1} < \operatorname{dist}_{(X_{n+1}, \overline{\operatorname{Int} P_{n+1}})}(0, P_{n+1})$. Finally, take ξ_{n+1} small enough such that (A_{n+1}) and (B_{n+1}) hold. The remaining properties directly follow from (12), (14) and the aforementioned lemma. This concludes the construction of the sequence $\{\chi_n\}_{n \in \mathbb{N}}$.

Now, we define

$$\Sigma = \bigcup_{n=1}^{\infty} \operatorname{Int}(P_n^{\xi_n}) \Big(= \bigcup_{n=1}^{\infty} \operatorname{Int}(P_n^{\varepsilon_{n+1}}) \Big).$$

Σ is a simply connected domain in U. Properties (C_n) and the fact that $\varepsilon_n < c_0/n^2$ give us that the sequence of minimal immersions $\{X_n\}$ is a Cauchy sequence, uniformly on compact sets of Σ, and so $\{X_n\}$ converges.

Let $\widehat{X} \colon \Sigma \to \mathbb{R}^3$ be the limit of $\{X_n\}$. $\widehat{X}$ has the following properties:

- $\widehat{X}$ is an immersion. Indeed, for any $z \in \Sigma$ there exists $n \in \mathbb{N}$ such that $z \in \operatorname{Int} P_n^{\xi_n}$. From Properties (F_i), $i = n+1, \ldots, k$ we get

  $$\lambda_{X_k}(z) \geq \alpha_k \lambda_{X_{k-1}}(z) \geq \cdots \geq \alpha_k \ldots \alpha_{n+1} \lambda_{X_n}(z) \geq \alpha_k \ldots \alpha_1 \lambda_{X_n}(z)$$

  for all $k > n$. Taking limit as $k \to \infty$, we deduce

  $$\lambda_{\widehat{X}}(z) \geq \frac{1}{2} \lambda_{X_n}(z) > 0, \tag{15}$$

  and so $\widehat{X}$ is an immersion.

- $\widehat{X}$ is minimal and conformal.

- Σ is complete with the metric induced by $\widehat{X}$. Indeed, if n is large enough, and taking (15) and (A_n) into account, one has

  $$\operatorname{dist}_{(\widehat{X}, \overline{\operatorname{Int} P_n^{\xi_n}})}(0, P_n^{\xi_n}) > \frac{1}{2} \operatorname{dist}_{(X_n, \overline{\operatorname{Int} P_n^{\xi_n}})}(0, P_n^{\xi_n}) > \frac{1}{2} \rho_n.$$

  The completeness is due to the fact that $\{\rho_n\}_{n \in \mathbb{N}}$ diverges.

- $\widehat{X}(\Sigma - \operatorname{Int} P) \subset T(X(P), \mu)$. Pick a point $z \in \Sigma - \operatorname{Int} P$. Then we know that z belongs to $\overline{\operatorname{Int} P_n - \operatorname{Int} P_{n-1}^{\varepsilon_n}}$, for some $n \in \mathbb{N}$, and so Property (F_n) implies $\operatorname{dist}_{\mathbb{R}^3}(X_n(z), \Gamma) < r_n$. Therefore, one has

$$\operatorname{dist}_{\mathbb{R}^3}(\widehat{X}(z), \Gamma) \leq \|\widehat{X}(z) - X_n(z)\| + r_n \leq \Big(\sum_{k=n}^{\infty} \|X_{k+1}(z) - X_k(z)\|\Big) + r_n;$$

  at this point we use Properties (C_k), for $k \geq n$, and then one obtains

$$\operatorname{dist}_{\mathbb{R}^3}(\widehat{X}(z), \Gamma) \leq \sum_{k=n}^{\infty} \varepsilon_k + r_n < \frac{\mu}{2}.$$

  Thus we have that $\widehat{X}(\Sigma - \operatorname{Int} P) \subset T(\Gamma, \mu/4)$. Moreover, our choice of Γ implies that $\widehat{X}(\Sigma - \operatorname{Int} P) \subset T(X(P), \mu/2 + r_1) \subset T(X(P), \mu)$.
- $\widehat{X}(P)$ is homologous to $X(P)$ in $T(X(P), \mu)$. Indeed, pick a natural number $n \in \mathbb{N}$, then from Properties C_k, $k = 1, \ldots, n$ we have that $X_n(P) \subset T(\Gamma, r_n)$. It is clear that $X_n(P)$ is homologous to $X_n(P_{n-1}^{\varepsilon_n})$ in $T(\Gamma, r_n)$, so using Property (E_n) and taking into account our choice of Γ we conclude that $X_n(P)$ and $X(P)$ are homologous in $T(\Gamma, r_n) \subset T(X(P), \mu)$. Since the curves $X_n(P)$ converge uniformly to $\widehat{X}(P)$ and all these curves have the same homological type as $X(P)$, then it is clear that $\widehat{X}(P)$ is also homologous to $X(P)$ in the tube $T(X(P), \mu)$.

This completes the proof of Theorem 1.

2.3. Proof of Lemma 3. All the arguments we need to prove this lemma are essentially contained in Theorem 1 and Lemma 2. First, we apply Theorem 1 to the immersion $X\colon \operatorname{Int} P \to \mathbb{R}^3$, the polygon P^ε and a constant $\mu > 0$, to be determined later. Hence, we obtain a complete minimal immersion $\widehat{X}\colon \Sigma \to \mathbb{R}^3$ which satisfies

(a.i) $\overline{\operatorname{Int} P^\varepsilon} \subset \Sigma \subset \overline{\Sigma} \subset \operatorname{Int} P$,

(a.ii) $\|X(z) - \widehat{X}(z)\| < \mu$ for all $z \in \operatorname{Int} P^\varepsilon$,

(a.iii) $\widehat{X}(\Sigma - \operatorname{Int} P^\varepsilon) \subset T(X(P^\varepsilon), \mu)$,

(a.iv) $\widehat{X}(P^\varepsilon)$ is homologous to $X(P^\varepsilon)$ in $T(X(P^\varepsilon), \mu)$.

As the immersion $\widehat{X}$ is complete, then we can find a new polygon $\widehat{P}$ satisfying

$$\overline{\operatorname{Int} P^\varepsilon} \subset \operatorname{Int} \widehat{P} \subset \overline{\operatorname{Int} \widehat{P}} \subset \operatorname{Int} P,$$

and so that the distance

$$\operatorname{dist}_{(\overline{\operatorname{Int} \widehat{P}}, \widehat{X})}(z, P^\varepsilon) > \frac{1}{\varepsilon} \quad \text{for all } z \in \widehat{P}. \tag{16}$$

Taking into account the hypotheses of the lemma, Property (a.iii) guarantees (for a small enough μ) that $\widehat{X}(\Sigma - \operatorname{Int} P^\varepsilon) \subset E - E_{-a}$ and $\widehat{X}(\Sigma - \operatorname{Int} P^\varepsilon) \subset T(\Gamma, c)$. The former inclusion yields that $\delta^H(\widehat{X}(\Sigma - \operatorname{Int} P^\varepsilon), \Gamma) < c$. Indeed, we proceed by contradiction. Suppose this is not true, then there exists a point $x \in \Gamma$ so that $\mathbb{B}(x, c) \cap \widehat{X}(\operatorname{Int} \widehat{P} - \operatorname{Int} P^\varepsilon)$ is empty. Among other things, this implies that $\widehat{X}(P^\varepsilon)$ can not be homologous to Γ in $T(\Gamma, c)$ which is contrary to Property (a.iv) (recall that $X(P^\varepsilon)$ is homologous to Γ.)

At this point, we are able to apply Lemma 2 to the following data:

$$X = \widehat{X}, \quad P = \widehat{P}, \quad O = \Sigma, \quad E = E_{-a}, \quad E' = E', \quad b_1 < \varepsilon/2, \quad b_2 = b.$$

Thus, we obtain a new polygon Q such that $\operatorname{Int} \widehat{P} \subset \overline{\operatorname{Int} \widehat{P}} \subset \operatorname{Int} Q \subset \overline{\operatorname{Int} Q} \subset \Sigma$ and a new minimal immersion $Y : \overline{\operatorname{Int} Q} \to \mathbb{R}^3$ with the following properties:

(b.i) $\|Y(z) - \widehat{X}(z)\| < b_1$ for all $z \in \overline{\operatorname{Int} \widehat{P}}$,

(b.ii) $Y(Q) \subset E' - E'_{-b}$,

(b.iii) $Y(\operatorname{Int} Q \setminus \operatorname{Int} \widehat{P}) \subset \mathbb{R}^3 - E_{-2b-a}$,

(b.iv) For any $z \in \operatorname{Int} Q \setminus \operatorname{Int} \widehat{P}$, one has

$$\begin{aligned}\|\widehat{X}(z) - Y(z)\| &< M(a, b, b_1, E, E') \\ &= \left[\left(2(2a+b) + \frac{1}{\kappa_1(\partial E)} + \delta^H(E, E')\right)^2 \right. \\ &\qquad \left. - \left(2(a+b) + \frac{1}{\kappa_1(\partial E)}\right)^2 + (2b)^2\right]^{\frac{1}{2}} + b_1,\end{aligned}$$

(b.v) $\widehat{X}(\widehat{P})$ and $Y(\widehat{P})$ are homologous in $T(\widehat{X}(\widehat{P}), M(a, b, b_1, E, E'))$.

To finish the proof, it suffices to check that Y is the immersion we are looking for. First, observe that Item (d.1) trivially holds.

In order to check (d.2), notice that Y converges to $\widehat{X}$, uniformly on $\overline{\operatorname{Int} \widehat{P}}$ as $b_1 \to 0$. So, Item (d.2) is an easy consequence of this fact and (16).

Item (d.3) directly follows from (b.ii). Moreover, Properties (a.ii), (b.i), and (b.iii) imply Item (d.4) (provided μ and b_1 are sufficiently small).

Similarly, if μ, $b_1 < \varepsilon/2$, then Item (d.5) is obtained from (a.ii) and (b.i).

Let us show that the immersion Y satisfies Item (d.6). We know that

$$\delta^H(\Gamma, \widehat{X}(Q - \operatorname{Int} P^\varepsilon)) < c.$$

Thus, Item (d.6) follows from (b.i) and (b.iv), provided that b_1 is small enough.

Finally, Item (d.7) in the lemma can be easily deduced from (a.iv) and (b.v), taking into account that $T(\widehat{X}(\widehat{P}), M(a, b, b_1, E, E'))$ and $T(X(P^\varepsilon), \mu)$ are contained in the tube $T(\Gamma, c + M(a, b, b_1, E, E'))$ and the fact that $X(P^\varepsilon)$ is homologous to Γ in that tube. This completes the proof of Lemma 3.

3. Proof of the main theorems

At this point we are able to prove the principal results of our paper. From now on C will represent a bounded, strictly convex regular domain of $\mathbb{R}^3$. Recall that strictly convex means that the mean and Gaussian curvatures are positive. If we call $(\mathcal{K}, \delta^H)$ as the metric space of compact sets of ∂C with the Hausdorff distance, then we will prove that the limit sets of properly immersed minimal disk are dense in $\mathcal{K}$. At the end of this section we will show that our results are sharp in the sense that neither strictly convexity nor analyticity can be removed from our assumptions.

Theorem 2. *Let C be a strictly convex bounded regular domain of space. For any smooth Jordan curve $\Gamma \subset \partial C$ and for any $\varepsilon > 0$ there exists a complete proper minimal immersion $\psi_{(\Gamma,\varepsilon)} : \mathbb{D} \to C$ so that $\delta^H(\psi_{(\Gamma,\varepsilon)}(\partial\mathbb{D}), \Gamma) < \varepsilon$.*

Proof. We know that there is $t_0 > 0$ so that C_t is well defined for any $t \in]-t_0, \infty[$. Furthermore, the normal projection $\mathcal{P}_t : \mathbb{R}^3 - C_t \to \partial(C_t)$ is a well defined smooth map.

We fix $\tau > 0$ such that $\tau < \min\{t_0, \varepsilon\}$. Now, define

$$M(n, c) \stackrel{\text{def}}{=} c\left(\frac{1}{n^2} + \sqrt{\frac{29\,c^2}{(n-1)^8} + \frac{4}{\kappa_1(\partial C)\,(n-1)^4}}\right).$$

Choose $c_1 > 0$ small enough such that $\sum_{n\geq 2} M(n, c_1) < \tau/4$. In particular,

$$\sum_{n=1}^{\infty} \frac{c_1^2}{n^4} < \sum_{n=2}^{\infty} M(n, c_1) < \tau/4 < t_0.$$

Using this constant c_1, we construct an expansive sequence $\{E^n\}_{n\in\mathbb{N}}$ of bounded convex regular domains in the following way: $E^n \stackrel{\text{def}}{=} C_{-t_n}$, where $t_n = \sum_{k=n}^{\infty} c_1^2/k^4$, $n \geq 1$. If we label $\Gamma_1 \stackrel{\text{def}}{=} \mathcal{P}_{-t_1}(\Gamma)$, it is obvious that $\Gamma_1 \subset \partial E^1$ is a Jordan curve.

We also take a decreasing sequence of positive reals $\{b_n\}_{n\in\mathbb{N}}$, satisfying $b_n < c_1^2/n^4$ for all $n \in \mathbb{N}$.

Finally, we define a sequence of real numbers $\{\mu_n\}_{n\in\mathbb{N}}$ in the following way: $\mu_1 = \tau/4$ and $\mu_n = \mu_{n-1} + M(n, c_1)$.

The next step consists of using Lemma 3 to construct (in a recursive way) a sequence

$$\chi_n = (X_n : \overline{\operatorname{Int} P_n} \to \mathbb{R}^3, P_n, \varepsilon_n, \xi_n),$$

where X_n are conformal minimal immersions with $X_n(0) = 0$, P_n are polygons, and $\{\varepsilon_n\}$, $\{\xi_n\}$, $\{\sigma_n\}$ are sequences of positive numbers converging to zero, verifying $\varepsilon_k < c_1/k^2$. Furthermore, the sequence $X_n : \overline{\operatorname{Int} P_n} \to \mathbb{R}^3$ must verify the following properties:

(I_n) $\overline{\mathrm{Int}\, P_{n-1}^{\xi_{n-1}}} \subset \mathrm{Int}\, P_{n-1}^{\varepsilon_n} \subset \overline{\mathrm{Int}\, P_{n-1}^{\varepsilon_n}} \subset \mathrm{Int}\, P_n^{\xi_n} \subset \overline{\mathrm{Int}\, P_n^{\xi_n}} \subset \mathrm{Int}\, P_n \subset \overline{\mathrm{Int}\, P_n} \subset \mathrm{Int}\, P_{n-1}$;

(II_n) $\|X_n(z) - X_{n-1}(z)\| < \varepsilon_n$ for all $z \in \mathrm{Int}\, P_{n-1}^{\varepsilon_n}$;

(III_n) $\lambda_{X_n}(z) \geq \alpha_n \lambda_{X_{n-1}}(z)$ for all $z \in \mathrm{Int}\, P_{n-1}^{\xi_{n-1}}$, where $\{\alpha_i\}_{i\in\mathbb{N}}$ is a sequence of real numbers such that $0 < \alpha_i < 1$ and $\left\{\prod_{i=1}^n \alpha_i\right\}_n$ converges to $1/2$;

(IV_n) $\frac{1}{\varepsilon_n} < \mathrm{dist}_{\left(\overline{\mathrm{Int}\, P_n^{\xi_n}}, X_n\right)}(P_{n-1}^{\xi_{n-1}}, P_n^{\xi_n})$;

(V_n) $X_n(z) \in E^n - (E^n)_{-b_n}$ for all $z \in P_n$;

(VI_n) $X_n(z) \in \mathbb{R}^3 \setminus (E^{n-1})_{-2(b_{n-1}+b_n)}$ for all $z \in \mathrm{Int}\, P_n \setminus \mathrm{Int}\, P_{n-1}^{\varepsilon_n}$;

(VII_n) $\delta^H(X_n(P_n), \Gamma) < \mu_n$;

(VIII_n) $\delta^H\left(X_n(\mathrm{Int}\, P_n - \mathrm{Int}\, P_{n-1}^{\varepsilon_n}), \Gamma\right) < \mu_n$;

(IX_n) $X_n(P_n)$ and Γ are homologous in $T(\Gamma, \mu_n)$.

To define χ_1, we consider D_1 a solution of Plateau's problem for the curve Γ_1. Let $X_1 : \overline{\mathbb{D}} \to D_1$ be a conformal parametrization of the minimal disk D_1. Then we choose a polygon $P_1 \subset \mathbb{D}$ sufficiently close to $\partial\mathbb{D}$ and a constant $\xi_1 > 0$ so that

- $\delta^H(X_1(\mathrm{Int}\, P_1 - \mathrm{Int}\, P_1^{\xi_1}), \Gamma_1) < b_1$ ($< \frac{\tau}{4}$). In particular, Properties (V_1) and (VII_1) hold.
- $X_1(P_1)$ is homologous to Γ in the tube $T(\Gamma, \mu_1)$.

Suppose that we have $\chi_1, \ldots, \chi_n$. In order to construct χ_{n+1}, we consider the following data:

$$E = E^n, \quad E' = E^{n+1}, \quad a = b_n, \quad c = \mu_n, \quad X = X_n, \quad P = P_n.$$

From Property (V_n) we know that $X(P) \subset E \setminus E_{-a}$, and Property ($\mathrm{VII}_n$) implies that $\delta^H(X(P), \Gamma) < c$. Furthermore, $X(P)$ is homologous to Γ in $T(\Gamma, c)$. Then it is straightforward that we can find a small enough positive constant $\varkappa$, such that Lemma 3 can be applied to the aforementioned data, and for any $\varepsilon \in]0, \varkappa[$.

Take a sequence $\{\widehat{\varepsilon}_m\} \searrow 0$, with $\widehat{\varepsilon}_m < \min\left\{\frac{c_1}{(n+1)^2}, \varkappa, b_{n+1}\right\}$ for all m. For each m, we consider Q_m and $Y_m : \overline{\mathrm{Int}\, Q_m} \to \mathbb{R}^3$ given by Lemma 3, for the above data and $\varepsilon = b = \widehat{\varepsilon}_m$. It is important to note that the constant $M(a, b, \widehat{\varepsilon}_m, E, E')$ in Lemma 3 is less that $M(n+1, c_1)$.

If m is large enough, Assertions (b.1) and (b.5) in Lemma 3 tell us that $\overline{\mathrm{Int}\, P_n^{\xi_n}} \subset \mathrm{Int}\, Q_m$ and the sequence $\{Y_m\}$ converges to X_n uniformly in $\overline{\mathrm{Int}\, P_n^{\xi_n}}$. In particular, $\{\lambda_{Y_m}\}$ converges uniformly to λ_{X_n} in $\overline{\mathrm{Int}\, P_n^{\xi_n}}$. Therefore there is an $m_0 \in \mathbb{N}$ such that

$$\overline{\mathrm{Int}\, P_n^{\xi_n}} \subset \mathrm{Int}\, P_n^{\hat{\varepsilon}_{m_0}} \subset \mathrm{Int}\, Q_{m_0}, \tag{17}$$

$$\lambda_{Y_{m_0}} \geq \alpha_{n+1} \lambda_{X_n} \quad \text{in } \mathrm{Int}\, P_n^{\xi_n}. \tag{18}$$

We define $X_{n+1} = Y_{m_0}$, $P_{n+1} = Q_{m_0}$, and $\varepsilon_{n+1} = \widehat{\varepsilon}_{m_0}$. From (17) and Statement (d.2) in Lemma 3, we infer that $\frac{1}{\varepsilon_{n+1}} < \mathrm{dist}_{(\overline{\mathrm{Int}\, P_{n+1}}, X_{n+1})}(P_n^{\xi_n}, P_{n+1})$. Finally, take ξ_{n+1} small enough such that (I_{n+1}) and (IV_{n+1}) hold.

The remainder properties directly follow from (17), (18) and Lemma 3. This concludes the construction of the sequence $\{\chi_n\}_{n\in\mathbb{N}}$.

Now, we extract some information from the properties of $\{\chi_n\}$. Actually, the limit of the sequence of minimal immersions will be the complete proper minimal immersion we are looking for.

At this point, we are able to define the immersion that proves the theorem. This immersion will be the limit of the sequence $\{X_n\}_{n\in\mathbb{N}}$. But before of this, we need to construct the domain of definition of the limit immersion. Properties (I_n), $n \in \mathbb{N}$, imply that the set

$$\Omega = \bigcup_{n=1}^{\infty} \mathrm{Int}\, P_n^{\varepsilon_{n+1}} = \bigcup_{n=1}^{\infty} \mathrm{Int}\, P_n^{\xi_n}$$

is an expansive union of simply connected domains, resulting in Ω being simply connected. Moreover, as $\mathrm{Int}\, P_n^{\xi_n} \subset \mathbb{D}$, then $\Omega \subset \mathbb{D}$. Using Riemann's mapping Theorem, we deduce that Ω is conformally equivalent to the unit disk.

Properties (II_n) tells that $\{X_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, uniformly on compact sets of Ω, and so, by using Harnak's theorem, it converges. Let $\psi_{(\Gamma,\varepsilon)} : \Omega \to \mathbb{R}^3$ be the limit of $\{X_n\}_{n\in\mathbb{N}}$. Then $\psi_{(\Gamma,\varepsilon)}$ has the following properties:

Claim 3.1. *$\psi_{(\Gamma,\varepsilon)}$ is a conformal minimal immersion.*

Proof. The proof of this claim is a direct consequence of Properties (III_n), $n \in \mathbb{N}$. □

Claim 3.2. *$\psi_{(\Gamma,\varepsilon)} : \Omega \to C$ is proper.*

Proof. Consider a compact subset $K \subset C$. Let n_0 be a natural number so that

$$K \subset (E^{n-1})_{-2(b_{n-1}+b_n)-\sum_{k\geq n}\varepsilon_k} \quad \text{for all } n \geq n_0.$$

From Properties (VI_n), we have $X_n(z) \in \mathbb{R}^3 - (E^{n-1})_{-2(b_{n-1}+b_n)}$ for all $z \in \mathrm{Int}\, P_n - \mathrm{Int}\, P_{n-1}^{\varepsilon_n}$. Moreover, taking into account (II_k), for $k \geq n$, we obtain

$$\psi_{(\Gamma,\varepsilon)}(z) \in \mathbb{R}^3 - (E^{n-1})_{-2(b_{n-1}+b_n)-\sum_{k\geq n}\varepsilon_k}.$$

Then, we have $\psi_{(\Gamma,\varepsilon)}^{-1}(K) \cap (\mathrm{Int}\, P_n - \mathrm{Int}\, P_{n-1}^{\varepsilon_n}) = \emptyset$ for $n \geq n_0$. This implies that $\psi_{(\Gamma,\varepsilon)}^{-1}(K) \subset \mathrm{Int}\, P_{n_0-1}^{\varepsilon_{n_0}}$, and so it is compact in Ω. □

Claim 3.3. *Ω is complete with the metric $\mathcal{S}_{\psi_{(\Gamma,\varepsilon)}}$.*

Proof. This is a trivial consequence of Properties (II_n) and (IV_n), $n \in \mathbb{N}$. □

Claim 3.4. *The limit set $\psi_{(\Gamma,\varepsilon)}(\partial\Omega)$ satisfies $\delta^H(\psi_{(\Gamma,\varepsilon)}(\partial\Omega), \Gamma) < \varepsilon$.*

Proof. First, we are going to prove that $\mathrm{dist}_{\mathbb{R}^3}(p, \Gamma) < \tau$, for all p in the limit set $\psi_{(\Gamma,\varepsilon)}(\partial\Omega)$.

Given $p \in \psi_{(\Gamma,\varepsilon)}(\partial\Omega)$, we know that there exists a sequence of points in Ω, $\{z_n\}_{n\in\mathbb{N}}$, with

$$\{\psi_{(\Gamma,\varepsilon)}(z_n)\}_{n\in\mathbb{N}} \to p.$$

Without loss of generality (up to re-index the sequence and take a subsequence) we can assume

$$z_n \in \mathrm{Int}\, P_n - \mathrm{Int}\, P_{n-1}^{\varepsilon_n} \tag{19}$$

$$z_n \in \mathrm{Int}\, P_n^{\varepsilon_{n+1}}. \tag{20}$$

From (19) and taking (VIII_n) into account we obtain

$$\mathrm{dist}_{\mathbb{R}^3}(X_n(z_n), \Gamma) \le \mu_n < \frac{\tau}{2}. \tag{21}$$

On the other hand, using (20) and the fact that z_n satisfies the conditions to apply Properties (II_n), for $k \ge n+1$, then we have

$$\|\psi_{(\Gamma,\varepsilon)}(z_n) - X_n(z_n)\| \le \sum_{k=n+1}^{\infty} \|X_k(z_n) - X_{k-1}(z_n)\| < \frac{\tau}{4}. \tag{22}$$

Combining (21) and (22) we trivially deduce that $\mathrm{dist}_{\mathbb{R}^3}(\psi_{(\Gamma,\varepsilon)}(z_n), \Gamma) < 3\tau/4$. Finally, we take limit as $n \to \infty$ and obtain $\mathrm{dist}_{\mathbb{R}^3}(p, \Gamma) \le 3\tau/4 < \varepsilon$ (recall that we have chosen $\tau < \varepsilon$).

Now, we have to prove that $\mathrm{dist}_{\mathbb{R}^3}(x, \psi_{(\Gamma,\varepsilon)}(\partial\Omega)) < \tau$, for all $x \in \Gamma$.

Pick a point $x \in \Gamma$. Property (VIII_n) once again tells us that

$$\delta^H(X_n(\mathrm{Int}\, P_n - \mathrm{Int}\, P_{n-1}^{\varepsilon_n}), \Gamma) \le \mu_n < \frac{\tau}{2}.$$

Assume that $\mathbb{B}(x, \tau/2) \cap X_n(P_{n-1}^{\varepsilon_n})$ is empty. It implies that $X_n(P_{n-1}^{\varepsilon_n})$ and Γ can not be homologous in $T(\Gamma, \tau/2)$. But Properties (IX_n) and (VIII_n) imply that $X_n(P_{n-1}^{\varepsilon_n})$ is homologous to Γ in $T(\Gamma, \mu_n) \subset T(\Gamma, \tau/2)$, which is absurd.

Hence, we deduce that there exists $z_n \in P_{n-1}^{\varepsilon_n}$ such that $\|X_n(z_n) - x\| < \tau/2$. Moreover, reasoning as in (22), we have $\|\psi_{(\Gamma,\varepsilon)}(z_n) - X_n(z_n)\| < \tau/4$. Then, we conclude

$$\|\psi_{(\Gamma,\varepsilon)}(z_n) - x\| < 3\tau/4 \quad \text{for all } n \in \mathbb{N}. \tag{23}$$

Notice that the sequence $\{\psi_{(\Gamma,\varepsilon)}(z_n)\}_{n\in\mathbb{N}}$ admits a convergent subsequence (recall that $\overline{C}$ is compact). Let p denote the limit of such subsequence. Inequality (23) yields that $\|p - x\| \le 3\tau/4$. In particular, we have $\mathrm{dist}_{\mathbb{R}^3}(x, \psi_{(\Gamma,\varepsilon)}(\partial\Omega)) < \varepsilon$. □

This completes the proof of Theorem 2. □

As we mentioned in the introduction, the above theorem has the following direct application:

Theorem 3. *Every bounded domain with regular boundary admits a complete properly immersed minimal disk.*

Proof. If R is a bounded regular domain, then there exists an open connected region $A \subset \partial R$ so that the mean and Gauss curvatures are positive on A. Consider R' a strictly convex regular domain $R' \subset R$ and such that $\emptyset \neq \partial R' \cap \partial R \subset A$, like in Figure 4. Hence, in order to get the minimal immersion we are looking for, it suffices to apply Theorem 2 to a Jordan curve Γ in $\partial R' \cap \partial R$ and an $\varepsilon > 0$ small enough. □

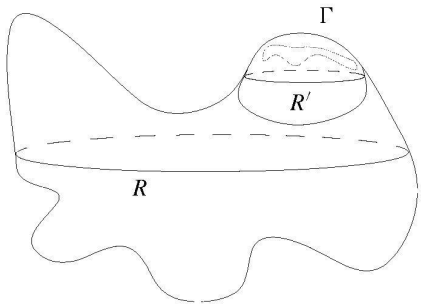

Figure 4. The domains R and R' and the curve Γ.

Theorem 3 implies that a bounded regular domain is not *universal for minimal surfaces.* A connected region of space which is open or the closure of an open set is *universal for minimal surfaces*, if every complete properly immersed minimal surface in the region is recurrent for Brownian motions. In particular, a bounded domain is universal if and only if it contains no complete properly immersed minimal surfaces.

The other interesting consequence follows from a well-known result in convex geometry. Next theorem essentially asserts that the Jordan curve Γ can be substituted by an arbitrary compact set in the statement of Theorem 2.

Theorem 4. *Let C be a regular strictly convex bounded domain, and consider a* connected compact set $K \subset \partial C$. *Then, for each $\varepsilon > 0$, there exists a complete proper minimal immersion $\varphi_{(K,\varepsilon)} : \mathbb{D} \to C$ satisfying that the Hausdorff distance $\delta^H(\varphi_{(K,\varepsilon)}(\partial\mathbb{D}), K) < \varepsilon$.*

Theorem 4 is a trivial consequence of Theorem 2 and the lemma below.

Lemma 4. *Let K be a connected compact set in ∂C, then for every $\nu > 0$ there exists an smooth Jordan curve Γ so that $\delta^H(K, \Gamma) < \nu$.*

Proof. Classical results about Hausdorff metric imply that there is a finite set $F \subset K$ such that $\delta^H(K, F) < \nu/3$. Label $K' = (K + \mathbb{B}(0, \nu/3)) \cap \partial C$, which is a connected open set in ∂C. Then, it is clear that we can find a compact simple piecewise smooth curve γ' in K' passing through all the points in F. If γ' is not closed, we can consider a curve parallel to γ', that we call γ'', and sufficiently close to γ' so that the Jordan curve γ obtained joining the extremes of γ' and γ'' satisfies $\delta^H(\gamma, K) < \nu/3$. To finish, we only need to approximate γ by a smooth Jordan curve satisfying $\delta^H(\gamma, \Gamma) < \nu/3$. □

Finally, we would like to show that our results are sharp in the following sense. If we remove from Theorem 2 the hypothesis of C being strictly convex, then the result fails. The most simple counterexample is open halfspace. If we consider a complete minimal disk properly immersed in an open halfspace then the limit set cannot be bounded, if not we will arrive to a contradiction by the maximum principle. Actually we can prove that:

Proposition 2. *Let D a domain of $\mathbb{R}^3$ satisfying that there exists an open region $U \subset \partial D$ where the mean curvature associated to the inward pointing normal is non positive. If U is a graph over a bounded convex domain of a plane, then there are no complete proper minimal disks in D whose limit set lies on U. In particular, Theorem 2 is not true for domains of this kind.*

Proof. The proof of this proposition is an easy application of the maximum principle. We proceed by contradiction. Assume there exists a complete proper minimal immersion $\psi : \mathbb{D} \to D$ so that the limit set $\psi(\partial\mathbb{D})$ is contained in U. From the hypotheses, we know that U is a graph over a plane Π. First, we translate U, orthogonally to Π, and toward the interior of D until it does not touch $\psi(\mathbb{D})$ (see Figure 5). Now continuously translate U toward $\psi(\mathbb{D})$ until the translated graph intersects $\psi(\mathbb{D})$ for the first time. This must occur because the immersion is proper. As the limit set is contained in U, then $\psi(\mathbb{D})$ is contained in the interior of the convex cylinder over the projection of U in Π. In particular the boundary of the graph never touches $\psi(\mathbb{D})$, then the translated graph U_0 and $\psi(\mathbb{D})$ have an interior contact point, and so the maximum principle leads to $U_0 = \psi(\mathbb{D})$ which is absurd. This contradiction proves the proposition. □

On the other hand, regularity is also necessary. Indeed, Martín, Meeks and Nadirashvili [4] have recently proved this result.

Theorem 5 ([4]). *Let $\mathcal{D}$ be any bounded open domain in $\mathbb{R}^3$. Then there exists a proper countable collection $\mathcal{F}$ of pairwise disjoint horizontal simple closed curves*

in $\mathcal{D}$ such that the complementary domain $\widetilde{\mathcal{D}} = \mathcal{D} - \mathcal{F}$ is universal for minimal surfaces with at least one annular end. In particular, any complete immersed minimal surface of finite genus in $\widetilde{\mathcal{D}}$ must have an uncountable number of ends.

Among other things, this means that Theorem 3 is sharp, too.

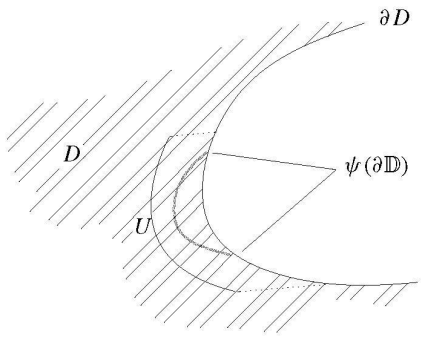

Figure 5. We translate U in the direction of the interior of D until it does not touch $\psi(\mathbb{D})$.

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