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Autor: Liu, Yu-Ru

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A prime analogue of the Erdös-Pomerance conjecture for elliptic curves

Yu-Ru Liu*

Abstract. Let E/\mathbb{Q} be an elliptic curve of rank ≥ 1 and $b \in E(\mathbb{Q})$ a rational point of infinite order. For a prime p of good reduction, let $g_b(p)$ be the order of the cyclic group generated by the reduction \bar{b} of b modulo p. We denote by $\omega(g_b(p))$ the number of distinct prime divisors of $g_b(p)$. Assuming the GRH, we show that the normal order of $\omega(g_b(p))$ is $\log\log p$. We also prove conditionally that there exists a normal distribution for the quantity

$$\frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}}.$$

The latter result can be viewed as an elliptic analogue of a conjecture of Erdös and Pomerance about the distribution of $\omega(f_a(n))$, where a is a natural number >1 and $f_a(n)$ the order of a modulo n.

Mathematics Subject Classification (2000). 11N37, 11G20.

Keywords. Prime divisors, order of cyclic groups, elliptic curves.

1. Introduction

For $n \in \mathbb{N} := \{1, 2, 3, \dots\}$, let $\omega(n)$ denote the number of distinct prime divisors of n. The Turán Theorem is about the second moment of $\omega(n)$ [23]; it states that for $x \in \mathbb{R}, x > 1$,

$$\sum_{n \le x} (\omega(n) - \log \log x)^2 \ll x \log \log x.$$

Turán's result implies an earlier theorem of Hardy and Ramanujan [8], which states that for any $\varepsilon>0$

$$\#\{n \le x \mid n \text{ satisfies } |\omega(n) - \log\log n| > \varepsilon \log\log n\}$$

is o(x) as $x \to \infty$. In other words, the normal order of $\omega(n)$ is $\log \log n$. The significance of the ' $\log \log n$ ' term is that it is about $\sum_{p \le n} \frac{\omega(p)}{p}$ where p runs over primes.

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The idea behind Turán's proof was essentially probabilistic. Further development of probabilistic ideas led Erdös and Kac [5] to prove a remarkable refinement of the Turán Theorem, namely, the existence of a normal distribution for $\omega(n)$. More precisely, they proved that for $\gamma \in \mathbb{R}$,

$$\lim_{x\to\infty}\frac{1}{x}\#\Big\{n\le x\ \big|\ n\ \text{ satisfies } \frac{\omega(n)-\log\log n}{\sqrt{\log\log n}}\le\gamma\,\Big\}=G(\gamma):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\gamma}e^{\frac{-t^2}{2}}dt.$$

The theorem of Erdös and Kac opened a door to the study of probabilistic number theory. In the early 1960s and subsequently the 1970s, the theory was refined by many authors, culminating in a generalized Erdös–Kac theorem proved independently by Kubilius [10] and Shapiro [20]. Their result is applicable to what are called 'strongly additive functions'. The interested reader can find a comprehensive treatment of it in the monograph of Elliott [3].

We can also consider functions that are not strongly additive, say the Euler's φ -function. Using the same principle of the work of Kubilius and Shapiro, the issue of $\omega(\varphi(n))$ devolves upon the estimation of the sums

$$\sum_{p \le x} \omega(p-1) \quad \text{and} \quad \sum_{p \le x} \omega^2(p-1),$$

where p denotes a rational prime. Sums of this type were estimated by Haselgrove [9] and Erdös and Pomerance [6]. They proved that

$$\sum_{p \le x} \omega(p-1) = \pi(x) \log \log x + O(\pi(x))$$

and

$$\sum_{p < x} \omega^2(p-1) = \pi(x)(\log\log x)^2 + O(\pi(x)\log\log x),$$

where $\pi(x)$ is the number of rational primes $\leq x$. Applying partial summation, we can derive from the above equalities that

$$\sum_{p \le n} \frac{\omega(p-1)}{p} = \frac{1}{2} (\log \log n)^2 + O(\log \log n)$$

and

$$\sum_{p \le n} \frac{\omega^2(p-1)}{p} = \frac{1}{3} (\log \log n)^3 + O\left((\log \log n)^2\right).$$

As a consequence we have the following result of Erdös and Pomerance [6], which states that

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ n \le x \mid n \text{ satisfies } \frac{\omega(\varphi(n)) - \frac{1}{2} (\log \log n)^2}{\frac{1}{\sqrt{3}} (\log \log n)^{3/2}} \le \gamma \right\} = G(\gamma).$$

In [6], Erdös and Pomerance also proposed the following question. Let a be a positive integer > 1. For any natural number n coprime to a, let $f_a(n)$ denote the order of a modulo n. Thus $f_a(n)$ is a divisor of $\varphi(n)$. Based on the belief that the difference between $\omega(\varphi(n))$ and $\omega(f_a(n))$ is 'small on average', Erdös and Pomerance conjectured that

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ n \le x \mid n \text{ satisfies } (a, n) = 1 \text{ and } \frac{\omega(f_a(n)) - \frac{1}{2}(\log\log n)^2}{\frac{1}{\sqrt{3}}(\log\log n)^{3/2}} \le \gamma \right\}$$
$$= \frac{\varphi(a)}{a} G(\gamma).$$

The conjecture remains open until today. Even a conditional result was only obtained recently by Murty and Saidak [17] under the assumption of the GRH (i.e., the Riemann Hypothesis for all Dedekind zeta functions of number fields). Later Li and Pomerance [13] also provided an alternative proof of the same result. The difficulty of this conjecture lies in the intervention of the distribution of primes in the non-abelian extensions $\mathbb{Q}(\zeta_q, \sqrt[q]{a})$ where q varies over rational primes and ζ_q is a primitive q-th root of unity.

Let us recall that $f_a(n)$ is the least common multiple of $\{f_a(p^\gamma) \mid p^\gamma \parallel n\}$ where p^γ is the exact power of p which divides n. Also $f_a(p^\gamma)$ divides $p^{\gamma-1}f_a(p)$. Thus similarly to the case of $\omega(\varphi(n))$, to study the conjecture of Erdös and Pomerance, it is sufficient to estimate the sums

$$\sum_{p \le x} \omega(f_a(p)) \quad \text{and} \quad \sum_{p \le x} \omega^2(f_a(p)).$$

Under the assumption of the GRH, Murty and Saidak proved that

$$\sum_{p \le x} \omega(f_a(p)) = \pi(x) \log \log x + O(\pi(x))$$

and

$$\sum_{p \le x} \omega^2(f_a(p)) = \pi(x) (\log \log x)^2 + O(\pi(x) \log \log x).$$

A conditional result of the conjecture follows.

In [17], Murty and Saidak also proved the following 'prime analogue' of the Erdös–Pomerance conjecture:

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \# \Big\{ p \le x \ \big| \ p \text{ satisfies } (a,p) = 1 \text{ and } \frac{\varpi(f_a(p)) - \log\log p}{\sqrt{\log\log p}} \le \gamma \Big\}$$
$$= G(\gamma).$$

In a sense, as we see from [17, §5, §7], there is not much difference between the study of $\omega(f_a(n))$ and $\omega(f_a(p))$, as the main technical difficulty of both problems depends on the study of $\omega(i_a(p))$, where $i_a(p) = (p-1)/f_a(p)$.

The purpose of this paper is to formulate an analogous Erdös–Pomerance conjecture for elliptic curves and provide a conditional proof of it. Let E/\mathbb{Q} be an elliptic curve of rank ≥ 1 . Let $b \in E(\mathbb{Q})$ be a rational point of infinite order. For a prime p of good reduction, let $g_b(p)$ be the order of $\langle \bar{b} \rangle$, the cyclic group generated by the reduction \bar{b} of b modulo p. The function $g_b(p)$ can be viewed as an elliptic analogue of $f_a(p)$. Thus, an analogous formulation of the conjecture of Erdös and Pomerance for elliptic curves is that there exists a normal distribution for the quantity

$$\frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}}.$$

We prove the following result.

Theorem 1. Let E/\mathbb{Q} be an elliptic curve of rank ≥ 1 and $b \in E(\mathbb{Q})$ a rational point of infinite order. For a prime p of good reduction, let $\langle \bar{b} \rangle$ be the cyclic group generated by the reduction \bar{b} of b modulo p and $g_b(p)$ its order. Assuming the GRH, we have

y the reduction b of b modulo p and
$$g_b(p)$$
 its order. Assuming
$$\sum_{\substack{p \leq x \\ p \text{ of good reduction}}} \left(\omega(g_b(p)) - \log\log x\right)^2 \ll \pi(x)\log\log x.$$

As a direct consequence of Theorem 1 we have

Corollary 2. Assuming the GRH, the normal order of $\omega(g_b(p))$ is $\log \log p$.

The following theorem is an analogous result of Murty and Saidak for elliptic curves.

Theorem 3. Let E/\mathbb{Q} , b, and $g_b(p)$ be defined as in Theorem 1. Let $\gamma \in \mathbb{R}$. Assuming the GRH, we have

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \# \Big\{ p \le x \ \big| \ p \text{ is of good reduction and } \ \frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}} \le \gamma \Big\}$$
$$= G(\gamma).$$

Thus, we obtain an elliptic analogue of a conjecture of Erdös and Pomerance in terms of primes.

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Notation. For $x \in \mathbb{R}$, x > 0, let f(x) and g(x) be two functions of x. If g(x) is positive and there exists a constant C > 0 such that $|f(x)| \le Cg(x)$, we write either $f(x) \ll g(x)$ or f(x) = O(g(x)). If both f(x) and g(x) are positive, we use $f(x) \asymp g(x)$ to denote that f(x) = O(g(x)) and g(x) = O(f(x)). If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$, we write f(x) = o(g(x)). Also, we use $\bar{\mathbb{Q}}$ and $\bar{\mathbb{F}}_p$ to denote some fixed algebraic closures of \mathbb{Q} and \mathbb{F}_p respectively.

2. Preliminaries

We first recall some theorems about elliptic curves that will be needed later. Let E/\mathbb{Q} be an elliptic curve of rank ≥ 1 . For a prime $l \in \mathbb{N}$, we denote by E[l] the l-torsion points. By adjoining to \mathbb{Q} the coordinates of the l-torsion points, we obtain $\mathbb{Q}(E[l])$, a finite Galois extension of \mathbb{Q} . Since

$$E[l] \cong (\mathbb{Z}/l\mathbb{Z}) \times (\mathbb{Z}/l\mathbb{Z})$$

(see [21, Corollary 6.4]), by choosing a basis, we have a natural injection

$$\Phi_l : \operatorname{Gal}(\mathbb{Q}(E[l])/\mathbb{Q}) \hookrightarrow \operatorname{GL}_2(\mathbb{Z}/l\mathbb{Z}).$$

In the following discussion we will abuse our notation by identifying an element $\gamma \in \text{Gal}(\mathbb{Q}(E[l])/\mathbb{Q})$ with its image $\Phi_l(\gamma) \in \text{GL}_2(\mathbb{Z}/l\mathbb{Z})$.

Let $b \in E(\mathbb{Q})$ be a rational point of infinite order. We denote by $l^{-1}b$ the set of elements $v \in E(\bar{\mathbb{Q}})$ such that

$$[l] v = \underbrace{v + v + \dots + v}_{l \text{ times}} = b.$$

Define $L_l = \mathbb{Q}(E[l], l^{-1}b)$, which is a finite extension of $\mathbb{Q}(E[l])$. We have the following theorem.

Theorem 4 (Bachmakov [1]). For a prime l, the Galois group $Gal(L_l/\mathbb{Q}(E[l]))$ can be identified with a subgroup of E[l] and is equal to E[l] for all but finitely many l.

The group $\operatorname{GL}_2(\mathbb{Z}/l\mathbb{Z})$ acts naturally on E[l] by matrix multiplication. We denote this action by * and we see that it induces a semidirect product $E[l] \rtimes \operatorname{GL}_2(\mathbb{Z}/l\mathbb{Z})$. Let G_l be the Galois group $\operatorname{Gal}(L_l/\mathbb{Q})$. From Theorem 4, for all but finitely many l, we have

$$G_l \cong E[l] \rtimes Gal(\mathbb{Q}(E[l])/\mathbb{Q}),$$

which is a subgroup of $E[l] \rtimes GL_2(\mathbb{Z}/l\mathbb{Z})$.

An element $(\tau, \gamma) \in G_l$ acts on E[l] and $l^{-1}b$ as follows: let $v_0 \in l^{-1}b$ be a fixed element; for $u \in E[l]$ and $v \in l^{-1}b$ we have

- $(\tau, \gamma) \cdot u := \gamma * u$.
- $(\tau, \gamma) \cdot v := v_0 + \gamma * (v v_0) + \tau$.

Notice that since $[l]v = [l]v_0 = b$, $(v - v_0) \in E[l]$. Thus, $\gamma * (v - v_0)$ is well defined. Also, since both $(v - v_0)$ and τ are in E[l], for $v \in l^{-1}b$, we have

$$[l]((\tau, \gamma) \cdot v) = [l]v_0 = b.$$

Thus, (τ, γ) is a well-defined action on the set $l^{-1}b$. Moreover, for $v \in l^{-1}b$, we have

$$(\tau, \gamma) \cdot v = v$$
 if and only if $(\gamma - I) * (v_0 - v) = \tau$,

where I is the 2×2 identity matrix.

Let p be a prime of good reduction. We denote by \overline{E} the reduction of E modulo p. Let $\overline{E}(\mathbb{F}_p)$ be the set of rational points of \overline{E} defined over the finite field \mathbb{F}_p . Let $b \in E(\mathbb{Q})$ be a rational point of infinite order and $\overline{b} \in \overline{E}(\mathbb{F}_p)$ the reduction of b modulo p. Let $\langle \overline{b} \rangle$ be the cyclic group generated by \overline{b} , which is a subgroup of $\overline{E}(\mathbb{F}_p)$. We denote by $g_b(p)$ the order of $\langle \overline{b} \rangle$. Thus $g_b(p)$ is a divisor of $\#\overline{E}(\mathbb{F}_p)$. We write

$$\#\overline{E}(\mathbb{F}_p) = g_b(p) \cdot i_b(p),$$

where $i_b(p)$ is the index of $\langle \overline{b} \rangle$ in $\overline{E}(\mathbb{F}_p)$. Let Δ be the discriminant of E. For $p \nmid l \Delta$, Lang and Trotter [12] gave a condition on the Frobenius element $(\tau_p, \gamma_p) \in G_l$ in order that $l \mid i_b(p)$. We review their arguments below.

Notice that $l \mid i_b(p)$ implies that $l \mid \#\overline{E}(\mathbb{F}_p)$. Since

$$\operatorname{tr} \gamma_p \equiv p + 1 - \#\overline{E}(\mathbb{F}_p) \pmod{l}$$

and

$$\det \gamma_p \equiv p \pmod{l}$$

(see [22, p. 172]), if $l \mid \#\overline{E}(\mathbb{F}_p)$, we have

$$1 - \operatorname{tr} \gamma_p + \det \gamma_p \equiv 0 \pmod{l}.$$

Thus $\gamma_p \in \operatorname{Gal}(\mathbb{Q}(E[l])/\mathbb{Q}) \subseteq \operatorname{GL}_2(\mathbb{Z}/l\mathbb{Z})$ has an eigenvalue 1.

We consider first the case when $\gamma_p=I$. We recall that the cyclic group generated by $\pi_p: x\mapsto x^p$ is dense in $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$. The group $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ acts on $w\in \overline{E}(\bar{\mathbb{F}}_p)$ coordinatewise. Thus for $w\in \overline{E}(\bar{\mathbb{F}}_p)$ we have

$$\pi_p \cdot w = w$$
 if and only if $w \in \overline{E}(\mathbb{F}_p)$.

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Let $w_1 \in E(\mathbb{Q}(E[l]))$. The Frobenius element $\gamma_p \in \operatorname{Gal}(\mathbb{Q}(E[l])/\mathbb{Q})$ acts on w_1 coordinatewise. This action is compatible with π_p in the following sense: let $\bar{w}_1 \in \overline{E}(\bar{\mathbb{F}}_p)$ be the reduction of w_1 modulo p; we have

$$\overline{\gamma_p \cdot w_1} = \pi_p \cdot \bar{w}_1.$$

Thus for $\gamma_p = I$ we have

$$\bar{w}_1 = \overline{\gamma_p \cdot w_1} = \pi_p \cdot \bar{w}_1.$$

It follows that $\bar{w}_1 \in \bar{E}(\mathbb{F}_p)$. Let $\bar{E}[l]$ denote the reduction of E[l] modulo p. Since $E[l] \subseteq E(\mathbb{Q}(E[l]))$, the above argument shows that

$$\overline{E}(\mathbb{F}_p) \supseteq \overline{E}[l] \cong (\mathbb{Z}/l\mathbb{Z}) \times (\mathbb{Z}/l\mathbb{Z}), \quad \text{provided that } p \nmid l\Delta$$

(see [21, Corollary 6.4]). Consider the subgroup $\langle \bar{b} \rangle$ in $\bar{E}(\mathbb{F}_p)$. Since $\langle \bar{b} \rangle$ is cyclic, it can not contain two $(\mathbb{Z}/l\mathbb{Z})$ factors. Thus, at least one of $(\mathbb{Z}/l\mathbb{Z})$ factors of $\bar{E}(\mathbb{F}_p)$ is contained in $\bar{E}(\mathbb{F}_p)/\langle \bar{b} \rangle$. Since $i_b(p)$ is the order of $\bar{E}(\mathbb{F}_p)/\langle \bar{b} \rangle$, we have $l \mid i_b(p)$. We conclude that for $\gamma_p = I$, l is a divisor of $i_b(p)$.

On the other hand, if γ_p has an eigenvalue 1 and $\gamma_p \neq I$, $\overline{E}(\mathbb{F}_p)$ can not contain a $(\mathbb{Z}/l\mathbb{Z}) \times (\mathbb{Z}/l\mathbb{Z})$ factor. Hence, the l-torsion points of $\overline{E}(\mathbb{F}_p)$, which is the kernel of the map $\gamma_p - I : E[l] \to E[l]$, form a cyclic subgroup. In other words, the l-primary part of $\overline{E}(\mathbb{F}_p)$ is of the form $\mathbb{Z}/l^{\alpha}\mathbb{Z}$ for some $\alpha \in \mathbb{N}$. Write

$$\overline{E}(\mathbb{F}_n) \cong \mathbb{Z}/l^{\alpha}\mathbb{Z} \times H,$$

where H is an abelian group with (|H|, l) = 1. We will abuse our notation by identifying an element in $\overline{E}(\mathbb{F}_p)$ with its image in $\mathbb{Z}/l^{\alpha}\mathbb{Z} \times H$. For $\overline{b} \in \overline{E}(\mathbb{F}_p)$, without loss of generality, we can assume that either $\overline{b} = (0, h)$ or $\overline{b} = (l^{\beta}, h)$ where $h \in H$ and $\beta > 0$.

Case 1. Suppose $\bar{b}=(0,h)$. Since (|H|,l)=1, the element $\bar{b}_l=(0,l^{-1}h)\in \bar{E}(\mathbb{F}_p)$ is well defined and $[l]\bar{b}_l=\bar{b}$.

Case 2. Suppose $\bar{b}=(l^{\beta},h)$. If $\beta=0$, the order of the cyclic group $\langle b \rangle$ is divisible by l^{α} , i.e., $l \nmid i_b(p)$. Hence, if $l \mid i_b(p)$, it implies that $\beta \geq 1$. Choosing $\bar{b}_l = (l^{\beta-1}, l^{-1}h) \in \bar{E}(\mathbb{F}_p)$, we have $[l]\bar{b}_l = \bar{b}$.

We conclude that if γ_p has an eigenvalue 1, $\gamma_p \neq 1$ and $l \mid i_b(p)$, there exists $\bar{b}_l \in \overline{E}(\mathbb{F}_p)$ such that $[l]\bar{b}_l = \bar{b}$. Let $b_l \in \overline{E}(\overline{\mathbb{Q}})$ such that the reduction of b_l modulo p is \bar{b}_l . Since $[l]\bar{b}_l = \bar{b}$, it follows that $b_l \in l^{-1}b$. Moreover, since $\bar{b}_l \in E(\mathbb{F}_p)$, we have

$$(\tau_p, \gamma_p) \cdot b_l = b_l,$$

which is equivalent to

$$(\gamma_p - I) * (v_0 - b_l) = \tau_p,$$

i.e., $\tau_p \in \text{Im}(\gamma_p - I)$.

Define a subset S_l of G_l as follows: an element (τ, γ) of G_l belongs to S_l if it satisfies one of the two following conditions:

- (1) $\gamma = I$ or
- (2) γ has an eigenvalue 1, $\ker((\gamma I) : E[l] \to E[l])$ is cyclic, and $\tau \in \operatorname{Im}(\gamma I)$. Notice that S_l is a union of conjugacy classes of G_l . Combining all the above discussions, we obtain the following result of Lang and Trotter.

Theorem 5 (Lang and Trotter [12]). Let $i_b(p)$ be the index of the cyclic group $\langle \bar{b} \rangle$ in $\bar{E}(\mathbb{F}_p)$. For a prime $l \in \mathbb{N}$, $p \nmid l \Delta$, the following two statements are equivalent:

- (1) $l | i_b(p)$.
- (2) $(\tau_p, \gamma_p) \in S_l$.

Another important ingredient of the proof of Theorems 1 and 3 is the Chebotarev density theorem. Let L/\mathbb{Q} be a finite Galois extension of degree n_L and discriminant d_L . We denote by G the Galois group of L/\mathbb{Q} and C a union of conjugacy classes of G. Let $\sigma_p \in G$ be a Frobenius element. Define

$$\pi_C(x, L/\mathbb{Q}) = \#\{p \le x \mid p \text{ is an unramified prime in } L/\mathbb{Q} \text{ and } \sigma_p \subseteq C\}.$$

We have

Theorem 6 (Lagarias and Odlyzko [11], Serre [19]). Assuming the GRH for the Dedekind zeta function of L, we have

$$\pi_C(x, L/\mathbb{Q}) = \frac{|C|}{|G|} \operatorname{li} x + O\left(|C| x^{\frac{1}{2}} \left(\frac{\log |d_L|}{n_L} + \log x\right)\right),$$

where $\operatorname{li} x = \int_2^x \frac{dt}{\log t}$.

The following theorem is useful for estimating the error term in the Chebotarev density theorem.

Theorem 7 (Serre [19]). Let L/\mathbb{Q} be a finite Galois extension of degree n_L and discriminant d_L . We have

$$\frac{n_L}{2} \sum_{q \text{ ramified}} \log q \leq \log |d_L| \leq (n_L - 1) \sum_{q \text{ ramified}} \log q + n_L \log n_L,$$

where the sum is over all primes q that are ramified in L.

3. Prime divisors of $i_b(p)$

We recall that $i_b(p)$ is the index of $\langle \overline{b} \rangle$ in $\overline{E}(\mathbb{F}_p)$. In this section, we consider the number of distinct prime divisors of $i_b(p)$. The following lemma is essential for the proof of Theorems 1 and 3. We use the notation \sum' to denote the sum over primes of good reduction.

Lemma 8. Assuming the GRH, we have

$$\sum_{p < x}' \omega^2(i_b(p)) \ll \pi(x).$$

Proof. Let $y = x^{\delta}$ with $0 < \delta < 1$ (a choice of δ will be made later). Define a truncation function ω_{y} of ω as follows:

$$\omega_y(i_b(p)) = \#\{l \le y \mid l \text{ is a prime and } l \mid i_b(p)\}.$$

For a prime $p \le x$, since

$$i_b(p) \le \#\overline{E}(\mathbb{F}_p) \le (p + 2\sqrt{p} + 1) \le 3x,$$

it follows that

$$\omega(i_b(p)) = \omega_{\nu}(i_b(p)) + O(1).$$

Hence we have

$$\begin{split} \sum_{p \leq x}' \omega^2(i_b(p)) &= \sum_{p \leq x}' \left(\omega_y(i_b(p)) + O(1) \right)^2 \ll \sum_{p \leq x}' \omega_y^2(i_b(p)) + O\left(\pi(x)\right) \\ &= \sum_{\substack{l_1, l_2 \leq y \\ l_1 \neq l_2}} \sum_{\substack{p \leq x \\ l_1 l_2 | l_b(p)}}' 1 + \sum_{\substack{l \leq y \\ l | l_b(p)}} \sum_{\substack{p \leq x \\ l | l_b(p)}}' 1 + O\left(\pi(x)\right), \end{split}$$

where l_1, l_2 , and l are rational primes. Consider the sum

$$\sum_{l \le y} \sum_{\substack{p \le x \\ l \mid i_b(p)}}^{\prime} 1.$$

Applying Theorems 5, 6 and 7 for all but finitely many primes l, under the GRH we have

$$\begin{split} \# \big\{ p &\leq x \mid p \text{ satisfies } l \mid i_b(p) \big\} \\ &= \operatorname{li} x \cdot \frac{|S_l|}{|G_l|} + O\Big(|S_l| \cdot x^{\frac{1}{2}} \cdot \Big(\sum_{q \text{ ramified}} \log q + + \log n_l + \log x\Big)\Big), \end{split}$$

where the sum is over all primes q that are ramified in L_l and $n_l = |G_l|$.

In the case of elliptic curves without complex multiplication (non-CM) Serre [18] proved that for all but finitely many primes l,

$$\operatorname{Gal}(\mathbb{Q}(E[l])/\mathbb{Q}) = \operatorname{GL}_2(\mathbb{Z}/l\mathbb{Z}).$$

Hence, for all but finitely many l, we have

$$|G_l|
times l^6$$
 and $|S_l|
times l^4$.

In the case of elliptic curves with complex multiplication (CM), from [7, p. 35–37], we have

$$|G_l| \times l^4$$
 and $|S_l| \times l^2$.

It is well known that q is ramified in L_l if and only if $q \mid l\Delta$ (see [2]). Hence, assuming the GRH, we have

$$\sum_{l \le y} \sum_{\substack{p \le x \\ l \mid l \mid b(p)}}^{\prime} 1 \ll \sum_{l \le y} \left(\frac{\pi(x)}{l^2} + O\left(l^4 x^{\frac{1}{2}} \log(l^6 x \Delta)\right) \right) \ll \pi(x) + O\left(x^{\frac{1}{2} + 5\delta + \varepsilon}\right),$$

where $\varepsilon > 0$ is arbitrarily small. Choosing $\delta = \frac{1}{11}$, we have

$$\sum_{l \le y} \sum_{\substack{p \le x \\ l \mid l_b(p)}}^{\prime} 1 \ll \pi(x).$$

Consider the sum

$$\sum_{\substack{l_1, l_2 \le y \\ l_1 \ne l_2}} \sum_{\substack{p \le x \\ l_1 l_2 \mid i_b(p)}}' 1.$$

The group homomorphisms

$$E[l_1l_2] \to E[l_1] \times E[l_2]$$
 and $\operatorname{GL}_2(\mathbb{Z}/l_1l_2\mathbb{Z}) \to \operatorname{GL}_2(\mathbb{Z}/l_1\mathbb{Z}) \times \operatorname{GL}_2(\mathbb{Z}/l_2\mathbb{Z}),$

which are induced by reduction modulo l_1 and l_2 respectively, are indeed isomorphisms. Moreover, these maps are compatible with the actions defined in Section 2. Since $|S_l|/|G_l| \approx 1/l^2$, by Theorems 5, 6 and 7 we have

$$\sum_{\substack{l_1, l_2 \le y \\ l_1 \ne l_2}} \sum_{\substack{p \le x \\ l_1 l_2 | i_b(p)}} 1 \ll \sum_{\substack{l_1, l_2 \le y \\ l_1 \ne l_2}} \left(\frac{\pi(x)}{(l_1 l_2)^2} + O\left((l_1 l_2)^4 x^{\frac{1}{2}} \log(l_1^6 l_2^6 x \Delta)\right) \right)$$

$$\ll \pi(x) + O\left(x^{\frac{1}{2} + 10\delta + \varepsilon}\right),$$

where $\varepsilon \to 0$ as $x \to \infty$. Choosing $\delta = \frac{1}{21}$, we have

$$\sum_{\substack{l_1, l_2 \le y \\ l_1 \ne l_2}} \sum_{\substack{p \le x \\ l_1 l_2 \mid l_1 l_2 \mid l_2 \neq p}} 1 \ll \pi(x).$$

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It follows that

$$\sum_{p\leq x}'\omega^2(i_b(p))\ll \pi(x).$$

This completes the proof of Lemma 8.

4. A Turán analogue of $\omega(g_b(p))$

In this section, we provide a proof of Theorem 1 which states that under the GRH, we have

$$\sum_{p \le x}' \left(\omega(g_b(p)) - \log \log x \right)^2 \ll \pi(x) \log \log x.$$

Our proof is a combination of Lemma 8 with the following theorem.

Theorem 9 (Miri and Murty [16], Liu [14]). Let E/\mathbb{Q} be an elliptic curve. We have (assuming the GRH if E is non-CM)

$$\sum_{p < x}' \left(\omega(\#\overline{E}(\mathbb{F}_p)) - \log \log x \right)^2 \ll \pi(x) \log \log x.$$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Since

$$\#\overline{E}(\mathbb{F}_p) = g_b(p) \cdot i_b(p),$$

we have

$$\omega(\#\overline{E}(\mathbb{F}_p)) \geq \omega(g_b(p)) \geq \omega(\#\overline{E}(\mathbb{F}_p)) - \omega(i_b(p)).$$

It follows that

$$\begin{split} \sum_{p \leq x}' \left(\omega(g_b(p)) - \log \log x \right)^2 &= \sum_{p \leq x}' \left(\omega(\#\overline{E}(\mathbb{F}_p)) + O\left(\omega(i_b(p))\right) - \log \log x \right)^2 \\ &\ll \sum_{p \leq x}' \left(\omega(\#\overline{E}(\mathbb{F}_p)) - \log \log x \right)^2 + \sum_{p \leq x}' \omega^2(i_b(p)). \end{split}$$

Combining Lemma 8 with Theorem 9 we obtain that under the GRH,

$$\sum_{p \le x}' \left(\omega(g_b(p)) - \log \log x \right)^2 \ll \pi(x) \log \log x.$$

This completes the proof of Theorem 1.

5. An Erdös–Kac analogue of $\omega(g_b(p))$

In this section, we give a proof of Theorem 3. More precisely, under the GRH we prove that there exists a normal distribution for the quantity

$$\frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}}.$$

Our proof is dependent on the following theorem.

Theorem 10 (Liu [15]). Let E/\mathbb{Q} be an elliptic curve. We have (assuming the GRH if E is non-CM)

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \# \Big\{ p \le x \mid p \text{ is of good reduction and } \frac{\omega(\# \overline{E}(\mathbb{F}_p)) - \log\log p}{\sqrt{\log\log p}} \le \gamma \Big\}$$
$$= G(\gamma).$$

Proof of Theorem 3. As in the proof of Theorem 1, we have

$$\begin{split} \frac{\omega(\#\overline{E}(\mathbb{F}_p)) - \log\log p}{\sqrt{\log\log p}} &\geq \frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}} \\ &\geq \frac{\omega(\#\overline{E}(\mathbb{F}_p)) - \log\log p}{\sqrt{\log\log p}} - \frac{\omega(i_b(p))}{\sqrt{\log\log p}}. \end{split}$$

For any $\varepsilon > 0$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, define the set

$$S(\varepsilon, \alpha, \beta) = \Big\{ p \ \big| \ p \text{ is of good reduction, } \ \alpha$$

Let $N(\varepsilon, \alpha, \beta)$ be the cardinality of $S(\varepsilon, \alpha, \beta)$. We have

$$N(\varepsilon, 0, x) < \pi(\sqrt{x}) + N(\varepsilon, \sqrt{x}, x).$$

Notice that

$$\sum_{p \le x}' \omega(i_b(p)) \ge \sum_{p \in S(\varepsilon, \sqrt{x}, x)} \omega(i_b(p)) \ge N(\varepsilon, \sqrt{x}, x) \cdot \varepsilon \sqrt{\log \log x - \log 2}.$$

Since $\omega^2(i_b(p)) \ge \omega(i_b(p))$, Lemma 8 implies that

$$N(\varepsilon, \sqrt{x}, x) \ll \frac{\pi(x)}{\sqrt{\log \log x}} = o(\pi(x)).$$

It follows that

$$N(\varepsilon, 0, x) = o(\pi(x)).$$

Thus for $\gamma \in \mathbb{R}$ we obtain

$$\begin{split} \# \Big\{ p &\leq x \; \big| \; p \; \text{is of good reduction and} \; \; \frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}} \leq \gamma \Big\} \\ &\leq \# \Big\{ p &\leq x \; \big| \; p \; \text{is of good reduction and} \\ & \frac{\omega(\#\overline{E}(\mathbb{F}_p)) - \log\log p}{\sqrt{\log\log p}} - \frac{\omega(i_b(p))}{\sqrt{\log\log p}} \leq \gamma \Big\} \\ &\leq \# \Big\{ p &\leq x \; \big| \; p \; \text{is of good reduction and} \\ & \frac{\omega(\#\overline{E}(\mathbb{F}_p)) - \log\log p}{\sqrt{\log\log p}} \leq \gamma + \varepsilon \Big\} + o\Big(\pi(x)\Big). \end{split}$$

Also we have

$$\begin{split} \# \Big\{ p \leq x \ \big| \ p \ \text{ is of good reduction and } & \frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}} \leq \gamma \Big\} \\ & \geq \# \Big\{ p \leq x \ \big| \ p \ \text{ is of good reduction and } & \frac{\omega(\#\overline{E}(\mathbb{F}_p)) - \log\log p}{\sqrt{\log\log p}} \leq \gamma \Big\}. \end{split}$$

Combine all of the above results with Theorem 10. As $x \to \infty$, for all $\varepsilon > 0$ we obtain

$$\begin{split} G(\gamma) & \leq \lim_{x \to \infty} \frac{1}{\pi(x)} \# \Big\{ p \leq x \ \Big| \ p \ \text{ is of good reduction and} \\ & \frac{\omega(g_b(p)) - \log\log p}{\sqrt{\log\log p}} \leq \gamma \Big\} \leq G(\gamma + \varepsilon). \end{split}$$

Since $G(\gamma)$ is a continuous function, for any $\varepsilon > 0$ we have

$$G(\gamma + \varepsilon) = G(\gamma) + O(\varepsilon).$$

Let $\varepsilon \to 0$. It follows that under the GRH,

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \# \Big\{ p \le x \mid p \text{ is of good reduction and } \frac{\omega(g_b(p)) - \log \log p}{\sqrt{\log \log p}} \le \gamma \Big\}$$

$$= G(\gamma).$$

This completes the proof of Theorem 3.

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Yu-Ru Liu, Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

E-mail: yrliu@math.uwaterloo.ca

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