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## Cohomogeneity one hypersurfaces of Euclidean Spaces

Francesco Mercuri, Fabio Podestà, José A. P. Seixas and Ruy Tojeiro

**Abstract.** We study isometric immersions  $f: M^n \rightarrow \mathbb{R}^{n+1}$  into Euclidean space of dimension  $n+1$  of a complete Riemannian manifold of dimension  $n$  on which a compact connected group of intrinsic isometries acts with principal orbits of codimension one. We give a complete classification if either  $n \geq 3$  and  $M^n$  is compact or if  $n \geq 5$  and the connected components of the flat part of  $M^n$  are bounded. We also provide several sufficient conditions for  $f$  to be a hypersurface of revolution.

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**Keywords.** Cohomogeneity one manifolds, hypersurfaces.

### 1. Introduction

An interesting problem in submanifold theory is to study isometric immersions  $f: M^n \rightarrow \mathbb{R}^N$  into Euclidean space of a connected complete Riemannian manifold of dimension  $n$  acted on by a closed connected subgroup of its isometry group  $\text{Iso}(M^n)$ . This study was initiated by Kobayashi [8], who proved that if  $N = n+1$  and  $M^n$  is compact and *homogeneous*, i.e.,  $\text{Iso}(M^n)$  acts transitively on  $M^n$ , then  $f(M^n)$  must be a round sphere.

In this paper we consider isometric immersions  $f: M^n \rightarrow \mathbb{R}^{n+1}$  of a complete Riemannian manifold  $M^n$  on which a *compact, connected* subgroup  $G$  of  $\text{Iso}(M^n)$  acts with maximal dimensional orbits of codimension one. We call  $f$  a *hypersurface of  $G$ -cohomogeneity one*. Observe that the group  $G$  may not be realizable as a group of extrinsic isometries of the ambient space. For instance, consider the cohomogeneity one action of  $\text{SO}(n)$  on  $\mathbb{R}^n$  and isometrically immerse  $\mathbb{R}^n$  into  $\mathbb{R}^{n+1}$  as a cylinder over a plane curve. However, such examples can only arise if  $f$  is not rigid. Recall that  $f$  is *rigid* if any other isometric immersion  $\tilde{f}: M^n \rightarrow \mathbb{R}^{n+1}$  differs from  $f$  by an isometry of  $\mathbb{R}^{n+1}$ .

Examples of cohomogeneity one hypersurfaces may be obtained as follows. Start with a cohomogeneity two compact subgroup  $G \subset \text{SO}(n+1)$ , so that the orbit space  $\mathbb{R}^{n+1}/G$  is a two dimensional manifold, possibly with boundary. Now consider

a curve that is either contained in the interior of  $\mathbb{R}^{n+1}/G$  or meets its boundary orthogonally. Then the inverse image of such a curve by the canonical projection onto the orbit space is a cohomogeneity one hypersurface. We shall call these examples the *standard examples*. Among them, the simplest ones are the *hypersurfaces of revolution*, which are invariant by the action of  $\mathrm{SO}_l(n+1)$ , the subgroup of  $\mathrm{SO}(n+1)$  that fixes a straight line  $l$ .

Our main result states that, under natural global assumptions, the standard examples comprise all cohomogeneity one hypersurfaces.

**Theorem 1.1.** *Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$  be a complete hypersurface of  $G$ -cohomogeneity one. Assume either that  $n \geq 3$  and  $M^n$  is compact or that  $n \geq 5$  and the connected components of the flat part of  $M^n$  are bounded. Then  $f$  is either rigid or a hypersurface of revolution. In particular,  $f$  is a standard example.*

We also provide several sufficient conditions for a hypersurface of  $G$ -cohomogeneity one as in Theorem 1.1 to be a hypersurface of revolution.

**Theorem 1.2.** *Under the assumptions of Theorem 1.1, any of the following conditions implies that  $f$  is a hypersurface of revolution:*

- (i) *there exists a principal orbit with positive curvature;*
- (ii) *there exists a principal orbit that is totally geodesic in  $M^n$ ;*
- (iii) *the principal orbits are umbilical in  $M^n$ ;*
- (iv)  *$n \neq 4$  and there exists a principal orbit that is homeomorphic to a sphere.*

Moreover, in this case  $G$  is isomorphic to one of the closed subgroups of  $\mathrm{SO}(n)$  that act transitively on  $S^{n-1}$ .

Theorem 1.2 generalizes and gives new (and shorter) proofs of various known results. Namely, it was proved under condition (iii) in [12] in the compact case for  $n \geq 4$  and later in [9] in the general case (even for  $n = 3, 4$ ). It was also proved in [4] (resp., [2]) in the compact case for  $n \geq 5$  (resp.,  $n \geq 4$ ) under the assumption that *all* orbits have positive (resp., constant) sectional curvature. We also point out that closed subgroups of  $\mathrm{SO}(n)$  that act transitively on the sphere are completely classified (cf. [7], p. 392).

## 2. The proofs

Given an isometric immersion  $f: M^n \rightarrow \mathbb{R}^{n+1}$ , let  $A_{\xi_p}$  denote the shape operator of  $f$  at  $p \in M^n$  with respect to a normal vector  $\xi_p \in T_p^\perp M^n$ , that is, the symmetric endomorphism of  $T_p M^n$  given by  $A_{\xi_p} X = -\tilde{\nabla}_X \xi$  for any  $X \in T_p M^n$ , where  $\xi$  is

a smooth local normal vector field extending  $\xi_p$  and  $\tilde{V}$  stands for the derivative of  $\mathbb{R}^{n+1}$ . Recall that the *relative nullity subspace* of  $f$  at  $p \in M^n$  is the kernel of  $A_{\xi_p}$ . It is well-known that on any open subset of  $M^n$  where the relative nullity subspaces of  $f$  have constant positive dimension, they define a smooth distribution whose leaves (called the *leaves of relative nullity*) are mapped by  $f$  onto open subsets of affine subspaces of  $\mathbb{R}^{n+1}$ .

Our approach to the study of hypersurfaces of cohomogeneity one is based on the following variant due to Ferus of a rigidity theorem of Sacksteder [14].

**Theorem 2.1.** *Let  $f, \tilde{f}: M^n \rightarrow \mathbb{R}^{n+1}$  be isometric immersions of a complete Riemannian manifold of dimension  $n \geq 3$ . If there exists no complete leaf of relative nullity of dimension  $n-1$  or  $n-2$  (in particular if  $M^n$  is compact), then the shape operators of  $f$  and  $\tilde{f}$  satisfy  $A(p) = \pm \tilde{A}(p)$  for every  $p \in M^n$ . As a consequence, if the subset of totally geodesic points of  $f$  does not disconnect  $M^n$  then  $f$  is rigid.*

The relation between the shape operators of  $f$  and  $\tilde{f}$  in the statement means, more precisely, that  $\tilde{A}_{\psi(\xi_p)} = \pm A_{\xi_p}$  for any  $p \in M^n$  and for any  $\xi_p \in T_p^\perp M_f^n$ , where  $\psi: T^\perp M_f^n \rightarrow T^\perp M_{\tilde{f}}^n$  is one of the two vector bundle isometries between the normal bundles of  $f$  and  $\tilde{f}$ .

By means of Theorem 2.1 we now derive the following result for hypersurfaces of  $G$ -cohomogeneity one, which is the main tool for the proofs of Theorems 1.1 and 1.2. We refer to [1] and the references therein for the basic facts on cohomogeneity one manifolds that are used in the sequel.

**Proposition 2.2.** *Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$  be a complete hypersurface of  $G$ -cohomogeneity one. If either  $f$  is rigid or there exists no complete leaf of relative nullity of  $f$  of dimension  $n-1$  or  $n-2$  (in particular if  $M^n$  is compact), then*

- (i)  *$B$ , the set of totally geodesic points of  $f$ , is  $G$ -invariant.*
- (ii) *There exists a Lie group homomorphism  $\Psi: G \rightarrow \mathrm{SO}(n+1)$  such that  $f \circ g = \Psi(g) \circ f$  for every  $g \in G$ , that is,  $f$  is  $G$ -equivariant.*
- (iii) *If  $\Sigma$  is a principal orbit of  $G$ , then  $f(\Sigma)$  is a principal orbit of the action of  $\tilde{G} = \Psi(G)$  on  $\mathbb{R}^{n+1}$ . In particular,  $f(\Sigma)$  is an isoparametric hypersurface of a sphere.*
- (iv) *If  $f(\Sigma)$  is a round sphere for some principal orbit  $\Sigma$  of  $G$ , then  $f$  is a hypersurface of revolution and  $\Psi$  is a monomorphism. In particular,  $G$  is isomorphic to one of the closed subgroups of  $\mathrm{SO}(n)$  that act transitively on  $S^{n-1}$ .*

*Proof.* Given  $g \in G$ , let  $A^g$  denote the shape operator of  $f \circ g$ . If  $f$  is rigid then  $A^g = A$  for every  $g \in G$ . We claim that this is also the case if there exists no complete

leaf of relative nullity of  $f$  of dimension  $n - 1$  or  $n - 2$ . In fact, on one hand we have

$$g_*(p) \circ A^g(p) = A(g(p)) \circ g_*(p) \text{ for each } p \in M. \quad (1)$$

This implies that for each fixed  $p \in M^n$  the map  $\phi_p: G \rightarrow \text{End}(T_p M^n)$  given by

$$\phi_p(g) = A^g(p) = (g_*(p))^{-1} \circ A(g(p)) \circ g_*(p)$$

is continuous. On the other hand, it follows from Theorem 2.1 that for each  $p \in M^n$  either  $A^g(p) = A(p)$  or  $A^g(p) = -A(p)$ . We obtain that  $\phi_p$  is a continuous map taking values in  $\{A(p), -A(p)\}$ . Since  $G$  is connected and  $\phi_p(I) = A(p)$ , our claim follows.

In particular, the set  $B^g$  of totally geodesic points of  $f \circ g$  coincides with  $B$  for every  $g \in G$ . In view of (1), this is equivalent to saying that  $B$  is  $G$ -invariant. Moreover, by the Fundamental Theorem of Hypersurfaces, for each  $g \in G$  there exists  $\tilde{g} \in \text{Iso}(\mathbb{R}^{n+1})$  such that  $f \circ g = \tilde{g} \circ f$ . It now follows from standard arguments (cf. [12]) that  $\Psi: G \rightarrow \text{Iso}(\mathbb{R}^{n+1})$ ,  $\Psi(g) = \tilde{g}$ , is a Lie-group homomorphism whose image lies in (a conjugacy class of)  $\text{SO}(n+1)$ , because it is compact (and hence has a fixed point) and connected. Assertion (iii) now follows from (ii).

Finally, if  $f(\Sigma)$  is a round sphere for some principal orbit  $\Sigma$  of  $G$  then, since  $G$  is connected, it must fix the line  $\ell$  orthogonal to the linear span of  $f(\Sigma)$ . Hence  $f$  is a hypersurface of revolution with  $\ell$  as axis. Moreover, the restriction of  $f$  to  $\Sigma$  must be injective. Since  $f \circ g = \Psi(g) \circ f$  for any  $g \in G$ , if  $\Psi(g) = I \in \text{SO}(n+1)$  for some  $g \in G$  we obtain that  $g(y) = y$  for all  $y \in \Sigma$ . Now, since  $\Sigma$  is a principal orbit, this implies that, for every  $y \in \Sigma$ ,  $g_*$  acts trivially on the normal space at  $y$  to the inclusion of  $\Sigma$  into  $M^n$ . As a consequence, if  $\gamma: \mathbb{R} \rightarrow M^n$  is a normal geodesic through  $y \in \Sigma$ , i.e., a complete geodesic that crosses  $\Sigma$  (and hence any other  $G$ -orbit) orthogonally, then  $g$  fixes any point of  $\gamma(\mathbb{R})$ . Since every point of  $M^n$  lies in a normal geodesic through a point of  $\Sigma$ , we obtain that  $g = I \in G$ , and the last assertion in (iv) follows.  $\square$

Our next result classifies complete hypersurfaces of  $G$ -cohomogeneity one with dimension  $n \geq 5$  that carry a complete leaf of relative nullity of dimension  $n - 2$ .

**Proposition 2.3.** *Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$ ,  $n \geq 5$ , be a complete hypersurface of  $G$ -cohomogeneity one. If there exists a complete leaf of relative nullity of dimension  $n - 2$  then  $M^n = S^2 \times \mathbb{R}^{n-2}$  and  $f$  splits as  $f = i \times \text{id}$ , where  $i: S^2 \rightarrow \mathbb{R}^3$  is an umbilical inclusion and  $\text{id}: \mathbb{R}^{n-2} \rightarrow \mathbb{R}^{n-2}$  is the identity map. In particular,  $f$  is rigid.*

*Proof.* Since  $M^n$  carries a complete leaf of relative nullity  $\mathcal{F}$ , it can not be compact. Thus the orbit space  $\Omega = M^n/G$  is homeomorphic to either  $\mathbb{R}$  or  $[0, \infty)$ . Moreover, if  $\pi: M^n \rightarrow \Omega$  denotes the canonical projection and  $\gamma: \mathbb{R} \rightarrow M^n$  is a normal

geodesic parameterized by arc-length, then  $\pi \circ \gamma$  maps  $\mathbb{R}$  homeomorphically onto  $\Omega$  in the first case, and it is a covering map of  $\mathbb{R} \setminus \{0\}$  onto the subset  $\Omega^0$  of internal points of  $\Omega$  in the latter. Set  $I = \gamma^{-1}(G(\mathcal{F}))$ . Since  $G(\mathcal{F})$  is a closed unbounded connected subset, using that  $G(\mathcal{F}) = G(\gamma(I))$  it follows easily that if  $I \neq \mathbb{R}$  then  $I = [a, \infty)$  for some  $a \in \mathbb{R}$  in the first case and  $I = (-\infty, -b] \cup [a, \infty)$  for some  $a, b > 0$  in the latter. Now observe that the type number of  $f$  (i.e., the rank of its shape operator) is everywhere equal to 2 on  $G(\mathcal{F})$ . This is because the relative nullity subspace coincides with the nullity of the curvature tensor at a point where the type number is at least 2, whence the subset where the type number is 2 is invariant under isometries. Let  $(t_0 - \varepsilon, t_0 + \varepsilon) \subset I$  be such that  $\Phi: (t_0 - \varepsilon, t_0 + \varepsilon) \times \Sigma_p \rightarrow \pi^{-1}((t_0 - \varepsilon, t_0 + \varepsilon))$  given by  $\Phi(t, g(p)) = g(\gamma(t))$ ,  $p = \gamma(t_0)$ , is a  $G$ -equivariant diffeomorphism. We call  $\Gamma = \pi^{-1}((t_0 - \varepsilon, t_0 + \varepsilon))$  a *tube* around  $\Sigma_p$ . We have a well-defined vector field  $\xi$  on  $\Gamma$  given by  $\xi(y) = g_*(\gamma(t))\gamma'(t)$  for  $y = g(\gamma(t))$ ,  $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ , and  $\xi(y)$  is orthogonal to  $\Sigma_{\gamma(t)}$  at  $y$ .

Now let  $\eta$  be a local unit normal vector field to  $f$  on  $\Gamma$  and  $A_\eta^f$  the shape operator of  $f$  with respect to  $\eta$ . Given a principal orbit  $\Sigma_q = G(q) \subset \Gamma$  of  $G$ , the vector fields  $\bar{\xi} = f_*(\xi|_{\Sigma_q})$  and  $\bar{\eta} = \eta|_{\Sigma_q}$  determine an orthonormal normal frame of the restriction  $f|_{\Sigma_q}: \Sigma_q \rightarrow \mathbb{R}^{n+1}$  of  $f$  to  $\Sigma_q$ . Denote by  $A_{\bar{\eta}}$  and  $A_{\bar{\xi}}$  the corresponding shape operators. Notice that  $A_{\bar{\xi}} = A_\xi^i$ , where  $i: \Sigma_q \rightarrow M^n$  is the inclusion of  $\Sigma_q$  into  $M^n$ . Thus  $A_{\bar{\xi}} \circ g_* = g_* \circ A_\xi$  for any  $g \in G$ , hence the eigenvalues of  $A_{\bar{\xi}}$  are constant. On the other hand,  $A_{\bar{\eta}} = \Pi \circ A_\eta$ , where  $\Pi$  is the orthogonal projection of  $TM^n$  onto  $T\Sigma_q$ . In particular,  $\text{rank } A_{\bar{\eta}} \leq \text{rank } A_\eta$ , so we have  $\text{rank } A_{\bar{\eta}} \leq 2$  on  $\Sigma_q$ . We have two cases to consider:

- (i)  $\text{rank } A_{\bar{\eta}} \leq 1$  on each principal orbit contained in  $\Gamma$ ;
- (ii)  $\text{rank } A_{\bar{\eta}} = 2$  on some principal orbit contained in  $\Gamma$ .

First we show that (i) can not occur. Assume otherwise. Then, it follows from Theorem 2 of [4] that the principal orbits in  $\Gamma$  are either isometric to Euclidean spheres or isometrically covered by Riemannian products  $\mathbb{R} \times S^{n-2}(a)$  (in what follows we suppose  $a = 1$ ). In the former case, for each principal orbit  $\Sigma_q \subset \Gamma$  it follows from the Gauss equation of the restriction  $f|_{\Sigma_q}: \Sigma_q \rightarrow \mathbb{R}^{n+1}$  that  $A_{\bar{\xi}}$  must be a multiple of the identity tensor, that is, the principal orbits in  $\Gamma$  are umbilical in  $M^n$ . This is in contradiction with Lemma 2.8 of [9], taking into account that  $n \geq 5$  and that  $f$  has type number 2 on  $\Gamma$ .

Suppose now that the principal orbits are covered by  $\mathbb{R} \times S^{n-2}$ . In this case, for any fixed principal orbit  $\Sigma_q \subset \Gamma$  there must exist an open subset  $U_0 \subset \Sigma_q$  where  $\text{rank } A_{\bar{\eta}} = 1$ . In fact, otherwise  $A_{\bar{\eta}}$  vanishes identically, hence the first normal spaces of  $f|_{\Sigma_q}$  (i.e., the subspaces of the normal spaces spanned by the image of the second fundamental form) have dimension one everywhere (notice that  $A_{\bar{\xi}}$  can not vanish anywhere, otherwise it would be identically zero and  $f|_{\Sigma_q}$  would be totally geodesic,

which is impossible). Then either  $f(\Sigma_q)$  is contained in an affine hyperplane  $\mathcal{H}$  of  $\mathbb{R}^{n+1}$  or the first normal spaces of  $f|_{\Sigma_q}$  are nonparallel along an open subset of  $\Sigma_q$ . Both possibilities lead to contradictions: the latter forces  $\Sigma_q$  to be flat (cf. [6], Theorem 1); in the former, since the shape operator of the isometric immersion  $f: \Sigma_q \rightarrow \mathcal{H}$  is  $A_{\bar{\xi}}$ , which has constant eigenvalues, it follows that  $f(\Sigma_q)$  is a round sphere, which is again impossible. We obtain that there exists an open subset  $U \subset \Gamma$  where  $\text{rank } A_{\bar{\eta}} = 1$  and  $U \cap \Sigma_q = U_0$ . Since the images of  $A_{\bar{\eta}}$  and  $A_{\eta}$  are related by  $\text{Im}(A_{\bar{\eta}}) = \Pi(\text{Im}(A_{\eta}))$ , and on  $U$  the dimensions of  $\text{Im}(A_{\bar{\eta}})$  and  $\text{Im}(A_{\eta})$  are 1 and 2, respectively, we must have  $\xi \in \text{Im}(A_{\eta})$  everywhere on  $U$ . Therefore, at any point  $x \in U$  we have that  $\ker A_{\eta}(x) \subset T_x \Sigma_x$ , and hence  $\ker A_{\eta}(x) = \ker A_{\bar{\eta}}(x)$ . It follows that the leaves of the distribution on  $U_0$  given by  $\ker A_{\bar{\eta}}$  are totally geodesic in  $\Sigma_q$  and  $\mathbb{R}^{n+1}$ . In particular, they are flat hypersurfaces of  $\Sigma_q$ . This is in contradiction with the fact that  $\Sigma_q$  is locally isometric to  $\mathbb{R} \times S^{n-2}$ . In fact, for any  $x \in U_0$  let  $W$  be an  $(n-2)$ -dimensional subspace of  $T_x(\Sigma_q)$  where the sectional curvatures of  $\Sigma_q$  are equal to 1 and let  $F_x$  be the totally geodesic flat hypersurface through  $x$ . Then  $S = W \cap T_x(F_x)$  has dimension at least 2, since  $n \geq 5$ . At each bidimensional subspace of  $S$ , the sectional curvature of  $\Sigma_q$  is 1, because  $S \subset W$  and, on the other hand, such a curvature must be zero, for  $S \subset T_x(F_x)$ . Therefore (i) is not possible, and we are left with (ii).

If  $\text{rank } A_{\bar{\eta}} = 2$  along a principal orbit  $\Sigma_q \subset \Gamma$ , then  $\text{rank } A_{\bar{\eta}} = 2$  on a possibly smaller tube around  $\Sigma_q$  contained in  $\Gamma$ , which we still denote by  $\Gamma$ . By Theorem 3 in [4], each principal orbit  $\Sigma_x$  contained in  $\Gamma$  is isometric to a Riemannian product  $S^2(a) \times S^{n-3}(b)$  of spheres and  $f|_{\Sigma_x}: \Sigma_x \rightarrow \mathbb{R}^{n+1}$  splits as a product  $f|_{\Sigma_x} = i_1 \times i_2: S^2(a) \times S^{n-3}(b) \rightarrow \mathbb{R}^3 \times \mathbb{R}^{n-2} = \mathbb{R}^{n+1}$ , where  $i_1: S^2(a) \rightarrow \mathbb{R}^3$  and  $i_2: S^{n-3}(b) \rightarrow \mathbb{R}^{n-2}$  are umbilical inclusions. Moreover,  $\{\bar{\eta}, \bar{\xi}\}$  is precisely the orthonormal normal frame of  $f|_{\Sigma_x}$  determined by the unit normal vector fields to the inclusions  $i_1$  and  $i_2$ , respectively. In particular,  $\bar{\xi}$  and  $\bar{\eta}$  are parallel with respect to the normal connection of  $f|_{\Sigma_x}$ . Hence,  $A_{\bar{\eta}}$  coincides with the restriction of  $A_{\eta}$  to  $T \Sigma_x$ , which in turn implies that  $\xi$  is an eigenvector of  $A_{\eta}$  along  $\Gamma$ . Now, since  $\text{rank } A_{\eta} = \text{rank } A_{\bar{\eta}} = 2$  on  $\Gamma$ , it follows that  $\xi \in \ker A_{\eta}$ . Therefore, the segments of normal geodesics in  $\Gamma$  are contained in the leaves of  $\ker A_{\eta}$ . Since these are assumed to be complete, we obtain that  $f$  has type number 2 on the whole  $M^n$  and that  $A_{\eta}$  is everywhere of the form  $A_{\eta} = \text{diag}(\varphi, \varphi, 0, \dots, 0)$ , where  $\varphi$  is nonzero and constant along each principal orbit and the  $\varphi$ -eigenspaces of  $A_{\eta}$  (or  $A_{\bar{\eta}}$ ) coincide with  $\ker A_{\bar{\xi}} = \ker A_{\xi}^i$ . Now, let  $X$  be a vector field such that  $A_{\eta}(X) = \varphi X$ . By the Codazzi equation

$$\nabla_X(A_{\eta}(\xi)) - A_{\eta}(\nabla_X \xi) = \nabla_{\xi}(A_{\eta}(X)) - A_{\eta}(\nabla_{\xi} X)$$

we get

$$-A_{\eta}(\nabla_X \xi) = \xi(\varphi)X + \varphi \nabla_{\xi} X - A_{\eta}(\nabla_{\xi} X) = \xi(\varphi)X,$$

where the last equality follows from  $\nabla_{\xi} X \in (\ker A_{\eta})^{\perp} = \ker(A_{\eta} - \varphi I)$ , using that  $\ker A_{\eta}$  is totally geodesic. Since  $\nabla_X \xi = -A_{\xi}^i(X) = -A_{\bar{\xi}}(X) = 0$ , it follows that  $\xi(\varphi) = 0$ . Therefore  $\varphi$  is a constant, which we may suppose to be 1. Standard arguments now show that  $M^n$  splits as  $M^n = S^2 \times \mathbb{R}^{n-2}$  (cf. [13]). By the main lemma in [10],  $f$  also splits as stated.  $\square$

*Proof of Theorem 1.1.* Suppose that  $f$  is not rigid. If  $M^n$  is compact and  $n \geq 3$ , it follows from Theorem 2.1 that  $B$ , the set of totally geodesic points of  $f$ , disconnects  $M^n$ . In order to get the same conclusion in the non-compact case, we must show that there does not exist a complete leaf of relative nullity of  $f$  of dimension  $\ell = n - 1$  or  $\ell = n - 2$ . For  $\ell = n - 1$  this follows from our assumption on the flat part of  $M^n$ . Proposition 2.3 takes care of the case  $\ell = n - 2$ .

Since  $B$  disconnects  $M^n$ , it must contain a regular point  $p$ . Then the (principal) orbit  $\Sigma$  through  $p$  is contained in  $B$ , because  $B$  is  $G$ -invariant by Proposition 2.2 (i). It follows from Lemma 3.14 of [5] that  $f(\Sigma)$  is contained in a hyperplane  $\mathcal{H}$  which is tangent to  $f$  along  $\Sigma$ , for  $\Sigma$  is connected. But  $f(\Sigma)$  is an isoparametric hypersurface of a sphere by Proposition 2.2 (iii), and hence  $f(\Sigma)$  must be a round hypersphere of  $\mathcal{H}$ . Proposition 2.2 (iv) now completes the proof.  $\square$

**Remark 2.4.** In case  $M^n$  is complete non-compact of dimension  $n \geq 5$ , the arguments in the beginning of the proof of Proposition 2.3 show, more precisely, that the conclusion of Theorem 1.1 fails only when every point of  $G(\gamma(I))$  is flat, where  $\gamma: \mathbb{R} \rightarrow M^n$  is a normal geodesic parameterized by arc-length and  $I$  is either  $[a, \infty)$  for some  $a \in \mathbb{R}$  or  $(-\infty, -b] \cup [a, \infty)$  for some  $a, b > 0$ , according to the orbit space being homeomorphic to  $\mathbb{R}$  or  $[0, \infty)$ , respectively. Notice that in the latter case  $M^n$  is flat outside a compact subset. We also point out that, since  $G$  is assumed to be compact, our assumption on the flat part of  $M^n$  is equivalent to  $M^n$  being *unflat at infinity* in the sense of [9].

*Proof of Theorem 1.2.* We already know from Theorem 1.1 that  $f$  is either rigid or a hypersurface of revolution. Thus, by Proposition 2.2 (iv) it suffices to prove that any of the conditions in the statement implies that  $f(\Sigma)$  is a round sphere for some principal orbit  $\Sigma$  of  $G$ .

Assume first that  $\Sigma$  is a positively curved principal orbit. By Proposition 2.2 (iii),  $f$  immerses  $\Sigma$  as a positively curved isoparametric hypersurface of some hypersphere of  $\mathbb{R}^{n+1}$ . It follows easily from the Cartan identities for isoparametric hypersurfaces of the sphere (cf. [4], Corollary 2) that  $f(\Sigma)$  is a round sphere.

As for condition (ii), if  $\Sigma$  is a totally geodesic principal orbit, then it is immersed by  $f$  as an isoparametric hypersurface of a sphere  $S^n$  whose first normal spaces in  $\mathbb{R}^{n+1}$  are one-dimensional. This can only happen if it is umbilical in  $S^n$ , and hence again a round hypersphere of  $S^n$ .

Now assume that (iii) holds. First notice that the position vector of  $f$  can not be tangent to  $f$  along  $f(\Sigma)$  for every principal orbit  $\Sigma$  of  $G$ , otherwise  $f$  would be a cone over an isoparametric hypersurface of the sphere, in contradiction with the completeness of  $M^n$ . Now, if  $\Sigma$  is a principal orbit along which the position vector is nowhere tangent to  $f$ , then the normal bundle of the restriction  $f|_{\Sigma}: \Sigma \rightarrow \mathbb{R}^{n+1}$  is spanned by the position vector and by  $f_*\xi$ , where  $\xi$  is a unit normal vector field to the inclusion of  $\Sigma$  into  $M^n$ . Since the shape operators of  $f|_{\Sigma}$  with respect to both vector fields are multiples of the identity tensor, it follows that  $f|_{\Sigma}$  is umbilical, and we obtain again that  $f(\Sigma)$  is a round sphere.

Finally, under condition (iv) the conclusion is a consequence of the following result.  $\square$

**Proposition 2.5.** *Let  $P^n \subset S^{n+1}$ ,  $n \geq 4$ , be an isoparametric hypersurface. If the universal covering of  $P^n$  is (homeomorphic to)  $S^n$ , then  $P^n$  is isometric to a Euclidean sphere.*

*Proof.* Let  $\lambda_1, \lambda_2, \dots, \lambda_g$  be the distinct (and constant) principal curvatures of  $P^n$ . Let  $m_1$  be the common multiplicity of the  $\lambda_k$ , when  $k$  is odd, and let  $m_2$  be the common multiplicity of the  $\lambda_k$ , when  $k$  is even. Denote by  $\beta_0, \beta_1, \beta_2, \dots, \beta_n$  the  $\mathbb{Z}_2$ -Betti numbers of  $P^n$ . Then we have (cf. [11]):

- (i) [F. Münzner]  $g \in \{1, 2, 3, 4, 6\}$ ;
- (ii)  $2n = g(m_1 + m_2)$ ;
- (iii) [E. Cartan] If  $g = 3$ , then  $m_1 = m_2 \in \{1, 2, 4, 8\}$ ;
- (iv) [U. Abresh] If  $g = 6$ , then  $m_1 = m_2 \in \{1, 2\}$ ;
- (v) [F. Münzner]  $\sum_{i=0}^n \beta_i = 2g$ .

Suppose first that  $n$ , the dimension of  $P^n$ , is odd. Then (ii) and (iv) imply that  $g \in \{1, 2, 3\}$ . Since  $n \geq 4$ , it follows from (iii) that  $g \in \{1, 2\}$ . If  $g = 2$ , then  $P^n$  is a Riemannian product of spheres and thus it cannot be covered by a sphere. Hence we must have  $g = 1$  and this implies that  $P^n$  is a Euclidean sphere.

Let now  $n$  be even, say  $n = 2q$ . Then the Euler characteristics of  $S^{2q}$  and  $P^n$  are related by  $\chi(S^{2q}) = m\chi(P^n)$ , where  $m$  is the number of sheets of the covering. Thus, either  $m = 1$  or  $m = 2$ , since  $\chi(S^{2q}) = 2$ . Suppose  $m = 2$ . Then  $\chi(P^n) = \sum_{i=0}^n (-1)^i \beta_i = 1$ , which implies, using Poincaré duality, that the Betti number  $\beta_q$  is odd. On the other hand, we get from (v) that  $\beta_q$  must be even. This contradiction tells us that  $m = 1$  and, again using (v), we obtain that  $g = 1$ . Therefore  $P^n$  is a Euclidean sphere.  $\square$

**Remark 2.6.** Proposition 2.5 is no longer true for  $n = 3$ , as shown by Cartan isoparametric hypersurfaces of  $S^4$  with three distinct principal curvatures [3], which are diffeomorphic to  $S^3/Q$ , where  $Q$  stands for the quaternion 8-group.

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