

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 80 (2005)

Artikel: Vanishing and non-vanishing for the first L_p -cohomology of groups
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DOI: <https://doi.org/10.5169/seals-60447>

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Vanishing and non-vanishing for the first L^p -cohomology of groups

Marc Bourdon, Florian Martin and Alain Valette

Abstract. We prove two results on the first L^p -cohomology $\bar{H}_{(p)}^1(\Gamma)$ of a finitely generated group Γ :

1) If $N \subset H \subset \Gamma$ is a chain of subgroups, with N non-amenable and normal in Γ , then $\bar{H}_{(p)}^1(\Gamma) = 0$ as soon as $\bar{H}_{(p)}^1(H) = 0$. This allows for a short proof of a result of W. Lück: if $N \triangleleft \Gamma$, N is infinite, finitely generated as a group, and Γ/N contains an element of infinite order, then $\bar{H}_{(2)}^1(\Gamma) = 0$.

2) If Γ acts isometrically, properly discontinuously on a proper CAT(−1) space X , with at least 3 limit points in ∂X , then for p larger than the critical exponent $e(\Gamma)$ of Γ in X , one has $\bar{H}_{(p)}^1(\Gamma) \neq 0$. As a consequence we extend a result of Y. Shalom: let G be a cocompact lattice in a rank 1 simple Lie group; if G is isomorphic to Γ , then $e(G) \leq e(\Gamma)$.

Mathematics Subject Classification (2000). 20J06, 43A07, 43A15, 57M07.

Keywords. Group cohomology, L^p -cohomology, CAT(−1) space, critical exponent.

1. Introduction

Let Γ be a countable group. Assume first that Γ admits a $K(\Gamma, 1)$ -space which is a simplicial complex X finite in every dimension. Let \tilde{X} be the universal cover of X . Fix $p \in [1, \infty[$. Denote by $\ell^p C^k$ the space of p -summable complex k -cochains on \tilde{X} , i.e. the ℓ^p -functions on the set C^k of k -simplices of \tilde{X} . The L^p -cohomology of Γ is the reduced cohomology of the complex

$$d_k: \ell^p C^k \rightarrow \ell^p C^{k+1},$$

where d_k is the simplicial coboundary operator; we denote it by

$$\bar{H}_{(p)}^k(\Gamma) = \text{Ker } d_k / \overline{\text{Im } d_{k-1}}.$$

As explained at the beginning of [Gro93], this definition only depends on Γ .

For $p = 2$, the space $\bar{H}_{(2)}^k(\Gamma)$ is a module over the von Neumann algebra of Γ , and its von Neumann dimension is the k -th L^2 -Betti number of Γ , denoted by $b_{(2)}^k(\Gamma)$; recall that $b_{(2)}^k(\Gamma) = 0$ if and only if $\bar{H}_{(2)}^k(\Gamma) = 0$.

For $k = 1$, it is possible to define the first L^p -cohomology of Γ under the mere assumption that Γ is finitely generated. Denote by $\mathcal{F}(\Gamma)$ the space of all complex-valued functions on Γ , and by λ_Γ the left regular representation of Γ on $\mathcal{F}(\Gamma)$. Define then the space of p -Dirichlet finite functions on Γ :

$$D_p(\Gamma) = \{f \in \mathcal{F}(\Gamma) \mid \lambda_\Gamma(g)f - f \in \ell^p(\Gamma) \text{ for every } g \in \Gamma\}.$$

If S is a finite generating set of Γ , define a norm on $D_p(\Gamma)/\mathbb{C}$ by:

$$\|f\|_{D_p}^p = \sum_{s \in S} \|\lambda_\Gamma(s)f - f\|_p^p.$$

Denote by $i: \ell^p(\Gamma) \rightarrow D_p(\Gamma)$ the inclusion. The first L^p -cohomology of Γ is

$$\bar{H}_{(p)}^1(\Gamma) = D_p(\Gamma) / \overline{i(\ell^p(\Gamma)) + \mathbb{C}}.$$

Let us recall briefly why this definition is coherent with the previous one. If Γ admits a finite $K(\Gamma, 1)$ -space X , we can choose one such that the 1-skeleton of \tilde{X} is a Cayley graph $\mathcal{G}(\Gamma, S)$ of Γ . This means that S is some finite generating subset of Γ , that $C^0 = \Gamma$, and that C^1 is the set \mathbb{E}_Γ of oriented edges:

$$\mathbb{E}_\Gamma = \{(x, sx) \mid x \in \Gamma, s \in S\}.$$

Then d_0 is the restriction to $\ell^p(\Gamma)$ of the coboundary operator

$$d_\Gamma: \mathcal{F}(\Gamma) \rightarrow \mathcal{F}(\mathbb{E}_\Gamma); \quad f \mapsto [(x, y) \mapsto f(y) - f(x)].$$

Since \tilde{X} is contractible, by Poincaré's lemma any closed cochain is exact, i.e. any element in $\text{Ker } d_1$ can be written as $d_\Gamma f$, for some $f \in D_p(\Gamma)$ defined up to an additive constant. This means that $d_\Gamma: D_p(\Gamma) \rightarrow \ell^p(\mathbb{E}_\Gamma)$ induces an isomorphism of Banach spaces $D_p(\Gamma)/\mathbb{C} \rightarrow \text{Ker } d_1$, which maps $i(\ell^p(\Gamma))$ to $\text{Im } d_0$. This shows the equivalence of both definitions of $\bar{H}_{(p)}^1(\Gamma)$.

Our first result is:

Theorem 1. *Let $N \subset H \subset \Gamma$ be a chain of groups, with H and Γ finitely generated, N infinite and normal in Γ .*

- 1) *If H is non-amenable and $\bar{H}_{(p)}^1(H) = 0$, then $\bar{H}_{(p)}^1(\Gamma) = 0$.*
- 2) *If $b_{(2)}^1(H) = 0$, then $b_{(2)}^1(\Gamma) = 0$.*

We do *not* know whether part 1) of Theorem 1 holds when H is amenable¹.

As an application of part 2) of Theorem 1, we will give a very short proof of the following result of W. Lück (Theorem 0.7 in [Lue97]):

Corollary 1. *Let Γ be a finitely generated group. Assume that Γ contains an infinite, normal subgroup N , which is finitely generated as a group, and such that Γ/N is not a torsion group. Then $b_{(2)}^1(\Gamma) = 0$.*

Using his theory of L^2 -Betti numbers for equivalence relations and group actions, D. Gaboriau was able to improve the previous result by merely assuming that Γ/N is infinite (see [Gab02], Théorème 6.8). It is a challenging, and vaguely irritating question, to find a purely group cohomological proof of Gaboriau's result.

As shown by Gaboriau's result, non-vanishing of $\bar{H}_{(2)}^1$ is an obstruction for the existence of finitely generated normal subgroups. We now present a non-vanishing result. Its proof is based on an idea due to G. Elek (see [Ele97], Theorem 2).

Let X be a proper CAT(−1) space (see [BH99] for the definitions), and let Γ be an infinite, finitely generated, properly discontinuous subgroup of isometries of X . Recall that the *critical exponent* of Γ is defined as

$$e(\Gamma) = \inf \left\{ s > 0 \mid \sum_{g \in \Gamma} e^{-s|go-o|} < +\infty \right\},$$

where o is any origin in X , and where $|\cdot - \cdot|$ denotes the distance in X . In many cases, $e(\Gamma) < +\infty$; in particular, this happens when the isometry group of X is co-compact (see Proposition 1.7 in [BM96]).

Theorem 2. *Assume that $e(\Gamma)$ is finite. If the limit set of Γ in ∂X has at least 3 points, then for $p > \max\{1, e(\Gamma)\}$ the Banach space $\bar{H}_{(p)}^1(\Gamma)$ is non zero.*

When Γ is in addition co-compact, Theorem 2 was already known to Pansu and Gromov (see [Pan89] and page 258 in [Gro93]).

Theorem 2 is optimal for the co-compact lattices in rank one semi-simple Lie group: for those $p > e(\Gamma)$ if and only if $\bar{H}_{(p)}^1(\Gamma) \neq 0$, thanks to a result of Pansu [Pan89]. Recall that $e(\Gamma) = 1$ for lattices Γ in $\mathrm{SO}(2, 1)$ (and exactly for those among rank one lattices). Since L^p -cohomology of groups is an invariant of isomorphism, by combining Pansu's result with Theorem 2, we obtain the following generalisation of a result of Shalom (Theorem 1.1 in [Sha00]):

Corollary 2. *Let G be a co-compact lattice in a rank one semi-simple Lie group (other than $\mathrm{SO}(2, 1)$). Assume that G is isomorphic to a properly discontinuous subgroup Γ of isometries of a proper CAT(−1) space X . Then $e(G) \leq e(\Gamma)$. \square*

¹When $p = 2$ and H is amenable, we appeal to the Cheeger–Gromov vanishing theorem [CG86]; to the best of our knowledge, there is no analogue of this result in L^p -cohomology for $p \neq 2$, although Gromov notices in Remark (A₂) of [Gro93], 8.A₁, that it should be the case.

Shalom established this by different methods in the special case where X is the symmetric space associated to $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$; his result also holds for non-cocompact lattices (when the Lie group is different from $\mathrm{SO}(2, 1)$). In [BCG99] the authors establish Corollary 2 in the case Γ is quasi-convex, this assumption simplifies their proof but they do not really need it.

The equality case in Corollary 2, which leads to a rigidity theorem, is studied in [Bou96] and [Yue96] and in [BCG99], when Γ is in addition quasi-convex. Again methods of proofs developed in [BCG99] should apply without the quasi-convex assumption.

2. Group cohomology; proof of Theorem 1

Let V be a topological Γ -module, i.e. a real or complex topological vector space endowed with a continuous, linear representation $\pi : \Gamma \times V \rightarrow V : (g, v) \mapsto \pi(g)v$. If H is a subgroup of Γ , we denote by $V|_H$ the space V viewed as an H -module for the restricted action, and by V^H the set of H -fixed points:

$$V^H = \{v \in V \mid \pi(h)v = v \text{ for all } h \in H\}.$$

We now introduce the space of 1-cocycles and 1-coboundaries on Γ , and the 1-cohomology with coefficients in V :

- $Z^1(\Gamma, V) = \{b : \Gamma \rightarrow V \mid b(gh) = b(g) + \pi(g)b(h) \text{ for all } g, h \in \Gamma\};$
- $B^1(\Gamma, V) = \{b \in Z^1(\Gamma, V) \mid \text{there exists } v \in V \text{ such that } b(g) = \pi(g)v - v \text{ for all } g \in \Gamma\};$
- $H^1(\Gamma, V) = Z^1(\Gamma, V)/B^1(\Gamma, V).$

Suppose that V is a Banach space. The space $Z^1(\Gamma, V)$ of 1-cocycles is a Fréchet space when endowed with the topology of pointwise convergence on Γ . The 1-reduced cohomology space with coefficients in V is

$$\overline{H^1}(\Gamma, V) = Z^1(\Gamma, V) / \overline{B^1(\Gamma, V)}.$$

Recall that V *almost has invariant vectors* if, for every finite subset F in Γ , and every $\epsilon > 0$, there exists a vector v of norm 1 in V , such that $\|\pi(g)v - v\| < \epsilon$ for every $g \in F$. The following result is due to Guichardet (Theorem 1 and Corollary 1 in [Gui72]).²

Proposition 1. *Let Γ be a countable group.*

²Strictly speaking, Guichardet proves this result for unitary Γ -modules; but his proof, only appealing to the Banach isomorphism theorem, carries over without change to Banach Γ -modules.

1) Let V be a Banach Γ -module with $V^\Gamma = 0$. The map

$$H^1(\Gamma, V) \rightarrow \overline{H^1}(\Gamma, V)$$

is an isomorphism if and only if V does not almost have invariant vectors.

2) Let $p \in [1, \infty[$. Assume that Γ is infinite. The map

$$H^1(\Gamma, \ell^p(\Gamma)) \rightarrow \overline{H^1}(\Gamma, \ell^p(\Gamma))$$

is an isomorphism if and only if Γ is non-amenable. \square

We will prove:

Proposition 2. Let $p \in [1, \infty[$. Let $N \subset H \subset \Gamma$ be a chain of groups, where Γ finitely generated and N is infinite and normal in Γ . If $H^1(H, \ell^p(H)) = 0$, then $H^1(\Gamma, \ell^p(\Gamma)) = 0$.

The following link between $\overline{H^1}_{(p)}(\Gamma)$ and $H^1(\Gamma, \ell^p(\Gamma))$ has been noticed by several people – see e.g. Lemma 3 in [BV97] (for $p = 2$ and Γ non-amenable), or in [Pul03] (in general). We give the easy argument for completeness.

Lemma 1. For finitely generated Γ , there are isomorphisms

$$D_p(\Gamma)/(i(\ell^p(\Gamma)) + \mathbb{C}) \simeq H^1(\Gamma, \ell^p(\Gamma)) \quad \text{and} \quad \overline{H^1}_{(p)}(\Gamma) \simeq \overline{H^1}(\Gamma, \ell^p(\Gamma)).$$

Proof. The map $D_p(\Gamma) \rightarrow Z^1(\Gamma, \ell^p(\Gamma))$: $f \mapsto [g \mapsto \lambda_\Gamma(g)f - f]$ is continuous, with kernel the space \mathbb{C} of constant functions, and the image of $i(\ell^p(\Gamma))$ is exactly $B^1(\Gamma, \ell^p(\Gamma))$. Moreover this map is onto because of the classical fact that $H^1(\Gamma, \mathcal{F}(\Gamma)) = 0$. \square

Before proving Proposition 2 (for which we will actually give two proofs), we explain how to deduce Theorem 1 from it.

Proof of Theorem 1 from Proposition 2. 1) In view of Lemma 1, the assumption of Theorem 1 reads $\overline{H^1}(H, \ell^p(H)) = 0$. Since H is non-amenable, by Proposition 1 we have $H^1(H, \ell^p(H)) = 0$. By Proposition 2 we deduce $H^1(\Gamma, \ell^p(\Gamma)) = 0$. By Lemma 1 again, we get the conclusion.

2) If H is non-amenable, the result is a particular case of the first part. If H is amenable, then so is N , and the result follows from the Cheeger–Gromov vanishing theorem [CG86]: if a group Γ contains an infinite, amenable, normal subgroup, then all L^2 -Betti numbers of Γ are zero. \square

Important remark. Cheeger and Gromov [CG86] defined L^2 -Betti numbers of a group Γ without any assumption on Γ , in particular not assuming Γ to be finitely generated. Using their definition, D. Gaboriau has shown us (private communication) a proof that $b_{(2)}^1(\Gamma) = 0$ always implies $\overline{H}^1(\Gamma, \ell^2(\Gamma)) = 0$. As a consequence, part 2) of Theorem 1 holds *even if H is not finitely generated*.

Our first proof of Proposition 2 will require the following lemma, which is classical for $p = 2$.

Lemma 2. *Let $p \in [1, \infty[$. Let H be a countable group. Let X be a countable set on which H acts freely. The following statements are equivalent:*

- i) H is amenable.
- ii) The permutation representation λ_X of H on $\ell^p(X)$, almost has invariant vectors.

Proof. We recall (see [Eym72]) that a group Γ is amenable if and only if it satisfies Reiter's condition (P_p) , i.e. for every finite subset $F \subset \Gamma$ and $\epsilon > 0$, there exists $f \in \ell^p(\Gamma)$ such that $f \geq 0$, $\|f\|_p = 1$, and $\|\lambda_\Gamma(g)f - f\|_p < \epsilon$ for $g \in F$. In particular $\ell^p(\Gamma)$ almost has invariant vectors.

So if H is amenable, then $\ell^p(X)$ almost has invariant vectors since it contains $\ell^p(H)$ as a sub-module. This proves (i) \Rightarrow (ii).

To prove (ii) \Rightarrow (i), we assume that $\ell^p(X)$ almost has invariant vectors and prove in 3 steps that H satisfies Reiter's property (P_1) , so is amenable. So fix a finite subset $F \subset H$, and $\epsilon > 0$; find $f \in \ell^p(X)$, $\|f\|_p = 1$, such that $\|\lambda_X(h)f - f\|_p < \frac{\epsilon}{2p}$ for $h \in F$.

1) Replacing f with $|f|$, we may assume that $f \geq 0$.

2) Set $g = f^p$, so that $g \in \ell^1(X)$, $\|g\|_1 = 1$, $g \geq 0$. For $h \in F$, we have:

$$\begin{aligned} \|\lambda_X(h)g - g\|_1 &= \sum_{x \in X} |f(h^{-1}x)^p - f(x)^p| \\ &\leq p \sum_{x \in X} |f(h^{-1}x) - f(x)| (f(h^{-1}x)^{p-1} + f(x)^{p-1}) \\ &\leq p \left(\sum_{x \in X} |f(h^{-1}x) - f(x)|^p \right)^{\frac{1}{p}} \left(\sum_{x \in X} (f(h^{-1}x)^{p-1} + f(x)^{p-1})^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\ &\leq p \|\lambda_X(h)f - f\|_p \left(2^{\frac{1}{p-1}} \sum_{x \in X} (f(h^{-1}x)^p + f(x)^p) \right)^{\frac{p-1}{p}} \\ &= 2p \|\lambda_X(h)f - f\|_p < \epsilon \end{aligned}$$

where we have used consecutively³ the inequalities

³The expert will recognize here the argument to pass from property (P_p) to property (P_1) , as in [Eym72].

- $|a^p - b^p| \leq p|a - b|(a^{p-1} + b^{p-1})$ for $a, b > 0$,
- Hölder's inequality,
- $(a + b)^{\frac{p}{p-1}} \leq 2^{\frac{1}{p}}(a^{\frac{p}{p-1}} + b^{\frac{p}{p-1}})$ for $a, b > 0$,

and the fact that $\|f\|_p = 1$.

3) Let $(x_n)_{n \geq 1}$ be a set of representatives for the orbits of H in X . Define a function g_n on H by $g_n(h) = g(hx_n)$, and set $G = \sum_{n=1}^{\infty} g_n$. Then $G \geq 0$ and $\|G\|_1 = \sum_{h \in H} \sum_{n=1}^{\infty} g(hx_n) = \sum_{x \in X} g(x) = 1$. Moreover, for $h \in F$:

$$\|\lambda_H(h)G - G\|_1 = \sum_{\gamma \in H} \left| \sum_{n=1}^{\infty} (g(h^{-1}\gamma x_n) - g(\gamma x_n)) \right| \leq \|\lambda_X(h)g - g\| < \epsilon$$

by the previous step. This establishes property (P_1) for H . \square

First proof of Proposition 2 (homological algebra)

Claim. $H^1(H, \ell^p(\Gamma)|_H) = 0$. Choosing representatives for the right cosets of H in Γ , we identify $\ell^p(\Gamma)|_H$ in an H -equivariant way with the ℓ^p -direct sum $\oplus \ell^p(H)$ of $[\Gamma : H]$ copies of $\ell^p(H)$. Since cohomology commutes with finite direct sums, the claim is clear if $[\Gamma : H] < \infty$. So assume that $[\Gamma : H] = \infty$. If $b \in Z^1(H, \ell^p(\Gamma)|_H)$, write $b = (b_k)_{k \geq 1}$ where $b_k \in Z^1(H, \ell^p(H))$ for every $k \geq 1$. By assumption, for each k , there is a function $f_k \in \ell^p(H)$ such that $b_k(h) = \lambda_H(h)f_k - f_k$ for every $h \in H$. Set

$$B_N(h) = (\lambda_H(h)f_1 - f_1, \dots, \lambda_H(h)f_N - f_N, 0, 0, \dots)$$

so that $B_N \in B^1(H, \ell^p(\Gamma)|_H)$ and B_N converges to b pointwise on H , for $N \rightarrow \infty$. This already shows that $\overline{H^1(H, \ell^p(\Gamma)|_H)} = 0$. Notice now that, by Proposition 1 (2), the assumption $H^1(H, \ell^p(H)) = 0$ implies that H is non-amenable. By Lemma 2 applied to $X = \Gamma$, this means that $\ell^p(\Gamma)|_H$ does not almost have invariant vectors. By Proposition 1 (1), we get $H^1(H, \ell^p(\Gamma)|_H) = 0$, proving the claim.

Recall from group cohomology (see e.g. 8.1 in [Gui80]) that, for any Γ -module V , there is an exact sequence

$$0 \rightarrow H^1(\Gamma/N, V^N) \xrightarrow{i_*} H^1(\Gamma, V) \xrightarrow{\text{Rest}_\Gamma^N} H^1(N, V|_N)^{\Gamma/N}$$

where $i : V^N \rightarrow V$ denotes the inclusion. In particular, if $V^N = 0$, then the restriction map

$$\text{Rest}_\Gamma^N : H^1(\Gamma, V) \rightarrow H^1(N, V|_N)$$

is injective. We apply this with $V = \ell^p(\Gamma)$ (noticing that $V^N = 0$ as N is infinite).

Consider then the composition of restriction maps

$$H^1(\Gamma, \ell^p(\Gamma)) \xrightarrow{\text{Rest}_\Gamma^H} H^1(H, \ell^p(\Gamma)|_H) \xrightarrow{\text{Rest}_H^N} H^1(N, \ell^p(\Gamma)|_N);$$

this composition is Rest_Γ^N , which is injective as we just saw. On the other hand, by the claim this composition is also the zero map. So $H^1(\Gamma, \ell^p(\Gamma)) = 0$, as was to be established. \square

Second proof of Proposition 2 (geometry). This proof works under the extra assumption that H is finitely generated. Fix finite generating sets T for H , S for Γ , with $T \subset S$, and consider the Cayley graph $\mathcal{G}(\Gamma, S)$ and its coboundary operator $d_\Gamma: \mathcal{F}(\Gamma) \rightarrow \mathcal{F}(\mathbb{E}_\Gamma)$. Then $D_p(\Gamma) = \{f \in \mathcal{F}(\Gamma) : d_\Gamma f \in \ell^p(\mathbb{E}_\Gamma)\}$. Similarly, let d_H be the coboundary operator associated with the Cayley graph $\mathcal{G}(H, T)$.

Fix $f \in D_p(\Gamma)$; the goal is to show that $f \in \ell^p(\Gamma) + \mathbb{C}$. Let $(g_i)_{i \in I}$ be a set of representatives for the right cosets of H in Γ , so that $\Gamma = \bigsqcup_{i \in I} Hg_i$. For $i \in I$, set $f_i(x) = f(xg_i)$ ($x \in H$). Then

$$\begin{aligned} \|d_H(f_i)\|_p^p &= \sum_{x \in H} \sum_{s \in T} |f(sxg_i) - f(xg_i)|^p \\ &\leq \sum_{x \in \Gamma} \sum_{s \in S} |f(sx) - f(x)|^p \\ &= \|d_\Gamma f\|^p < \infty, \end{aligned}$$

i.e. $f_i \in D_p(H)$. Using our assumption and Lemma 1, we may write

$$f_i = h_i + u_i$$

where $h_i \in \ell^p(H)$ and $u_i \in \mathbb{C}$. Define functions h and u on Γ by $h(xg_i) = h_i(x)$ and $u(xg_i) = u_i$ ($x \in H$).

First claim. $h \in \ell^p(\Gamma)$. Indeed, since H is non-amenable (by Proposition 1), there exists a constant $C > 0$ (depending only on p, H, T) such that for every $i \in I$:

$$\|h_i\|_p \leq C \|d_H(h_i)\|_p.$$

Then summing over i we obtain

$$\begin{aligned} \|h\|_p^p &= \sum_{i \in I} \|h_i\|_p^p \\ &\leq C^p \sum_{i \in I} \|d_H(f_i)\|_p^p = C^p \sum_{i \in I} \sum_{x \in H} \sum_{s \in T} |h_i(sx) - h_i(x)|^p \\ &= C^p \sum_{i \in I} \sum_{x \in H} \sum_{s \in T} |f_i(sx) - f_i(x)|^p = C^p \sum_{x \in \Gamma} \sum_{s \in T} |f(sx) - f(x)|^p \end{aligned}$$

$$\begin{aligned} &\leq C^p \sum_{x \in \Gamma} \sum_{s \in S} |f(sx) - f(x)|^p \\ &= C^p \|d_\Gamma(f)\|_p^p < \infty. \end{aligned}$$

Second claim. u is constant. Indeed, since $f = h + u$, and $d_\Gamma(f), d_\Gamma(h) \in \ell^p(\mathbb{E}_\Gamma)$, we have $d_\Gamma(u) \in \ell^p(\mathbb{E}_\Gamma)$. In particular this implies, for fixed indices $i, j \in I$:

$$\begin{aligned} \infty &> \sum_{x \in N} |u((g_j g_i^{-1})x g_i) - u(x g_i)|^p = \sum_{x \in N} |u((g_j g_i^{-1})x g_i) - u_i|^p \\ &= \sum_{x \in N} |u(x (g_j g_i^{-1}) g_i) - u_i|^p \end{aligned}$$

since N is normal in Γ . The latter sum is equal to

$$\sum_{x \in N} |u_j - u_i|^p < \infty.$$

Since N is infinite, this forces $u_i = u_j$, i.e. u is constant.

The first and the second claim together prove Proposition 2. \square

3. Some results of W. Lück

The following result was obtained by Lück in [Lue94], Theorem 2.1. We recall his short, elegant argument.

Lemma 3. *Let N be a finitely generated group, and let α be an automorphism of N . Let $H = N \rtimes_\alpha \mathbb{Z}$ be the corresponding semi-direct product. Then $b_{(2)}^1(H) = 0$.*

Proof. The proof depends on two classical properties of the L^2 -Betti numbers for a finitely generated group Γ :

- $b_2^1(\Gamma) \leq d(\Gamma)$, where $d(\Gamma)$ denotes the minimal number of generators of Γ ;
- if Λ is a subgroup of finite index d in Γ , then $b_{(2)}^k(\Lambda) = d \cdot b_{(2)}^k(\Gamma)$.

Let then $p: H \rightarrow \mathbb{Z}$ denote the quotient map; for $n \geq 1$, set $H_n = p^{-1}(n\mathbb{Z})$, a subgroup of index n in H . Then:

$$n \cdot b_{(2)}^1(H) = b_{(2)}^1(H_n) \leq d(H_n) \leq d(N) + 1.$$

Since this holds for every $n \geq 1$, the lemma follows. \square

Proof of Corollary 1. Since Γ/N is not a torsion group, we find a subgroup H of Γ , containing N , such that H/N is infinite cyclic. Since N is finitely generated, we have $b_{(2)}^1(H) = 0$, by Lemma 3. The result follows then immediately from Theorem 1. \square

Example 1. We point out that Lemma 3 has no analogue in L^p -cohomology, with $p \neq 2$. To see it, let M be a 3-dimensional, compact, hyperbolic manifold which fibers over the circle. Denote by Σ_g the fiber of that fibration: this is a closed Riemann surface of genus $g \geq 2$. Then the fundamental group $\Gamma = \pi_1(M)$ admits a semi-direct product decomposition $\Gamma = \pi_1(\Sigma_g) \rtimes \mathbb{Z}$, so that $\bar{H}_{(2)}^1(\Gamma) = 0$ by Lemma 2. However

$$\inf\{p \geq 1 : \bar{H}_{(p)}^1(\Gamma) \neq 0\} = 2,$$

as was proven by Pansu [Pan89].

4. Proof of Theorem 2

Denote by ∂X the (Gromov) boundary of X . Let $\Lambda = \overline{\Gamma o} \cap \partial X$ be the limit set of Γ in ∂X (the closure of Γo is taken in the compact set $X \cup \partial X$).

Since X is a $\text{CAT}(-1)$ space, its boundary carries a natural metric d (called a *visual metric*) which can be defined as follows (see [Bou95], Théorème 2.5.1); for every ξ and η in ∂X :

$$d(\xi, \eta) = e^{-(\xi|\eta)},$$

where $(\cdot | \cdot)$ denotes the Gromov product on ∂X based on o , namely

$$(\xi|\eta) = \lim_{(x,y) \rightarrow (\xi,\eta)} \frac{1}{2}(|o-x| + |o-y| - |x-y|).$$

Observe that there exists a constant B such that for every $g \in \Gamma$ there is a point ξ in ∂X with $d(go, [o, \xi]) \leq B$. Indeed this property does not depend on the choice of the origin o . So we choose o on a bi-infinite geodesic (η_1, η_2) . Then go belongs to $(g\eta_1, g\eta_2)$. Now since X is Gromov-hyperbolic, one of the two points $g\eta_1$ or $g\eta_2$ satisfies the claim.

Let u be a Lipschitz function of $(\partial X, d)$ which is non-constant on Λ ; such functions do exist since Λ is not reduced to a point. Following G. Elek [Ele97], let f be the function on Γ defined by $f(g) = u(\xi_g)$, where ξ_g is a point in ∂X such that $d(g^{-1}o, [o, \xi_g]) \leq B$.

Claim. $f \in D_p(\Gamma)$ for $p > \max\{1, e(\Gamma)\}$. Indeed we have

$$\begin{aligned} \|f\|_{D_p}^p &= \sum_{s \in S} \sum_{g \in \Gamma} |f(sg) - f(g)|^p = \sum_{s \in S} \sum_{g \in \Gamma} |u(\xi_{sg}) - u(\xi_g)|^p \\ &\leq C \sum_{s \in S} \sum_{g \in \Gamma} [d(\xi_{sg}, \xi_g)]^p = C \sum_{g \in \Gamma} \sum_{s \in S} e^{-p(\xi_{sg}|\xi_g)} \\ &\leq D \sum_{g \in \Gamma} e^{-p|g^{-1}o - o|} < +\infty, \end{aligned}$$

where C, D are constants depending only on u, B and S . The details for the first inequality in the last line are the following. Observe that $|(sg)^{-1}o - g^{-1}o| = |s^{-1}o - o|$ is bounded above by an absolute constant. This implies that if x_g and x_{sg} respectively denote the points on $[o, \xi_g)$ and $[o, \xi_{sg})$ whose distance from o is equal to $|g^{-1}o - o|$, then $|x_g - x_{sg}|$ is bounded above by an absolute constant. Now with the triangle inequality

$$|x - y| \leq |x - x_{sg}| + |x_{sg} - x_g| + |x_g - y|,$$

and from the definition of the Gromov product, it follows that

$$(\xi_{sg}|\xi_g) \geq \frac{1}{2}(|o - x_{sg}| + |o - x_g| - |x_{sg} - x_g|),$$

so that $(\xi_{sg}|\xi_g)$ is bounded below by $|g^{-1}o - o|$ plus an absolute additive constant. This proves the claim.

Since Λ has at least 3 points, the group Γ is non-amenable (namely it is well-known that Λ is a minimal set, and that an amenable group stabilises one or two points in ∂X). So by Proposition 1 and by Lemma 1, we must prove that f does not belong to $i(\ell^p(\Gamma)) + \mathbb{C}$. Assume it does, then $f(g)$ tends to a constant number when the length of g in Γ tends to $+\infty$. This contradicts the fact that u is non-constant on Λ . \square

Acknowledgements. We thank G. Courtois, A. Karlsson, G. Mislin and M. Puls for useful comments on the first draft of this paper.

Note added in proof. The following example, suggested by F. Paulin, shows that Corollary 2 fails for lattices in $\mathrm{SO}(2, 1)$. Start with the free group \mathbb{F}_2 on two generators. Embed it as a lattice G in $\mathrm{SO}(2, 1)$, so that $e(G) = 1$. On the other hand, let X_λ be the regular tree of degree 4, with edge length $\lambda > 0$. This is a proper $\mathrm{CAT}(-1)$ space. Let \mathbb{F}_2 act as a properly discontinuous group Γ of isometries of X_λ , by viewing X_λ as the Cayley tree of \mathbb{F}_2 . Then $e(\Gamma) = \frac{\log 3}{\lambda}$, which is less than 1 for λ large enough.

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Received January 13, 2004

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