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## Vanishing and non-vanishing for the first $L^p$ -cohomology of groups

Marc Bourdon, Florian Martin and Alain Valette

**Abstract.** We prove two results on the first  $L^p$ -cohomology  $\overline{H}_{(p)}^1(\Gamma)$  of a finitely generated group  $\Gamma$ :

1) If  $N \subset H \subset \Gamma$  is a chain of subgroups, with  $N$  non-amenable and normal in  $\Gamma$ , then  $\overline{H}_{(p)}^1(\Gamma) = 0$  as soon as  $\overline{H}_{(p)}^1(H) = 0$ . This allows for a short proof of a result of W. Lück: if  $N \triangleleft \Gamma$ ,  $N$  is infinite, finitely generated as a group, and  $\Gamma/N$  contains an element of infinite order, then  $\overline{H}_{(2)}^1(\Gamma) = 0$ .

2) If  $\Gamma$  acts isometrically, properly discontinuously on a proper CAT(−1) space  $X$ , with at least 3 limit points in  $\partial X$ , then for  $p$  larger than the critical exponent  $e(\Gamma)$  of  $\Gamma$  in  $X$ , one has  $\overline{H}_{(p)}^1(\Gamma) \neq 0$ . As a consequence we extend a result of Y. Shalom: let  $G$  be a cocompact lattice in a rank 1 simple Lie group; if  $G$  is isomorphic to  $\Gamma$ , then  $e(G) \leq e(\Gamma)$ .

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### 1. Introduction

Let  $\Gamma$  be a countable group. Assume first that  $\Gamma$  admits a  $K(\Gamma, 1)$ -space which is a simplicial complex  $X$  finite in every dimension. Let  $\tilde{X}$  be the universal cover of  $X$ . Fix  $p \in [1, \infty[$ . Denote by  $\ell^p C^k$  the space of  $p$ -summable complex  $k$ -cochains on  $\tilde{X}$ , i.e. the  $\ell^p$ -functions on the set  $C^k$  of  $k$ -simplices of  $\tilde{X}$ . The  $L^p$ -cohomology of  $\Gamma$  is the reduced cohomology of the complex

$$d_k: \ell^p C^k \rightarrow \ell^p C^{k+1},$$

where  $d_k$  is the simplicial coboundary operator; we denote it by

$$\overline{H}_{(p)}^k(\Gamma) = \text{Ker } d_k / \overline{\text{Im } d_{k-1}}.$$

As explained at the beginning of [Gro93], this definition only depends on  $\Gamma$ .

For  $p = 2$ , the space  $\overline{H}_{(2)}^k(\Gamma)$  is a module over the von Neumann algebra of  $\Gamma$ , and its von Neumann dimension is the  $k$ -th  $L^2$ -Betti number of  $\Gamma$ , denoted by  $b_{(2)}^k(\Gamma)$ ; recall that  $b_{(2)}^k(\Gamma) = 0$  if and only if  $\overline{H}_{(2)}^k(\Gamma) = 0$ .

For  $k = 1$ , it is possible to define the first  $L^p$ -cohomology of  $\Gamma$  under the mere assumption that  $\Gamma$  is finitely generated. Denote by  $\mathcal{F}(\Gamma)$  the space of all complex-valued functions on  $\Gamma$ , and by  $\lambda_\Gamma$  the left regular representation of  $\Gamma$  on  $\mathcal{F}(\Gamma)$ . Define then the space of  $p$ -Dirichlet finite functions on  $\Gamma$ :

$$D_p(\Gamma) = \{f \in \mathcal{F}(\Gamma) \mid \lambda_\Gamma(g)f - f \in \ell^p(\Gamma) \text{ for every } g \in \Gamma\}.$$

If  $S$  is a finite generating set of  $\Gamma$ , define a norm on  $D_p(\Gamma)/\mathbb{C}$  by:

$$\|f\|_{D_p}^p = \sum_{s \in S} \|\lambda_\Gamma(s)f - f\|_p^p.$$

Denote by  $i: \ell^p(\Gamma) \rightarrow D_p(\Gamma)$  the inclusion. The first  $L^p$ -cohomology of  $\Gamma$  is

$$\overline{H}_{(p)}^1(\Gamma) = D_p(\Gamma) / \overline{i(\ell^p(\Gamma)) + \mathbb{C}}.$$

Let us recall briefly why this definition is coherent with the previous one. If  $\Gamma$  admits a finite  $K(\Gamma, 1)$ -space  $X$ , we can choose one such that the 1-skeleton of  $\tilde{X}$  is a Cayley graph  $\mathcal{G}(\Gamma, S)$  of  $\Gamma$ . This means that  $S$  is some finite generating subset of  $\Gamma$ , that  $C^0 = \Gamma$ , and that  $C^1$  is the set  $\mathbb{E}_\Gamma$  of oriented edges:

$$\mathbb{E}_\Gamma = \{(x, sx) \mid x \in \Gamma, s \in S\}.$$

Then  $d_0$  is the restriction to  $\ell^p(\Gamma)$  of the coboundary operator

$$d_\Gamma: \mathcal{F}(\Gamma) \rightarrow \mathcal{F}(\mathbb{E}_\Gamma); \quad f \mapsto [(x, y) \mapsto f(y) - f(x)].$$

Since  $\tilde{X}$  is contractible, by Poincaré’s lemma any closed cochain is exact, i.e. any element in  $\text{Ker } d_1$  can be written as  $d_\Gamma f$ , for some  $f \in D_p(\Gamma)$  defined up to an additive constant. This means that  $d_\Gamma: D_p(\Gamma) \rightarrow \ell^p(\mathbb{E}_\Gamma)$  induces an isomorphism of Banach spaces  $D_p(\Gamma)/\mathbb{C} \rightarrow \text{Ker } d_1$ , which maps  $i(\ell^p(\Gamma))$  to  $\text{Im } d_0$ . This shows the equivalence of both definitions of  $\overline{H}_{(p)}^1(\Gamma)$ .

Our first result is:

**Theorem 1.** *Let  $N \subset H \subset \Gamma$  be a chain of groups, with  $H$  and  $\Gamma$  finitely generated,  $N$  finite and normal in  $\Gamma$ .*

- 1) *If  $H$  is non-amenable and  $\overline{H}_{(p)}^1(H) = 0$ , then  $\overline{H}_{(p)}^1(\Gamma) = 0$ .*
- 2) *If  $b_{(2)}^1(H) = 0$ , then  $b_{(2)}^1(\Gamma) = 0$ .*

We do *not* know whether part 1) of Theorem 1 holds when  $H$  is amenable<sup>1</sup>.

As an application of part 2) of Theorem 1, we will give a very short proof of the following result of W. Lück (Theorem 0.7 in [Lue97]):

**Corollary 1.** *Let  $\Gamma$  be a finitely generated group. Assume that  $\Gamma$  contains an infinite, normal subgroup  $N$ , which is finitely generated as a group, and such that  $\Gamma/N$  is not a torsion group. Then  $b_{(2)}^1(\Gamma) = 0$ .*

Using his theory of  $L^2$ -Betti numbers for equivalence relations and group actions, D. Gaboriau was able to improve the previous result by merely assuming that  $\Gamma/N$  is infinite (see [Gab02], Théorème 6.8). It is a challenging, and vaguely irritating question, to find a purely group cohomological proof of Gaboriau’s result.

As shown by Gaboriau’s result, non-vanishing of  $\bar{H}_{(2)}^1$  is an obstruction for the existence of finitely generated normal subgroups. We now present a non-vanishing result. Its proof is based on an idea due to G. Elek (see [Ele97], Theorem 2).

Let  $X$  be a proper CAT(−1) space (see [BH99] for the definitions), and let  $\Gamma$  be an infinite, finitely generated, properly discontinuous subgroup of isometries of  $X$ . Recall that the *critical exponent* of  $\Gamma$  is defined as

$$e(\Gamma) = \inf \left\{ s > 0 \mid \sum_{g \in \Gamma} e^{-s|go-o|} < +\infty \right\},$$

where  $o$  is any origin in  $X$ , and where  $|\cdot - \cdot|$  denotes the distance in  $X$ . In many cases,  $e(\Gamma) < +\infty$ ; in particular, this happens when the isometry group of  $X$  is co-compact (see Proposition 1.7 in [BM96]).

**Theorem 2.** *Assume that  $e(\Gamma)$  is finite. If the limit set of  $\Gamma$  in  $\partial X$  has at least 3 points, then for  $p > \max\{1, e(\Gamma)\}$  the Banach space  $\bar{H}_{(p)}^1(\Gamma)$  is non zero.*

When  $\Gamma$  is in addition co-compact, Theorem 2 was already known to Pansu and Gromov (see [Pan89] and page 258 in [Gro93]).

Theorem 2 is optimal for the co-compact lattices in rank one semi-simple Lie group: for those  $p > e(\Gamma)$  if and only if  $\bar{H}_{(p)}^1(\Gamma) \neq 0$ , thanks to a result of Pansu [Pan89]. Recall that  $e(\Gamma) = 1$  for lattices  $\Gamma$  in  $SO(2, 1)$  (and exactly for those among rank one lattices). Since  $L^p$ -cohomology of groups is an invariant of isomorphism, by combining Pansu’s result with Theorem 2, we obtain the following generalisation of a result of Shalom (Theorem 1.1 in [Sha00]):

**Corollary 2.** *Let  $G$  be a co-compact lattice in a rank one semi-simple Lie group (other than  $SO(2, 1)$ ). Assume that  $G$  is isomorphic to a properly discontinuous subgroup  $\Gamma$  of isometries of a proper CAT(−1) space  $X$ . Then  $e(G) \leq e(\Gamma)$ .  $\square$*

<sup>1</sup>When  $p = 2$  and  $H$  is amenable, we appeal to the Cheeger–Gromov vanishing theorem [CG86]; to the best of our knowledge, there is no analogue of this result in  $L^p$ -cohomology for  $p \neq 2$ , although Gromov notices in Remark (A<sub>2</sub>) of [Gro93], 8.A<sub>1</sub>, that it should be the case.

Shalom established this by different methods in the special case where  $X$  is the symmetric space associated to  $\mathrm{SO}(n, 1)$  or  $\mathrm{SU}(n, 1)$ ; his result also holds for non-cocompact lattices (when the Lie group is different from  $\mathrm{SO}(2, 1)$ ). In [BCG99] the authors establish Corollary 2 in the case  $\Gamma$  is quasi-convex, this assumption simplifies their proof but they do not really need it.

The equality case in Corollary 2, which leads to a rigidity theorem, is studied in [Bou96] and [Yue96] and in [BCG99], when  $\Gamma$  is in addition quasi-convex. Again methods of proofs developed in [BCG99] should apply without the quasi-convex assumption.

## 2. Group cohomology; proof of Theorem 1

Let  $V$  be a topological  $\Gamma$ -module, i.e. a real or complex topological vector space endowed with a continuous, linear representation  $\pi : \Gamma \times V \rightarrow V : (g, v) \mapsto \pi(g)v$ . If  $H$  is a subgroup of  $\Gamma$ , we denote by  $V|_H$  the space  $V$  viewed as an  $H$ -module for the restricted action, and by  $V^H$  the set of  $H$ -fixed points:

$$V^H = \{v \in V \mid \pi(h)v = v \text{ for all } h \in H\}.$$

We now introduce the space of 1-cocycles and 1-coboundaries on  $\Gamma$ , and the 1-cohomology with coefficients in  $V$ :

- $Z^1(\Gamma, V) = \{b : \Gamma \rightarrow V \mid b(gh) = b(g) + \pi(g)b(h) \text{ for all } g, h \in \Gamma\}$ ;
- $B^1(\Gamma, V) = \{b \in Z^1(\Gamma, V) \mid \text{there exists } v \in V \text{ such that } b(g) = \pi(g)v - v \text{ for all } g \in \Gamma\}$ ;
- $H^1(\Gamma, V) = Z^1(\Gamma, V)/B^1(\Gamma, V)$ .

Suppose that  $V$  is a Banach space. The space  $Z^1(\Gamma, V)$  of 1-cocycles is a Fréchet space when endowed with the topology of pointwise convergence on  $\Gamma$ . The 1-reduced cohomology space with coefficients in  $V$  is

$$\overline{H^1}(\Gamma, V) = Z^1(\Gamma, V)/\overline{B^1(\Gamma, V)}.$$

Recall that  $V$  *almost has invariant vectors* if, for every finite subset  $F$  in  $\Gamma$ , and every  $\epsilon > 0$ , there exists a vector  $v$  of norm 1 in  $V$ , such that  $\|\pi(g)v - v\| < \epsilon$  for every  $g \in F$ . The following result is due to Guichardet (Theorem 1 and Corollary 1 in [Gui72]).<sup>2</sup>

**Proposition 1.** *Let  $\Gamma$  be a countable group.*

<sup>2</sup>Strictly speaking, Guichardet proves this result for unitary  $\Gamma$ -modules; but his proof, only appealing to the Banach isomorphism theorem, carries over without change to Banach  $\Gamma$ -modules.

1) Let  $V$  be a Banach  $\Gamma$ -module with  $V^\Gamma = 0$ . The map

$$H^1(\Gamma, V) \rightarrow \overline{H^1}(\Gamma, V)$$

is an isomorphism if and only if  $V$  does not almost have invariant vectors.

2) Let  $p \in [1, \infty[$ . Assume that  $\Gamma$  is infinite. The map

$$H^1(\Gamma, \ell^p(\Gamma)) \rightarrow \overline{H^1}(\Gamma, \ell^p(\Gamma))$$

is an isomorphism if and only if  $\Gamma$  is non-amenable. □

We will prove:

**Proposition 2.** Let  $p \in [1, \infty[$ . Let  $N \subset H \subset \Gamma$  be a chain of groups, where  $\Gamma$  finitely generated and  $N$  is infinite and normal in  $\Gamma$ . If  $H^1(H, \ell^p(H)) = 0$ , then  $H^1(\Gamma, \ell^p(\Gamma)) = 0$ .

The following link between  $\overline{H^1}_{(p)}(\Gamma)$  and  $H^1(\Gamma, \ell^p(\Gamma))$  has been noticed by several people – see e.g. Lemma 3 in [BV97] (for  $p = 2$  and  $\Gamma$  non-amenable), or in [Pul03] (in general). We give the easy argument for completeness.

**Lemma 1.** For finitely generated  $\Gamma$ , there are isomorphisms

$$D_p(\Gamma)/(i(\ell^p(\Gamma)) + \mathbb{C}) \simeq H^1(\Gamma, \ell^p(\Gamma)) \quad \text{and} \quad \overline{H^1}_{(p)}(\Gamma) \simeq \overline{H^1}(\Gamma, \ell^p(\Gamma)).$$

*Proof.* The map  $D_p(\Gamma) \rightarrow Z^1(\Gamma, \ell^p(\Gamma)): f \mapsto [g \mapsto \lambda_\Gamma(g)f - f]$  is continuous, with kernel the space  $\mathbb{C}$  of constant functions, and the image of  $i(\ell^p(\Gamma))$  is exactly  $B^1(\Gamma, \ell^p(\Gamma))$ . Moreover this map is onto because of the classical fact that  $H^1(\Gamma, \mathcal{F}(\Gamma)) = 0$ . □

Before proving Proposition 2 (for which we will actually give two proofs), we explain how to deduce Theorem 1 from it.

*Proof of Theorem 1 from Proposition 2.* 1) In view of Lemma 1, the assumption of Theorem 1 reads  $\overline{H^1}(H, \ell^p(H)) = 0$ . Since  $H$  is non-amenable, by Proposition 1 we have  $H^1(H, \ell^p(H)) = 0$ . By Proposition 2 we deduce  $H^1(\Gamma, \ell^p(\Gamma)) = 0$ . By Lemma 1 again, we get the conclusion.

2) If  $H$  is non-amenable, the result is a particular case of the first part. If  $H$  is amenable, then so is  $N$ , and the result follows from the Cheeger–Gromov vanishing theorem [CG86]: if a group  $\Gamma$  contains an infinite, amenable, normal subgroup, then all  $L^2$ -Betti numbers of  $\Gamma$  are zero. □

**Important remark.** Cheeger and Gromov [CG86] defined  $L^2$ -Betti numbers of a group  $\Gamma$  without any assumption on  $\Gamma$ , in particular not assuming  $\Gamma$  to be finitely generated. Using their definition, D. Gaboriau has shown us (private communication) a proof that  $b_{(2)}^1(\Gamma) = 0$  always implies  $\overline{H}^1(\Gamma, \ell^2(\Gamma)) = 0$ . As a consequence, part 2) of Theorem 1 holds *even if  $H$  is not finitely generated*.

Our first proof of Proposition 2 will require the following lemma, which is classical for  $p = 2$ .

**Lemma 2.** *Let  $p \in [1, \infty[$ . Let  $H$  be a countable group. Let  $X$  be a countable set on which  $H$  acts freely. The following statements are equivalent:*

- i)  $H$  is amenable.
- ii) The permutation representation  $\lambda_X$  of  $H$  on  $\ell^p(X)$ , almost has invariant vectors.

*Proof.* We recall (see [Eym72]) that a group  $\Gamma$  is amenable if and only if it satisfies Reiter’s condition  $(P_p)$ , i.e. for every finite subset  $F \subset \Gamma$  and  $\epsilon > 0$ , there exists  $f \in \ell^p(\Gamma)$  such that  $f \geq 0$ ,  $\|f\|_p = 1$ , and  $\|\lambda_\Gamma(g)f - f\|_p < \epsilon$  for  $g \in F$ . In particular  $\ell^p(\Gamma)$  almost has invariant vectors.

So if  $H$  is amenable, then  $\ell^p(X)$  almost has invariant vectors since it contains  $\ell^p(H)$  as a sub-module. This proves (i)  $\Rightarrow$  (ii).

To prove (ii)  $\Rightarrow$  (i), we assume that  $\ell^p(X)$  almost has invariant vectors and prove in 3 steps that  $H$  satisfies Reiter’s property  $(P_1)$ , so is amenable. So fix a finite subset  $F \subset H$ , and  $\epsilon > 0$ ; find  $f \in \ell^p(X)$ ,  $\|f\|_p = 1$ , such that  $\|\lambda_X(h)f - f\|_p < \frac{\epsilon}{2p}$  for  $h \in F$ .

- 1) Replacing  $f$  with  $|f|$ , we may assume that  $f \geq 0$ .
- 2) Set  $g = f^p$ , so that  $g \in \ell^1(X)$ ,  $\|g\|_1 = 1$ ,  $g \geq 0$ . For  $h \in F$ , we have:

$$\begin{aligned} \|\lambda_X(h)g - g\|_1 &= \sum_{x \in X} |f(h^{-1}x)^p - f(x)^p| \\ &\leq p \sum_{x \in X} |f(h^{-1}x) - f(x)|(f(h^{-1}x)^{p-1} + f(x)^{p-1}) \\ &\leq p \left( \sum_{x \in X} |f(h^{-1}x) - f(x)|^p \right)^{\frac{1}{p}} \left( \sum_{x \in X} (f(h^{-1}x)^{p-1} + f(x)^{p-1})^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\ &\leq p \|\lambda_X(h)f - f\|_p \left( 2^{\frac{1}{p-1}} \sum_{x \in X} (f(h^{-1}x)^p + f(x)^p) \right)^{\frac{p-1}{p}} \\ &= 2p \|\lambda_X(h)f - f\|_p < \epsilon \end{aligned}$$

where we have used consecutively<sup>3</sup> the inequalities

<sup>3</sup>The expert will recognize here the argument to pass from property  $(P_p)$  to property  $(P_1)$ , as in [Eym72].

- $|a^p - b^p| \leq p|a - b|(a^{p-1} + b^{p-1})$  for  $a, b > 0$ ,
- Hölder's inequality,
- $(a + b)^{\frac{p}{p-1}} \leq 2^{\frac{1}{p}}(a^{\frac{p}{p-1}} + b^{\frac{p}{p-1}})$  for  $a, b > 0$ ,

and the fact that  $\|f\|_p = 1$ .

3) Let  $(x_n)_{n \geq 1}$  be a set of representatives for the orbits of  $H$  in  $X$ . Define a function  $g_n$  on  $H$  by  $g_n(h) = g(hx_n)$ , and set  $G = \sum_{n=1}^{\infty} g_n$ . Then  $G \geq 0$  and  $\|G\|_1 = \sum_{h \in H} \sum_{n=1}^{\infty} g(hx_n) = \sum_{x \in X} g(x) = 1$ . Moreover, for  $h \in F$ :

$$\|\lambda_H(h)G - G\|_1 = \sum_{\gamma \in H} \left| \sum_{n=1}^{\infty} (g(h^{-1}\gamma x_n) - g(\gamma x_n)) \right| \leq \|\lambda_X(h)g - g\| < \epsilon$$

by the previous step. This establishes property  $(P_1)$  for  $H$ . □

**First proof of Proposition 2 (homological algebra)**

*Claim.*  $H^1(H, \ell^p(\Gamma)|_H) = 0$ . Choosing representatives for the right cosets of  $H$  in  $\Gamma$ , we identify  $\ell^p(\Gamma)|_H$  in an  $H$ -equivariant way with the  $\ell^p$ -direct sum  $\oplus \ell^p(H)$  of  $[\Gamma : H]$  copies of  $\ell^p(H)$ . Since cohomology commutes with finite direct sums, the claim is clear if  $[\Gamma : H] < \infty$ . So assume that  $[\Gamma, H] = \infty$ . If  $b \in Z^1(H, \ell^p(\Gamma)|_H)$ , write  $b = (b_k)_{k \geq 1}$  where  $b_k \in Z^1(H, \ell^p(H))$  for every  $k \geq 1$ . By assumption, for each  $k$ , there is a function  $f_k \in \ell^p(H)$  such that  $b_k(h) = \lambda_H(h)f_k - f_k$  for every  $h \in H$ . Set

$$B_N(h) = (\lambda_H f_1 - f_1, \dots, \lambda_N(h)f_N - f_N, 0, 0, \dots)$$

so that  $B_N \in B^1(H, \ell^p(\Gamma)|_H)$  and  $B_N$  converges to  $b$  pointwise on  $H$ , for  $N \rightarrow \infty$ . This already shows that  $\overline{H^1(H, \ell^p(\Gamma)|_H)} = 0$ . Notice now that, by Proposition 1 (2), the assumption  $H^1(H, \ell^p(H)) = 0$  implies that  $H$  is non-amenable. By Lemma 2 applied to  $X = \Gamma$ , this means that  $\ell^p(\Gamma)|_H$  does not almost have invariant vectors. By Proposition 1 (1), we get  $H^1(H, \ell^p(\Gamma)|_H) = 0$ , proving the claim.

Recall from group cohomology (see e.g. 8.1 in [Gui80]) that, for any  $\Gamma$ -module  $V$ , there is an exact sequence

$$0 \rightarrow H^1(\Gamma/N, V^N) \xrightarrow{i_*} H^1(\Gamma, V) \xrightarrow{\text{Rest}_\Gamma^N} H^1(N, V|_N)^{\Gamma/N}$$

where  $i : V^N \rightarrow V$  denotes the inclusion. In particular, if  $V^N = 0$ , then the restriction map

$$\text{Rest}_\Gamma^N : H^1(\Gamma, V) \rightarrow H^1(N, V|_N)$$

is injective. We apply this with  $V = \ell^p(\Gamma)$  (noticing that  $V^N = 0$  as  $N$  is infinite).

Consider then the composition of restriction maps

$$H^1(\Gamma, \ell^p(\Gamma)) \xrightarrow{\text{Rest}_\Gamma^H} H^1(H, \ell^p(\Gamma)|_H) \xrightarrow{\text{Rest}_H^N} H^1(N, \ell^p(\Gamma)|_N);$$

this composition is  $\text{Rest}_\Gamma^N$ , which is injective as we just saw. On the other hand, by the claim this composition is also the zero map. So  $H^1(\Gamma, \ell^p(\Gamma)) = 0$ , as was to be established.  $\square$

**Second proof of Proposition 2 (geometry).** This proof works under the extra assumption that  $H$  is finitely generated. Fix finite generating sets  $T$  for  $H$ ,  $S$  for  $\Gamma$ , with  $T \subset S$ , and consider the Cayley graph  $\mathcal{G}(\Gamma, S)$  and its coboundary operator  $d_\Gamma : \mathcal{F}(\Gamma) \rightarrow \mathcal{F}(\mathbb{E}_\Gamma)$ . Then  $D_p(\Gamma) = \{f \in \mathcal{F}(\Gamma) : d_\Gamma f \in \ell^p(\mathbb{E}_\Gamma)\}$ . Similarly, let  $d_H$  be the coboundary operator associated with the Cayley graph  $\mathcal{G}(H, T)$ .

Fix  $f \in D_p(\Gamma)$ ; the goal is to show that  $f \in \ell^p(\Gamma) + \mathbb{C}$ . Let  $(g_i)_{i \in I}$  be a set of representatives for the right cosets of  $H$  in  $\Gamma$ , so that  $\Gamma = \bigsqcup_{i \in I} Hg_i$ . For  $i \in I$ , set  $f_i(x) = f(xg_i)$  ( $x \in H$ ). Then

$$\begin{aligned} \|d_H(f_i)\|_p^p &= \sum_{x \in H} \sum_{s \in T} |f(sxg_i) - f(xg_i)|^p \\ &\leq \sum_{x \in \Gamma} \sum_{s \in S} |f(sx) - f(x)|^p \\ &= \|d_\Gamma f\|_p^p < \infty, \end{aligned}$$

i.e.  $f_i \in D_p(H)$ . Using our assumption and Lemma 1, we may write

$$f_i = h_i + u_i$$

where  $h_i \in \ell^p(H)$  and  $u_i \in \mathbb{C}$ . Define functions  $h$  and  $u$  on  $\Gamma$  by  $h(xg_i) = h_i(x)$  and  $u(xg_i) = u_i$  ( $x \in H$ ).

*First claim.*  $h \in \ell^p(\Gamma)$ . Indeed, since  $H$  is non-amenable (by Proposition 1), there exists a constant  $C > 0$  (depending only on  $p, H, T$ ) such that for every  $i \in I$ :

$$\|h_i\|_p \leq C \|d_H(h_i)\|_p.$$

Then summing over  $i$  we obtain

$$\begin{aligned} \|h\|_p^p &= \sum_{i \in I} \|h_i\|_p^p \\ &\leq C^p \sum_{i \in I} \|d_H(f_i)\|_p^p = C^p \sum_{i \in I} \sum_{x \in H} \sum_{s \in T} |h_i(sx) - h_i(x)|^p \\ &= C^p \sum_{i \in I} \sum_{x \in H} \sum_{s \in T} |f_i(sx) - f_i(x)|^p = C^p \sum_{x \in \Gamma} \sum_{s \in T} |f(sx) - f(x)|^p \end{aligned}$$

$$\begin{aligned} &\leq C^p \sum_{x \in \Gamma} \sum_{s \in S} |f(sx) - f(x)|^p \\ &= C^p \|d_\Gamma(f)\|_p^p < \infty. \end{aligned}$$

*Second claim.*  $u$  is constant. Indeed, since  $f = h + u$ , and  $d_\Gamma(f), d_\Gamma(h) \in \ell^p(\mathbb{E}_\Gamma)$ , we have  $d_\Gamma(u) \in \ell^p(\mathbb{E}_\Gamma)$ . In particular this implies, for fixed indices  $i, j \in I$ :

$$\begin{aligned} \infty > \sum_{x \in N} |u((g_j g_i^{-1})x g_i) - u(x g_i)|^p &= \sum_{x \in N} |u((g_j g_i^{-1})x g_i) - u_i|^p \\ &= \sum_{x \in N} |u(x (g_j g_i^{-1}) g_i) - u_i|^p \end{aligned}$$

since  $N$  is normal in  $\Gamma$ . The latter sum is equal to

$$\sum_{x \in N} |u_j - u_i|^p < \infty.$$

Since  $N$  is infinite, this forces  $u_i = u_j$ , i.e.  $u$  is constant.

The first and the second claim together prove Proposition 2. □

### 3. Some results of W. Lück

The following result was obtained by Lück in [Lue94], Theorem 2.1. We recall his short, elegant argument.

**Lemma 3.** *Let  $N$  be a finitely generated group, and let  $\alpha$  be an automorphism of  $N$ . Let  $H = N \rtimes_\alpha \mathbb{Z}$  be the corresponding semi-direct product. Then  $b_{(2)}^1(H) = 0$ .*

*Proof.* The proof depends on two classical properties of the  $L^2$ -Betti numbers for a finitely generated group  $\Gamma$ :

- $b_{(2)}^1(\Gamma) \leq d(\Gamma)$ , where  $d(\Gamma)$  denotes the minimal number of generators of  $\Gamma$ ;
- if  $\Lambda$  is a subgroup of finite index  $d$  in  $\Gamma$ , then  $b_{(2)}^k(\Lambda) = d \cdot b_{(2)}^k(\Gamma)$ .

Let then  $p: H \rightarrow \mathbb{Z}$  denote the quotient map; for  $n \geq 1$ , set  $H_n = p^{-1}(n\mathbb{Z})$ , a subgroup of index  $n$  in  $H$ . Then:

$$n \cdot b_{(2)}^1(H) = b_{(2)}^1(H_n) \leq d(H_n) \leq d(N) + 1.$$

Since this holds for every  $n \geq 1$ , the lemma follows. □

*Proof of Corollary 1.* Since  $\Gamma/N$  is not a torsion group, we find a subgroup  $H$  of  $\Gamma$ , containing  $N$ , such that  $H/N$  is infinite cyclic. Since  $N$  is finitely generated, we have  $b_{(2)}^1(H) = 0$ , by Lemma 3. The result follows then immediately from Theorem 1.  $\square$

**Example 1.** We point out that Lemma 3 has no analogue in  $L^p$ -cohomology, with  $p \neq 2$ . To see it, let  $M$  be a 3-dimensional, compact, hyperbolic manifold which fibers over the circle. Denote by  $\Sigma_g$  the fiber of that fibration: this is a closed Riemann surface of genus  $g \geq 2$ . Then the fundamental group  $\Gamma = \pi_1(M)$  admits a semi-direct product decomposition  $\Gamma = \pi_1(\Sigma_g) \rtimes \mathbb{Z}$ , so that  $\bar{H}_{(2)}^1(\Gamma) = 0$  by Lemma 2. However

$$\inf\{p \geq 1 : \bar{H}_{(p)}^1(\Gamma) \neq 0\} = 2,$$

as was proven by Pansu [Pan89].

**4. Proof of Theorem 2**

Denote by  $\partial X$  the (Gromov) boundary of  $X$ . Let  $\Lambda = \overline{\Gamma o} \cap \partial X$  be the limit set of  $\Gamma$  in  $\partial X$  (the closure of  $\Gamma o$  is taken in the compact set  $X \cup \partial X$ ).

Since  $X$  is a CAT(-1) space, its boundary carries a natural metric  $d$  (called a *visual metric*) which can be defined as follows (see [Bou95], Théorème 2.5.1); for every  $\xi$  and  $\eta$  in  $\partial X$ :

$$d(\xi, \eta) = e^{-(\xi|\eta)},$$

where  $(\cdot|\cdot)$  denotes the Gromov product on  $\partial X$  based on  $o$ , namely

$$(\xi|\eta) = \lim_{(x,y) \rightarrow (\xi,\eta)} \frac{1}{2}(|o-x| + |o-y| - |x-y|).$$

Observe that there exists a constant  $B$  such that for every  $g \in \Gamma$  there is a point  $\xi$  in  $\partial X$  with  $d(g o, [o, \xi]) \leq B$ . Indeed this property does not depend on the choice of the origin  $o$ . So we choose  $o$  on a bi-infinite geodesic  $(\eta_1, \eta_2)$ . Then  $g o$  belongs to  $(g\eta_1, g\eta_2)$ . Now since  $X$  is Gromov-hyperbolic, one of the two points  $g\eta_1$  or  $g\eta_2$  satisfies the claim.

Let  $u$  be a Lipschitz function of  $(\partial X, d)$  which is non-constant on  $\Lambda$ ; such functions do exist since  $\Lambda$  is not reduced to a point. Following G. Elek [Ele97], let  $f$  be the function on  $\Gamma$  defined by  $f(g) = u(\xi_g)$ , where  $\xi_g$  is a point in  $\partial X$  such that  $d(g^{-1} o, [o, \xi_g]) \leq B$ .

*Claim.*  $f \in D_p(\Gamma)$  for  $p > \max\{1, e(\Gamma)\}$ . Indeed we have

$$\begin{aligned} \|f\|_{D_p}^p &= \sum_{s \in S} \sum_{g \in \Gamma} |f(sg) - f(g)|^p = \sum_{s \in S} \sum_{g \in \Gamma} |u(\xi_{sg}) - u(\xi_g)|^p \\ &\leq C \sum_{s \in S} \sum_{g \in \Gamma} [d(\xi_{sg}, \xi_g)]^p = C \sum_{g \in \Gamma} \sum_{s \in S} e^{-p(\xi_{sg}|\xi_g)} \\ &\leq D \sum_{g \in \Gamma} e^{-p|g^{-1}o - o|} < +\infty, \end{aligned}$$

where  $C, D$  are constants depending only on  $u, B$  and  $S$ . The details for the first inequality in the last line are the following. Observe that  $|(sg)^{-1}o - g^{-1}o| = |s^{-1}o - o|$  is bounded above by an absolute constant. This implies that if  $x_g$  and  $x_{sg}$  respectively denote the points on  $[o, \xi_g)$  and  $[o, \xi_{sg})$  whose distance from  $o$  is equal to  $|g^{-1}o - o|$ , then  $|x_g - x_{sg}|$  is bounded above by an absolute constant. Now with the triangle inequality

$$|x - y| \leq |x - x_{sg}| + |x_{sg} - x_g| + |x_g - y|,$$

and from the definition of the Gromov product, it follows that

$$(\xi_{sg}|\xi_g) \geq \frac{1}{2}(|o - x_{sg}| + |o - x_g| - |x_{sg} - x_g|),$$

so that  $(\xi_{sg}|\xi_g)$  is bounded below by  $|g^{-1}o - o|$  plus an absolute additive constant. This proves the claim.

Since  $\Lambda$  has at least 3 points, the group  $\Gamma$  is non-amenable (namely it is well-known that  $\Lambda$  is a minimal set, and that an amenable group stabilises one or two points in  $\partial X$ ). So by Proposition 1 and by Lemma 1, we must prove that  $f$  does not belong to  $i(\ell^p(\Gamma)) + \mathbb{C}$ . Assume it does, then  $f(g)$  tends to a constant number when the length of  $g$  in  $\Gamma$  tends to  $+\infty$ . This contradicts the fact that  $u$  is non-constant on  $\Lambda$ . □

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**Note added in proof.** The following example, suggested by F. Paulin, shows that Corollary 2 fails for lattices in  $SO(2, 1)$ . Start with the free group  $\mathbb{F}_2$  on two generators. Embed it as a lattice  $G$  in  $SO(2, 1)$ , so that  $e(G) = 1$ . On the other hand, let  $X_\lambda$  be the regular tree of degree 4, with edge length  $\lambda > 0$ . This is a proper CAT(-1) space. Let  $\mathbb{F}_2$  act as a properly discontinuous group  $\Gamma$  of isometries of  $X_\lambda$ , by viewing  $X_\lambda$  as the Cayley tree of  $\mathbb{F}_2$ . Then  $e(\Gamma) = \frac{\log 3}{\lambda}$ , which is less than 1 for  $\lambda$  large enough.

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