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Cohomogeneity one hypersurfaces of Euclidean Spaces

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Abstract. We study isometric immersions $f: M^n \rightarrow \mathbb{R}^{n+1}$ into Euclidean space of dimension $n+1$ of a complete Riemannian manifold of dimension n on which a compact connected group of intrinsic isometries acts with principal orbits of codimension one. We give a complete classification if either $n \geq 3$ and M^n is compact or if $n \geq 5$ and the connected components of the flat part of M^n are bounded. We also provide several sufficient conditions for f to be a hypersurface of revolution.

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1. Introduction

An interesting problem in submanifold theory is to study isometric immersions $f: M^n \rightarrow \mathbb{R}^N$ into Euclidean space of a connected complete Riemannian manifold of dimension n acted on by a closed connected subgroup of its isometry group $\text{Iso}(M^n)$. This study was initiated by Kobayashi [8], who proved that if $N = n+1$ and M^n is compact and *homogeneous*, i.e., $\text{Iso}(M^n)$ acts transitively on M^n , then $f(M^n)$ must be a round sphere.

In this paper we consider isometric immersions $f: M^n \rightarrow \mathbb{R}^{n+1}$ of a complete Riemannian manifold M^n on which a *compact, connected* subgroup G of $\text{Iso}(M^n)$ acts with maximal dimensional orbits of codimension one. We call f a *hypersurface of G -cohomogeneity one*. Observe that the group G may not be realizable as a group of extrinsic isometries of the ambient space. For instance, consider the cohomogeneity one action of $\text{SO}(n)$ on \mathbb{R}^n and isometrically immerse \mathbb{R}^n into \mathbb{R}^{n+1} as a cylinder over a plane curve. However, such examples can only arise if f is not rigid. Recall that f is *rigid* if any other isometric immersion $\tilde{f}: M^n \rightarrow \mathbb{R}^{n+1}$ differs from f by an isometry of \mathbb{R}^{n+1} .

Examples of cohomogeneity one hypersurfaces may be obtained as follows. Start with a cohomogeneity two compact subgroup $G \subset \text{SO}(n+1)$, so that the orbit space \mathbb{R}^{n+1}/G is a two dimensional manifold, possibly with boundary. Now consider

a curve that is either contained in the interior of \mathbb{R}^{n+1}/G or meets its boundary orthogonally. Then the inverse image of such a curve by the canonical projection onto the orbit space is a cohomogeneity one hypersurface. We shall call these examples the *standard examples*. Among them, the simplest ones are the *hypersurfaces of revolution*, which are invariant by the action of $\mathrm{SO}_l(n+1)$, the subgroup of $\mathrm{SO}(n+1)$ that fixes a straight line l .

Our main result states that, under natural global assumptions, the standard examples comprise all cohomogeneity one hypersurfaces.

Theorem 1.1. *Let $f: M^n \rightarrow \mathbb{R}^{n+1}$ be a complete hypersurface of G -cohomogeneity one. Assume either that $n \geq 3$ and M^n is compact or that $n \geq 5$ and the connected components of the flat part of M^n are bounded. Then f is either rigid or a hypersurface of revolution. In particular, f is a standard example.*

We also provide several sufficient conditions for a hypersurface of G -cohomogeneity one as in Theorem 1.1 to be a hypersurface of revolution.

Theorem 1.2. *Under the assumptions of Theorem 1.1, any of the following conditions implies that f is a hypersurface of revolution:*

- (i) *there exists a principal orbit with positive curvature;*
- (ii) *there exists a principal orbit that is totally geodesic in M^n ;*
- (iii) *the principal orbits are umbilical in M^n ;*
- (iv) *$n \neq 4$ and there exists a principal orbit that is homeomorphic to a sphere.*

Moreover, in this case G is isomorphic to one of the closed subgroups of $\mathrm{SO}(n)$ that act transitively on S^{n-1} .

Theorem 1.2 generalizes and gives new (and shorter) proofs of various known results. Namely, it was proved under condition (iii) in [12] in the compact case for $n \geq 4$ and later in [9] in the general case (even for $n = 3, 4$). It was also proved in [4] (resp., [2]) in the compact case for $n \geq 5$ (resp., $n \geq 4$) under the assumption that all orbits have positive (resp., constant) sectional curvature. We also point out that closed subgroups of $\mathrm{SO}(n)$ that act transitively on the sphere are completely classified (cf. [7], p. 392).

2. The proofs

Given an isometric immersion $f: M^n \rightarrow \mathbb{R}^{n+1}$, let A_{ξ_p} denote the shape operator of f at $p \in M^n$ with respect to a normal vector $\xi_p \in T_p^\perp M^n$, that is, the symmetric endomorphism of $T_p M^n$ given by $A_{\xi_p} X = -\tilde{\nabla}_X \xi$ for any $X \in T_p M^n$, where ξ is

a smooth local normal vector field extending ξ_p and $\tilde{\nabla}$ stands for the derivative of \mathbb{R}^{n+1} . Recall that the *relative nullity subspace* of f at $p \in M^n$ is the kernel of A_{ξ_p} . It is well-known that on any open subset of M^n where the relative nullity subspaces of f have constant positive dimension, they define a smooth distribution whose leaves (called the *leaves of relative nullity*) are mapped by f onto open subsets of affine subspaces of \mathbb{R}^{n+1} .

Our approach to the study of hypersurfaces of cohomogeneity one is based on the following variant due to Ferus of a rigidity theorem of Sacksteder [14].

Theorem 2.1. *Let $f, \tilde{f}: M^n \rightarrow \mathbb{R}^{n+1}$ be isometric immersions of a complete Riemannian manifold of dimension $n \geq 3$. If there exists no complete leaf of relative nullity of dimension $n-1$ or $n-2$ (in particular if M^n is compact), then the shape operators of f and \tilde{f} satisfy $A(p) = \pm \tilde{A}(p)$ for every $p \in M^n$. As a consequence, if the subset of totally geodesic points of f does not disconnect M^n then f is rigid.*

The relation between the shape operators of f and \tilde{f} in the statement means, more precisely, that $\tilde{A}_{\psi(\xi_p)} = \pm A_{\xi_p}$ for any $p \in M^n$ and for any $\xi_p \in T_p^\perp M_f^n$, where $\psi: T^\perp M_f^n \rightarrow T^\perp M_{\tilde{f}}^n$ is one of the two vector bundle isometries between the normal bundles of f and \tilde{f} .

By means of Theorem 2.1 we now derive the following result for hypersurfaces of G -cohomogeneity one, which is the main tool for the proofs of Theorems 1.1 and 1.2. We refer to [1] and the references therein for the basic facts on cohomogeneity one manifolds that are used in the sequel.

Proposition 2.2. *Let $f: M^n \rightarrow \mathbb{R}^{n+1}$ be a complete hypersurface of G -cohomogeneity one. If either f is rigid or there exists no complete leaf of relative nullity of f of dimension $n-1$ or $n-2$ (in particular if M^n is compact), then*

- (i) B , the set of totally geodesic points of f , is G -invariant.
- (ii) There exists a Lie group homomorphism $\Psi: G \rightarrow \text{SO}(n+1)$ such that $f \circ g = \Psi(g) \circ f$ for every $g \in G$, that is, f is G -equivariant.
- (iii) If Σ is a principal orbit of G , then $f(\Sigma)$ is a principal orbit of the action of $\tilde{G} = \Psi(G)$ on \mathbb{R}^{n+1} . In particular, $f(\Sigma)$ is an isoparametric hypersurface of a sphere.
- (iv) If $f(\Sigma)$ is a round sphere for some principal orbit Σ of G , then f is a hypersurface of revolution and Ψ is a monomorphism. In particular, G is isomorphic to one of the closed subgroups of $\text{SO}(n)$ that act transitively on S^{n-1} .

Proof. Given $g \in G$, let A^g denote the shape operator of $f \circ g$. If f is rigid then $A^g = A$ for every $g \in G$. We claim that this is also the case if there exists no complete

leaf of relative nullity of f of dimension $n - 1$ or $n - 2$. In fact, on one hand we have

$$g_*(p) \circ A^g(p) = A(g(p)) \circ g_*(p) \text{ for each } p \in M. \quad (1)$$

This implies that for each fixed $p \in M^n$ the map $\phi_p: G \rightarrow \text{End}(T_p M^n)$ given by

$$\phi_p(g) = A^g(p) = (g_*(p))^{-1} \circ A(g(p)) \circ g_*(p)$$

is continuous. On the other hand, it follows from Theorem 2.1 that for each $p \in M^n$ either $A^g(p) = A(p)$ or $A^g(p) = -A(p)$. We obtain that ϕ_p is a continuous map taking values in $\{A(p), -A(p)\}$. Since G is connected and $\phi_p(I) = A(p)$, our claim follows.

In particular, the set B^g of totally geodesic points of $f \circ g$ coincides with B for every $g \in G$. In view of (1), this is equivalent to saying that B is G -invariant. Moreover, by the Fundamental Theorem of Hypersurfaces, for each $g \in G$ there exists $\tilde{g} \in \text{Iso}(\mathbb{R}^{n+1})$ such that $f \circ g = \tilde{g} \circ f$. It now follows from standard arguments (cf. [12]) that $\Psi: G \rightarrow \text{Iso} \mathbb{R}^{n+1}$, $\Psi(g) = \tilde{g}$, is a Lie-group homomorphism whose image lies in (a conjugacy class of) $\text{SO}(n+1)$, because it is compact (and hence has a fixed point) and connected. Assertion (iii) now follows from (ii).

Finally, if $f(\Sigma)$ is a round sphere for some principal orbit Σ of G then, since G is connected, it must fix the line ℓ orthogonal to the linear span of $f(\Sigma)$. Hence f is a hypersurface of revolution with ℓ as axis. Moreover, the restriction of f to Σ must be injective. Since $f \circ g = \Psi(g) \circ f$ for any $g \in G$, if $\Psi(g) = I \in \text{SO}(n+1)$ for some $g \in G$ we obtain that $g(y) = y$ for all $y \in \Sigma$. Now, since Σ is a principal orbit, this implies that, for every $y \in \Sigma$, g_* acts trivially on the normal space at y to the inclusion of Σ into M^n . As a consequence, if $\gamma: \mathbb{R} \rightarrow M^n$ is a normal geodesic through $y \in \Sigma$, i.e., a complete geodesic that crosses Σ (and hence any other G -orbit) orthogonally, then g fixes any point of $\gamma(\mathbb{R})$. Since every point of M^n lies in a normal geodesic through a point of Σ , we obtain that $g = I \in G$, and the last assertion in (iv) follows. \square

Our next result classifies complete hypersurfaces of G -cohomogeneity one with dimension $n \geq 5$ that carry a complete leaf of relative nullity of dimension $n - 2$.

Proposition 2.3. *Let $f: M^n \rightarrow \mathbb{R}^{n+1}$, $n \geq 5$, be a complete hypersurface of G -cohomogeneity one. If there exists a complete leaf of relative nullity of dimension $n - 2$ then $M^n = S^2 \times \mathbb{R}^{n-2}$ and f splits as $f = i \times \text{id}$, where $i: S^2 \rightarrow \mathbb{R}^3$ is an umbilical inclusion and $\text{id}: \mathbb{R}^{n-2} \rightarrow \mathbb{R}^{n-2}$ is the identity map. In particular, f is rigid.*

Proof. Since M^n carries a complete leaf of relative nullity \mathcal{F} , it can not be compact. Thus the orbit space $\Omega = M^n/G$ is homeomorphic to either \mathbb{R} or $[0, \infty)$. Moreover, if $\pi: M^n \rightarrow \Omega$ denotes the canonical projection and $\gamma: \mathbb{R} \rightarrow M^n$ is a normal

geodesic parameterized by arc-length, then $\pi \circ \gamma$ maps \mathbb{R} homeomorphically onto Ω in the first case, and it is a covering map of $\mathbb{R} \setminus \{0\}$ onto the subset Ω^0 of internal points of Ω in the latter. Set $I = \gamma^{-1}(G(\mathcal{F}))$. Since $G(\mathcal{F})$ is a closed unbounded connected subset, using that $G(\mathcal{F}) = G(\gamma(I))$ it follows easily that if $I \neq \mathbb{R}$ then $I = [a, \infty)$ for some $a \in \mathbb{R}$ in the first case and $I = (-\infty, -b] \cup [a, \infty)$ for some $a, b > 0$ in the latter. Now observe that the type number of f (i.e., the rank of its shape operator) is everywhere equal to 2 on $G(\mathcal{F})$. This is because the relative nullity subspace coincides with the nullity of the curvature tensor at a point where the type number is at least 2, whence the subset where the type number is 2 is invariant under isometries. Let $(t_0 - \varepsilon, t_0 + \varepsilon) \subset I$ be such that $\Phi: (t_0 - \varepsilon, t_0 + \varepsilon) \times \Sigma_p \rightarrow \pi^{-1}((t_0 - \varepsilon, t_0 + \varepsilon))$ given by $\Phi(t, g(p)) = g(\gamma(t))$, $p = \gamma(t_0)$, is a G -equivariant diffeomorphism. We call $\Gamma = \pi^{-1}((t_0 - \varepsilon, t_0 + \varepsilon))$ a *tube* around Σ_p . We have a well-defined vector field ξ on Γ given by $\xi(y) = g_*(\gamma(t))\gamma'(t)$ for $y = g(\gamma(t))$, $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, and $\xi(y)$ is orthogonal to $\Sigma_{\gamma(t)}$ at y .

Now let η be a local unit normal vector field to f on Γ and A_η^f the shape operator of f with respect to η . Given a principal orbit $\Sigma_q = G(q) \subset \Gamma$ of G , the vector fields $\bar{\xi} = f_*(\xi|_{\Sigma_q})$ and $\bar{\eta} = \eta|_{\Sigma_q}$ determine an orthonormal normal frame of the restriction $f|_{\Sigma_q}: \Sigma_q \rightarrow \mathbb{R}^{n+1}$ of f to Σ_q . Denote by $A_{\bar{\eta}}$ and $A_{\bar{\xi}}$ the corresponding shape operators. Notice that $A_{\bar{\xi}} = A_\xi^i$, where $i: \Sigma_q \rightarrow M^n$ is the inclusion of Σ_q into M^n . Thus $A_{\bar{\xi}} \circ g_* = g_* \circ A_\xi$ for any $g \in G$, hence the eigenvalues of $A_{\bar{\xi}}$ are constant. On the other hand, $A_{\bar{\eta}} = \Pi \circ A_\eta$, where Π is the orthogonal projection of $T M^n$ onto $T \Sigma_q$. In particular, $\text{rank } A_{\bar{\eta}} \leq \text{rank } A_\eta$, so we have $\text{rank } A_{\bar{\eta}} \leq 2$ on Σ_q . We have two cases to consider:

- (i) $\text{rank } A_{\bar{\eta}} \leq 1$ on each principal orbit contained in Γ ;
- (ii) $\text{rank } A_{\bar{\eta}} = 2$ on some principal orbit contained in Γ .

First we show that (i) can not occur. Assume otherwise. Then, it follows from Theorem 2 of [4] that the principal orbits in Γ are either isometric to Euclidean spheres or isometrically covered by Riemannian products $\mathbb{R} \times S^{n-2}(a)$ (in what follows we suppose $a = 1$). In the former case, for each principal orbit $\Sigma_q \subset \Gamma$ it follows from the Gauss equation of the restriction $f|_{\Sigma_q}: \Sigma_q \rightarrow \mathbb{R}^{n+1}$ that $A_{\bar{\xi}}$ must be a multiple of the identity tensor, that is, the principal orbits in Γ are umbilical in M^n . This is in contradiction with Lemma 2.8 of [9], taking into account that $n \geq 5$ and that f has type number 2 on Γ .

Suppose now that the principal orbits are covered by $\mathbb{R} \times S^{n-2}$. In this case, for any fixed principal orbit $\Sigma_q \subset \Gamma$ there must exist an open subset $U_0 \subset \Sigma_q$ where $\text{rank } A_{\bar{\eta}} = 1$. In fact, otherwise $A_{\bar{\eta}}$ vanishes identically, hence the first normal spaces of $f|_{\Sigma_q}$ (i.e., the subspaces of the normal spaces spanned by the image of the second fundamental form) have dimension one everywhere (notice that $A_{\bar{\xi}}$ can not vanish anywhere, otherwise it would be identically zero and $f|_{\Sigma_q}$ would be totally geodesic,

which is impossible). Then either $f(\Sigma_q)$ is contained in an affine hyperplane \mathcal{H} of \mathbb{R}^{n+1} or the first normal spaces of $f|_{\Sigma_q}$ are nonparallel along an open subset of Σ_q . Both possibilities lead to contradictions: the latter forces Σ_q to be flat (cf. [6], Theorem 1); in the former, since the shape operator of the isometric immersion $f: \Sigma_q \rightarrow \mathcal{H}$ is $A_{\bar{\xi}}$, which has constant eigenvalues, it follows that $f(\Sigma_q)$ is a round sphere, which is again impossible. We obtain that there exists an open subset $U \subset \Gamma$ where $\text{rank } A_{\bar{\eta}} = 1$ and $U \cap \Sigma_q = U_0$. Since the images of $A_{\bar{\eta}}$ and A_{η} are related by $\text{Im}(A_{\bar{\eta}}) = \Pi(\text{Im}(A_{\eta}))$, and on U the dimensions of $\text{Im}(A_{\bar{\eta}})$ and $\text{Im}(A_{\eta})$ are 1 and 2, respectively, we must have $\xi \in \text{Im}(A_{\eta})$ everywhere on U . Therefore, at any point $x \in U$ we have that $\ker A_{\eta}(x) \subset T_x \Sigma_x$, and hence $\ker A_{\eta}(x) = \ker A_{\bar{\eta}}(x)$. It follows that the leaves of the distribution on U_0 given by $\ker A_{\bar{\eta}}$ are totally geodesic in Σ_q and \mathbb{R}^{n+1} . In particular, they are flat hypersurfaces of Σ_q . This is in contradiction with the fact that Σ_q is locally isometric to $\mathbb{R} \times S^{n-2}$. In fact, for any $x \in U_0$ let W be an $(n-2)$ -dimensional subspace of $T_x(\Sigma_q)$ where the sectional curvatures of Σ_q are equal to 1 and let F_x be the totally geodesic flat hypersurface through x . Then $S = W \cap T_x(F_x)$ has dimension at least 2, since $n \geq 5$. At each bidimensional subspace of S , the sectional curvature of Σ_q is 1, because $S \subset W$ and, on the other hand, such a curvature must be zero, for $S \subset T_x(F_x)$. Therefore (i) is not possible, and we are left with (ii).

If $\text{rank } A_{\bar{\eta}} = 2$ along a principal orbit $\Sigma_q \subset \Gamma$, then $\text{rank } A_{\bar{\eta}} = 2$ on a possibly smaller tube around Σ_q contained in Γ , which we still denote by Γ . By Theorem 3 in [4], each principal orbit Σ_x contained in Γ is isometric to a Riemannian product $S^2(a) \times S^{n-3}(b)$ of spheres and $f|_{\Sigma_x}: \Sigma_x \rightarrow \mathbb{R}^{n+1}$ splits as a product $f|_{\Sigma_x} = i_1 \times i_2: S^2(a) \times S^{n-3}(b) \rightarrow \mathbb{R}^3 \times \mathbb{R}^{n-2} = \mathbb{R}^{n+1}$, where $i_1: S^2(a) \rightarrow \mathbb{R}^3$ and $i_2: S^{n-3}(b) \rightarrow \mathbb{R}^{n-2}$ are umbilical inclusions. Moreover, $\{\bar{\eta}, \bar{\xi}\}$ is precisely the orthonormal normal frame of $f|_{\Sigma_x}$ determined by the unit normal vector fields to the inclusions i_1 and i_2 , respectively. In particular, $\bar{\xi}$ and $\bar{\eta}$ are parallel with respect to the normal connection of $f|_{\Sigma_x}$. Hence, $A_{\bar{\eta}}$ coincides with the restriction of A_{η} to $T\Sigma_x$, which in turn implies that ξ is an eigenvector of A_{η} along Γ . Now, since $\text{rank } A_{\eta} = \text{rank } A_{\bar{\eta}} = 2$ on Γ , it follows that $\xi \in \ker A_{\eta}$. Therefore, the segments of normal geodesics in Γ are contained in the leaves of $\ker A_{\eta}$. Since these are assumed to be complete, we obtain that f has type number 2 on the whole M^n and that A_{η} is everywhere of the form $A_{\eta} = \text{diag}(\varphi, \varphi, 0, \dots, 0)$, where φ is nonzero and constant along each principal orbit and the φ -eigenspaces of A_{η} (or $A_{\bar{\eta}}$) coincide with $\ker A_{\bar{\xi}} = \ker A_{\xi}^{\perp}$. Now, let X be a vector field such that $A_{\eta}(X) = \varphi X$. By the Codazzi equation

$$\nabla_X(A_{\eta}(\xi)) - A_{\eta}(\nabla_X \xi) = \nabla_{\xi}(A_{\eta}(X)) - A_{\eta}(\nabla_{\xi} X)$$

we get

$$-A_{\eta}(\nabla_X \xi) = \xi(\varphi)X + \varphi \nabla_{\xi} X - A_{\eta}(\nabla_{\xi} X) = \xi(\varphi)X,$$

where the last equality follows from $\nabla_{\xi} X \in (\ker A_{\eta})^{\perp} = \ker(A_{\eta} - \varphi I)$, using that $\ker A_{\eta}$ is totally geodesic. Since $\nabla_X \xi = -A_{\xi}^i(X) = -A_{\bar{\xi}}(X) = 0$, it follows that $\xi(\varphi) = 0$. Therefore φ is a constant, which we may suppose to be 1. Standard arguments now show that M^n splits as $M^n = S^2 \times \mathbb{R}^{n-2}$ (cf. [13]). By the main lemma in [10], f also splits as stated. \square

Proof of Theorem 1.1. Suppose that f is not rigid. If M^n is compact and $n \geq 3$, it follows from Theorem 2.1 that B , the set of totally geodesic points of f , disconnects M^n . In order to get the same conclusion in the non-compact case, we must show that there does not exist a complete leaf of relative nullity of f of dimension $\ell = n - 1$ or $\ell = n - 2$. For $\ell = n - 1$ this follows from our assumption on the flat part of M^n . Proposition 2.3 takes care of the case $\ell = n - 2$.

Since B disconnects M^n , it must contain a regular point p . Then the (principal) orbit Σ through p is contained in B , because B is G -invariant by Proposition 2.2 (i). It follows from Lemma 3.14 of [5] that $f(\Sigma)$ is contained in a hyperplane \mathcal{H} which is tangent to f along Σ , for Σ is connected. But $f(\Sigma)$ is an isoparametric hypersurface of a sphere by Proposition 2.2 (iii), and hence $f(\Sigma)$ must be a round hypersphere of \mathcal{H} . Proposition 2.2 (iv) now completes the proof. \square

Remark 2.4. In case M^n is complete non-compact of dimension $n \geq 5$, the arguments in the beginning of the proof of Proposition 2.3 show, more precisely, that the conclusion of Theorem 1.1 fails only when every point of $G(\gamma(I))$ is flat, where $\gamma: \mathbb{R} \rightarrow M^n$ is a normal geodesic parameterized by arc-length and I is either $[a, \infty)$ for some $a \in \mathbb{R}$ or $(-\infty, -b] \cup [a, \infty)$ for some $a, b > 0$, according to the orbit space being homeomorphic to \mathbb{R} or $[0, \infty)$, respectively. Notice that in the latter case M^n is flat outside a compact subset. We also point out that, since G is assumed to be compact, our assumption on the flat part of M^n is equivalent to M^n being *unflat at infinity* in the sense of [9].

Proof of Theorem 1.2. We already know from Theorem 1.1 that f is either rigid or a hypersurface of revolution. Thus, by Proposition 2.2 (iv) it suffices to prove that any of the conditions in the statement implies that $f(\Sigma)$ is a round sphere for some principal orbit Σ of G .

Assume first that Σ is a positively curved principal orbit. By Proposition 2.2 (iii), f immerses Σ as a positively curved isoparametric hypersurface of some hypersphere of \mathbb{R}^{n+1} . It follows easily from the Cartan identities for isoparametric hypersurfaces of the sphere (cf. [4], Corollary 2) that $f(\Sigma)$ is a round sphere.

As for condition (ii), if Σ is a totally geodesic principal orbit, then it is immersed by f as an isoparametric hypersurface of a sphere S^n whose first normal spaces in \mathbb{R}^{n+1} are one-dimensional. This can only happen if it is umbilical in S^n , and hence again a round hypersphere of S^n .

Now assume that (iii) holds. First notice that the position vector of f can not be tangent to f along $f(\Sigma)$ for every principal orbit Σ of G , otherwise f would be a cone over an isoparametric hypersurface of the sphere, in contradiction with the completeness of M^n . Now, if Σ is a principal orbit along which the position vector is nowhere tangent to f , then the normal bundle of the restriction $f|_\Sigma: \Sigma \rightarrow \mathbb{R}^{n+1}$ is spanned by the position vector and by $f_*\xi$, where ξ is a unit normal vector field to the inclusion of Σ into M^n . Since the shape operators of $f|_\Sigma$ with respect to both vector fields are multiples of the identity tensor, it follows that $f|_\Sigma$ is umbilical, and we obtain again that $f(\Sigma)$ is a round sphere.

Finally, under condition (iv) the conclusion is a consequence of the following result. \square

Proposition 2.5. *Let $P^n \subset S^{n+1}$, $n \geq 4$, be an isoparametric hypersurface. If the universal covering of P^n is (homeomorphic to) S^n , then P^n is isometric to a Euclidean sphere.*

Proof. Let $\lambda_1, \lambda_2, \dots, \lambda_g$ be the distinct (and constant) principal curvatures of P^n . Let m_1 be the common multiplicity of the λ_k , when k is odd, and let m_2 be the common multiplicity of the λ_k , when k is even. Denote by $\beta_0, \beta_1, \beta_2, \dots, \beta_n$ the \mathbb{Z}_2 -Betti numbers of P^n . Then we have (cf. [11]):

- (i) [F. Münzner] $g \in \{1, 2, 3, 4, 6\}$;
- (ii) $2n = g(m_1 + m_2)$;
- (iii) [E. Cartan] If $g = 3$, then $m_1 = m_2 \in \{1, 2, 4, 8\}$;
- (iv) [U. Abresch] If $g = 6$, then $m_1 = m_2 \in \{1, 2\}$;
- (v) [F. Münzner] $\sum_{i=0}^n \beta_i = 2g$.

Suppose first that n , the dimension of P^n , is odd. Then (ii) and (iv) imply that $g \in \{1, 2, 3\}$. Since $n \geq 4$, it follows from (iii) that $g \in \{1, 2\}$. If $g = 2$, then P^n is a Riemannian product of spheres and thus it cannot be covered by a sphere. Hence we must have $g = 1$ and this implies that P^n is a Euclidean sphere.

Let now n be even, say $n = 2q$. Then the Euler characteristics of S^{2q} and P^n are related by $\chi(S^{2q}) = m\chi(P^n)$, where m is the number of sheets of the covering. Thus, either $m = 1$ or $m = 2$, since $\chi(S^{2q}) = 2$. Suppose $m = 2$. Then $\chi(P^n) = \sum_{i=0}^n (-1)^i \beta_i = 1$, which implies, using Poincaré duality, that the Betti number β_q is odd. On the other hand, we get from (v) that β_q must be even. This contradiction tells us that $m = 1$ and, again using (v), we obtain that $g = 1$. Therefore P^n is a Euclidean sphere. \square

Remark 2.6. Proposition 2.5 is no longer true for $n = 3$, as shown by Cartan isoparametric hypersurfaces of S^4 with three distinct principal curvatures [3], which are diffeomorphic to S^3/Q , where Q stands for the quaternion 8-group.

References

- [1] A. V. Alekseevsky, D. V. Alekseevsky, Riemannian G -manifold with one dimensional orbit space. *Ann. Global Anal. Geom.* **11** (1993), 197–211. Zbl 0810.53043 MR 1237453
- [2] A. C. Asperti, F. Mercuri, M. H. Noronha, Cohomogeneity one manifolds and hypersurfaces of revolution. *Boll. Un. Mat. Ital. B* **11** (1997), 199–215. Zbl 0882.53006 MR 1456261
- [3] E. Cartan, Familles de surfaces isoparamétriques dans les espaces à courbure constante. *Ann. Mat. Pura Appl.* **17** (1938), 177–191. Zbl 0020.06505
- [4] H. P. Castro, M. H. Noronha, Homogeneous submanifolds of codimension two. *Geom. Dedicata* **78** (1999), 89–110. Zbl 0935.53023 MR 1722173
- [5] M. Dajczer, D. Gromoll, Rigidity of complete Euclidean hypersurfaces. *J. Differential Geom.* **31** (1990), 401–416. Zbl 0667.53003 MR 1037409
- [6] M. Dajczer, R. Tojeiro, Submanifolds with nonparallel first normal bundle. *Canad. Math. Bull.* **37** (1994), 330–337. Zbl 0812.53015 MR 1289768
- [7] J.-H. Eschenburg, E. Heintze, On the classification of polar representations. *Math. Z.* **232** (1999), 391–398. Zbl 0943.22014 MR 1719714
- [8] S. Kobayashi, Compact homogeneous hypersurfaces. *Trans. Amer. Math. Soc.* **88** (1958), 137–143. Zbl 0081.38102 MR 0096284
- [9] F. Mercuri, J. A. P. Seixas, Hypersurfaces of cohomogeneity one and hypersurfaces of revolution. *Differential Geom. Appl.* **20** (2004), 255–239. Zbl 1046.53043 MR 2038557
- [10] J. D. Moore, Isometric immersions of Riemannian products. *J. Differential Geom.* **5** (1971), 159–168. Zbl 0213.23804 MR 0307128
- [11] R. Palais, C.-L. Terng, *Critical Point Theory and Submanifold Geometry*. Lecture Notes in Math. 1353, Springer-Verlag, Berlin 1988. Zbl 0658.49001 MR 0972503
- [12] F. Podestà, A. Spiro, Cohomogeneity one manifolds and hypersurfaces of the Euclidean space. *Ann. Global Anal. Geom.* **13** (1995), 169–184. Zbl 0827.53007 MR 1336212
- [13] P. Ryan, Homogeneity and some curvature conditions for hypersurfaces. *Tôhoku Math. J.* **21** (1969), 363–388. Zbl 0185.49904 MR 0253243
- [14] R. Sacksteder, The rigidity of hypersurfaces. *J. Math. Mech.* **11** (1962), 929–939. Zbl 0108.34702 MR 0144286

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