

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 80 (2005)

Artikel: Transcendental submanifolds of RP^n
Autor: Akbulut, Selman / King, Henry
DOI: <https://doi.org/10.5169/seals-60449>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 15.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Transcendental submanifolds of $\mathbb{R}\mathbb{P}^n$

Selman Akbulut and Henry King*

Abstract. In this paper we give examples of closed smooth submanifolds of $\mathbb{R}\mathbb{P}^n$ which are isotopic to nonsingular projective subvarieties of $\mathbb{R}\mathbb{P}^n$ but they can not be isotopic to the real parts of nonsingular complex projective subvarieties of $\mathbb{C}\mathbb{P}^n$.

Mathematics Subject Classification (2000). 57R55, 14.

Keywords. Real algebraic sets.

0. Introduction

Let $j: \mathbb{R}^n \hookrightarrow \mathbb{R}\mathbb{P}^n$ be the canonical imbedding as a chart. Real algebraic sets in \mathbb{R}^n are not in general real algebraic sets in $\mathbb{R}\mathbb{P}^n$. The Zariski closure of the image of an algebraic set (under j) usually has extra components at infinity. An algebraic subset of \mathbb{R}^n which remains an algebraic set in $\mathbb{R}\mathbb{P}^n$ is called a *projectively closed algebraic set* ([AK1]). Not every algebraic set is projectively closed. In general, isotoping a submanifold of the projective space $\mathbb{R}\mathbb{P}^n$ to an algebraic subset is a much harder problem than the corresponding problem in the affine case \mathbb{R}^n . In this paper we produce a transcendental submanifold of $\mathbb{R}\mathbb{P}^n$ in the sense of [AK5]. That is, we find a smooth submanifold of $\mathbb{R}\mathbb{P}^n$ which is isotopic to a nonsingular projective algebraic subset, but which can not be isotoped to the real part of any complex nonsingular algebraic subset of $\mathbb{C}\mathbb{P}^n$. This results generalizes the affine examples of [AK5] to the projective case. We want to thank MSRI for giving us the opportunity to work together.

1. Preliminaries

By a *closed* (sub)manifold we mean a compact (sub)manifold without boundary.

Let V be a real (or complex) algebraic set defined over \mathbb{R} , and let $R = \mathbb{Z}_2$ (or $R = \mathbb{Z}$). Then we can define algebraic homology groups $H_*^A(V; R)$ to be the subgroup of $H_*(V; R)$ generated by the compact real (or complex) algebraic subsets of V

*First named author is partially supported by NSF grant DMS 9971440

(cf. [AK1]). We define $H_A^*(V; R)$ to be the Poincaré duals of the groups $H_*^A(V; R)$ when defined. By the resolution of singularities theorem ([H]), $H_*^A(V; R)$ is also the subgroup generated by the classes $g_*([S])$ where $g: S \rightarrow V$ is an entire rational function, S is a compact nonsingular real (or complex) algebraic set and $[S]$ is the fundamental class of S . Therefore even when V is real, we can define $H_*^A(V; \mathbb{Z})$ to be the subgroup generated by $g_*([S])$ where $g: S \rightarrow V$ is an entire rational function from an oriented compact nonsingular real algebraic set and $[S]$ is the fundamental class of S .

Now let $V \subset \mathbb{R}P^n$ be a nonsingular projective real algebraic set of dimension v , and suppose its complexification $V_{\mathbb{C}} \subset \mathbb{C}P^n$ is nonsingular. Let $j: V \hookrightarrow V_{\mathbb{C}}$ denote the inclusion. Define $\bar{H}_{2k}^A(V_{\mathbb{C}}; \mathbb{Z})$ to be the subgroup of $H_{2k}^A(V_{\mathbb{C}}; \mathbb{Z})$ generated by irreducible complex algebraic subsets defined over \mathbb{R} with k -dimensional real parts. In other words it is generated by the complexification of k -dimensional real algebraic subsets of V in $V_{\mathbb{C}}$. Again by the resolution theorem, $\bar{H}_{2k}^A(V_{\mathbb{C}}; \mathbb{Z})$ is generated by the classes $g_*([L_{\mathbb{C}}])$, where $L_{\mathbb{C}}$ is an irreducible nonsingular complex projective algebraic set defined over \mathbb{R} with nonempty real part and $g: L_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$ is a regular map defined over \mathbb{R} . Let $\bar{H}_A^{2k}(V_{\mathbb{C}}; \mathbb{Z})$ denote the Poincaré dual of $\bar{H}_{2v-2k}^A(V_{\mathbb{C}}; \mathbb{Z})$. Define

$$\bar{H}_{\mathbb{C}\text{-alg}}^{2k}(V; \mathbb{Z}) = j^* \bar{H}_A^{2k}(V_{\mathbb{C}}; \mathbb{Z}).$$

Let $\bar{H}_{\mathbb{C}\text{-alg}}^{2k}(V; \mathbb{Z}_2)$ to be the mod 2 reduction of $\bar{H}_{\mathbb{C}\text{-alg}}^{2k}(V; \mathbb{Z})$ (under the obvious coefficient homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}_2$). Define the natural subgroup of $H_A^{2k}(V; \mathbb{Z}_2)$

$$H_A^k(V; \mathbb{Z}_2)^2 = \{ \alpha^2 \mid \alpha \in H_A^k(V; \mathbb{Z}_2) \}.$$

Recall that Theorem A (b) of [AK5] relates these groups to each other:

Theorem 1. *For all k the following holds: $\bar{H}_{\mathbb{C}\text{-alg}}^{2k}(V; \mathbb{Z}_2) = H_A^k(V; \mathbb{Z}_2)^2$.*

Let $M \subset V$ be a closed smooth submanifold of a nonsingular algebraic set V . The problem of whether M is isotopic to a nonsingular algebraic subset of V is an old one. If we allow stabilization (replacing V by $V \times \mathbb{R}^k$ for sufficiently large k) the problem becomes solvable ([N] and [T] for the affine case, [K] for the projective case, and [AK2] for the general case where there are obstructions).

If we don't allow stabilization the problem becomes much harder, in which the complexification of V begins to play an important role. In [AK3] it was shown that every closed smooth $M \subset \mathbb{R}^n$ can be isotoped to a nonsingular (topological) component of an algebraic subset of \mathbb{R}^n . More generally in [AK4] it was shown that any immersed submanifold of \mathbb{R}^n can be isotoped to a nonsingular algebraic subset of \mathbb{R}^n if and only if M is cobordant through immersions to an algebraic subset of \mathbb{R}^n . Here we need a very special case of these theorems, the one which allows us to isotop some submanifolds of $\mathbb{R}P^n$ (the examples in the next section) to projectively closed

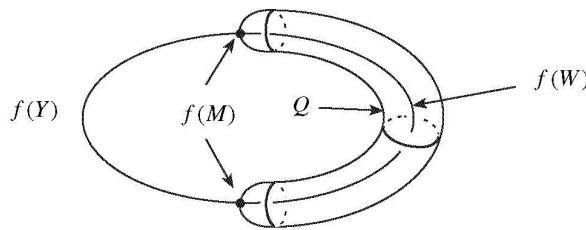
algebraic subsets. Obviously the following lemmas hold in more general contexts, but to make the examples of the main theorem in the next section transparent, we chose to state them in this special form, which is enough to prove the theorem.

Lemma 2. *Every closed codimension one submanifold of \mathbb{R}^k can be C^∞ approximated by a nonsingular projectively closed algebraic subset.*

Proof. This for example is proven in [K] (also Theorem 2.8.2 of [AK1]), but can also be seen from Seifert’s original proof [S] by noting that the highest degree terms of the polynomial he constructs are a constant times $|x|^{2n}$ (clearly the zeros of such polynomials are projectively closed algebraic sets). \square

Lemma 3. *Let $M^m \subset Y^{m+1}$ be closed smooth manifolds with M separating Y . Let $f : Y \rightarrow \mathbb{R}^k$ be a smooth imbedding. Then $f(M)$ is isotopic to a nonsingular projectively closed subvariety of \mathbb{R}^k . In particular, by viewing $f(M)$ as a submanifold of $\mathbb{R}\mathbb{P}^k$ via the natural inclusion $\mathbb{R}^k \subset \mathbb{R}\mathbb{P}^k$, $f(M)$ is isotopic to a nonsingular projective algebraic subvariety of $\mathbb{R}\mathbb{P}^k$.*

Proof. By [AK3] we can isotope $f(Y)$ to a nonsingular topological component Y' of a real algebraic subvariety of \mathbb{R}^k . Let $W \subset Y$ be one of the codimension zero components of $Y - M$, and let $Q^{k-1} \subset \mathbb{R}^k$ be the boundary of a small tubular neighborhood of $f(W)$ in \mathbb{R}^k . By Lemma 2 above we can isotope Q to a projectively closed nonsingular algebraic subvariety Q' of \mathbb{R}^k which is C^1 close to Q . Then $V := Y' \cap Q'$ gives the desired variety. V is isotopic to $f(M)$ by transversality. V is projectively closed since Q' is projectively closed. \square



2. Transcendental submanifolds

In [AK5] we constructed smooth submanifolds $M \subset \mathbb{R}^n$ which are isotopic to nonsingular algebraic subsets of \mathbb{R}^n , but not isotopic to the real parts of nonsingular complex algebraic subsets of $\mathbb{C}\mathbb{P}^n$. So, this means that either every algebraic model of M in \mathbb{R}^n develops extra components at infinity when its Zariski closure is taken in $\mathbb{R}\mathbb{P}^n$ (i.e. it

is not projectively closed), or M admits nonsingular algebraic models in $\mathbb{R}P^n$ but the complexifications all such models necessarily contain singular points in $\mathbb{C}P^n$. The following theorem eliminates the first possibility, hence it gives genuine topological obstructions to moving smooth submanifolds of $\mathbb{R}P^n$ to nonsingular algebraic sets in the strong sense (i.e. no singularities in complexification).

Theorem 4. *There are closed smooth submanifolds $M \subset \mathbb{R}P^n$ which can be approximated (via a small isotopy) by nonsingular subvarieties of $\mathbb{R}P^n$, but they can not be isotoped to the real parts of nonsingular complex algebraic subvarieties of $\mathbb{C}P^n$ defined over \mathbb{R} .*

Proof. By [MM] for any s there is an m such that we have an imbedding $\mathbb{R}P^m \subset \mathbb{R}^{2m-s}$. Let $M = \mathbb{R}P^m \times S^1 \subset \mathbb{R}^{2m-s} \times \mathbb{R}^3 = \mathbb{R}^{2m+3-s} \subset \mathbb{R}P^n$, where $n = 2m + 3 - s$. Let $Y = \mathbb{R}P^m \times S^2$ in $\mathbb{R}P^n$, so by Lemma 2 above M can be isotoped to a nonsingular projectively closed algebraic subset V^v of $\mathbb{R}P^n$ where $v = m + 1$.

We claim that when $s \geq 3$ when m is even, and $s \geq 5$ when $m = 4k + 1$, V can not be the real part of a nonsingular complex algebraic subset $V_{\mathbb{C}}$ of $\mathbb{C}P^n$ (defined over \mathbb{R}). Suppose such a $V_{\mathbb{C}}$ exists. By a version of the Lefschetz hyperplane theorem due to Larsen [Hr], for $i \leq 2v - n = s - 1$, the restriction induces an isomorphism

$$H^i(\mathbb{C}P^n; \mathbb{Z}) \xrightarrow{\cong} H^i(V_{\mathbb{C}}; \mathbb{Z}).$$

In particular when $i \leq s - 1$, the group $\bar{H}_{\mathbb{C}\text{-alg}}^i(V; \mathbb{Z})$ lies in the image of $H^i(\mathbb{R}P^n; \mathbb{Z})$ under restriction. But since V lies in a chart \mathbb{R}^n of $\mathbb{R}P^n$, from the diagram

$$\begin{CD} H^i(V_{\mathbb{C}}; \mathbb{Z}_2) @>j^*>> H^i(V; \mathbb{Z}_2) \\ @A\cong AA @AA \\ H^i(\mathbb{C}P^n; \mathbb{Z}_2) @>j^*>> H^i(\mathbb{R}P^n; \mathbb{Z}_2) \end{CD}$$

we conclude that $\bar{H}_{\mathbb{C}\text{-alg}}^i(V; \mathbb{Z}_2) = 0$ for $0 < i \leq s - 1$. On the other hand by [AK1] the Stiefel–Whitney classes of V are represented by real algebraic subsets since V is a nonsingular real algebraic set. Hence, when m is even $w_1(V) = \alpha \times 1$ is algebraic, also when $m = 4k + 1$ then $w_2(V) = \alpha^2 \times 1$ is algebraic, where α is the generator of $H^1(\mathbb{R}P^m; \mathbb{Z}_2)$. So when m is even and $s \geq 3$, or when $m = 4k + 1$ and $s \geq 5$ we get a contradiction to Theorem 1 above, for example when m is even

$$0 \neq \alpha^2 \times 1 = w_1^2(M) \in H_A^1(V; \mathbb{Z}_2)^2 = \bar{H}_{\mathbb{C}\text{-alg}}^2(V; \mathbb{Z}_2) = 0.$$

We should point out that the above mentioned theorem of [MM] actually implies that there are imbeddings $\mathbb{R}P^m \subset \mathbb{R}^{2m-s}$ for the pairs (m, s) with m even and $s \geq 3$, or $m = 4k + 1$ and $s \geq 5$. This is what is used in the proof. \square

Remark 5. Recall that on any nonsingular real algebraic variety structure V of a smooth manifold M , the Stiefel–Whitney classes, and mod 2 reductions of Pontryagin classes of M are algebraic in V ([AK1], [AK5]). So the examples in the above theorem generalize in many directions. For example we can take M^v to be any closed smooth manifold which admits a separating imbedding into a closed manifold Y^{v+1} , such that $Y^{v+1} \subset \mathbb{R}^{2v-q}$ with $2i \leq q$, and $\alpha \in H^i(M; \mathbb{Z}_2)$ such that $\alpha^2 \neq 0$ and α lies in the subring of the cohomology group generated by Stiefel–Whitney and Pontryagin classes.

We thank the referees for helpful comments.

References

- [AK1] S. Akbulut and H. King, The topology of real algebraic sets with isolated singularities. *Ann. of Math.* **113** (1981), 425–446. Zbl 0494.57004 MR 0621011
- [AK2] S. Akbulut and H. King, *Topology of real algebraic sets*. Math. Sci. Res. Inst. Publ. 25, Springer-Verlag, New York 1992. Zbl 0808.14045 MR 1225577
- [AK3] S. Akbulut and H. King, On approximating submanifolds by algebraic sets and a solution to the Nash conjecture. *Invent. Math.* **107** (1992), 87–98. Zbl 0781.14038 MR 1135465
- [AK4] S. Akbulut and H. King, Algebraicity of immersions in \mathbb{R}^n . *Topology* **31** (1992), 701–712. Zbl 0788.57019 MR 1191374
- [AK5] S. Akbulut and H. King, Transcendental submanifolds of \mathbb{R}^n . *Comment. Math. Helv.* **68** (1993), 308–318. Zbl 0806.57017 MR 1214234
- [Hr] R. Hartshorne, Equivalence relations on algebraic cycles and subvarieties of small codimension. *Proc. Sympos. Pure Math.* **29** (1975), 129–164. Zbl 0314.14001 MR 0369359
- [H] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero I, II. *Ann. of Math.* **79** (1964), 109–203; *ibid* 205–326. Zbl 0122.38603 MR 0199184
- [K] H. King, Approximating submanifolds of real projective space by varieties. *Topology* **15** (1976), 81–85. Zbl 0316.57015 MR 0396572
- [MM] M. Mahowald and J. Milgram, Embedding real projective spaces. *Ann. of Math.* **87** (1968), 411–422. Zbl 0172.25602 MR 0226656
- [N] J. Nash, Real algebraic manifolds. *Ann. of Math.* **56** (1952), 405–421. Zbl 0048.38501 MR 0050928
- [S] H. Seifert, Algebraische Approximation von Mannigfaltigkeiten. *Math. Z.* **41** (1936), 1–17. Zbl 0013.36902
- [T] A. Tognoli, Su una congettura di Nash. *Ann. Sci. Norm. Sup. Pisa* **27** (1973), 167–185. Zbl 0263.57011 MR 0396571

Received April 26, 2004

Selman Akbulut, Department of Mathematics, University of Maryland, MD, 20742, U.S.A.
E-mail: akbulut@math.msu.edu

Henry King, Department of Mathematics, Michigan State University, MI, 48824, U.S.A.
E-mail: hck@math.md.edu