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Commentarii Mathematici Helvetici

Commutator length of symplectomorphisms

Michael Entov*

Abstract. Each element x of the commutator subgroup $[G, G]$ of a group G can be represented as a product of commutators. The minimal number of factors in such a product is called the commutator length of x. The commutator length of G is defined as the supremum of commutator lengths of elements of $[G, G]$.

We show that for certain closed symplectic manifolds (M, ω), including complex projective spaces and Grassmannians, the universal cover $\widetilde{\mathrm{Ham}}(M, \omega)$ of the group of Hamiltonian symplectomorphisms of (M, ω) has infinite commutator length. In particular, we present explicit examples of elements in $\widetilde{\mathrm{Ham}}(M, \omega)$ that have arbitrarily large commutator length – the estimate on their commutator length depends on the multiplicative structure of the quantum cohomology of (M, ω). By a different method we also show that in the case $c_1(M) = 0$ the group $\widetilde{\mathrm{Ham}}(M, \omega)$ and the universal cover $\widetilde{\mathrm{Symp}}_0(M, \omega)$ of the identity component of the group of symplectomorphisms of (M, ω) have infinite commutator length.

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Keywords. Commutator length, quasimorphism, symplectic manifold, Hamiltonian symplectomorphism, quantum cohomology, Floer homology.

Contents

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1. An overview of the main results

1.1. Basic definitions and preliminaries

Let G be a group. Each element x of the commutator subgroup $[G, G] \subset G$ can be represented as a product of commutators. The minimal number of factors in such a product is called the *commutator length of* x and will be denoted by $\mathrm{cl}\,(x)$ (sometimes this quantity is also called "the genus of x"). Define the *commutator length of* G as $\mathrm{cl}\,(G) := \sup_{x\in[G,G]} \mathrm{cl}\,(x)$. Define the *stable commutator norm* $\|x\|_{\mathrm{cl}}$ of $x \in [G, G]$ as $\|x\|_{\mathrm{cl}} := \lim_{k\to+\infty} \mathrm{cl}\,(x^k)/k$. A simple exercise shows that such a limit always exists.

For results on commutator length of finite-dimensional Lie groups see e.g. [8], [17]. In this paper we will study the commutator lengths of the universal covers of the infinite-dimensional Lie groups $\mathrm{Symp}_0\,(M,\omega)$ and $\mathrm{Ham}\,(M,\omega)$, associated with a closed connected symplectic manifold (M,ω).

Here are the definitions of these groups. Let (M,ω) be a closed connected symplectic manifold. By $\mathrm{Symp}_0(M,\omega)$ we denote the identity component of the group of symplectomorphisms of (M,ω). Its universal cover will be denoted by $\widetilde{\mathrm{Symp}}_0(M,\omega)$.

The group $\mathrm{Ham}(M,\omega)$ of *Hamiltonian symplectomorphisms* of (M,ω) is defined as follows. A function $H : S^1 \times M \to \mathbb{R}$ (called *Hamiltonian*) defines a time-dependent *Hamiltonian vector field* $X(t,x), (t,x) \in S^1 \times M$, on M by the formula[1]

$$dH^t(\cdot) = \omega(X^t, \cdot), \tag{1}$$

where $H^t = H(t,\cdot)$, $X^t = X(t,\cdot)$, and the formula holds pointwise on M for every $t \in S^1$. The flow of a Hamiltonian vector field preserves the symplectic form. A *Hamiltonian symplectomorphism* φ of (M,ω) is a symplectomorphism of M which can be represented as the time-1 map of the flow (called *Hamiltonian flow*) of a (time-dependent) Hamiltonian vector field. If this Hamiltonian vector field is defined by a Hamiltonian function H we say that φ and the whole Hamiltonian flow are *generated* by H.

Observe that different Hamiltonian functions can generate the same (time-dependent) Hamiltonian vector field and therefore the same Hamiltonian flow and the same Hamiltonian symplectomorphism. Namely, given a Hamiltonian vector field X generated by a Hamiltonian $H : S^1 \times M \to \mathbb{R}$ one can write any other Hamiltonian generating X as $H' = H + h(t)$ for some function $h : S^1 \to \mathbb{R}$. In order to deal with this non-uniqueness we introduce the following normalization.

Definition 1.1.1. Suppose M is closed. A Hamiltonian $H : S^1 \times M \to \mathbb{R}$ is called *normalized* if $\int_M H^t\omega^n = 0$, for any $t \in S^1$.

Thus any Hamiltonian vector field (or Hamiltonian flow) is generated by a unique normalized Hamiltonian function.

Hamiltonian symplectomorphisms of (M,ω) form a group $\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}_0(M,\omega)$. In particular, if Hamiltonian flows $\{\varphi^t\}, \{\psi^t\}$ are generated, respectively, by Hamiltonians F, G, then the flow $\{\varphi^t \cdot \psi^t\}$ is generated by the Hamiltonian function $F \sharp G(t,x) := F(t,x) + G(t, (\varphi^t)^{-1}(x))$ and the flow $\{(\varphi^t)^{-1}\}$ is generated by the Hamiltonian function $\bar{F}(t,x) = -F(t, \varphi^t(x))$.

Denote by $\widetilde{\mathrm{Ham}}(M,\omega)$ the universal cover of $\mathrm{Ham}(M,\omega)$. One can view $\widetilde{\mathrm{Ham}}(M,\omega)$ as a subgroup of $\widetilde{\mathrm{Symp}}_0(M,\omega)$ [2], [9]. A theorem of A. Banyaga [2] shows that

$$\begin{aligned} \widetilde{\mathrm{Ham}}(M,\omega) = {} & [\widetilde{\mathrm{Ham}}(M,\omega), \widetilde{\mathrm{Ham}}(M,\omega)] = \\ & [\widetilde{\mathrm{Symp}}_0(M,\omega), \widetilde{\mathrm{Symp}}_0(M,\omega)]. \end{aligned} \tag{2}$$

Definition 1.1.2. We will denote by φ_H the Hamiltonian symplectomorphism generated by a Hamiltonian function H. The Hamiltonian flow generated by H is

[1] Note our sign convention. Also note the signs in the formula (4) for the action functional and in the formula (8) for the Conley–Zehnder index.

a path connecting Id and φ_H which defines a lift of φ_H in $\widetilde{\mathrm{Ham}}\,(M,\omega)$. This lift will be denoted by $\widetilde{\varphi}_H$.

Obviously, $\widetilde{\varphi}_F^{-1} = \widetilde{\varphi}_{\bar{F}}$, $\widetilde{\varphi}_F \cdot \widetilde{\varphi}_G = \widetilde{\varphi}_{F\sharp G}$ and if F is time-independent then $\widetilde{\varphi}_{kF} = \widetilde{\varphi}_F^k$ for any $k \in \mathbb{Z}$.

A *ball* in (M^{2n},ω) is the image of a symplectic embedding into M of a round $2n$-dimensional open ball in the standard symplectic $\mathbb{R}^{2n}$. We say that a ball $B \subset M$ is *displaced* by $\varphi \in \mathrm{Ham}\,(M,\omega)$ if

$$\varphi(B) \cap \mathrm{Closure}\,(B) = \emptyset.$$

A ball $B \subset M$ is called *displaceable* if it can be displaced by some $\varphi \in \mathrm{Ham}\,(M,\omega)$.

Definition 1.1.3. Let $B \subset M$ be a ball. Identify B symplectically with the ball $B(0,R)$ of a radius R with the center at zero in the standard symplectic $\mathbb{R}^{2n}$. Define a time-independent Hamiltonian $H_B : M \to \mathbb{R}$ as follows. Outside B set $H_B(x) \equiv -1$. Inside B set $H_B : B(0,R) \to \mathbb{R}$ as $H_B(x) := \zeta(\|x\|)$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^{2n}$ and $\zeta : \mathbb{R} \to \mathbb{R}$ is a smooth function which is equal to a positive constant near 0 and to -1 outside of $[-R/2, R/2]$, and which is chosen in such a way that $\int_M H_B\,\omega^n = 0$ (thus H is a normalized Hamiltonian).

Finally, we say that an almost complex structure J on (M,ω) is *compatible* with ω if $\omega(\cdot, J\cdot)$ is a Riemannian metric on M. Almost complex structures compatible with ω form a non-empty contractible space [18]. The first Chern class of TM for any J compatible with ω is the same and will be denoted by $c_1(M)$.

1.2. The results

Let $M = \mathbb{C}P^n$ be the complex projective space equipped with the standard Fubini-Study symplectic form ω. Let $A = [\mathbb{C}P^1] \in H_2(\mathbb{C}P^n)$ be the homology class of a projective line in $\mathbb{C}P^n$.

Theorem 1.2.1. *Let $B \subset \mathbb{C}P^n$ be a ball and let $F : S^1 \times \mathbb{C}P^n \to \mathbb{R}$ be a normalized Hamiltonian so that φ_F displaces B. Let g be a positive integer. Then for any*

$$w > \int_0^1 \sup_M F(t,\cdot)dt + \left[\frac{gn}{n+1}\right]\omega(A)$$

one has $\mathrm{cl}\,(\widetilde{\varphi}_{F\sharp wH_B}) > g$ in the group $\widetilde{\mathrm{Ham}}\,(\mathbb{C}P^n,\omega)$. Hence

$$\mathrm{cl}\,(\widetilde{\mathrm{Ham}}\,(\mathbb{C}P^n,\omega)) = +\infty.$$

The statement $\mathrm{cl}\,(\widetilde{\mathrm{Ham}}\,(\mathbb{C}P^n,\omega)) = +\infty$ can be strengthened: one can show that the infinite supremum of commutator lengths is actually reached on some cyclic subgroup of $\widetilde{\mathrm{Ham}}\,(\mathbb{C}P^n,\omega)$.

Theorem 1.2.2 (L. Polterovich). *Let $B \subset \mathbb{C}P^n$ be a displaceable ball. Then for any $w > 0$*

$$\|\widetilde{\varphi}_{wH_B}\|_{\mathrm{cl}} \geq \frac{w(n+1)}{n\omega(A)} > 0.$$

A closed connected symplectic manifold (M, ω) is called *symplectically aspherical* if ω vanishes on any spherical homology class.

Theorem 1.2.3 (M. Entov–L. Polterovich). *Let (M, ω) be symplectically aspherical. Let $B \subset \mathbb{C}P^n$ be a ball and let $F : S^1 \times \mathbb{C}P^n \to \mathbb{R}$ be a normalized Hamiltonian so that φ_F displaces B. Then for any*

$$w > \int_0^1 \sup_M F(t, \cdot) dt$$

one has $\mathrm{cl}\,(\widetilde{\varphi}_{F\sharp wH_B}) > 1$ *in the group* $\widetilde{\mathrm{Ham}}\,(M, \omega)$.

A closed connected symplectic surface (M, ω) of positive genus is symplectically aspherical and $\mathrm{Ham}\,(M, \omega) = \widetilde{\mathrm{Ham}}\,(M, \omega)$ (see [40]). In this case Theorem 1.2.3 provides examples of elements of $\mathrm{Ham}\,(M, \omega)$ with commutator length greater than 1.

Theorem 1.2.1 has a direct analogue for complex Grassmannians. We will state here only the following conclusion. Let $M = Gr\,(r, n)$, $1 \leq r \leq n-1$, be the Grassmannian of complex r-dimensional subspaces in $\mathbb{C}^n$. It carries a natural symplectic form ω (see Example 2.6.3). Let A be the generator of $H_2(Gr\,(r, n), \mathbb{Z}) \cong \mathbb{Z}$ such that $\omega(A) > 0$.

Theorem 1.2.4. *For a displaceable ball $B \subset Gr\,(r, n)$ and any $w > 0$*

$$\|\widetilde{\varphi}_{wH_B}\|_{\mathrm{cl}} \geq \frac{wr(n-r)}{n\omega(A)} > 0.$$

in $\widetilde{\mathrm{Ham}}\,(Gr\,(r, n), \omega)$. *Hence* $\mathrm{cl}\,(\widetilde{\mathrm{Ham}}\,(Gr\,(r, n), \omega)) = +\infty$.

The proofs of the results above can be found in Section 2.7. They are based on the Floer theory.

Observe that these results provide an estimate on the commutator length of a certain *individual* element $\varphi \in \widetilde{\mathrm{Ham}}\,(M, \omega)$ and the estimates on $\|\varphi\|_{\mathrm{cl}}$ are obtained as a *consequence of estimates on the commutator length of each individual* φ^k, $k \in \mathbb{Z}$. Another way of estimating $\|\varphi\|_{\mathrm{cl}}$ is related to the notion of *a quasimorphism*. This approach gives no information on the number $\mathrm{cl}\,(\varphi^k)$ for an *individual* k – it reflects only the *asymptotics* of those numbers as $k \to +\infty$.

Recall that *a quasimorphism* on a group G is a function $\mathfrak{f} : G \to \mathbb{R}$ which satisfies the homomorphism equation up to a bounded error: there exists $R > 0$

such that

$$|\mathfrak{f}(ab) - \mathfrak{f}(a) - \mathfrak{f}(b)| \leq R$$

for all $a, b \in G$. A quasimorphism $\mathfrak{f}$ is called *homogeneous* if $\mathfrak{f}(a^m) = m\mathfrak{f}(a)$ for all $a \in G$ and $m \in \mathbb{Z}$. A simple exercise shows that if there exists a homogeneous quasimorphism which does not vanish on $\phi \in [G, G]$ then $\|\phi\|_{\mathrm{cl}} > 0$ and hence $\mathrm{cl}\,(G) = +\infty$. Conversely, according to a theorem of C. Bavard [4], if $\|\phi\|_{\mathrm{cl}} > 0$ then there exists a homogeneous quasimorphism on G which does not vanish on ϕ.

Theorem 1.2.5. *Let (M, ω) be a closed connected symplectic manifold with $c_1(M) = 0$ and let $G = \widetilde{\mathrm{Symp}}_0\,(M, \omega)$. Then there exists a homogeneous quasimorphism $\mathfrak{f} : G \to \mathbb{R}$ which does not vanish on $[G, G] = \widetilde{\mathrm{Ham}}\,(M, \omega)$ and therefore*

$$\mathrm{cl}\,(\widetilde{\mathrm{Symp}}_0\,(M, \omega)) = \mathrm{cl}\,(\widetilde{\mathrm{Ham}}\,(M, \omega)) = +\infty.$$

In particular, for any (not necessarily displaceable) ball $B \subset M$ one has $\mathfrak{f}(\widetilde{\varphi}_{H_B}) \neq 0$ and hence

$$\|\widetilde{\varphi}_{H_B}\|_{\mathrm{cl}} > 0$$

both in $\widetilde{\mathrm{Ham}}\,(M, \omega)$ and in $\widetilde{\mathrm{Symp}}_0(M, \omega)$.

Example 1.2.6. Here are examples of closed symplectic manifolds with $c_1(M) = 0$:

1) Symplectic tori.

2) Complex Kähler manifolds with $c_1 = 0$. These are Ricci-flat manifolds that include complex K3-surfaces, hyper-Kähler manifolds, Calabi-Yau manifolds of complex dimension 3 and certain smooth complete intersections in $\mathbb{C}P^n$, $n \geq 2$. The last example can be explicitly described as follows. If $M \subset \mathbb{C}P^n$ is a smooth complex submanifold of complex dimension k, which is the intersection of $n - k$ hypersurfaces of degrees $d_1, \ldots, d_{n-k}$, then $c_1(M) = 0$ as soon as $\sum_{i=1}^{n-k} d_i = n+1$.

Theorem 1.2.5 is proven in Section 9. Our construction of $\mathfrak{f}$ directly generalizes the one of J. Barge and E. Ghys [3] who proved a similar result for the group $\mathrm{Symp}^c\,(B^{2n})$ of compactly supported symplectomorphisms of the standard symplectic ball B^{2n}. A similar construction (albeit used for different purposes) is presented in [11].

Remark 1.2.7. 1) According to the theorem of C. Bavard [4] mentioned above, Theorems 1.2.2 and 1.2.4 yield the existence of a homogeneous quasimorphism on $\widetilde{\mathrm{Ham}}\,(M, \omega)$ for complex projective spaces and Grassmannians. However the proof of Bavard's result [4] uses Hahn–Banach theorem and does not provide any explicit construction of a quasimorphism on $\widetilde{\mathrm{Ham}}\,(M, \omega)$. Such an explicit construction has been carried out in [13] for complex projective spaces, Grassmannians and certain other symplectic manifolds. This in turn yields an alternative proof of Theorems 1.2.2 and 1.2.4. In fact for complex projective spaces the quasimorphism on $\widetilde{\mathrm{Ham}}\,(\mathbb{C}P^n,\omega)$ constructed in [13] descends to a homogeneous quasimorphism on $\mathrm{Ham}\,(\mathbb{C}P^n,\omega)$, yielding, in particular,

$$\|\varphi_{H_B}\|_{\mathrm{cl}} > 0$$

in $\mathrm{Ham}\,(\mathbb{C}P^n,\omega)$ for a displaceable ball $B \subset \mathbb{C}P^n$ (see [13]). Hence

$$\mathrm{cl}\,(\mathrm{Ham}\,(\mathbb{C}P^n,\omega)) = +\infty.$$

2) Using completely different methods J.-M. Gambaudo and E. Ghys [16] constructed homogeneous quasimorphisms on $G = \mathrm{Symp}_0^c\,(M,\omega)$ which do not vanish on $[G,G] = \mathrm{Ham}^c\,(M,\omega)$ for any two-dimensional symplectic manifold (M,ω). Thus the commutator lengths of both groups $\mathrm{Symp}_0^c\,(M,\omega)$, $\mathrm{Ham}^c\,(M,\omega)$ are infinite.

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2. Commutator length of elements in $\widetilde{\mathrm{Ham}}\,(M,\omega)$

In this chapter we state general results concerning estimates on the commutator length of individual elements of $\widetilde{\mathrm{Ham}}\,(M,\omega)$ in terms of the Floer theory and give examples of such elements with a large commutator length. In Sections 2.1–2.4 we introduce the necessary preliminaries about quantum and Floer (co)homology. The main results are stated in Sections 2.5. In Section 2.6 we state practical applications of the main results and use them to prove Theorems 1.2.1–1.2.4.

2.1. Strongly semi-positive symplectic manifolds

Let Π denote the group of non-torsion spherical homology classes of M, i.e. the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M,\mathbf{Z})/Tors$.

A closed symplectic manifold (M^{2n},ω) is called *strongly semi-positive* [12], if for every $A \in \Pi$

$$2-n \le c_1(A) < 0 \Longrightarrow \omega(A) \le 0.$$

In [48] such manifolds were said to satisfy the "Assumption W^+".

As in [12], for technical reasons the closed symplectic manifold (M, ω) is assumed to be strongly semi-positive whenever we use Floer or quantum (co)homology, though in view of the recent developments (see [15], [25], [26], [27]) it is likely that this assumption can be removed.

The class of strongly semi-positive manifolds includes all symplectic manifolds of dimension less or equal to 4. A particularly important subclass of strongly semi-positive manifolds is formed by *spherically monotone* symplectic manifolds, i.e. the ones with $c_1|_\Pi = \kappa\,[\omega]|_\Pi$ for some $\kappa > 0$. Complex projective spaces, complex Grassmannians and flag spaces and symplectically aspherical manifolds are spherically monotone.

2.2. Novikov ring, quantum cohomology and the Euler class

The *rational Novikov ring* Λ_ω of a symplectic manifold (M, ω) is built from the rational group ring of the group Π by means of the symplectic form ω [21]. Namely, an element of Λ_ω is a formal sum

$$\lambda = \sum_{A\in\Pi} \lambda_A e^A$$

with rational coefficients $\lambda_A \in \mathbb{Q}$ which satisfies the condition

$$\sharp\{A \in \Pi \,|\, \lambda_A \neq 0, \omega(A) \leq c\} < \infty \quad \text{for every } c > 0.$$

The natural multiplication turns Λ_ω into a ring.

The grading on Λ_ω, defined by the condition $\deg(e^A) = 2c_1(A)$, fits with the ring structure. Sometimes we will need to consider the opposite grading on Λ_ω so that the degree of (e^A) is $-2c_1(A)$. The latter grading does not fit with the multiplicative structure of Λ_ω. When using the second grading we will say that Λ_ω is *anti-graded*, while using the first grading we will call Λ_ω simply *graded*. The set of elements of degree i of the *graded* Λ_ω will be denoted by Λ^i_ω.

The *quantum cohomology* of M is a module over Λ_ω defined as a graded tensor product $QH^*(M, \omega) = H^*(M, \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_\omega$ over $\mathbb{Q}$ (here Λ_ω is taken in the graded version). More importantly the group $QH^*(M, \omega)$ carries a delicate multiplicative structure called the *quantum product* – we refer the reader to [43], [44], [51] for the definitions. The quantum product is associative, Λ_ω-linear and respects the grading on $QH^*(M, \omega)$. The cohomology class $\mathbf{1} \in H^0(M)$, Poincaré-dual to the fundamental class $[M]$, is a unit element in $QH^*(M, \omega)$. We will denote by $m \in H^{2n}(M^{2n}, \mathbb{Q})$ the singular cohomology class Poincaré-dual to a point.

Similarly, *quantum homology* is defined as a graded tensor product

$$QH_*(M, \omega) = H_*(M, \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_\omega$$

over $\mathbb{Q}$, where Λ_ω is taken in the *anti-graded* version. Thus $QH_*(M, \omega)$ is a graded module over the anti-graded ring Λ_ω. There is a natural evaluation pairing

$$(\cdot,\cdot) : QH^k(M, \omega) \times QH_k(M, \omega) \to \Lambda^0_\omega.$$

Namely, let $\alpha e^A \in QH^k(M,\omega)$, $\alpha \in H^i(M,\mathbb{Q})$, $e^A \in \Lambda_\omega^{k-i}$, and $\beta e^B \in QH_k(M,\omega)$, $\beta \in H_j(M,\mathbb{Q})$, $e^B \in \Lambda_\omega^{j-k}$ (recall that elements of Λ_ω^{j-k} have degree $k-j$ in the anti-graded version of Λ_ω). Then

$$(\alpha e^A, \beta e^B) = \alpha(\beta) e^{A+B},$$

where $\alpha(\beta) \in \mathbb{Q}$ is the result of classical evaluation of cohomology class on homology (in particular, $(\alpha e^A, \beta e^B) = 0$ if $i \neq j$ above). This is a non-degenerate pairing. Since $\mathbb{Q} \subset \Lambda_\omega^0$ the universal coefficient theorem implies that the pairing $(\cdot,\cdot)$ defines an *isomorphism of groups*:

$$QH^k(M,\omega) \cong \mathrm{Hom}(QH_k(M,\omega), \Lambda_\omega^0),$$

where $\mathrm{Hom}(QH_k(M,\omega), \Lambda_\omega^0)$ is the group of all *group homomorphisms* $QH_k(M,\omega) \to \Lambda_\omega^0$.

The Poincaré isomorphism $PD : QH^*(M,\omega) \to QH_{2n-*}(M,\omega)$ is defined by the formula:

$$PD(\sum_{A\in\Pi} \alpha_A e^A) := \sum_{A\in\Pi} PD(\alpha_A) e^A, \quad \alpha_A \in H_*(M,\mathbb{Q}),$$

where in the right-hand side PD denotes the classical Poincaré isomorphism between singular homology and cohomology. One can check that $PD : QH^*(M,\omega) \to QH_{2n-*}(M,\omega)$ is the homomorphism of Λ_ω-modules (recall that $QH_*(M,\omega)$ is a graded module over the *anti-graded* ring Λ_ω).

The Poincaré isomorphism and the evaluation pairing $(\cdot,\cdot)$ together define a nondegenerate pairing $\langle\cdot,\cdot\rangle$:

$$\langle\cdot,\cdot\rangle : QH^*(M,\omega) \times QH^{2n-*}(M,\omega) \to \Lambda_\omega^0,$$

where

$$\langle\alpha,\beta\rangle = (\alpha, PD(\beta)).$$

In particular, if $\alpha \in QH^i(M,\omega)$, $\beta \in QH^j(M,\omega)$ and $i+j \neq 2n$ then $\langle\alpha,\beta\rangle = 0$.

The pairing $\langle\cdot,\cdot\rangle$ can be also expressed in terms of the quantum multiplication (see [36], [46], cf. Section 7). Namely, for any $\alpha, \beta \in QH^*(M,\omega)$

$$\langle\alpha,\beta\rangle = (\alpha * \beta, [M]),$$

where $\alpha*\beta \in QH^*(M,\omega)$ is the quantum product of α and β and the fundamental class $[M]$ is viewed as an element of $QH_{2n}(M,\omega)$. In other words, if $\alpha*\beta$ is represented as $\alpha*\beta = \sum_b \lambda_b b$, $b \in H^*(M,\mathbb{Q})$, $\lambda_b \in \Lambda_\omega$, then $\langle\alpha,\beta\rangle$ is the Λ_ω^0-component of the coefficient $\lambda_m \in \Lambda_\omega$ at the singular cohomology class m.

Pick a basis $\{e_i\}$, $e_i \in H^{k_i}(M,\mathbb{Q})$, of $H^*(M,\mathbb{Q})$ over $\mathbb{Q}$. Then $\{e_i\}$ is also a basis of $QH^*(M,\omega)$ over Λ_ω. Consider the basis $\{F_i\}$, $F_i \in H_i(M,\mathbb{Q})$, of $H_*(M,\mathbb{Q})$ over $\mathbb{Q}$ dual to $\{e_i\}$, i.e. $e_i(F_j) = \delta_{ij}$. Then $\{F_i\}$ is also a basis of $QH_*(M,\omega)$ over Λ_ω. Set $\bar{e}_i = PD(F_i) \in QH^{2n-i}(M,\omega)$ for every i. The isomorphism PD is Λ_ω-linear and therefore $\{\bar{e}_i\}$ is a basis of $QH^*(M,\omega)$ over Λ_ω. Thus

$$\langle e_i, \bar{e}_j\rangle = \delta_{ij}. \tag{3}$$

Any two Λ_ω-bases $\{e_i\}$, $\{\bar{e}_i\}$, $e_i \in QH^{k_i}(M,\omega)$, $\bar{e}_i \in QH^{2n-k_i}(M,\omega)$, of $QH^*(M,\omega)$ satisfying (3) will be called *Poincaré-dual* to each other. Given such bases $\{e_i\}$, $\{\bar{e}_i\}$, set

$$E = \sum_i (-1)^{k_i} e_i * \bar{e}_i.$$

One can easily check that $E \in QH^{2n}(M,\omega)$, called the *Euler class*, does not depend on the choice of the Poincaré-dual bases $\{e_i\}$, $\{\bar{e}_i\}$. The component of degree $2n$ of $E = \sum_i (-1)^{\deg e_i} e_i * \bar{e}_i$ is equal to $\chi(M)m \in H^{2n}(M,\mathbb{Q})$, where $\chi(M)$ is the Euler characteristic.

2.3. Floer cohomology: basic definitions

Consider pairs (γ, f), where $\gamma : S^1 \to M$ is a contractible curve and $f : D^2 \to M$ is a disk spanning γ, i.e. $f|_{\partial D^2} = \gamma$. Two pairs (γ, f) and (γ, f') are called *equivalent* if the connected sum $f \sharp f'$ represents a torsion class in $H_2(M, \mathbf{Z})$.

Given a (time-dependent) Hamiltonian function H on M denote by $\mathcal{P}(H)$ the space of equivalence classes $\hat{\gamma} = [\gamma, f]$ of pairs (γ, f), where $\gamma : S^1 \to M$ is a contractible time-1 periodic trajectory of the Hamiltonian flow of H. The theorems proving Arnold's conjecture imply that any Hamiltonian symplectomorphism of M has a closed contractible 1-periodic orbit and therefore $\mathcal{P}(H)$ is always non-empty (see [15], [25]; in the semi-positive case the conjecture was first proven in [21]).

For an element $\hat{\gamma} = [\gamma, f] \in \mathcal{P}(H)$ define its *action:*

$$\mathcal{A}_H(\hat{\gamma}) := -\int_D f^*\omega - \int_0^1 H(t, \gamma(t))dt. \tag{4}$$

If one views $\mathcal{A}_H$ as a functional on the space of all equivalence classes $[\gamma, f]$ for all contractible loops γ then $\mathcal{P}(H)$ is precisely the set of critical points of $\mathcal{A}_H$ on such a space.

The group Π acts on $\mathcal{P}(H)$: if $\hat{\gamma} = [\gamma, f] \in \mathcal{P}(H)$, $A \in \Pi$ then

$$A : \hat{\gamma} \mapsto A\sharp\hat{\gamma} := [\gamma, A\sharp f]. \tag{5}$$

Note that

$$\mathcal{A}_H(A\sharp\hat{\gamma}) = \mathcal{A}_H(\hat{\gamma}) - \omega(A) \tag{6}$$

for any $\hat{\gamma} \in \mathcal{P}(H)$, $A \in \Pi$.

Denote by Spec (H) the set of values of $\mathcal{A}_H$ on the elements of $\mathcal{P}(H)$.

Proposition 2.3.1 ([22], [33], [47]). Spec $(H) \subset \mathbb{R}$ *is a set of measure zero.*

Later, in Section 6, we will also need the following product version of the previous definitions. Given l Hamiltonians $H = (H_1, \ldots, H_l)$ on M we denote by $\mathcal{P}(H)$ the set of equivalence classes $[\hat{\gamma}_1, \ldots, \hat{\gamma}_l]$ of tuples $(\hat{\gamma}_1, \ldots, \hat{\gamma}_l)$, where

$\hat{\gamma}_i = [\gamma_i, f_i] \in \mathcal{P}(H_i)$ and the equivalence relation is given by

$$(\hat{\gamma}_1, \ldots, \hat{\gamma}_l) \sim (A_1 \sharp \hat{\gamma}_1, \ldots, A_l \sharp \hat{\gamma}_l),$$

whenever $A_i \in \Pi$ and $A_1 + \ldots + A_l$ is a torsion class. The group Π acts on $\mathcal{P}(H)$:

$$A : [\hat{\gamma}_1, \ldots, \hat{\gamma}_l] \mapsto [A \sharp \hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_l]. \tag{7}$$

For $\hat{\gamma} := [\hat{\gamma}_1, \ldots, \hat{\gamma}_l] \in \mathcal{P}(H)$ set $\mathcal{A}_H(\hat{\gamma}) = \mathcal{A}_{H_1}(\hat{\gamma}_1) + \ldots + \mathcal{A}_{H_l}(\hat{\gamma}_l)$. As before,

$$\mathcal{A}_H(A \sharp \hat{\gamma}) = \mathcal{A}_H(\hat{\gamma}) - \omega(A)$$

for any $\hat{\gamma} \in \mathcal{P}(H)$, $A \in \Pi$.

The infinite-dimensional "Morse theory" of the action functional gives rise to the *Floer homology* – an analogue of the classical Morse homology (for details see e.g. [21], [45]).

Namely, let H be a (time-dependent) Hamiltonian and let J be an almost complex structure on M compatible with ω. For a generic pair (H, J) – which will be called *a regular Floer pair* – one can define a chain complex $CF_*(H, J)$. It is a graded module over the *anti-graded* ring Λ_ω (see Section 2.2) equipped with a Λ_ω-linear differential that decreases the grading by 1. As a Λ_ω-module it is generated by elements of $\mathcal{P}(H)$ and the module structure fits with the action of Π on $\mathcal{P}(H)$ defined by the formula (5). Namely, if $\hat{\gamma} \in \mathcal{P}(H)$ and $e^A \in \Lambda_\omega$, then the product $e^A \cdot \hat{\gamma}$ in the module $CF_*(H, J)$ over Λ_ω is equal to $A \sharp \hat{\gamma}$.

Elements $\hat{\gamma} \in \mathcal{P}(H)$ are graded by the Conley–Zehnder index $\mu(\hat{\gamma})$ [10]. We follow the sign convention which in the case of a small time-independent Morse Hamiltonian H gives the following relation: if γ, f are constant maps to a critical point x of H and $\mu_{Morse}(x)$ is the Morse index of x then

$$\mu([\gamma, f]) = 2n - \mu_{Morse}(x). \tag{8}$$

The Conley–Zehnder index satisfies the property

$$\mu(A \sharp \hat{\gamma}) = \mu(\hat{\gamma}) - 2c_1(A),$$

so that the grading on $CF_*(H, J)$ fits indeed with the structure of $CF_*(H, J)$ as a graded module over the *anti-graded* ring Λ_ω.

The homology $HF_*(H, J)$ of the chain complex $CF_*(H, J)$ is called *the Floer homology group* – it is a graded module over the *anti-graded* ring Λ_ω. Using Λ^0_ω-valued cochains one can build an appropriate dual cochain complex $CF^*(H, J)$ whose cohomology $HF^*(H, J)$ is called *the Floer cohomology group* of (M, ω) – it is a graded module over the *graded* ring Λ_ω. Similarly to the situation with quantum (co)homology (see Section 2.2) there exists a non-degenerate evaluation pairing $(\cdot, \cdot) : HF^*(H, J) \times HF_*(H, J) \to \Lambda^0_\omega$ (see e.g. [36]), which, according to the universal coefficient theorem, leads to a *group* isomorphism

$$HF^k(H, J) = \mathrm{Hom}(HF_k(H, J), \Lambda^0_\omega),$$

where $\mathrm{Hom}(HF_k(H, J), \Lambda^0_\omega)$ denotes the group of all *group homomorphisms* $HF_k(H, J) \to \Lambda^0_\omega$. As in the classical Morse theory chains in $CF_*(H, J)$ can be viewed as cochains in $CF^*(\bar{H}, J)$ leading to the Poincaré isomorphism of Λ^0_ω-modules:

$$PD : HF^*(H, J) \to HF_{2n-*}(\bar{H}, J),$$

which, as in the situation with quantum (co)homology (see Section 2.2), can be extended to an isomorphism of Λ_ω-modules. As an additive module over Λ_ω the cohomology $HF^*(H, J)$ is isomorphic to $H^*(M, \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_\omega$.

The Floer cohomology $HF^*(H, J)$ carries a remarkable multiplicative structure called *the pair-of-pants product* (see [36], [46]). This multiplicative structure is isomorphic to the multiplicative structure on the quantum cohomology of (M, ω). Namely, S. Piunikhin, D. Salamon and M. Schwarz [36] defined for each regular Floer pair (H, J) a Λ_ω*-linear ring isomorphism preserving the grading*:

$$\Psi_{H,J} : QH^*(M, \omega) \to HF^*(H, J).$$

Similarly (see [36]) there exists a Λ_ω-linear grading-preserving isomorphism between quantum and Floer homology which together with $\Psi_{H,J}$ intertwines the Poincaré isomorphism and evaluation pairing in quantum and Floer (co)homology. The Floer (co)homology groups for different regular Floer pairs (H, J) are related by natural isomorphisms which are compatible with the Piunikhin–Salamon–Schwarz identifications of Floer and quantum (co)homology.

2.4. Spectral numbers

At first let (H, J) be a regular Floer pair. Then $(\bar{H}, J)$ is a regular Floer pair as well. Under our sign convention – see (1) and (4) – the Floer-Morse gradient-type connecting trajectories, defining the differential in $CF_*(H, J)$, correspond to the *downward* gradient flow of the action functional $\mathcal{A}_H$ (and if H is a sufficiently small time-independent Morse function they also correspond to the *upward* gradient flow of H).

Consider now a real filtration on $CF_*(H, J)$ defined by $\mathcal{A}_H$. Namely, let $CF_*^{(-\infty,s]}(H, J)$, $-\infty \le s \le +\infty$, be spanned *over* $\mathbb{Q}$ by all $\hat{\gamma} \in \mathcal{P}(H)$ with $\mathcal{A}_H(\hat{\gamma}) \le s$. Clearly, $CF_*^{(-\infty,s]}(H, J)$ is invariant with respect to the differential of $CF_*(H, J)$ and thus is a chain subcomplex. Note that $CF_*^{(-\infty,s]}(H, J)$ is a (possibly infinite-dimensional) $\mathbb{Q}$-subspace of $CF_*(H, J)$ and *not* a Λ_ω-submodule of $CF_*(H, J)$ since multiplication by an element of Λ_ω may increase the action. Denote by $HF_*^s(H, J)$ the image of the homology of $CF_*^{(-\infty,s]}(H, J)$ in $HF_*(H, J)$ under the inclusion $CF_*^{(-\infty,s]}(H, J) \hookrightarrow CF_*(H, J)$. Again note that $HF_*^s(H, J)$ is only a $\mathbb{Q}$-subspace of $HF_*(H, J)$ and *not* its Λ_ω-submodule.

Definition 2.4.1. Given a non-zero class $\alpha \in QH^*(M, \omega)$ set

$$c(\alpha, H) := \inf \{ s \mid PD(\Psi_{\bar{H},J}(\alpha)) \in HF^s_{2n-*}(H, J) \}.$$

The idea behind the definition is due to C. Viterbo [50] who originally used it in the context of Morse homology and finite-dimensional generating functions for Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$. It has taken a considerable work involving new techniques to extend this idea to the Floer theory with various versions of Floer homology viewed as infinite-dimensional analogs of Morse homology and with the action functional viewed as an "infinite-dimensional generating function". In the case of a cotangent bundle and the Floer theory for Lagrangian intersections it was done by Y.-G. Oh [31], [32], then in the case of a closed symplectically aspherical symplectic manifold and the Floer theory for Hamiltonian symplectomorphisms by M.Schwarz [47], and in the case of a general closed symplectic manifold and the Floer theory for Hamiltonian symplectomorphisms it was done by Y.-G. Oh in his recent papers [33], [34], where Definition 2.4.1 is taken from.

The properties of spectral numbers that we need are summarized in the following proposition.

Proposition 2.4.2. *Let $\alpha \in QH^*(M,\omega)$ be non-zero.*

1) *If H belongs to a regular Floer pair then $c(\alpha, H)$ is finite. The number $c(\alpha, H)$ does not depend on the choice of J in a regular Floer pair (H, J) and depends continuously on H with respect to the C^0-norm on the space of (time-dependent) Hamiltonian functions. Thus one can define $c(\alpha, H)$ for any H (i.e. not necessarily belonging to a regular Floer pair). The resulting quantity depends C^0-continuously on H.*

2) *$c(\lambda e^A \alpha, H) = c(\alpha, H) - \omega(A)$ for any H, $A \in \Pi$ and any non-zero $\lambda \in \mathbb{Q}$.*

3) *If $H' = H + h(t)$ for some function $h : S^1 \to \mathbb{R}$ then $c(\alpha, H') = c(\alpha, H) - \int_0^1 h(t)dt$.*

4) *Pick any $\epsilon > 0$ and any Floer chain representing the Floer homology class $PD(\Psi_{\bar{H},J}(\alpha)) \in HF_{2n-*}(H, J)$. Decompose the chain into a sum $\sum_i \lambda_i \hat{\gamma}_i$, $\lambda_i \in \mathbb{Q}$, $\hat{\gamma}_i \in \mathcal{P}(H)$. Then this sum has to contain a non-zero term $\lambda_i \hat{\gamma}_i$ with $\mathcal{A}_H(\hat{\gamma}_i) \geq c(\alpha, H) - \epsilon$. Moreover, one can prove that $c(\alpha, H)$ belongs to the spectrum* Spec (H).

5) *If $\alpha \in H^*(M, \mathbb{Q}) \subset QH^*(M,\omega)$ is a* **singular** *cohomology class then*

$$-\int_0^1 \sup_M H^t dt \leq c(\alpha, H) \leq -\int_0^1 \inf_M H^t dt. \tag{9}$$

6) *If a quantum product $\alpha_1 * \alpha_2$ is non-zero and Hamiltonians H_1, H_2 are normalized then*

$$c(\alpha_1 * \alpha_2, H_1 \sharp H_2) \leq c(\alpha_1, H_1) + c(\alpha_2, H_2).$$

7) *Assume that H_1, H_2 are normalized Hamiltonians such that $\widetilde{\varphi}_{H_1} = \widetilde{\varphi}_{H_2}$ in $\widetilde{\mathrm{Ham}}(M,\omega)$. Then $c(\alpha, H_1) = c(\alpha, H_2)$. Thus for any $\widetilde{\varphi} \in \widetilde{\mathrm{Ham}}(M,\omega)$ one can define*

$$c(\alpha, \widetilde{\varphi}) = c(\alpha, H),$$

where H is any normalized Hamiltonian such that $\widetilde{\varphi} = \widetilde{\varphi}_H$.

8) *The function $\widetilde{\varphi} \mapsto c(\alpha, \widetilde{\varphi})$ is invariant under conjugation in $\widetilde{\mathrm{Ham}}\,(M, \omega)$.*

9) *If (M, ω) is symplectically aspherical then $c(\mathbf{1}, \widetilde{\varphi}) = -c(m, \widetilde{\varphi}^{-1})$.*

The proof of the proposition will be outlined in Section 3. The last equality in part 9 is a partial case of a more general statement relating spectral numbers of inverse elements in $\widetilde{\mathrm{Ham}}\,(M, \omega)$ – see [13].

2.5. The main theorems

Theorem 2.5.1. *Let (M, ω) be a closed connected strongly semi-positive symplectic manifold. Let H be a normalized Hamiltonian and let $E \in QH^*(M, \omega)$ be the Euler class. Assume that for a positive integer g the class $E^g \in QH^*(M, \omega)$ is non-zero and $c(E^g, H) > 0$. Then $\mathrm{cl}\,(\widetilde{\varphi}_H) > g$ in the group $\widetilde{\mathrm{Ham}}\,(M, \omega)$.*

The proof of Theorem 2.5.1 occupies Sections 4–8. Its general overview and final steps can be found in Section 8.

In the case when (M, ω) is symplectically aspherical and the Euler characteristic $\chi(M)$ is zero the class E vanishes and Theorem 2.5.1 cannot be applied. However the problem can be fixed by means of the following result.

Theorem 2.5.2 (L. Polterovich)**.** *Let (M, ω) be a closed connected symplectically aspherical manifold. Let H be a normalized Hamiltonian. Assume that $c(m, \widetilde{\varphi}) > 0$. Then $\mathrm{cl}\,(\widetilde{\varphi}) > 1$ in the group $\widetilde{\mathrm{Ham}}\,(M, \omega)$.*

The proof, unlike in the case of Theorem 2.5.1, involves only a short calculation using basic properties of spectral numbers – see Section 3.2.

Normalized Hamiltonian functions giving rise to positive spectral numbers are produced by the following construction.

Proposition 2.5.3 (Y. Ostrover, [35])**.** *Let (M, ω) be a closed connected strongly semi-positive symplectic manifold. Let $B \subset M$ be a ball and let $F : S^1 \times M \to \mathbb{R}$ be a normalized Hamiltonian such that φ_F displaces B. Then for any $\alpha \neq 0$*

$$c(\alpha, F \sharp w H_B) = c(\alpha, F) + w$$

and thus $c(\alpha, F \sharp w H_B) > 0$ for any sufficiently large w.

For a proof of Proposition 2.5.3 see Section 3.3. Proposition 2.5.3 combined with Theorems 2.5.1 yields the following immediate corollary.

Corollary 2.5.4. *Let (M, ω) a be closed, connected and strongly semi-positive symplectic manifold. Let $E \in QH^*(M, \omega)$ be the Euler class. Assume that the class*

$E^g \in QH^*(M,\omega)$ is non-zero for any non-negative g. Then $\mathrm{cl}(\widetilde{\mathrm{Ham}}\,(M,\omega))=+\infty$.

Remark 2.5.5. It is shown in [13] that if (M,ω) is spherically monotone and E is *invertible* in the ring $QH^*(M,\omega)$ then there exists a non-trivial homogeneous quasimorphism on $\widetilde{\mathrm{Ham}}\,(M,\omega)$ and hence $\mathrm{cl}\,(\widetilde{\mathrm{Ham}}\,(M,\omega)) = +\infty$.

2.6. Applications of Theorem 2.5.1

We say that the class $E^g \in QH^*(M,\omega) = H^*(M,\mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_\omega$ is *split* if it can be represented as

$$E^g = \alpha \otimes_{\mathbb{Q}} e^C, \tag{10}$$

$\alpha \in H^*(M,\mathbb{Q})$, $e^C \in \Lambda_\omega$.

Corollary 2.6.1. *Let (M,ω) a be closed, connected and strongly semi-positive symplectic manifold. Let $B \subset M$ be a ball and let $F : S^1 \times M \to \mathbb{R}$ be a normalized Hamiltonian such that φ_F displaces B.*

1) *Assume that the class $E^g \in QH^*(M,\omega)$ is non-zero and split as $E^g = \alpha \otimes_{\mathbb{Q}} e^C$ for a* **particular** *$g > 0$. Then for any*

$$w > \int_0^1 \sup_M F(t,\cdot)dt + \omega(C)$$

one has $\mathrm{cl}\,(\widetilde{\varphi}_{F\sharp wH_B}) > g$ in the group $\widetilde{\mathrm{Ham}}\,(M,\omega)$.

2) *Assume that for* **each** *$g > 0$ the class E^g is non-zero and split as $E^g = \alpha_g \otimes_{\mathbb{Q}} e^{C_g}$. Set $I_g := \omega(C_g)$ and assume that $I := \lim_{g\to\infty} I_g/g > 0$. Then for any $w > 0$*

$$\|\widetilde{\varphi}_{wH_B}\|_{\mathrm{cl}} \geq w/I.$$

Below we present two examples, where the multiplicative structure of the quantum cohomology ring of (M,ω) is known. In these cases the classes E^g are non-zero and split and the quantities $I_g = \omega(C_g)$ are computable.

Example 2.6.2. Let $M = \mathbb{C}P^n$ and let ω be the standard Fubini-Study form. Let A, a be respectively, the generators of $H_2(\mathbb{C}P^n)$ and $H^2(\mathbb{C}P^n)$ which are positive with respect to ω. Set $q = e^A \in \Lambda_\omega$. Then the (rational) quantum cohomology ring of $(\mathbb{C}P^n,\omega)$ is isomorphic to the graded polynomial ring

$$\frac{\mathbb{Q}[a,q,q^{-1}]}{\{a^{n+1} = q,\ q\cdot q^{-1} = 1\}},$$

where the degree of a is two and the degrees of q and q^{-1} are, respectively, $2n+2$ and $-2n-2$ [43], [44], [51].

Since the class A generates $\Pi = H_2(\mathbb{C}P^n)$ and $c_1(A) = n+1$ we can easily check that the Euler class E actually belongs to the singular cohomology $H^{2n}(\mathbb{C}P^n,\mathbb{Q})$

and can be written as $E = \chi(\mathbb{C}P^n)m$. Now in the quantum cohomology ring of $\mathbb{C}P^n$ one obviously has $\chi(\mathbb{C}P^n)m = \chi(\mathbb{C}P^n)a^n$ and thus

$$\chi(\mathbb{C}P^n)m^g = \chi(\mathbb{C}P^n)a^{gn} = \chi(\mathbb{C}P^n)a^l \otimes_{\mathbb{Q}} e^{kA} \in H^*(\mathbb{C}P^n, \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_\omega,$$

where $gn = k(n+1) + l$ with $k, l \in \mathbf{Z}, 0 \leq l \leq n$. Therefore $E^g = (\chi(\mathbb{C}P^n)m)^g$ is split for any g and

$$I_g = k\omega(A),$$

where $k = [gn/n+1]$. This yields

$$I := \lim_{g\to\infty} \frac{I_g}{g} = \lim_{g\to\infty} \frac{[gn/n+1]\,\omega(A)}{g} = \frac{n}{n+1}\omega(A). \tag{11}$$

Example 2.6.3. Consider a generalization of the previous example: let $M = Gr\,(r,n)$, $1 \leq r \leq n-1$, be the Grassmannian of complex r-dimensional subspaces in $\mathbb{C}^n$. The manifold $Gr\,(r,n)$ carries a natural symplectic form ω – one can view the Grassmannian as the result of symplectic reduction for the Hamiltonian action of $U(r)$ (by multiplication from the right) on the space of complex $r \times n$ matrices which carries a natural symplectic structure.

The linear basis of the cohomology group $H^*(Gr\,(r,n), \mathbb{Q})$ is formed by the Schubert classes σ^I, where $I = (i_1, \ldots, i_k)$ is a partition such that $r \geq i_1 \geq \ldots \geq i_k > 0$, $k \leq n-r$. Given two numbers i, j, $0 \leq i \leq r$, $0 \leq j \leq n-r$, denote by $\{i \times j\}$ the partition $(i, i, \ldots, i)$ with i repeated j times. (If $i = 0$ or $j = 0$ then $\{i \times j\}$ is an empty partition.) In particular, the top-dimensional cohomology class $m \in H^{2r(n-r)}(Gr\,(r,n), \mathbb{Q})$ can be viewed as $\sigma^{\{r\times(n-r)\}}$.

Let A be the generator of $H_2(Gr\,(r,n), \mathbf{Z}) \cong \mathbf{Z}$ such that $\omega(A) > 0$. The quantum cohomology $QH^*(Gr\,(r,n), \omega)$ is isomorphic, as a group, to $\mathbb{Q}[\sigma^I] \otimes_{\mathbb{Q}} \mathbb{Q}[q, q^{-1}]$ where $q = e^A$ and I runs over the set of all partitions as above. The multiplicative structure of the quantum cohomology ring of $(Gr\,(r,n), \omega)$ is described in [6], [49], [52]. In particular, one can show that the Euler class for in the quantum cohomology of $(Gr\,(r,n), \omega)$ belongs to the singular cohomology [1], [5]:

$$E = \chi(Gr\,(r,n))m. \tag{12}$$

Moreover there exists an algorithm [7] which allows to compute explicitly the structural coefficients of the quantum cohomology ring $QH^*(Gr\,(r,n), \omega)$ with respect to the basis $\{\sigma^I\}$. Using this algorithm A. Postnikov obtained the following result [41].

Proposition 2.6.4. *Fix $g \geq 1$ and consider the quantum cohomology class $m^g \in QH^*(Gr\,(r,n))$. At least one of the following two statements about the numbers r, n, g is always true:*

(I) *$gr = an + b$, $a = [gr/n]$, $0 \leq b \leq r$.*

(II) *$g(n-r) = cn + d$, $c = [g(n-r)/r]$, $0 \leq d \leq n-r$.*

In the case (I)

$$m^g = \sigma^{\{b\times(n-r)\}} \otimes_{\mathbb{Q}} q^{a(n-r)}.$$

In the case (II)

$$m^g = \sigma^{\{r\times d\}} \otimes_{\mathbb{Q}} q^{cr}.$$

Recalling that $c_1(A) = n \neq 0$, $\chi(Gr\,(r,n)) > 0$ and applying (12) together with Proposition 2.6.4 we get that $E^g = (\chi(Gr\,(r,n))m)^g$ is non-zero and split for any g with

$$I_g = a(n-r)\omega(A) = \left[\frac{gr}{n}\right](n-r)\omega(A)$$

in the case (I) and

$$I_g = cr\omega(A) = \left[\frac{g(n-r)}{n}\right]r\omega(A)$$

in the case (II). Thus

$$I := \lim_{g\to\infty} \frac{I_g}{g} = \frac{r(n-r)}{n}\cdot\omega(A). \tag{13}$$

2.7. Proofs of Corollary 2.6.1 and Theorems 1.2.1–1.2.4

Proof of Corollary 2.6.1. Since $E^g = \alpha \otimes_{\mathbb{Q}} e^C$ parts 2 and 5 of Proposition 2.4.2 yield

$$c(E^g, F) = c(\alpha, F) - \omega(C) \geq -\int_0^1 \sup_M F^t dt - \omega(C).$$

But according to Proposition 2.5.3,

$$c(\alpha, F\sharp wH_B) = c(\alpha, F) + w.$$

Combining the inequality above with the hypothesis

$$w > \int_0^1 \sup_M F(t,\cdot)dt + \omega(C),$$

we immediately get

$$c(E^g, F\sharp wH_B) > 0.$$

Therefore, according to Theorem 2.5.1, $\mathrm{cl}\,(\widetilde{\varphi}_{F\sharp wH_B}) > g$. Part 1 of Corollary 2.6.1 is proven.

To show part 2 of Corollary 2.6.1 we start with the following simple observations. Recall that

$$\widetilde{\varphi}_{F\sharp kwH_B} = \widetilde{\varphi}_F \cdot \widetilde{\varphi}_{kwH_B}$$

for any $k \in \mathbb{Z}$ and w. This, together with the definition of commutator length, yields

$$\mathrm{cl}\,(\widetilde{\varphi}_F) = \mathrm{cl}\,(\widetilde{\varphi}_F^{-1}) \geq |\mathrm{cl}\,(\widetilde{\varphi}_{F\sharp kwH_B}) - \mathrm{cl}\,(\widetilde{\varphi}_{kwH_B})|.$$

Since H_B is time-independent, $\widetilde{\varphi}^k_{wH_B} = \widetilde{\varphi}_{kwH_B}$ for any integer k. Combining all these observations, we see that

$$\mathrm{cl}\,(\widetilde{\varphi}_F) \geq |\mathrm{cl}\,(\widetilde{\varphi}_{F\sharp kwH_B}) - \mathrm{cl}\,(\widetilde{\varphi}^k_{wH_B})| \geq \mathrm{cl}\,(\widetilde{\varphi}_{F\sharp kwH_B}) - \mathrm{cl}\,(\widetilde{\varphi}^k_{wH_B})$$

for any positive integer k and therefore

$$\mathrm{cl}\,(\widetilde{\varphi}^k_{wH_B}) \geq \mathrm{cl}\,(\widetilde{\varphi}_{F\sharp kwH_B}) - \mathrm{cl}\,(\widetilde{\varphi}_F).$$

Dividing the last inequality by k and taking the limit as k goes to $+\infty$ one gets

$$\|\widetilde{\varphi}_{wH_B}\|_{\mathrm{cl}} = \lim_{k\to+\infty} \frac{\mathrm{cl}\,(\widetilde{\varphi}_{F\sharp kwH_B})}{k}. \tag{14}$$

Now pick an arbitrary small $\delta > 0$. Then

$$\lim_{g\to+\infty} (g\, I\,(1+\delta) - I_g) = +\infty.$$

Hence for any sufficiently large integer g

$$g\, I\,(1+\delta) > \int_0^1 \sup_M F(t,\cdot)dt + I_g.$$

Introducing a new integral parameter k such that

$$g = \left[\frac{kw}{I\,(1+\delta)}\right]$$

we see that for any sufficiently large integer k

$$kw > \int_0^1 \sup_M F(t,\cdot)dt + I_{[kw/I(1+\delta)]}.$$

Thus, according to the already proven part 1, for all sufficiently large integers k

$$\mathrm{cl}\,(\widetilde{\varphi}_{F\sharp kwH_B}) > \left[\frac{kw}{I(1+\delta)}\right].$$

Together with (14) this yields

$$\|\widetilde{\varphi}_{wH_B}\|_{\mathrm{cl}} \geq \lim_{k\to+\infty} \frac{[kw/I(1+\delta)]}{k} = \frac{w}{I(1+\delta)}.$$

This is true for any sufficiently small $\delta > 0$ and thus

$$\|\widetilde{\varphi}_{wH_B}\|_{\mathrm{cl}} \geq w/I.$$

Part 2 of the corollary is proven. □

Proof of Theorem 1.2.1. Theorem 1.2.1 follows immediately from part 1 of Corollary 2.6.1 and formula (11) of Example 2.6.2. □

Proof of Theorem 1.2.2. The theorem follows from part 2 of Corollary 2.6.1 and Example 2.6.2. □

Proof of Theorem 1.2.3. As in the proof of part 1 of Corollary 2.6.1 we easily get that if $w > \int_0^1 \sup_M F^t dt$ then $c(m, F\sharp wH_B) > 0$ and, according to Theorem 2.5.2, $\mathrm{cl}\,(\widetilde{\varphi}_{F\sharp kwH_B}) > 1$. The theorem is proven.

Note that for a symplectically aspherical (M, ω) the quantum and the singular cohomology rings coincide and $E = \chi(M)m$. Therefore if $\chi(M) \neq 0$ one can prove the theorem by applying part 1 of Corollary 2.6.1 instead of Theorem 2.5.2. □

Proof of Theorem 1.2.4. Theorem 1.2.4 follows from part 2 of Corollary 2.6.1 and formula (13) of Example 2.6.3. □

3. Properties of spectral numbers – proofs

The goal of this section is to outline the proof of Proposition 2.4.2. At the end of the section we will also prove Proposition 2.5.3.

3.1. Outline of the proof of Proposition 2.4.2 (cf. [13], [34])

Part 1. Part 1 is proven in [34]. For symplectically aspherical manifolds it was first proven in [47]; for a singular cohomology class α and a general symplectic manifold it was proven before in [33]. A proof for spherically monotone symplectic manifolds can be found in [13]. □

Part 2. According to part 1, spectral numbers depend continuously on the Hamiltonian. Therefore without loss of generality we may assume that H belongs to a regular Floer pair (H, J). Since $\Psi_{\bar{H},J}$ is linear over Λ_ω, one can pass from Floer chains representing $PD(\Psi_{\bar{H},J}(\alpha))$ to Floer chains representing $PD(\Psi_{\bar{H},J}(\lambda e^A \alpha))$ and backwards by multiplying them by λe^A or, respectively, by its inverse $\lambda^{-1}e^{-A}$. According to (6), the multiplication by λe^A decreases the level of a chain, with respect to the filtration on $CF_*(H, J)$, by $\omega(A)$ and the multiplication by $\lambda^{-1}e^{-A}$ increases it by the same quantity yielding the necessary property. □

Part 3. Follows immediately from the definitions. □

Part 4. Using the continuity of spectral numbers with respect to Hamiltonian we may assume without loss of generality that H belongs to a regular Floer pair (H, J). Suppose the property does not hold. Then for some $\epsilon > 0$ the Floer homology class $PD(\Psi_{\bar{H},J}(\alpha)) \in HF_{2n-*}(H, J)$ can be represented by a chain of the form $\sum b_i \hat{\gamma}_i$ (the sum may be infinite), with $b_i \in \mathbb{Q}$, $\hat{\gamma}_i \in \mathcal{P}(H)$, so that $\mathcal{A}_H(\hat{\gamma}_i) \leq c(\alpha, H) - \epsilon$ for all i. But then $PD(\Psi_{\bar{H},J}(\alpha)) \in HF^s_{2n-*}(H, J)$ for $s = c(\alpha, H) - \epsilon$ which contradicts the definition of $c(\alpha, H)$. Part 3 is proven. The statement $c(\alpha, H) \in \mathrm{Spec}\,(H)$ is proven in [34] (for a singular class α it was proven in [33]). □

Part 5. This is proven in [47] for symplectically aspherical manifolds. The general case is proven in [33]. □

Part 6. Part 6 is proven in [47] (see Proposition 4.1 there) for symplectically aspherical manifolds (in that case the quantum cohomology coincides with the singular one and the quantum product is the usual cup-product). In view of the results from [36] (also see [12]) the same proof works also for an arbitrary strongly semi-positive manifold. □

Part 7. Part 7 is proven in the general case by the same argument that was used in [47] in the symplectically aspherical case. Namely let $\{H_\tau\}_{0\le\tau\le 1}$ be a homotopy H_1 and H_2 in the class of normalized Hamiltonians generating $\widetilde{\varphi}_{H_1} = \widetilde{\varphi}_{H_2}$. Similarly to the situation in the symplectically aspherical case (see Proposition 3.1 in [47]), there exists a natural one-to-one correspondence between $\mathcal{P}(H_1)$ and $\mathcal{P}(H_\tau)$ for any τ and hence a one-to-one correspondence between $\mathrm{Spec}\,(H_1)$ and $\mathrm{Spec}\,(H_\tau)$. Moreover, a computation similar to Lemma 3.3 in [47] shows that $\mathrm{Spec}\,(H_1) = \mathrm{Spec}\,(H_\tau)$ for any τ. According to Proposition 2.3.1, the set $\mathrm{Spec}\,(H_1) \subset \mathbb{R}$ is of measure zero. Now part 4 of the proposition says that for any τ the number $c(\alpha, H_\tau)$ belongs to the same set $\mathrm{Spec}\,(H_1)$ of measure zero. On the other hand, according to part 1, $c(\alpha, H_\tau)$ depends continuously on τ. Therefore it is a constant function of τ and $c(\alpha, H_1) = c(\alpha, H_2)$, which proves part 7 of the proposition. □

Part 8. Take two normalized Hamiltonians H_1 and H_2. In view of Part 7, it suffices to prove that $c(\alpha, H_1) = c(\alpha, H_2 \sharp H_1 \sharp \bar{H}_2)$. Set

$$H' := H_2 \sharp H_1 \sharp \bar{H}_2,$$

$$H'' := H_1(t, \varphi_{H_2}^{-1}(x)).$$

Then $\widetilde{\varphi}_{H'} = \widetilde{\varphi}_{H''}$. Also, if H_1, H_2 are normalized then H' and H'' are normalized as well. Therefore, in view of part 7, one has $c(\alpha, H') = c(\alpha, H'')$. On the other hand, there exists a natural isomorphism between the Floer complexes of H_1 and H'' preserving the filtration. Hence $c(\alpha, H_1) = c(\alpha, H'')$ and therefore $c(\alpha, H_1) = c(\alpha, H')$. This finishes the proof of part 8. □

Part 9. This is proven in [47]. □

3.2. Proof of Theorem 2.5.2

Part 6 of Proposition 2.4.2 yields

$$c(m, aba^{-1}b^{-1}) \leq c(m, a) + c(\mathbf{1}, ba^{-1}b^{-1}). \tag{15}$$

On the other hand, in view of parts 8 and 9 of Proposition 2.4.2,

$$c(\mathbf{1}, ba^{-1}b^{-1}) = c(\mathbf{1}, a^{-1}) = -c(m, a). \tag{16}$$

Combining (15) and (16) we get

$$c(m, aba^{-1}b^{-1}) \leq 0. \tag{17}$$

Hence if $c(m, \widetilde{\varphi}) > 0$ then $\mathrm{cl}\,(\widetilde{\varphi}) > 1$ in $\widetilde{\mathrm{Ham}}\,(M, \omega)$ and the theorem is proven. □

3.3. Proof of Proposition 2.5.3

Since φ_F displaces B and φ_{wH_B} is supported in B, the fixed points of $\widetilde{\varphi}_{F\sharp wH_B}$ coincide with the fixed points of F. An easy calculation shows that if $\hat{\gamma} \in \mathcal{P}(F)$ then the value of $\mathcal{A}_{F\sharp wH_B}$ on the element of $\mathcal{P}(F\sharp wH_B)$, corresponding to $\hat{\gamma}$, is equal to $\mathcal{A}_F(\hat{\gamma}) + w$. Hence

$$\mathrm{Spec}\,(F\sharp wH) = \mathrm{Spec}\,(F) + w. \tag{18}$$

Parts 1 and 4 of Proposition 2.4.2 say that $c(\alpha, F\sharp wH_B) \subset \mathrm{Spec}\,(F\sharp wH)$ and $c(\alpha, F\sharp wH_B)$ changes continuously with w. But, according to Proposition 2.3.1, for any w the set $\mathrm{Spec}\,(F\sharp wH)$ is of measure zero. Together with (18) this yields

$$c(\alpha, F\sharp wH_B) = c(\alpha, F\sharp H_B) + w.$$

The proposition is proven. □

4. A function $\Upsilon_{l,g}$ on sets of conjugacy classes and K-area

In this section we start to prepare the tools needed for the proof of Theorem 2.5.1. Our setup throughout the section will be an arbitrary connected Lie group G equipped with a bi-invariant Finsler pseudo-metric. We will define a function on a set of conjugacy classes. This function will be used to measure the distance from a given element of the group (or, more precisely, from its conjugacy class) to the set of elements whose commutator length is bounded by a certain fixed number. Then we will show that the function can be described in the language of connections on the trivial G-bundle over an oriented compact surface with boundary.

Our main example will be the group $\widetilde{\mathrm{Ham}}\,(M, \omega)$. The pseudo-metric on this group is defined as the lift of the Hofer metric from $\mathrm{Ham}\,(M, \omega)$.

4.1. A function $\Upsilon_{l,g}$ on sets of conjugacy classes

We are going to define a function $\Upsilon_{l,g}$ on sets of l conjugacy classes by modifying the definition of a similar function Υ_l in [12].

Namely, given a connected Lie group G and a positive integer k denote by $G^{(k)} \subset G$ the subset formed by all products of k commutators in G. Set $G^{(0)} = Id$. Each $G^{(k)}$ is invariant under conjugation and contains inverses of all its elements. Obviously,

$$\{Id\} \subset G^{(1)} \subset G^{(2)} \subset \ldots \subset [G,G],$$

$$\bigcup_{k\geq 1} G^{(k)} = [G,G]$$

and

$$\mathrm{cl}\,(x) = \min\{k \,|\, x \in G^{(k)}\}$$

for $x \in [G,G]$.

Now suppose the Lie algebra $\mathfrak{g}$ of G admits a bi-invariant norm. Such a norm defines a bi-invariant Finsler norm $\|\cdot\|$ on TG (which smoothly depends on a point in G). This allows us to measure lengths of paths in G: if $\gamma : [0,1] \to G$ is a smooth path then

$$\mathrm{length}\,(\gamma) := \int_0^1 \|\gamma'(t)\| dt.$$

Thus one can define a *bi-invariant pseudo-metric* ρ on G called *Finsler pseudo-metric*: if $\phi, \psi \in G$ then

$$\rho(\phi,\psi) := \inf_{\gamma} \mathrm{length}\,(\gamma)$$

where the infimum is taken over all paths γ connecting ϕ and ψ. It is a pseudo-metric in the sense that it satisfies the same conditions as a genuine metric (it is non-negative, symmetric and satisfies the triangle inequality) but may be degenerate: $\rho(x,y)$ may be zero if $x \neq y$.

Definition 4.1.1. Given a bi-invariant Finsler pseudo-metric ρ on G and a tuple $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_l)$ of conjugacy classes in G define

$$\Upsilon_{l,g}\,(\mathcal{C}) = \inf_{x\in G^{(g)},\varphi_i\in\mathcal{C}_i} \rho\,(Id, x\cdot\prod_{i=1}^{l}\varphi_i).$$

In particular, in the notation of [12] one has $\Upsilon_{l,0}\,(\mathcal{C}) = \Upsilon_l\,(\mathcal{C})$, where $\Upsilon_l\,(\mathcal{C})$ is defined as: $\Upsilon_l\,(\mathcal{C}) := \inf_{\varphi_i\in\mathcal{C}_i} \rho\,(Id, \prod_{i=1}^{l}\varphi_i)$. (Obviously, $\Upsilon_{l,g}\,(\mathcal{C})$ does not depend on the order of classes $\mathcal{C}_i$ in $\mathcal{C}$).

Now consider a particular case when $l = 1$ and observe that conjugate group elements have the same commutator length. Definition 4.1.1 yields

Proposition 4.1.2. *Let $\mathcal{C}$ be a conjugacy class in G that lies inside $[G,G]$. Then the following conditions are equivalent:*
(i) $\mathrm{cl}\,(x) = g$ *for any* $x \in \mathcal{C}$*;*
(ii) $\Upsilon_{1,k}(\mathcal{C}) = 0$ *for* $k \geq g$ *and* $\Upsilon_{1,k}(\mathcal{C}) > 0$ *for* $1 \leq k < g$.

4.2. K-area and $\Upsilon_{l,g}$

We will now present an alternative way to describe the function $\Upsilon_{l,g}$.

Let Σ be a connected oriented Riemann surface of genus g with $l \geq 1$ infinite cylindrical ends $\Sigma_1, \ldots, \Sigma_l$. Fix an orientation-preserving identification $\Phi_i : [0,+\infty) \times S^1 \to \Sigma_i$, $1 \leq i \leq l$, of each end $\Sigma_i \subset \Sigma$ with the standard oriented cylinder $[0,+\infty) \times S^1$.

Fix a volume form Ω on Σ so that $\int_\Sigma \Omega = 1$. According to Moser's theorem [30], any two such volume forms coinciding at infinity can be mapped into each other by a compactly supported diffeomorphism of Σ.

Let G be a connected Lie group whose tangent bundle is equipped with a bi-invariant Finsler norm that defines a bi-invariant Finsler pseudo-metric ρ on G (see Section 4.1). Identify the Lie algebra $\mathfrak{g}$ of G with the space of right-invariant vector fields on G. Assume that G acts effectively on a connected manifold F, i.e. $G \to \mathit{Diff}(F)$ is a monomorphism.

Below we will not use any results from the Lie theory and thus all our considerations will hold even if G is an infinite-dimensional Lie group, like $G = \widetilde{\mathrm{Ham}}\,(M,\omega)$ which will be our main example.

Consider the trivial G-bundle $\pi : P \to \Sigma$, with the fiber F. Let L^∇ denote the curvature of a connection ∇ on the bundle $\pi : P \to \Sigma$.[2] If the fiber $\pi^{-1}(x)$ is identified with F then to a pair of vectors $v, w \in T_x\Sigma$ the curvature tensor associates an element $L^\nabla(v,w) \in \mathfrak{g}$. Here we use the fact that G acts effectively on F. If no identification of $\pi^{-1}(x)$ with F is fixed then $L^\nabla(v,w) \in \mathfrak{g}$ is defined up to the adjoint action of G on $\mathfrak{g}$. Thus if the norm $\|\cdot\|$ on TG is the bi-invariant, $\|L^\nabla(v,w)\|$ does not depend on the identification of $\pi^{-1}(x)$ with F.

Definition 4.2.1. Define $\|L^\nabla\|$ as

$$\|L^\nabla\| = \max_{v,w} \frac{\|L^\nabla(v,w)\|}{|\Omega(v,w)|},$$

where the maximum is taken over all pairs $(v,w) \in T\Sigma \times T\Sigma$ such that $\Omega(v,w) \neq 0$.

Given a connection ∇ on $\pi : P \to \Sigma$ its holonomy along a loop based at $x \in \Sigma$ can be viewed as an element of G acting on F provided that the bundle is trivialized over x. If the trivialization of the bundle is allowed to vary then the

[2] We work only with G-connections, i.e. the connections whose parallel transports belong to the structural group G.

holonomy is defined up to conjugation in the group G. Observe that the action of the gauge group does not change $\|L^\nabla\|$.

Now let $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_l)$ be conjugacy classes in G.

Definition 4.2.2. Let $\mathcal{L}(\mathcal{C})$ denote the set of connections ∇ on $\pi : P \to \Sigma$ which are flat over the set $\bigcup_{i=1}^{l} \Phi_i([K, +\infty) \times S^1)$ for some $K > 0$ and such that for any $i = 1, \dots, l$ and any $s \geq K$ the holonomy of ∇ along the oriented path $\Phi_i(s \times S^1) \subset \Sigma_i$, with respect to a fixed trivialization of π, does not depend on s and lies in $\mathcal{C}_i$.

Definition 4.2.3. The number $0 < K\text{-}area_{l,g}(\mathcal{C}) \leq +\infty$ is defined as

$$K\text{-}area_{l,g}(\mathcal{C}) = \sup_{\nabla \in \mathcal{L}(\mathcal{C})} \|L^\nabla\|^{-1}.$$

Obviously, since none of the ends of Σ is preferred over the others, the quantity $K\text{-}area_{l,g}(\mathcal{C}_1, \dots, \mathcal{C}_l)$ does not depend on the order of the conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_l$.

Now let $\Upsilon_{l,g}$ be defined by means of the same pseudo-metric on G which was used in the definition of K-area.

Proposition 4.2.4.

$$\Upsilon_{l,g}(\mathcal{C}) = \frac{1}{K\text{-}area_{l,g}(\mathcal{C})}.$$

(If the K-area is infinite its inverse is assumed to be zero.)

4.3. Proof of Proposition 4.2.4

As in the case of genus zero (see [12]), one needs to deal with *certain systems of paths.*

Definition 4.3.1. An *(l,g)-system of paths* is a tuple

$$(a, b) = (a_1, \dots, a_l, b_1, \bar{b}_1, \dots, b_g, \bar{b}_g)$$

of $l + 2g$ smooth paths $a_1, \dots, a_l, b_1, \bar{b}_1, \dots, b_g, \bar{b}_g : [0,1] \to G$ such that

1) $\prod_{i=1}^{l} a_i(0) \cdot \prod_{k=1}^{g} b_k(0) \cdot \bar{b}_k(0) = Id$;

2) $b_k(1)$ and $\bar{b}_k^{-1}(1)$ are conjugate for each $k = 1, \dots, g$.

The length(a, b) of an (l,g)-system of paths (a, b) is defined as the sum of lengths of the paths that form the system, where the length of a path is measured with respect to the chosen Finsler pseudo-metric on G.

Definition 4.3.2. Let $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_l)$ be conjugacy classes in G. Define $\mathcal{G}_{l,g}(\mathcal{C})$ as the set of all (l,g)-systems of paths (a, b) such that $a_i(1) \in \mathcal{C}_i$, $i = 1, \dots, l$.

The proof of Proposition 4.2.4 follows immediately from the next two propositions:

Proposition 4.3.3.

$$\frac{1}{K\text{-}area_{l,g}(\mathcal{C})} = \inf_{(a,b)\in\mathcal{G}_{l,g}(\mathcal{C})} \text{length}\,(a,b).$$

Proposition 4.3.4.

$$\Upsilon_{l,g}(\mathcal{C}) = \inf_{(a,b)\in\mathcal{G}_{l,g}(\mathcal{C})} \text{length}\,(a,b).$$

Proposition 4.3.4 extends Proposition 3.5.3 in [12]. We will prove it below.

Proposition 4.3.3 can be proved by exactly the same methods as in Section 10 of [12]. Here we will discuss only the most important point which needs to be understood as one carries over the proof from [12] to the current situation.

Without loss of generality one can view Σ as a compact surface with boundary (rescale the infinite cylindrical ends of Σ to make them finite and then take the closure of the open surface). The (l,g)-system of paths associated to a connection appears if we cut open each of the g handles of Σ to get a surface Σ' of genus zero with $l+2g$ boundary components. Then, in the same way as in Section 10 of [12], we construct $l+2g$ paths $(a_1,\dots,a_l,b_1,\bar{b}_1,\dots,b_g,\bar{b}_g)$, with each path corresponding to a boundary component of Σ', so that

$$a_1(0)\cdot\ldots\cdot a_l(0)\cdot b_1(0)\cdot\bar{b}_1(0)\cdot\ldots\cdot b_g(0)\cdot\bar{b}_g(0) = Id$$

(cf. Definition 3.5.1 in [12]). In particular, each pair $b_k,\bar{b}_k$, $k=1,\dots,g$, corresponds to the two boundary components $T_k,\overline{T}_k$ of Σ' that come from the cut along a circle in the k-th handle of Σ. The fact that each $b_k(1)$ is conjugate to $\bar{b}_k^{-1}(1)$ comes from the difference of the trivializations of the principal bundle $\Sigma\times G\to\Sigma$ over T_k and $\overline{T}_k$. Indeed, recall that in the construction of the paths $(a_1,\dots,a_l,b_1,\bar{b}_1,\dots,b_g,\bar{b}_g)$ (see Section 11.2.2 of [12]) the trivializations of $\Sigma\times G\to\Sigma$ near the boundary components are not arbitrary. However – see Section 11.2.2 of [12] – as one identifies the oriented circles $T_k\cong\overline{T}_k^{-1}$, $k=1,\dots,g$, one can always assume that the gauge transformation that identifies the trivializations over T_k and $\overline{T}_k^{-1}$ is a *constant* map $T_k\to G$. For an appropriate global trivialization of the bundle $\Sigma\times G\to\Sigma$ the constant image of $T_k\to G$ is simply a holonomy of the connection over the closed path $b_k\circ\bar{b}_k^{-1}$ (the composition sign stands for the composition in the space of paths) that goes once around the k-th handle of Σ. Thus $(a_1,\dots,a_l,b_1,\bar{b}_1,\dots,b_g,\bar{b}_g)$ is indeed an (l,g)-system of paths. The rest of the argument from [12] can be carried over to our case in a straightforward manner. This finishes our discussion of the proof of Proposition 4.3.3. $\square$

Proof of Proposition 4.3.4. Pick any g commutators $X_1,\dots,X_g$ in G, where $X_k = Y_kZ_kY_k^{-1}Z_k^{-1}$, $k=1,\dots,g$. Consider all (l,g)-systems of paths $(a,b)\in\mathcal{G}_{l,g}(\mathcal{C})$ in which all paths, except for a_1, are constant so that:

- the image of each constant path a_i, $i = 2, \dots, l$, is an element $\phi_i \in \mathcal{C}_i$;
- for each $k = 1, \dots, g$ the image of the constant path b_k is Y_k and the image of the constant path $\bar{b}_k$ is $Z_k Y_k^{-1} Z_k^{-1}$.

Set $a_1(1) = \phi_1 \in \mathcal{C}_1$. Any such (l, g)-system of paths (a, b) has to satisfy the condition

$$Id = \left(\prod_{i=1}^{l} a_i(0)\right)\left(\prod_{k=1}^{g} b_k(0)\bar{b}_k(0)\right) = a_1(0) \cdot \prod_{i=2}^{l} \phi_i \cdot \prod_{k=1}^{g} Y_k Z_k Y_k^{-1} Z_k^{-1}$$

and thus

$$\text{length}\,(a, b) = \text{length}\,(a_1) =$$

$$= \rho\,(Id, a_1(1) \cdot a_1^{-1}(0)) = \rho\,(Id, \left(\prod_{i=1}^{l} \phi_i\right)\left(\prod_{k=1}^{g} X_k\right)).$$

Therefore

$$\inf_{(a,b) \in \mathcal{G}_{l,g}\,(\mathcal{C})} \text{length}\,(a, b) \le \inf_{\phi \in \mathcal{C}, \psi \in G^{(g)}} \rho\,(Id, \left(\prod_{i=1}^{l} \phi_i\right) \cdot \psi) = \Upsilon_{l,g}\,(\mathcal{C}).$$

Let us now prove the opposite inequality. Pick an arbitrary $(a, b) \in \mathcal{G}_{l,g}\,(\mathcal{C})$ so that $\bar{b}_k(1) = Z_k \cdot b_k^{-1}(1) \cdot Z_k^{-1}$ for some $Z_k \in G$, $k = 1, \dots, g$. Then

$$\begin{aligned} \text{length}\,(a, b) \; = & \sum_{i=1}^{l} \rho\,(Id, a_i(0) \cdot a_i^{-1}(1)) + \\ & + \sum_{k=1}^{g} \rho\,(Id, b_k(0) \cdot b_k^{-1}(1)) \; + \\ & + \sum_{k=1}^{g} \rho\,(Id, Z_k \cdot b_k(1) \cdot Z_k^{-1} \cdot \bar{b}_k(0)). \end{aligned} \tag{19}$$

The right-hand side of (19) can be estimated from below by means of the triangular inequality as

$$\text{length}\,(a, b) \ge \rho\,(Id, \Xi), \tag{20}$$

where

$$\Xi = \left(\prod_{i=1}^{l} a_i(0) \cdot a_i^{-1}(1)\right)\left(\prod_{k=1}^{g} b_k(0) \cdot b_k^{-1}(1) \cdot Z_k \cdot b_k(1) \cdot Z_k^{-1} \cdot \bar{b}_k(0)\right).$$

Set $A_i = \prod_{j=1}^{i} a_j(0)$ and observe that

$$\prod_{i=1}^{l} a_i(0) \cdot a_i^{-1}(1) = \left(\prod_{i=1}^{l} A_i a_i^{-1}(1) A_i^{-1}\right) \cdot A_l = \left(\prod_{i=1}^{l} \phi_i^{-1}\right) \cdot A_l,$$

for some $\phi = (\phi_1, \dots, \phi_l) \in \mathcal{C}$. Denote by X_k the commutator $X_k = b_k^{-1}(1) \cdot Z_k \cdot b_k(1) \cdot Z_k^{-1}$. One can write

$$\Xi = \left(\prod_{i=1}^{l} \phi_i^{-1} \right) \cdot \Xi_1, \tag{21}$$

where

$$\Xi_1 = A_l \cdot \prod_{k=1}^{g} b_k(0) X_k \bar{b}_k(0). \tag{22}$$

Note that by the definition of an (l, g)-system of paths

$$A_l = \bar{b}_g^{-1}(0) \cdot b_g^{-1}(0) \cdot \ldots \cdot \bar{b}_1^{-1}(0) \cdot b_1^{-1}(0).$$

Also note that for any commutator X and any $b \in G$ one has $Xb = bX'$ for some other commutator X'. Using these observations we can move the commutators in the product in (22), possibly changing them into other commutators, and rewrite Ξ_1 as:

$$\Xi_1 = A_l \cdot \left(\prod_{k=1}^{g} X'_k \right) \cdot A_l^{-1}$$

for some commutators X'_k, $k = 1, \dots, g$. Thus $\Xi_1 \in G^{(g)}$. Now substituting (21) into (20) one gets

$$\text{length}\,(a, b) \geq \rho\,(Id, \left(\prod_{i=1}^{l} \phi_i^{-1} \right) \cdot \Xi_1),$$

where $\phi \in \mathcal{C}, \Xi_1 \in G^{(g)}$, and therefore

$$\Upsilon_{l,g}\,(\mathcal{C}) = \inf_{\phi \in \mathcal{C}, \psi \in G^{(g)}} \rho\,(Id, \left(\prod_{i=1}^{l} \phi_i^{-1} \right) \cdot \psi) \leq \inf_{(a,b) \in \mathcal{G}_{l,g}\,(\mathcal{C})} \text{length}\,(a, b),$$

which finishes the proof of the proposition. □

5. Hofer metric on the group $\mathrm{Ham}\,(M, \omega)$, Hamiltonian connections and weak coupling

We are going to apply the constructions from the previous section to the group $G = \widetilde{\mathrm{Ham}}\,(M, \omega)$, where (M, ω) is a closed connected symplectic manifold. In this case there exists an additional construction – *weak coupling* – that allows to estimate K-area from above and $\Upsilon_{l,g}$ from below.

Throughout this section we assume that (M, ω) is an arbitrary closed connected symplectic manifold.

The group $G = \widetilde{\mathrm{Ham}}\,(M, \omega)$ can be viewed as an infinite-dimensional Lie group: it can be equipped with the structure of an infinite-dimensional manifold so that the group product and taking the inverse of an element become smooth operations [42]. The Lie algebra $\mathfrak{g}$ of $\widetilde{\mathrm{Ham}}\,(M,\omega)$, which is also the Lie algebra of $\mathrm{Ham}\,(M,\omega)$, can be identified with the Poisson-Lie algebra of all functions on M with the zero mean value.

The norm $\|H\| = \max_M |H|$ on $\mathfrak{g}$ is bi-invariant and defines a bi-invariant Finsler pseudo-metric on $G = \widetilde{\mathrm{Ham}}\,(M,\omega)$ which we will call *Hofer pseudo-metric.* It is a lift of a famous genuine metric on $\mathrm{Ham}\,(M,\omega)$, called *Hofer metric* [20], [23], [37]. (The original metric introduced by Hofer was defined by the norm $\|H\| = \max_M H - \min_M H$ and is equivalent to the metric we use).

Let Σ be a connected oriented Riemann surface of genus g with $l \geq 1$ infinite cylindrical ends and a fixed area form Ω of total area 1 as in Section 4.2. Consider the trivial bundle $\Sigma \times M \to \Sigma$ with the fiber $F = (M,\omega)$ and the structure group $\mathrm{Ham}\,(M,\omega)$. Let us briefly recall the following basic definitions (see [19] for details).

Definition 5.0.5. A closed 2-form $\tilde{\omega}$ on the total space $\pi : \Sigma \times M$ of the bundle $\Sigma \times M \to \Sigma$ is called *fiber compatible* if its restriction on each fiber of π is ω.

Let us fix a trivialization of the bundle $\pi : \Sigma\times M \to \Sigma$ and let $pr_M : \Sigma\times M \to M$ be the natural projection. The *weak coupling construction* [19], which goes back to W.Thurston, gives rise to the following fact.

Proposition 5.0.6. ([19]). *Let $\tilde{\omega}$ be a closed fiber compatible form on $\Sigma\times M$ that coincides with $pr_M^*\omega$ at infinity. Then for a sufficiently small $\varepsilon > 0$ there exists a smooth family of closed 2-forms $\{\Omega_\tau\}$, $\tau \in [0,\varepsilon]$, on $\Sigma \times M$ with the following properties:*

(i) $\Omega_0 = \pi^*\Omega$*;*

(ii) *if one rescales the ends of Σ so that Σ becomes a compact surface with boundary then $[\Omega_\tau] = \tau[\tilde{\omega}] + [\pi^*\Omega]$, where the cohomology classes are taken in the relative cohomology group $H^2(\Sigma \times M, \partial\Sigma \times M)$;*

(iii) *the restriction of Ω_τ on each fiber of π is a multiple of the symplectic form on that fiber;*

(iv) *Ω_τ is symplectic for any $\tau \in (0,\varepsilon]$.*

Definition 5.0.7. Define $\mathrm{size}\,(\tilde{\omega})$ as the supremum of all ε for which there exists a family $\{\Omega_\tau\}$, $\tau \in [0,\varepsilon]$, satisfying the properties (i)-(iv) listed above.

Any fiber compatible closed form $\tilde{\omega}$ defines a connection ∇ on $\pi : \Sigma \times M \to \Sigma$ whose parallel transports (with respect to a fixed trivialization) belong to $\mathrm{Ham}\,(M,\omega)$. Such a connection is called *Hamiltonian.* Conversely, any Hamiltonian connection ∇ on $\Sigma \times M \to \Sigma$ can be defined by means of a unique fiber compatible closed 2-form $\tilde{\omega}_\nabla$ such that the 2-form on Σ obtained from $\tilde{\omega}_\nabla^{n+1}$ by fiber integration is 0. The curvature of a Hamiltonian connection ∇ can be viewed as a 2-form associating to each pair $v, w \in T_x\Sigma$ of tangent vectors a normalized Hamiltonian function $H_{v,w}$ on the fiber $\pi^{-1}(x)$. The form $\tilde{\omega}_\nabla$ restricted to the horizontal lifts of vectors $v, w \in T_x\Sigma$ at a point $y \in \pi^{-1}(x)$ coincides with $H_{v,w}(y)$ [19].

Let $H = (H_1, \ldots, H_l)$ be (time-dependent) *normalized* Hamiltonians on M. Let $\mathcal{C}_H = (\mathcal{C}_{H_1}, \ldots, \mathcal{C}_{H_l})$ be the conjugacy classes in $\widetilde{\mathrm{Ham}}\,(M,\omega)$ containing, respectively, the elements $\widetilde{\varphi}_{H_1}, \ldots, \widetilde{\varphi}_{H_l}$.

Fix a trivialization of $\Sigma \times M \to \Sigma$. Let Φ_i, $i =, 1 \ldots, l$, and $K > 0$ be as in Definition 4.2.2. Denote by $\widetilde{\mathcal{L}}(\mathcal{C}_H)$ the set of all Hamiltonian connections ∇ on $\Sigma \times M \to \Sigma$ such that, with respect to the fixed trivialization, the holonomy of ∇ along the path $\tau \mapsto \Phi_i(s \times e^{2\pi i\tau})$, $0 \le \tau \le t$, is $\varphi^t_{H_i}$ for any $s \ge K$, $t \in [0,1]$ and $i = 1, \ldots, l$. Define $\mathcal{F}(\mathcal{C}_H)$ as the set of all the forms $\tilde{\omega}_\nabla$, $\nabla \in \widetilde{\mathcal{L}}(\mathcal{C}_H)$.

Definition 5.0.8. Define the number $0 < \mathrm{size}_g\,(H) \le +\infty$ as

$$\mathrm{size}_g\,(H) = \sup_{\tilde{\omega} \in \mathcal{F}(\mathcal{C}_H)} \mathrm{size}\,(\tilde{\omega}).$$

The following theorem can be proven by exactly the same arguments as the similar results in [38], [39] (cf. [12]).

Proposition 5.0.9. *Let K-area and $\Upsilon_{l,g}$ be measured with respect to the bi-invariant Hofer pseudo-metric on $G = \widetilde{\mathrm{Ham}}\,(M,\omega)$. Let $H = (H_1, \ldots, H_l)$ be normalized Hamiltonians. Then*

$$K\text{-}area_{l,g}\,(\mathcal{C}_H) \le \mathrm{size}_g\,(H),$$

and, in view of Proposition 4.2.4,

$$\Upsilon_{l,g}\,(\mathcal{C}_H) \ge 1/\mathrm{size}_g\,(H).$$

(If $\mathrm{size}_g\,(H) = \infty$ we set $1/\mathrm{size}_g\,(H) = 0$.)

6. Pseudo-holomorphic curves and an estimate on $\Upsilon_{l,g}$

In this section we make a crucial step towards the proof of Theorem 2.5.1: we obtain an estimate below on $\Upsilon_{l,g}$ for $\widetilde{\mathrm{Ham}}\,(M,\omega)$ based on the existence of some pseudo-holomorphic curves. The relation between existence of such curves and the hypothesis of Theorem 2.5.1 will be discussed in Section 7.

Throughout the section (M,ω) can be assumed to be an arbitrary closed connected symplectic manifold.

6.1. The spaces $\mathcal{T}(H)$, $\mathcal{T}_\tau(H)$, $\mathcal{T}_{\tau,J}(H)$ of almost complex structures

With Σ as in the previous section, consider again the trivial (and trivialized) bundle $\pi : \Sigma \times M \to \Sigma$. Let j be a complex structure on Σ compatible with the area form Ω. Without loss of generality we may assume that the identifications $\Phi_i : [0, +\infty) \times S^1 \to \Sigma_i$, $1 \le i \le l$, are chosen in such a way that near infinity the complex structure on the ends gets identified with the standard complex structure on the cylinder $[0, +\infty) \times S^1$. Let J be an almost complex structure on M compatible with ω.

We say that an almost complex structure $\tilde{J}$ on $\Sigma \times M$ is *J-fibered* if the following conditions are fulfilled:

- $\tilde{J}$ preserves the tangent spaces to the fibers of π;
- the restriction of $\tilde{J}$ on any fiber of π is an almost complex structure compatible with the symplectic form ω on that fiber;
- the restriction of $\tilde{J}$ to any fiber $\pi^{-1}(x)$ for x outside of some compact subset of Σ is J.

Let $H = (H_1, \ldots, H_l)$ be (time-dependent) Hamiltonians on M. Pick $\hat{\gamma} \in \mathcal{P}(H)$. Let us also pick a cut-off function $\beta : \mathbb{R} \to [0, 1]$ such that $\beta(s)$ vanishes for $s \le \epsilon$ and $\beta(s) = 1$ for $s \ge 1 - \epsilon$ for some small $\epsilon > 0$. For each section $u : \Sigma \to \Sigma \times M$ set

$$u_i := pr_M \circ u \circ \Phi_i : [0, +\infty) \times S^1 \to M.$$

For each $i = 1, \ldots, l$ consider the non-homogeneous Cauchy–Riemann equation

$$\partial_s u_i + J(u_i)\partial_t u_i - \beta(s)\nabla_u H_i(t, u_i) = 0, \tag{23}$$

where the gradient is taken with respect to the Riemannian metric $\omega(\cdot, J\cdot)$ on M. According to [18], the solutions of such an equation correspond exactly to the pseudo-holomorphic sections of $\pi^{-1}(\Sigma_i) \to \Sigma_i$ with respect to some unique J-fibered almost complex structure on $\pi^{-1}(\Sigma_i)$.

Definition 6.1.1. Let $\tilde{J}$ be an almost complex structure on $\Sigma \times M$ and let $H = (H_1, \ldots, H_l)$ be Hamiltonians as above. We shall say that $\tilde{J}$ is *H-compatible* if there exists an almost complex structure $J = J(\tilde{J})$ on M compatible with ω such that the following conditions hold:

- $\tilde{J}$ is J-fibered.
- $\pi \circ \tilde{J} = j \circ \pi$.
- There exists a number K such that for each $i = 1, ..., l$ and each $\tilde{J}$-holomorphic section u of π the restriction of $u_i : [0, +\infty) \times S^1 \to M$ to $[K, +\infty) \times S^1$ is a solution of the non-homogeneous Cauchy–Riemann equation (23) for $J = J(\tilde{J})$.

Denote by $\mathcal{T}(H)$ the space of all H-compatible almost complex structures on $\Sigma \times M$.

Let $\mathcal{C}_H = (\mathcal{C}_{H_1}, \ldots, \mathcal{C}_{H_l})$ be the conjugacy classes in $\widetilde{\mathrm{Ham}}\,(M, \omega)$ as in Section 5.

Definition 6.1.2. Consider all the families $\{\Omega_{\tilde{\omega}_\nabla,\tau}\}$, that arise from the weak coupling construction associated with $\tilde{\omega}_\nabla$, $\nabla \in \widetilde{\mathcal{L}}(\mathcal{C}_H)$ (see Section 5). Given a number $\tau_0 \in (0, \mathrm{size}_g(H))$ consider the set $\mathcal{Q}_{\tau_0}$ of all the symplectic forms $\Omega_{\tilde{\omega}_\nabla,\tau_0}$ from the families $\{\Omega_{\tilde{\omega}_\nabla,\tau}\}$ as above (i.e. we consider only those families which are defined for the value τ_0 of the parameter τ and pick the form $\Omega_{\tilde{\omega}_\nabla,\tau_0}$ from each such family). Denote by $\mathcal{T}_{\tau_0}(H)$ the set of all the almost complex structures in $\mathcal{T}(H)$ which are compatible with some symplectic form from $\mathcal{Q}_{\tau_0}$. For an almost complex structure J on M compatible with ω denote by $\mathcal{T}_{\tau,J}(H)$ the set of all $\tilde{J} \in \mathcal{T}_\tau(H)$ such that $J = J(\tilde{J})$.

6.2. Moduli spaces $\mathcal{M}_g(\hat{\gamma}, H, \tilde{J})$ and the number $s_g(H)$

In [12] we defined a moduli space $\mathcal{M}(\hat{\gamma}, H, \tilde{J})$ of certain pseudo-holomorphic curves of genus zero. Here we briefly outline the definition modifying it in a straightforward manner from the case of genus zero to the case of an arbitrary genus.

In the setup of the previous section let

$$\hat{\gamma} = \Big[[\gamma_1, f_1], \dots, [\gamma_l, f_l]\Big] \in \mathcal{P}(H),$$

where $[\gamma_i, f_i] \in \mathcal{P}(H_i)$, $i = 1, \dots, l$.

Given an almost complex structure $\tilde{J} \in \mathcal{T}(H)$ consider $\tilde{J}$-holomorphic sections $u : \Sigma \to \Sigma \times M$ such that the ends $u_i([0, +\infty) \times S^1)$, $i = 1, \dots, l$, of the surface $pr_M \circ u(\Sigma) \subset M$ converge uniformly at infinity respectively to the periodic orbits $\gamma_1, \dots, \gamma_l$ and such that $pr_M \circ u(\Sigma)$ capped off with the discs $f_1(D^2), \dots, f_l(D^2)$ is a closed surface representing a torsion integral homology class in M. Denote the space of such pseudo-holomorphic sections u by $\mathcal{M}_g(\hat{\gamma}, H, \tilde{J})$ (the index g indicates the genus of the $\tilde{J}$-holomorphic curves we consider).

Definition 6.2.1 (cf. [12]). We will say that a real number c is *g-durable* if there exists

- a sequence $\{\tau_k\} \nearrow \mathrm{size}_g(H)$,
- a sequence $\{\hat{\gamma}_k\}$, $\hat{\gamma}_k \in \mathcal{P}(H)$, such that $\lim_{k\to+\infty} \mathcal{A}_H(\hat{\gamma}_k) = c$,
- a sequence $\{\tilde{J}_{\tau_k}\}$, $\tilde{J}_{\tau_k} \in \mathcal{T}_{\tau_k}(H)$,

so that for all k the spaces $\mathcal{M}_g(\hat{\gamma}_k, H, \tilde{J}_{\tau_k})$ are non-empty.

Denote by $s_g(H)$ the supremum of all g-durable numbers. If $\mathrm{spec}_g(H)$ is empty let $s_g(H) = -\infty$.

6.3. Estimating $\Upsilon_{l,g}$ by actions of periodic orbits

Let (M, ω) be a closed connected symplectic manifold. Consider the function $\Upsilon_{l,g}$ on conjugacy classes in $\widetilde{\mathrm{Ham}}(M, \omega)$ defined by means of the Hofer pseudo-metric on $\widetilde{\mathrm{Ham}}\,(M,\omega)$ (see Section 5).

Proposition 6.3.1. *Suppose the Hamiltonians* $H = (H_1, \ldots, H_l)$ *are normalized. Then* $\Upsilon_{l,g}\,(\mathcal{C}_H) \geq s_g(H)$.

For the proof of Proposition 6.3.1 see Section 6.4. Proposition 6.3.1 is a generalization of Theorem 1.3.1 from [12] to the case of pseudo-holomorphic curves of an arbitrary genus.

6.4. The proof of Proposition 6.3.1

The proof virtually repeats the proof of Theorem 1.3.1 from [12] in the case of genus zero.

Without loss of generality we may assume that $s_g(H) > 0$ (otherwise the proposition is trivial). Fix an arbitrary small $\epsilon > 0$ so that $s_g(H) - \epsilon > 0$. Then there exists a g-durable number c such that

$$s_g(H) \geq c \geq s_g(H) - \epsilon > 0.$$

Now, according to Definition 6.2.1, for any sufficiently small $\delta > 0$ there exist

- a number τ_0, $\mathrm{size}_g\,(H) - \delta \leq \tau_0 < \mathrm{size}_g\,(H)$,
- a Hamiltonian connection $\nabla \in \widetilde{\mathcal{L}}(\mathcal{C}_H)$ and the corresponding 2-form $\tilde{\omega}_\nabla$ on $\Sigma \times M$ such that $\tau_0 \leq \mathrm{size}\,(\tilde{\omega}_\nabla)$,
- a symplectic form $\{\Omega_{\tilde{\omega}_\nabla,\tau_0}\}$ from a weak coupling deformation,
- an almost complex structure $\tilde{J}_{\tau_0} \in \mathcal{T}_{\tau_0}(H)$ compatible with $\{\Omega_{\tilde{\omega}_\nabla,\tau_0}\}$,
- $\hat{\gamma} \in \mathcal{P}(H)$ such that $\mathcal{A}_H(\hat{\gamma}) \geq c - \delta > 0$,

so that the space $\mathcal{M}_g(\hat{\gamma}, H, \tilde{J}_{\tau_0})$ is non-empty.

Pick a map $u \in \mathcal{M}_g(\hat{\gamma}, H, \tilde{J}_{\tau_0})$. Then

$$0 \leq \int_{u(\Sigma)} \Omega_{\tilde{\omega}_\nabla,\tau_0}$$

because $\tilde{J}_{\tau_0}$ is compatible with the symplectic form $\Omega_{\tilde{\omega}_\nabla,\tau_0}$ and the surface $u(\Sigma) \subset \Sigma \times M$ is a pseudo-holomorphic curve with respect to $\tilde{J}_{\tau_0}$. On the other hand,

$$\int_{u(\Sigma)} \Omega_{\tilde{\omega}_\nabla,\tau_0} = \int_{u(\Sigma)} \tau_0\tilde{\omega}_\nabla + \int_{u(\Sigma)} \pi^*\Omega$$

because of the cohomological condition satisfied by a weak coupling deformation (see condition (iii) in Proposition 5.0.6). Thus

$$0 \leq \int_{u(\Sigma)} \tau_0\tilde{\omega}_\nabla + \int_{u(\Sigma)} \pi^*\Omega. \tag{24}$$

Next we recall the following lemma (see Lemma 5.0.1 in [12]) whose proof does not depend on the genus of Σ.

Lemma 6.4.1.

$$\int_{u(\Sigma)} \tilde{\omega}_\nabla = -\mathcal{A}_H(\hat{\gamma}). \tag{25}$$

Now, using Lemma 6.4.1 and the fact that the total Ω-area of Σ is 1, one can rewrite (24) as

$$\tau_0 \leq \frac{1}{\mathcal{A}_H(\hat{\gamma})}$$

and hence

$$\text{size}_g(H) - \delta \leq \tau_0 \leq \frac{1}{\mathcal{A}_H(\hat{\gamma})} \leq \frac{1}{c-\delta}.$$

Since this is true for any $\delta > 0$,

$$\text{size}_g(H) \leq \frac{1}{c} \leq \frac{1}{s_g(H)-\epsilon}.$$

In view of Proposition 5.0.9,

$$\Upsilon_{l,g}(\mathcal{C}_H) \geq s_g(H) - \epsilon.$$

Since $\epsilon > 0$ was chosen arbitrarily,

$$\Upsilon_{l,g}(\mathcal{C}_H) \geq s_g(H)$$

and the proposition is proven. □

7. Pair-of-pants product on Floer cohomology and the moduli spaces $\mathcal{M}_g(\hat{\gamma}, H, \tilde{J})$

The goal of this section is to relate the moduli spaces $\mathcal{M}_g(\hat{\gamma}, H, \tilde{J})$ and the number $s_g(H)$ to the hypothesis of Theorem 2.5.1. This will be done as in [12] by means of the multiplicative structure on Floer and quantum cohomology which will be used to guarantee that certain moduli spaces $\mathcal{M}_g(\hat{\gamma}, H, \tilde{J})$ with $\mathcal{A}_H(\hat{\gamma}) > 0$ are non-empty and for that reason $s_g(H) > 0$.

Throughout this section the symplectic manifold (M^{2n}, ω) is assumed to be strongly semi-positive.

Let Σ be a Riemann surface of genus g as above with $l = 1$ cylindrical end. Let (H, J) be a regular Floer pair. Fix a number τ, $0 < \tau < \text{size}_g(H)$.

Proposition 7.0.2 ([36], [46], cf. [12]). *For a generic $\tilde{J}_\tau \in \mathcal{T}_{\tau,J}(H)$ and any $\hat{\gamma} \in \mathcal{P}(H)$ with the Conley–Zehnder index $\mu(\hat{\gamma}) = 2n(1-g)$ the space $\mathcal{M}_g(\hat{\gamma}, H, \tilde{J}_\tau)$ is either empty or an oriented compact zero-dimensional manifold.*

Given such a generic $\tilde{J}_\tau$ we will say that the pair $(H, \tilde{J}_\tau)$ is *regular*. For a regular pair $(H, \tilde{J}_\tau)$, $\tilde{J}_\tau \in \mathcal{T}_{\tau,J}(H)$, and an element $\hat{\gamma} \in \mathcal{P}(H)$, $\mu(\hat{\gamma}) = 2n(1-g)$, count the curves from the compact oriented zero-dimensional moduli space $\mathcal{M}_g(\hat{\gamma}, H, \tilde{J}_\tau)$ with their signs. The resulting Gromov–Witten number will be denoted by $\#\mathcal{M}_g(\hat{\gamma}, H, \tilde{J}_\tau)$. Take the sum

$$\sum_{\hat{\gamma}} \#\mathcal{M}_g(\hat{\gamma}, H, \tilde{J}_\tau)\hat{\gamma}, \tag{26}$$

over all $\hat{\gamma} \in \mathcal{P}(H)$ such that $\mu(\hat{\gamma}) = 2n(1-g)$. The sum in (26) represents an integral chain $\theta_{\Sigma,H,\tilde{J}_\tau}$ in the chain complex $CF_*(H, J)$ or, from the Poincaré-dual point of view, an integral cochain $\theta^{\Sigma,H,\tilde{J}_\tau}$ in the cochain complex $CF^*(\bar{H}, J)$.

The following proposition from [12] is a minor generalization of Theorem 3.1 in [36]: we use a bigger class of admissible almost complex structures but the proof can be carried out in exactly the same way (see [12] for a discussion and an outline of the proof).

Proposition 7.0.3. *Let (H, J) be a regular Floer pair. For any τ, $0 < \tau < \mathrm{size}_g(H)$, and a regular pair $(H, \tilde{J}_\tau)$ as above the cochain $\theta^{\Sigma,H,\tilde{J}_\tau}$ defines a cocycle in the cochain complex $CF^*(\bar{H}, J)$. The corresponding cohomology class*

$$\Theta^{H,J,g} \in HF^*(\bar{H}, J)$$

is of degree $2ng$ and does not depend on τ and $\tilde{J}_\tau \in \mathcal{T}_{\tau,J}(H)$. Likewise the chains $\theta_{\Sigma,H,\tilde{J}_\tau}$ in $CF_*(H, J)$ are cycles and represent a homology class

$$\Theta_{H,J,g} \in HF_*(H, J),$$

of degree $2n(1-g)$ which does not depend on τ and $\tilde{J}_\tau \in \mathcal{T}_{\tau,J}(H)$ and is Poincaré-dual to $\Theta^{H,J,g}$.

Proposition 7.0.4. *Let $E \in QH^{2n}(M, \omega)$ be the Euler class and let (H, J) be a regular Floer pair. Then*

$$\Psi^{-1}_{\bar{H},J}(\Theta^{H,J,g}) = E^g \in QH^{2ng}(M, \omega).$$

Postponing the proof of Proposition 7.0.4 we first state the main result of this section.

Proposition 7.0.5. *Suppose that H belongs to a regular Floer pair (H, J) and that the class $E^g \in QH^{2ng}(M, \omega)$ is non-zero. Then $s_g(H) \geq c(E^g, H)$.*

Proof of Proposition 7.0.5. According to Proposition 7.0.4,

$$PD(\Psi_{\bar{H},J}(E^g)) = \Theta_{H,J,g}$$

and, according to Proposition 7.0.3, for any τ, $0 < \tau < \mathrm{size}_g(H)$, the Floer homology class $\Theta_{H,J,g}$ can be represented by a chain

$$\theta_{\Sigma,H,\tilde{J}_\tau} = \sum_{\hat{\gamma}} \#\mathcal{M}_g(\hat{\gamma}, H, \tilde{J}_\tau)\hat{\gamma}, \tag{27}$$

from $CF_*(H,J)$, where the sum in (27) is taken over all $\hat{\gamma} \in \mathcal{P}(H)$ such that $\mu(\hat{\gamma}) = 2n(1-g)$. According to part 4 of Proposition 2.4.2, for any $\epsilon > 0$ the sum (27) has to contain a non-zero term $\#\mathcal{M}_g(\hat{\gamma}, H, \tilde{J}_\tau)\hat{\gamma}$ such that $\mathcal{A}_H(\hat{\gamma}) > c(E^g, H) - \epsilon$. Recalling Definition 6.2.1 we see that the number $c(E^g, H)$ is g-durable and therefore $s_g(H)$, which is the supremum of all g-durable numbers, is no less than $c(E^g, H)$. The proposition is proven. □

Proof of Proposition 7.0.4. For brevity denote $V_k = QH_k(M,\omega)$, $\bar{V}^k = QH^k(M,\omega) = \mathrm{Hom}(V_k, \Lambda^0_\omega)$, $V = \oplus V_k$, $\bar{V} = \oplus \bar{V}^k$. Without loss of generality we can identify the Floer cohomology with $\bar{V}$ and the Floer homology with V by means of the Piunikhin–Salamon–Schwarz isomorphisms as above. Consider the spaces of the form $V^{\otimes l_1} \otimes \bar{V}^{\otimes l_2}$ where $\otimes$ stands for tensor product over Λ^0_ω.

The pair-of-pants product on the Floer cohomology can be viewed as a cohomological operation on $HF^*(H,J)$ associated to a surface of genus zero with two "entering" and one "exiting" cylindrical ends. In fact it is a part of a more general series of cohomological operations on $HF^*(H,J)$ defined by means of the moduli spaces $\mathcal{M}_g\,(\hat{\gamma}, H, \tilde{J})$ for surfaces of an arbitrary genus g and with an arbitrary number of cylindrical ends. This was first proven [46] in a simpler setup but can be shown in our case in exactly the same way (cf. [36], also see [12]).

More precisely, every Riemann surface of a genus $g \geq 0$ with $l_1 \geq 0$ positively oriented ends ("entrances") and $l_2 \geq 0$ negatively oriented ends ("exits") gives rise to a certain element

$$\Xi_{l_1,l_2,g} \in V^{\otimes l_1} \otimes \bar{V}^{\otimes l_2},$$

which depends only on l_1, l_2, g. In case of a closed surface, when $l_1 = l_2 = 0$, the element $\Xi_{0,0,g} \in \Lambda^0_\omega$ is an *integer number* – it comes as a result of counting certain *closed* pseudo-holomorphic curves with *integral multiplicities* (see [36], [46]).

Each element $\Xi_{l_1,l_2,g}$ can be viewed as a polylinear map over Λ^0_ω, denoted by $\xi_{l_1,l_2,g}$, which sends a tuple of l_1 quantum cohomology classes $(f_1, \ldots, f_{l_1})$ from $\bar{V}$ to an element of $\bar{V}^{\otimes l_2}$. Namely, each monomial term

$$\alpha_1 \otimes \ldots \otimes \alpha_{l_1} \otimes g_1 \otimes \ldots \otimes g_{l_2},$$

$\alpha_i \in V$, $g_i \in \bar{V}$, in $\Xi_{l_1,l_2,g}$ sends $f_1 \otimes \ldots \otimes f_{l_1} \in \bar{V}^{\otimes l_1}$ to

$$\prod_{i=1}^{l_1} (f_i, \alpha_i) \cdot g_1 \otimes \ldots \otimes g_{l_2},$$

where $(\cdot,\cdot)$ is the evaluation pairing with values in Λ^0_ω (see Section 2.2). In particular, if $f_i \in V^k$ and $\alpha_i \in V_j$ then $(f_i, \alpha_i) = 0$ unless $k = j$.

We will use the Poincaré isomorphism $PD : V \to \bar{V}$ to raise and lower indices of tensors:

$$PD : V^{\otimes l_1} \otimes \bar{V}^{\otimes l_2} \to V^{\otimes \{l_1 - 1\}} \otimes \bar{V}^{\otimes \{l_2+1\}}.$$

Here the l_1-th factor V in the product $V^{\otimes l_1} \otimes \bar{V}^{\otimes l_2}$ is transformed by PD into $\bar{V}$.

The basic "topological field theory" properties of $\xi_{l_1,l_2,g}$, $\Xi_{l_1,l_2,g}$ and the relation between them and the ring structure of the quantum cohomology can be described by the following proposition:

Proposition 7.0.6 ([36], [46]). *The following properties hold for any l_1, l_2, l_3, g, g_1, g_2:*

1) $\xi_{l_1,l_2,g_1} \circ \xi_{l_2,l_3,g_2} = \xi_{l_1,l_3,g_1+g_2}$ *("topological field theory property").*

2) $PD(\Xi_{l_1,l_2,g}) = \Xi_{l_1-1,l_2+1,g}$*, if* $l_1 \geq 1$.

3) $\Xi_{0,1,0} = \mathbf{1} \in \bar{V}^0$ *and* $\Xi_{1,0,0}$ *is the class* $PD(\mathbf{1}) \in V_{2n}$*, Poincaré-dual to* $\mathbf{1}$ *in* V*. Thus the map* $\xi_{1,0,0} : \bar{V} \to \Lambda^0_\omega$ *is the evaluation pairing with* $PD(\mathbf{1}) \in V_{2n}$*: its sends each* $\beta = \sum_i \beta_i \in \bar{V}$*,* $\beta_i \in \bar{V}^i$*, to* $\xi_{1,0,0}(\beta) = (\beta_{2n}, PD(\mathbf{1}))$.

4) $\xi_{1,1,0} = Id : \bar{V} \to \bar{V}$.

5) $\xi_{2,1,0} : \bar{V} \otimes \bar{V} \to \bar{V}$ *is the quantum (pair-of-pants) multiplication.*

The class $\Psi^{-1}_{H,J}(\Theta^{H,J,g}) \in QH^*(M,\omega)$ is precisely the element $\Xi_{0,1,g} \in \bar{V}$ corresponding to a surface of genus g with one "exiting" end (see [36], [46]). On the other hand, such a surface can be obtained by taking a surface with g "entrances" and one "exit" and gluing to each "entrance" a surface of genus 1 with one "exit". In view of parts 1 and 5 of Proposition 7.0.6 we only need to prove the following lemma.

Lemma 7.0.7. $\Xi_{0,1,1} = E$.

Proof of Lemma 7.0.7. We will present the cohomology class $\Xi_{0,1,1}$, viewed as a Λ^0_ω-linear map $V \to \Lambda^0_\omega$, as a composition of two polylinear forms *over* Λ_ω. This will allow us to compute $\Xi_{0,1,1}$ by means of Λ_ω-bases of $\bar{V}$.

Let $\{e_i\}$, $e_i \in \bar{V}^{k_i}$, be a Λ_ω-basis of $\bar{V}$ and let $\{F_i\}$, $F_i \in V_{k_i}$, be the dual Λ_ω-basis of V so that $(e_i, F_j) = \delta_{ij}$. Set $\bar{e}_i := PD(F_i) \in \bar{V}^{2n-k_i}$. Then $\{\bar{e}_i\}$ is a Λ_ω-basis of $\bar{V}$ Poincaré-dual to the basis $\{e_i\}$ (see Section 2.2).

Now recall part 4 of Proposition 7.0.6 and observe that the map $\xi_{1,1,0} = Id : \bar{V} \to \bar{V}$ is obviously Λ_ω-linear. Thus $\Xi_{1,1,0}$ is, in fact, an element of $V \otimes_{\Lambda_\omega} \bar{V}$, where $\otimes_{\Lambda_\omega}$ stands for the graded tensor product *over* Λ_ω as opposed to $\otimes$ which denotes the tensor product over Λ^0_ω. (Recall that as far as the grading is concerned, $\bar{V}$ and V are graded modules over, respectively, the graded and the anti-graded versions of Λ_ω – see Section 2.2). Hence, using the chosen bases of V and $\bar{V}$ over Λ_ω, one can write

$$\Xi_{1,1,0} = Id = \sum_i F_i \otimes e_i \in V \otimes_{\Lambda_\omega} \bar{V}.$$

Since the Poincaré isomorphism PD is Λ_ω-linear, the fact that $\Xi_{1,1,0} \in V \otimes_{\Lambda_\omega} \bar{V}$ together with part 2 of Proposition 7.0.6 tells us that $\Xi_{0,2,0} \in \bar{V} \otimes_{\Lambda_\omega} \bar{V}$. With respect to the chosen basis of $\bar{V}$ over Λ_ω it can be written as

$$\Xi_{0,2,0} = \sum_i \bar{e}_i \otimes e_i \in \bar{V} \otimes_{\Lambda_\omega} \bar{V}. \tag{28}$$

The quantum multiplication $\xi_{2,1,0} : \bar{V} \times \bar{V} \to \bar{V}$ is also Λ_ω-linear because it is associative [24], [28], [43], [44]. Thus $\Xi_{1,1,0} \in V \otimes_{\Lambda_\omega} V \otimes_{\Lambda_\omega} \bar{V}$.

Part 1 of Proposition 7.0.6 tells us that $\Xi_{0,1,1}$ can be found if we compose the operation associated to a surface of genus 0 with two "exits" and the operation associated with a surface of genus 0 with 2 "entrances" and 1 "exit". Since, as we have explained above, both operations are, in fact, Λ_ω-linear, we can use (28) together with parts 1 and 5 of Proposition 7.0.6 and express $\Xi_{0,1,1} \in \bar{V}$ in terms of the chosen basis of $\bar{V}$ *over* Λ_ω:

$$\Xi_{0,1,1} = \sum_i \bar{e}_i * e_i = \sum_i (-1)^{\deg e_i} e_i * \bar{e}_i = E \in \bar{V}.$$

This finishes the proof of Lemma 7.0.7 and Proposition 7.0.4 are proven. □

Remark 7.0.8. Recall from Section 2.2 that the component of degree $2n$ of E is equal to $\chi(M)m \in H^{2n}(M, \mathbb{Q})$. Thus, applying parts 1 and 3 of Proposition 7.0.6 one gets that $\Xi_{0,0,1} = \chi(M)$ – see Corollary 5.4.12 from [46].

8. Proof of Theorem 2.5.1

If H belongs to a regular Floer pair then Proposition 4.1.2 (in the case $l = 1$) and Proposition 7.0.5 yield

$$\Upsilon_{1,g}(\mathcal{C}_H) \geq c(E^g, H).$$

Since both sides depend continuously on H, the inequality is in fact true for an arbitrary H. Thus if $c(E^g, H)$ then $\Upsilon_{1,g}(\mathcal{C}_H) > 0$. In view of Proposition 4.1.2 and (2), it shows that $\mathrm{cl}\,(\widetilde{\varphi}_H) > g$ and the theorem is proven.

For the readers benefit we now quickly review the course of the proof. We used the function $\Upsilon_{1,g}$ on conjugacy classes in $\widetilde{\mathrm{Ham}}\,(M, \omega)$ to measure the distance, with respect to the Hofer pseudo-metric, from the conjugacy class $\mathcal{C}_H$ of an element $\widetilde{\varphi}_H \in \widetilde{\mathrm{Ham}}\,(M, \omega)$ to the set of elements whose commutator length does not exceed g. We passed to the description of $\Upsilon_{1,g}$ in terms of K-area and used the weak coupling construction to estimate $\Upsilon_{1,g}(\mathcal{C}_H)$ from below by a non-negative (but possibly zero) number $1/\mathrm{size}_g\,(H)$. Thus, as soon as $1/\mathrm{size}_g\,(H)$ is non-zero, the commutator length of $\widetilde{\varphi}_H$ is greater than g.

In order to guarantee that $\mathrm{size}_g\,(H) < +\infty$ and $1/\mathrm{size}_g\,(H) > 0$ we first showed that any number $0 < \tau < \mathrm{size}_g\,(H)$ can be estimated from above by $1/\mathcal{A}_H(\hat{\gamma})$ if for a connected oriented surface Σ of genus g with one cylindrical end the moduli space $\mathcal{M}(\hat{\gamma}, H, \tilde{J}_\tau)$ is non-empty for a certain almost complex structure $\tilde{J}_\tau$ compatible with a symplectic form corresponding to the parameter τ in some weak coupling deformation.

Then we discussed how the condition $c(E^g, H) > 0$ would guarantee the existence of elements $\hat{\gamma}$ as above and provide a uniform positive bound from below on their actions as τ tends to $\mathrm{size}_g(H)$ – such a positive bound would also estimate the number $1/\mathrm{size}_g(H)$ from below and makes sure it is non-zero. In order to deduce the existence of the bound from the condition $c(E^g, H) > 0$, assume without loss of generality that H belongs to a regular Floer pair. Pick a generic $\tilde{J}_\tau$ so that all the moduli space $\mathcal{M}_g(\hat{\gamma}, H, \tilde{J}_\tau)$ of expected dimension zero are either empty or smooth, compact and oriented. Consider all the elements $\hat{\gamma} \in \mathcal{P}(H)$ which give rise to such $\mathcal{M}_g(\hat{\gamma}, H, \tilde{J}_\tau)$. The sum $\sum \#\mathcal{M}_g(\hat{\gamma}, H, \tilde{J}_\tau)\hat{\gamma}$ over all such $\hat{\gamma}$ is a Floer cycle which represents a Floer homology class corresponding, under the composition of Poincaré and Piunikhin–Salamon–Schwarz isomorphisms, to the quantum cohomology class E^g. The condition $c(E^g, H) > 0$ guarantees that the sum $\sum \#\mathcal{M}_g(\hat{\gamma}, H, \tilde{J}_\tau)\hat{\gamma}$ contains at least one non-zero term corresponding to some $\hat{\gamma}$ with a positive action arbitrarily close to $c(E^g, H)$ from below. Since τ can be chosen arbitrarily close to $\mathrm{size}_g(H)$ this construction provides the necessary positive estimate from below on $1/\mathrm{size}_g(H)$ – as long as $c(E^g, H) > 0$.

Thus if $c(E^g, H) > 0$ the distance $\Upsilon_{1,g}(\mathcal{C}_H)$ is positive and therefore the commutator length of $\widetilde{\varphi}_H$ is greater than g. □

9. Proof of Theorem 1.2.5

As in [3] the construction of $\mathfrak{f}$ will involve two main ingredients: a homogeneous quasimorphism τ on the universal cover $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ of the group $\mathrm{Sp}(2n, \mathbb{R})$ and a collection of functions $F_{\{\phi_t\}} : M \to \widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ that will be defined for any path $\{\phi_t\}$ in $\mathrm{Symp}_0(M, \omega)$. The only place where one needs to make adjustments to the argument from [3] is the construction of $F_{\{\phi_t\}}$. The quasimorphism τ that we will use is the same as in [3] – we recall its definition below.

9.1. The quasimorphism τ on $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$

Let $\mathbb{R}^{2n}$ be the standard linear symplectic space with the symplectic form $dp \wedge dq$ on it. Let $\Lambda(\mathbb{R}^{2n})$ be the *Lagrange Grassmannian* of $\mathbb{R}^{2n}$, i.e. the space of all Lagrangian planes in $\mathbb{R}^{2n}$. It is a compact manifold that can be identified with $U(n)/O(n)$. The map associating to a unitary matrix the square of its determinant descends to a map: $\det^2 : \Lambda(\mathbb{R}^{2n}) \to S^1$. Set the Lagrangian p-coordinate plane $L_0 \subset \mathbb{R}^{2n}$ as a base point in $\Lambda(\mathbb{R}^{2n})$. Fix a lift $\widetilde{L}_0$ of L_0 in the universal cover $\widetilde{\Lambda}(\mathbb{R}^{2n})$ of $\Lambda(\mathbb{R}^{2n})$ corresponding to the constant path in $\Lambda(\mathbb{R}^{2n})$ identically equal to L_0. Let $\widetilde{\det}^2 : \widetilde{\Lambda}(\mathbb{R}^{2n}) \to \mathbb{R}$ be the lift of $\det^2$ such that $\widetilde{\det}^2(\widetilde{L}_0) = 0$. Roughly speaking, to any path of Lagrangian subspaces starting at L_0 the function $\widetilde{\det}^2$ associates its rotation number in $\Lambda(\mathbb{R}^{2n})$ (with the whole construction depending on our choice of the Darboux basis in $\mathbb{R}^{2n}$).

The group $\mathrm{Sp}(2n,\mathbb{R})$ acts transitively on $\Lambda(\mathbb{R}^{2n})$ and the universal cover $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ acts on $\widetilde{\Lambda}(\mathbb{R}^{2n})$. The fundamental group of $\Lambda(\mathbb{R}^{2n})$ is isomorphic to $\mathbf{Z}$ and the map induced by the projection $U(n) \to U(n)/O(n) = \Lambda(\mathbb{R}^{2n})$ on $\pi_1(U(n))$ is the multiplication by two. On the other hand $U(n)$ is a deformation retract of $\mathrm{Sp}\,(2n,\mathbb{R})$. These identifications of $\pi_1(\widetilde{\Lambda}(\mathbb{R}^{2n}))$ and $\pi_1(\mathrm{Sp}\,(2n,\mathbb{R}))$ with $\mathbf{Z}$ by means of $\pi_1(U(n))$ lead to the following proposition.

Proposition 9.1.1. *A subset of $\widetilde{\mathrm{Sp}}\,(2n,\mathbb{R})$ is bounded if and only if the function*

$$\widetilde{\Phi} \mapsto \widetilde{\det}^2(\widetilde{\Phi}(\widetilde{L}_0))$$

is bounded on it.

Now for $\widetilde{\Phi} \in \widetilde{\mathrm{Sp}}\,(2n,\mathbb{R})$ set

$$\tau_{\det}(\widetilde{\Phi}) := \widetilde{\det}^2(\widetilde{\Phi}(L_0))$$

and define

$$\tau(\widetilde{\Phi}) := \lim_{k\to+\infty} \frac{\tau_{\det}(\widetilde{\Phi}^k)}{k}.$$

Proposition 9.1.2 ([3]). *The function $\tau : \widetilde{\mathrm{Sp}}\,(2n,\mathbb{R}) \to \mathbb{R}$ is continuous and, moreover, it is a homogeneous quasimorphism on $\widetilde{\mathrm{Sp}}\,(2n,\mathbb{R})$.*

9.2. Construction and basic properties of $F_{\{\phi_t\}}$

Let μ be the measure on M^{2n} defined by the volume form ω^n. Pick an almost complex structure J on M compatible with ω. The Riemannian metric $\omega(\cdot, J\cdot)$ and the form ω can be viewed as the real and the imaginary part of a Hermitian metric on M.

Lemma 9.2.1. *Let (M^{2n},ω) be a closed symplectic manifold with $c_1(M) = 0$. Then there exists a compact triangulated subset $Y \subset M$ of codimension at least 3 such that the tangent bundle of M admits a unitary trivialization over $M \setminus Y$.*

Proof of Lemma 9.2.1. Start with a piecewise smooth triangulation of M. Since M is orientable and $c_1(M) = 0$ there exists a unitary trivialization of TM over the 2-skeleton of M. Consider the barycentric star decomposition associated with the triangulation. Let Y be the $(2n-3)$-skeleton of the barycentric star cell decomposition. Then the 2-skeleton of the original triangulation of M is a deformation retract of $M \setminus Y$. Therefore there exists a unitary trivialization of TM over $M \setminus Y$. The lemma is proven. $\square$

Remark 9.2.2. Obviously the unitary trivialization from Lemma 9.2.1 is also symplectic: it symplectically identifies each tangent space of $M \setminus Y$ with the standard symplectic space $\mathbb{R}^{2n}$.

Let $\mathcal{F} \to M$ be the bundle of all unitary frames of TM. Use the Riemannian metric on M and a metric on $U(n)$ to define a metric on the total space $\mathcal{F}$. Any unitary trivialization of TM over $M \setminus Y$ can be viewed as a section f of $\mathcal{F} \to M$ over $M \setminus Y$. The Riemannian metrics on $\mathcal{F}$ and M allow to define the norm $\|df(x)\|$ of the differential df at any point $x \in M \setminus Y$.

Lemma 9.2.3. *One can choose f so that $\|df(x)\|$ is uniformly bounded on $M \setminus Y$.*

We omit the proof of this technical result.

Fix a unitary trivialization of TM over $M \setminus Y$ as in the lemma above. Pick an element $\widetilde{\phi} \in \widetilde{\mathrm{Symp}}_0(M, \omega)$ and represent it by a path $\{\phi_t\}_{0 \le t \le 1}$, $\phi_0 = Id$, in $\mathrm{Symp}_0(M, \omega)$. Consider the sets

$$Y_{\{\phi_t\}} := \bigcup_{k \in \mathbb{Z}} \bigcup_{0 \le t \le 1} \phi_t^{-k}(Y), \quad X_{\{\phi_t\}} := M \setminus Y_{\{\phi_t\}}.$$

Since Y is of codimension at least 3, $Y_{\{\phi_t\}}$ has measure zero and therefore $X_{\{\phi_t\}}$ is a subset of full measure in M.

As it was stated in Remark 9.2.2, our trivialization of TM over $M \setminus Y$ symplectically identifies each tangent space of $M \setminus Y$ with the standard symplectic space $\mathbb{R}^{2n}$. Therefore one can view the differentials of the symplectomorphisms ϕ_t, $0 \le t \le 1$, at any point of $X_{\{\phi_t\}}$ as elements of the group $\mathrm{Sp}(2n, \mathbb{R})$. Thus to any point $x \in X_{\{\phi_t\}}$ one can associate a path $\{d\phi_t(x)\}_{0 \le t \le 1}$ in $\mathrm{Sp}(2n, \mathbb{R})$ starting at the identity. Denote by $F_{\{\phi_t\}}(x)$ the element in $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ represented by this path. Set $F_{\{\phi_t\}} = Id$ over $M \setminus X_{\{\phi_t\}} = Y_{\{\phi_t\}}$.

Lemma 9.2.4. *For any path $\{\phi_t\}$ representing $\widetilde{\phi} \in \widetilde{\mathrm{Symp}}_0(M, \omega)$ the function $\tau(F_{\{\phi_t\}}(\cdot))$ is integrable on M.*

Proof of Lemma 9.2.4. For a point $x \in X_{\{\phi_t\}}$ denote

$$V(x) := \widetilde{\det}^2(F_{\{\phi_t\}}(x)).$$

According to Proposition 9.1.2, τ is continuous. Therefore in order to show that the function $\tau(F_{\{\phi_t\}}(\cdot))$ is integrable on M it suffices to show that $F_{\{\phi_t\}} : M \to \widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ has a bounded image in $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$. In view of Proposition 9.1.1 this would follow if we show that the function $V(x) := \widetilde{\det}^2(F_{\{\phi_t\}}(x))$ is bounded on $X_{\{\phi_t\}}$.

The function V can be described as follows. Let $\Lambda \to M$ be the fiber bundle whose fiber over a point $x \in M$ is the Lagrange Grassmannian $\Lambda(T_xM)$ of the symplectic tangent space T_xM. Consider the section $L : M \setminus Y \to \Lambda$ of the bundle $\Lambda \to M$ that associates to each $x \in M$ the Lagrangian plane $L_0 \subset \Lambda(T_xM) = \Lambda(\mathbb{R}^{2n})$, where T_xM is identified with the standard symplectic linear space $\mathbb{R}^{2n}$ by means of the symplectic (unitary) trivialization of TM over $M \setminus Y$ as above. For a given $x \in X_{\{\phi_t\}}$ the path $\{\phi_t\}$ acting on $L(x)$ determines, by means of the trivialization, a path in $\Lambda(\mathbb{R}^{2n})$ to which the map $\widetilde{\det}^2$ associates a number. This number is exactly $V(x)$.

Assume that $V(x)$ is not bounded. Then, since $F_{\{\phi_t\}}$ is continuous on $X_{\{\phi_t\}}$, there must exist a sequence $\{x_k\}$ in $X_{\{\phi_t\}}$ which converges to a point $y \in M \setminus X_{\{\phi_t\}}$ such that $\lim_{k\to+\infty} |V(x_k)| = +\infty$. As $k \to +\infty$, the paths $\gamma_k(t) := \phi_t(x_k)$, $0 \le t \le 1$, converge uniformly to a path $\gamma_{lim}(t) := \phi_t(y)$, $0 \le t \le 1$, lying in $Y_{\{\phi_t\}}$.

The total space Λ of the bundle $\Lambda \to M$ is compact. Therefore, possibly choosing a subsequence of $\{x_k\}$, we may assume without loss of generality that the Lagrangian subspaces $L(x_k) \in \Lambda(T_{x_k}M) \subset \Lambda$ converge to a Lagrangian subspace $L(y) \in \Lambda(T_yM)$ and thus the paths $\{d\phi_t(L(x_k))\}_{0\le t\le 1}$ in Λ converge uniformly to a path $\{d\phi_t(L(y))\}_{0\le t\le 1}$.

The total space $\mathcal{F}$ of the bundle $\mathcal{F} \to M$ is compact and the section $f : M\setminus Y \to \mathcal{F}$, defining our fixed unitary trivialization, has uniformly bounded derivatives. Therefore, using Arzela-Ascoli theorem and possibly passing to a subsequence, we may assume without loss of generality that the sequence of maps $f \circ \gamma_k : [0,1] \to \mathcal{F}$ converges C^0-uniformly to a *continuous* map $g \circ \gamma_{lim} : [0,1] \to \mathcal{F}$, where g is a *continuous* section of $\mathcal{F} \to M$ over the smooth embedded curve $\gamma_{lim}([0,1])$.

Now each $V(x_k)$ is the rotation number of the path $\{d\phi_t(L(x_k))\}_{0\le t\le 1}$ of Lagrangian planes in the bundle $\gamma_k^* TM$ over $[0,1]$ with respect to the moving frame $f \circ \gamma_k(t)$ defining the trivialization of that bundle. Since the sequence $\{L(x_k)\}$ converges to $L(y)$ and $\{f \circ \gamma_k\}$ converges uniformly to $g \circ \gamma_{lim}$, the rotation numbers $V(x_k)$ converge to a *finite* number $V_{lim}(y)$, which is the rotation number of the path $\{d\phi_t(L(y))\}_{0\le t\le 1}$ of Lagrangian planes in the bundle $\gamma_{lim}^* TM$ over $[0,1]$ with respect to the continuously moving frame $g \circ \gamma_{lim}(t)$ defining a continuous trivialization of the bundle. This is in contradiction with our original assumption that $\lim_{k\to+\infty} |V(x_k)| = +\infty$. The lemma is proven. $\square$

Lemma 9.2.5. *Suppose that paths $\{\phi_t\}_{0\le t\le 1}$, $\{\psi_t\}_{0\le t\le 1}$, $\phi_0 = \psi_0 = Id$, in* $\mathrm{Symp}_0(M,\omega)$ *represent the same element in* $\widetilde{\mathrm{Symp}}_0(M,\omega)$. *Then there exists a subset $X_{\{\phi_t\},\{\psi_t\}} \subset X_{\{\phi_t\}} \cap X_{\{\psi_t\}}$ of the full measure in M such that for any $x \in X_{\{\phi_t\},\{\psi_t\}}$*

$$F_{\{\phi_t\}}(x) = F_{\{\psi_t\}}(x) \in \widetilde{\mathrm{Sp}}(2n,\mathbb{R}).$$

Proof of Lemma 9.2.5. Since the paths $\{\phi_t\}$, $\{\psi_t\}$ define the same element in $\widetilde{\mathrm{Symp}}_0(M,\omega)$ there exists a homotopy $\{\varphi_{s,t}\}$, $0 \le s \le 1$, $0 \le t \le 1$, between them which keeps the endpoints fixed. Consider now the set

$$Z_{\{\phi_t\},\{\psi_t\}} := \bigcup_{k\in\mathbb{Z}} \bigcup_{0\le s,t\le 1} \varphi_{s,t}^{-k}(Y).$$

Since, according to its definition (see Lemma 9.2.1), Y is a subset of codimension at least 3 in M, the set $Z_{\{\phi_t\},\{\psi_t\}}$ is of codimension at least 1 and therefore has measure zero. The set $X_{\{\phi_t\},\{\psi_t\}} := M\setminus Z_{\{\phi_t\},\{\psi_t\}}$ is the one we have been looking for. The lemma is proven. □

Lemma 9.2.6. *Let $\{\phi_t\}$, $\{\psi_t\}$, $0\le t\le 1$, be paths in $\mathrm{Symp}_0(M,\omega)$ starting at the identity. Then there exists a set $X'_{\{\phi_t\},\{\psi_t\}}$ of full measure in M such that the following quantities are well-defined and equal for any $x\in X'_{\{\phi_t\},\{\psi_t\}}$:*

$$F_{\{\phi_t\psi_t\}}(x) = F_{\{\phi_t\}}(\psi_1(x))F_{\{\psi_t\}}(x). \tag{29}$$

Proof of Lemma 9.2.6. Consider the set

$$Z'_{\{\phi_t\},\{\psi_t\}} := \bigcup_{k\in\mathbb{Z}} \bigcup_{0\le s,t\le 1} (\phi_t\psi_s)^{-k}(Y).$$

Since Y is of codimension at least 3 in M, the set $Z'_{\{\phi_t\},\{\psi_t\}}$ has measure zero and therefore $X'_{\{\phi_t\},\{\psi_t\}} := M\setminus Z'_{\{\phi_t\},\{\psi_t\}}$ is a set of full measure in M. Also

$$X'_{\{\phi_t\},\{\psi_t\}} \subset X_{\{\phi_t\psi_t\}} \cap X_{\{\phi_t\}} \cap X_{\{\psi_t\}}$$

and therefore for any $x\in X'_{\{\phi_t\},\{\psi_t\}}$ both sides of the equality (29) are well-defined. For any such x the element $F_{\{\phi_t\psi_t\}}(x)\in\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ can be represented by a path

$$\{d(\phi_t\psi_t)(x)\} = \{d\phi_t(\psi_t(x))\cdot d\psi_t(x)\},$$

$0\le t\le 1$, in $\mathrm{Sp}(2n,\mathbb{R})$. But the path $\{d\phi_t(\psi_t(x))\}_{0\le t\le 1}$ is homotopic with the fixed endpoints to the path $\{d\phi_t(\psi_1(x))\}_{0\le t\le 1}$ in $\mathrm{Sp}(2n,\mathbb{R})$: the homotopy is given by the 2-parametric family

$$d\phi_t(\psi_s(x)),\quad 0\le t\le 1,\quad t\le s\le 1,$$

of elements of $\mathrm{Sp}(2n,\mathbb{R})$, which is well-defined since $\phi_t\psi_s(x)\notin Y$ for all s,t and $x\in X'_{\{\phi_t\},\{\psi_t\}}$. Thus for any $x\in X'_{\{\phi_t\},\{\psi_t\}}$ the paths $\{d\phi_t(\psi_t(x))\}_{0\le t\le 1}$ and $\{d\phi_t(\psi_1(x))\}_{0\le t\le 1}$ represent the same element in $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ and hence

$$F_{\{\phi_t\psi_t\}}(x) = F_{\{\phi_t\}}(\psi_1(x))F_{\{\psi_t\}}(x).$$

The lemma is proven. □

9.3. Final steps in the construction of $\mathfrak{f}$

From this stage on the construction of $\mathfrak{f}$ proceeds in exactly the same way as in [3]. Namely, let $\tilde{\phi}\in\widetilde{\mathrm{Symp}}_0(M,\omega)$ be represented by a path $\{\phi_t\}_{0\le t\le 1}$, $\phi_0 = Id$, in $\mathrm{Symp}_0(M,\omega)$. For $x \in X_{\{\phi_t\}}$ and $k \geq 1$ define

$$\tau_k(\{\phi_t\},x) = \tau\Big(\prod_{i=0}^{k-1} F_{\{\phi_t\}}(\phi_1^i(x))\Big).$$

According to the definition of $X_{\{\phi_t\}}$, the number $\tau_k(\{\phi_t\},x)$ is well-defined. Since τ is a quasimorphism there exists a constant C such that for any l and x

$$\begin{aligned}\tau_k(\{\phi_t\},x) + \tau_l(\{\phi_t\},\phi_1^k(x)) - C \leq \tau_{k+l}(\{\phi_t\},x) \leq \\ \leq \tau_k(\{\phi_t\},x) + \tau_l(\{\phi_t\},\phi_1^k(x)) + C.\end{aligned} \tag{30}$$

Lemma 9.3.1. *For any k the function $\tau_k(\{\phi_t\},\cdot)$ is integrable on M.*

Proof of Lemma 9.3.1. Proceed by induction. For $k=1$ the statement follows from Lemma 9.2.4. To prove the step of the induction use (30) with $l=1$ and recall that ϕ_1 is a symplectomorphism and therefore preserves the measure. □

Lemma 9.3.2. *The integral $\int_M \tau_k(\{\phi_t\},\cdot)\,d\mu$ does not depend on the choice of the path $\{\phi_t\}$ representing $\widetilde{\phi} \in \widetilde{\mathrm{Symp}}_0(M,\omega)$.*

Proof of Lemma 9.3.2. Let $\{\phi_t\}$, $\{\psi_t\}$ be two paths representing the same $\widetilde{\phi} \in \widetilde{\mathrm{Symp}}_0(M,\omega)$. Applying Lemma 9.2.5 one gets that the functions $\tau_k(\{\phi_t\},\cdot)$ and $\tau_k(\{\psi_t\},\cdot)$ differ on a set of measure zero in M and therefore have the same integral. □

Thus we can define $T_k(\widetilde{\phi}) := \int_M \tau_k(\{\phi_t\},\cdot)\,d\mu$ for any $\{\phi_t\}$ representing $\widetilde{\phi}$. In view of (30),

$$T_k(\widetilde{\phi}) + T_l(\widetilde{\phi}) - C\mu(M) \leq T_{k+l}(\widetilde{\phi}) \leq T_k(\widetilde{\phi}) + T_l(\widetilde{\phi}) + C\mu(M) \tag{31}$$

for any l. Therefore there exists a limit

$$\mathfrak{f}(\widetilde{\phi}) := \lim_{k\to+\infty} T_k(\widetilde{\phi})/k.$$

This is the definition of $\mathfrak{f}$. Now we will prove that $\mathfrak{f}$ is a non-trivial homogeneous quasimorphism.

9.4. $\mathfrak{f}$ is a quasimorphism

Formula (31) implies that

$$|\ \mathfrak{f}(\widetilde{\phi}) - T_1(\widetilde{\phi})\ | \leq C\mu(M) \tag{32}$$

for some constant C that does not depend on $\widetilde{\phi}$. Now let $\{\phi_t\}$, $\{\psi_t\}$, $0 \leq t \leq 1$, be paths in $\mathrm{Symp}_0(M,\omega)$ starting at the identity and representing, respectively,

$\widetilde{\phi}, \widetilde{\psi} \in \widetilde{\mathrm{Symp}}_0\,(M,\omega)$. The equality (29) holds on a set $X'_{\{\phi_t\},\{\psi_t\}}$ of full measure and therefore

$$T_1(\widetilde{\phi}\widetilde{\psi}) = \int_M \tau\Big(F_{\{\phi_t\psi_t\}}(x)\Big)d\mu = \int_M \tau\Big(F_{\{\phi_t\}}\,(\psi_1(x))F_{\{\widetilde{\psi}\}}\,(x)\Big)d\mu.$$

Since ψ_1 is a symplectomorphism,

$$\int_M \tau\Big(F_{\{\phi_t\}}\,(\psi_1(x))\Big)d\mu = \int_M \tau\Big(F_{\{\phi_t\}}\,(x)\Big)d\mu,$$

and since τ is a quasimorphism,

$$\Bigg|\int_M \tau\Big(F_{\{\phi_t\}}\,(\psi_1(x))F_{\{\psi_t\}}\,(x)\Big)d\mu - \int_M \tau\Big(F_{\{\psi_t\}}\,(x)\Big)d\mu - \\ - \int_M \tau\Big(F_{\{\phi_t\}}\,(x)\Big)d\mu\Bigg| \le C_1\mu(M)$$

for some constant C_1 independent of $\widetilde{\phi}, \widetilde{\psi}$. Therefore

$$|T_1(\widetilde{\phi}\widetilde{\psi}) - T_1(\widetilde{\psi}) - T_1(\widetilde{\phi})| \le C_1\mu(M)$$

for any $\widetilde{\phi}, \widetilde{\psi}$. Thus T_1 is a quasimorphism. In view of (32), $\mathfrak{f}$ is a quasimorphism as well.

9.5. $\mathfrak{f}$ is homogeneous

Represent each $\widetilde{\phi}^k$, $1 \le k \le m$, by a path $\{\phi^k_t\}_{0\le t\le 1}$, $\phi_0 = Id$. Observe that, according to (29),

$$F_{\{\phi^m_t\}}\,(x) = \prod_{i=0}^{m-1} F_{\{\phi_t\}}\,(\phi^i_1(x))$$

for any x from a set of full measure in M. For any such x

$$\tau_k(\{\phi^m_t\}, x) = \tau\Bigg(\prod_{i=0}^{k-1} F_{\{\phi^m_t\}}\,(\phi_1^{mi}(x))\Bigg) = \tau\Bigg(\prod_{i=0}^{k-1}\prod_{j=0}^{m-1} F_{\{\phi_t\}}\,(\phi_1^{mi+j}(x))\Bigg) = \\ = \tau\Bigg(\prod_{l=0}^{mk-1} F_{\{\phi_t\}}\,(\phi^l_1(x))\Bigg) = \tau_{mk}(\{\phi^m_t\}, x)$$

and therefore

$$T_k(\widetilde{\phi}^m) = T_{mk}(\widetilde{\phi}).$$

Hence

$$\mathfrak{f}(\widetilde{\phi}^m) = \lim_{k\to+\infty} T_k(\widetilde{\phi}^m)/k = \lim_{k\to+\infty} T_{mk}(\widetilde{\phi})/k = \\ = m\cdot\lim_{k\to+\infty} T_{mk}(\widetilde{\phi})/mk = m\mathfrak{f}(\widetilde{\phi})$$

and therefore $\mathfrak{f}$ is homogeneous.

9.6. $\mathfrak{f}$ does not vanish on $\widetilde{\varphi}_{H_B} \in \widetilde{\mathrm{Ham}}\,(M,\omega)$

In order to prove that $\mathfrak{f}$ does not vanish on $\widetilde{\mathrm{Ham}}\,(M,\omega)$ consider the same Hamiltonian symplectomorphism that J. Barge and E. Ghys used to show that their homogeneous quasimorphism on $\mathrm{Symp}^c\,(B^{2n})$ does not vanish (see [3], p. 263).

Namely, consider a ball $B \subset M^{2n} \setminus Y$. Assume without loss of generality that our chosen trivialization of TM over $M \setminus Y$ coincides on B with the trivialization defined by the identification of B with a ball in the standard symplectic $\mathbb{R}^{2n}$. Consider the Hamiltonian symplectomorphism φ_{H_B} generated by H_B and its lift $\widetilde{\varphi}_{H_B}$ in $\widetilde{\mathrm{Ham}}\,(M,\omega)$. Following the construction in [3] one easily sees that value of our quasimorphism $\mathfrak{f}$ on $\widetilde{\varphi}_{H_B}$ is the same as the value of the Barge–Ghys quasimorphism on φ_{H_B}. According to [3], the latter value is non-zero. Hence $\mathfrak{f}$ does not vanish on $\widetilde{\mathrm{Ham}}\,(M,\omega)$.

This finishes the proof of the claim that $\mathfrak{f}$ is a homogeneous quasimorphism on $\widetilde{\mathrm{Symp}}_0\,(M,\omega)$ that does not vanish on $\widetilde{\varphi}_{H_B} \in \widetilde{\mathrm{Ham}}\,(M,\omega)$. Theorem 1.2.5 is proven. $\square$

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