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On $(-P \cdot P)$ -constant deformations of Gorenstein surface singularities

Tomohiro Okuma

Abstract. Let $\pi: X \rightarrow T$ be a small deformation of a normal Gorenstein surface singularity $X_0 = \pi^{-1}(0)$ over the complex number field \mathbb{C} . Suppose that T is a neighborhood of the origin of \mathbb{C} and that X_0 is not log-canonical. We show that if a topological invariant $-P_t \cdot P_t$ of $X_t = \pi^{-1}(t)$ is constant, then, after a suitable finite base change, π admits a simultaneous resolution $f: M \rightarrow X$ which induces a locally trivial deformation of each maximal string of rational curves at an end of the exceptional set of $M_0 \rightarrow X_0$; in particular, if X_0 has a star-shaped resolution graph, then π admits a weak simultaneous resolution (in other words, π is an equisingular deformation).

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Keywords. Deformation, Gorenstein surface singularity, simultaneous resolution.

1. Introduction

We continue the study of a family of Gorenstein surface singularities preserving a certain topological invariant ([15]). Let (X_0, x_0) be a normal complex Gorenstein surface singularity and $\pi: X \rightarrow T$ a flat deformation of (X_0, x_0) , where T is a reduced complex space. Let $f: M \rightarrow X$ be a proper modification with the exceptional set E . Then $f: M \rightarrow X$ is called a *very weak* simultaneous resolution if $\pi \circ f$ is flat and $f_t: M_t \rightarrow X_t$ is a resolution of X_t for all $t \in T$. Laufer proved [11, Theorem 4.3] that the constancy of a *topological* invariant $-K \cdot K$ in the deformation π implies the existence of a simultaneous canonical model (which is also called a simultaneous RDP resolution); then he obtained the following

Theorem 1.1 (Laufer [11, Theorem 5.7]). *π admits a very weak simultaneous resolution after a finite base change if and only if $-K_t \cdot K_t$ is constant, where K_t is the canonical divisor on the minimal resolution space of $X_t = \pi^{-1}(t)$.*

However, the structure of the exceptional divisor in a very weak simultaneous resolution can vary greatly. Let us recall a strong kind of simultaneous resolution;

$f: M \rightarrow X$ is called a *weak* simultaneous resolution if it is a very weak simultaneous resolution and the morphism $E \rightarrow T$ induced by $\pi \circ f$ is a locally trivial deformation. If a weak simultaneous resolution of π exists, then π is called an equisingular deformation [20]. It is shown [11, Theorem 6.4] that π admits a weak simultaneous resolution if and only if each singularity (X_t, x_t) is homeomorphic to (X_0, x_0) . But, at present, there is no statement about the existence of weak simultaneous resolutions similar to Theorem 1.1.

In this paper, we deal with deformations of Gorenstein surface singularities preserving the topological invariant $-P \cdot P$, where P denotes the nef-part of the Zariski decomposition of the log-canonical divisor on a good resolution [21]. We shall show that such a family has a simultaneous resolution with some nice properties; it is a weak simultaneous resolution in a special case. Assume that T is a sufficiently small neighborhood of the origin of the complex number field \mathbb{C} and the (X_0, x_0) is not a log-canonical singularity. In [14], we obtained that if the topological invariant $-P_t \cdot P_t$ is constant, then π admits a simultaneous log-canonical model; it is a log-version of Laufer's result mentioned before Theorem 1.1. In [15], we proved that the constancy of $-P_t \cdot P_t$ implies not only the log-version above, but also the existence of a simultaneous resolution $f: M \rightarrow X$, after a finite base change, such that each $f_t: M_t \rightarrow X_t$ is a resolution with the exceptional divisor having only normal crossings, and f_t is minimal among resolutions with such properties. Our new result in this paper gives a geometric characterization of $(-P \cdot P)$ -constant deformations that clarifies what structure of the exceptional set is preserved. We prove the following

Theorem 1.2. *Assume that $-P_t \cdot P_t$ is constant. Then, after a finite base change, there exists a section $\gamma: T \rightarrow X$ of π such that each $\gamma(t)$ is a non-log-canonical singularity and a simultaneous resolution $f: M \rightarrow X$ which satisfy the following conditions:*

- (1) *for each $t \in T$, $f_t: M_t \rightarrow X_t$ is a resolution with the exceptional divisor having only normal crossings, and f_t is minimal among resolutions with such properties;*
- (2) *if E denotes the reduced divisor such that $f(E) = \gamma(T)$, then the restriction E_t of E is the reduced divisor supported on $f^{-1}(\gamma(t))$;*
- (3) *there exists a reduced divisor $S \leq E$ such that S_t is the sum of all maximal strings of rational curves at the ends of E_t for each $t \in T$ and that $\pi \circ f|_S: S \rightarrow T$ is a locally trivial deformation.*

Any singular point on $X_t \setminus \{\gamma(t)\}$ is a rational double point of type A_n .

Corollary 1.3. *Assume that $-P_t \cdot P_t$ is constant and that the resolution graph of (X_0, x_0) is star-shaped. Then each X_t has only one singular point x_t and π is an equisingular deformation.*

In case where X_t has only a singularity x_t , an outline of the proof of Theo-

rem 1.2 is as follows. Let $f: M \rightarrow X$ be a resolution which satisfies the condition (1) of Theorem 1.2 and $g: Y \rightarrow X$ the simultaneous log-canonical model (the existence of them follows from [14] and [15], respectively). Denote by E and F the exceptional divisor of f and g , respectively. First, we shall show that there exists a morphism $h: M \rightarrow Y$ such that $f = g \circ h$. Let $P = h^*(K_Y + F)$ and $N = K_M + E - P$. Next, we verify that the restriction $P_t + N_t$ is the Zariski decomposition of the log-canonical divisor on M_t . Then it follows that $S := \text{Supp}(N)$ satisfies the condition (3) of Theorem 1.2.

In [3], Ishii proved that for a small deformation of any normal surface singularity, the constancy of the invariant $-K \cdot K$ implies the existence of the simultaneous canonical model of the deformation. We hope that Theorem 1.2 may be generalized to the non-Gorenstein case.

Thanks are due to Professor Jonathan Wahl for his helpful advice. Thanks are also due to the referee for valuable comments.

Notation and terminology

We denote by \mathbb{Z} , \mathbb{N} and \mathbb{Q} , the set of integers, the set of positive integers and the set of rational numbers, respectively. Let X be a normal variety. For a \mathbb{Q} -divisor $D = \sum d_i D_i$ on X , where D_i are distinct prime divisors, we write $D_{red} = \sum_{d_i \neq 0} D_i$. We say that a resolution $f: M \rightarrow X$ of X is *semigood* (resp. *good*) if the exceptional set of f is a divisor having only normal crossings (resp. simple normal crossings). Let $g: Y \rightarrow X$ be a partial resolution and E the reduced exceptional divisor of g . Then g is called a *canonical model* of X if Y has only canonical singularities and K_Y is g -ample; it is called a *log-canonical model* of X if the pair (Y, E) has only log-canonical singularities and $K_Y + E$ is g -ample.

2. Preliminaries

In this section, we review some results on surface singularities needed later. A *minimal semigood* (resp. *minimal good*) resolution of a normal surface singularity is the smallest resolution among all semigood (resp. good) resolutions. The minimal semigood resolution is obtained from the minimal good resolution by contracting each (-1) -curve intersecting one component twice. The *weighted dual graph* of a normal surface singularity is that of the exceptional divisor on the minimal good resolution of the singularity.

Let (X, x) be a normal surface singularity and $f: (M, A) \rightarrow (X, x)$ the minimal semigood resolution with the exceptional divisor A . Let K be a canonical divisor on M and $A = \bigcup_{i=1}^t A_i$ the decomposition into irreducible components. We call a divisor (resp. \mathbb{Q} -divisor) on M supported in A a *cycle* (resp. *\mathbb{Q} -cycle*). For any divisors D and E on M , the intersection number $D \cdot E$ is defined as $\nu(D) \cdot \nu(E)$,

where $\nu(D)$ denotes a \mathbb{Q} -cycle determined by $(\nu(D) - D) \cdot A_i = 0$ for $1 \leq i \leq t$. Let $P + N$ be the Zariski decomposition of $K + A$: N is an effective \mathbb{Q} -cycle such that $P = K + A - N$ is f -nef and $P \cdot A_i = 0$ for all $A_i \leq N_{red}$ (see [17, Theorem A.1]). The intersection number $-P \cdot P$ is a topological invariant of the singularity (X, x) , and its fundamental properties are stated in [21].

Definition 2.1. Let $S = \sum_{i=1}^n A_i$ be a chain of nonsingular rational curves. We call S a string at an end of A if $A_i \cdot A_{i+1} = 1$ for $1 \leq i \leq n-1$, and these account for all intersections in A among the A_i 's, except that A_n intersects exactly one other curve. Let $S^* = \sum_{i=1}^n a_i A_i$ be a \mathbb{Q} -cycle such that $S^* \cdot A_1 = -1$ and $S^* \cdot A_i = 0$ ($i > 1$). Note that $a_i > 0$ for $i = 1, \dots, n$.

Lemma 2.2. *In the situation above, we have the inequalities*

$$a_{n-j+1} \leq j a_{n-j} / (j + 1), \quad j = 1, \dots, n - 1.$$

Hence $a_1 > a_2 > \dots > a_n$.

Proof. Let $-b_i = A_i \cdot A_i$. Then $b_i \geq 2$. By the definition of S^* , we have $a_{k-1} - b_k a_k + a_{k+1} = 0$ for $1 \leq k \leq n$, where $a_0 = 1$ and $a_{n+1} = 0$. It is clear that $a_n \leq a_{n-1}/2$. Now use induction on j . □

Proposition 2.3 (Wahl [21, Proposition 2.3, (2.7)]). *Suppose (X, x) is not a quotient, simple elliptic, or cusp singularity. Let $\{S_1, \dots, S_p\}$ be the set of all maximal strings at the ends of A . Then $N = \sum_{i=1}^p S_i^*$.*

Lemma 2.4 (see [13, Lemma 1.8]). *If (X, x) is not a rational double point, then $[N] = 0$, where $[N]$ denotes the integral part of N .*

The m -th L^2 -plurigenus of (X, x) is expressed as

$$\delta_m(X, x) = \dim_{\mathbb{C}} \mathcal{O}_X(mK_X) / f_* \mathcal{O}_M(mK + (m - 1)A)$$

(see [22, pp. 67–68]). $\delta_1(X, x)$ is equal to the geometric genus $p_g(X, x)$.

Theorem 2.5 (see [13]). *There exists a bounded function $v(m)$ such that*

$$\delta_{m+1}(X, x) = -(P \cdot P)m^2/2 - (K \cdot P)m/2 + p_g(X, x) + v(m)$$

for $m \geq 0$. *If (X, x) is a Gorenstein singularity with $p_g(X, x) \geq 1$, then the function $v(m)$ is determined by the weighted dual graph of the maximal strings at the ends of A .*

Assume that (X, x) is not a log-canonical singularity, or equivalently that $\nu(P) \neq 0$ (see [21, Remark 2.4], [6, §9]). Let $g: Y \rightarrow X$ be the log-canonical model and F the exceptional divisor of g . Then we obtain a morphism $h: M \rightarrow Y$, which is the minimal resolution of the singularities of Y , and $P \sim_{\mathbb{Q}} h^*(K_Y + F)$ (see

[15, §3]). Let C be a reduced cycle which is the sum of the components A_i such that $P \cdot A_i = 0$. Then C is exactly the exceptional divisor for h , and contains no (-1) -curves. Let C_0 be the sum of the components $A_i \leq C$ such that $A_i \cdot A_i = -2$.

Definition 2.6. Let \bar{X} be a normal surface obtained by contracting the cycle C_0 on M . Then \bar{X} has only rational double points. We call the natural morphism $\bar{X} \rightarrow X$ an RDP good resolution of the singularity (X, x) .

Lemma 2.7. *The natural morphism $h': \bar{X} \rightarrow Y$ is the canonical model of Y .*

Proof. Since a rational double point is a canonical singularity, it suffices to show that $K_{\bar{X}}$ is h' -ample. Let $\varphi: M \rightarrow \bar{X}$ be the contraction. Then for any irreducible curve $\ell \subset \varphi(C)$, we have $K_{\bar{X}} \cdot \ell = K \cdot \varphi_*^{-1}\ell > 0$, where $\varphi_*^{-1}\ell$ denotes the strict transform of ℓ . Hence $K_{\bar{X}}$ is h' -ample. \square

The following theorem gives another construction of the RDP good resolution.

Theorem 2.8 (see [15, Theorem 3.2]). *Let r be a positive integer such that rN is a cycle, and let $f': (M', A') \rightarrow (X, x)$ be any semigood resolution. Then there exists a positive integer $\beta(X, x)$ determined by the weighted dual graph of (X, x) such that for any $m \geq \beta(X, x)$, the blowing-up of X with respect to the sheaf $f'_*\mathcal{O}_{M'}(K_{M'} + mr(K_{M'} + A'))$ is the RDP good resolution of (X, x) .*

3. Simultaneous resolution

Let (X_0, x_0) be a normal Gorenstein surface singularity and $\pi: X \rightarrow T$ a deformation of $X_0 = \pi^{-1}(0)$, where T is an open neighborhood of the origin of \mathbb{C} . Then each X_t is normal and Gorenstein. We assume that (X_0, x_0) is not log-canonical. The aim of this section is to show that a simultaneous RDP good resolution of π is obtained as the canonical model of a simultaneous log-canonical model of π .

For any morphism $h: W \rightarrow X$, we denote by W_t the fiber $(\pi \circ h)^{-1}(t)$ and by h_t the restriction $h|_{W_t}: W_t \rightarrow X_t$.

Definition 3.1 (cf. Laufer [11, V]). Let $f: M \rightarrow X$ be a resolution of the singularities of X and E the exceptional set of f . We call $f: M \rightarrow X$ a weak simultaneous resolution if each f_t is a resolution of X_t and $\pi \circ f|_E: E \rightarrow T$ is a locally trivial deformation of the exceptional divisor of M_0 .

We assume that T is sufficiently small so that $\pi|_{X \setminus X_0}: X \setminus X_0 \rightarrow T \setminus \{0\}$ admits a weak simultaneous resolution. We note that if π admits a weak simultaneous resolution along a section $\gamma: T \rightarrow X$ of π , then the weighted dual graph of $(X_t, \gamma(t))$ is the same as that of (X_0, x_0) (see [11, VI]).

Let us review some results on simultaneous partial resolutions studied in [14]

and [15]. Let $g: Y \rightarrow X$ be the log-canonical model of X and F the reduced exceptional divisor of g .

Definition 3.2 (cf. [14, Definition 4.1 and Lemma 4.2]). We call the morphism g a simultaneous log-canonical model of π if for any $t \in T$ the restriction $g_t: Y_t \rightarrow X_t$ is the log-canonical model of X_t and F_t is a reduced divisor supported on the exceptional set of g_t .

Let $f(t): \tilde{X}_t \rightarrow X_t$ be the minimal semigood resolution, A_t the exceptional divisor and K_t the canonical divisor on \tilde{X}_t . Let $A_{t,p}$ be the connected component of A_t which blows down to a singular point $p \in X_t$. Let $P_{t,p} + N_{t,p}$ be the Zariski decomposition of $K_t + A_{t,p}$, where $N_{t,p}$ is a \mathbb{Q} -divisor supported in $A_{t,p}$. We put $N_t := \sum_p N_{t,p}$ and $P_t \cdot P_t := \sum_p P_{t,p} \cdot P_{t,p}$.

Theorem 3.3 (see [14, Theorem 4.11]). *The following conditions are equivalent:*

- (1) g is the simultaneous log-canonical model of π ;
- (2) $-P_t \cdot P_t$ is constant.

The next lemma follows from Theorem 3.3, [14, Remark 4.3], [15, Lemma 4.2] and [5, Proposition 2.2].

Lemma 3.4. *Suppose that $-P_t \cdot P_t$ is constant. Then there exists a section $\gamma: T \rightarrow X$ of π such that $(X_t, \gamma(t))$ is a non-log-canonical singularity and any singularity on $X_t \setminus \{\gamma(t)\}$ is a rational double point for each $t \in T$ (note that $g(F) = \gamma(T)$).*

The idea for the proof of the next lemma is due to Tomari [19].

Lemma 3.5. *Suppose that $-P_t \cdot P_t$ is constant. Let $\alpha: W \rightarrow Y$ be a morphism such that $g \circ \alpha$ is a semigood resolution of X , and let B be the exceptional set of $g \circ \alpha$. Then $\alpha_* \mathcal{O}_W(m(K_W + B) - B) = \mathcal{O}_Y(m(K_Y + F) - F)$ for any $m \in \mathbb{N}$.*

Proof. Let $L^W = K_W + B$ and $L^Y = K_Y + F$. Since X is Gorenstein and L^Y is g -ample, there exists a \mathbb{Q} -Cartier effective divisor F' supported on F such that $-F' \sim_{\mathbb{Q}} L^Y$. It is clear that $\alpha_* \mathcal{O}_W(mL^W - B) \subset \mathcal{O}_Y(mL^Y - F)$. To prove the converse, we may assume that Y is Stein. So it suffices to show the following

$$H^0(W, \mathcal{O}_W(mL^W - B)) \supset \alpha^* H^0(Y, \mathcal{O}_Y(mL^Y - F)).$$

Let $\omega \in H^0(Y, \mathcal{O}_Y(mL^Y - F))$. Then $\text{div}(\omega) + mL^Y - F \geq 0$. Let n be a positive integer such that $nF \geq F'$. Then

$$\text{div}(\omega) + mL^Y - (1/n)F' \geq \text{div}(\omega) + mL^Y - F \geq 0.$$

Note that the left hand side is a \mathbb{Q} -Cartier divisor. Since L^Y is log-canonical, there exists an exceptional effective divisor Δ such that $L^W = \alpha^* L^Y + \Delta$. By

Lemma 3.4, we see that $Y \setminus F$ has only canonical singularities (see [16, Theorem 2.6]). Thus $\text{Supp}(\Delta + \alpha^*F) = B$. It follows from the inequality above that

$$\text{div}(\alpha^*\omega) + mL^W \geq m\Delta + (1/n)\alpha^*F'.$$

Since $\text{Supp}(m\Delta + (1/n)\alpha^*F') = B$ and the left hand side is an integral divisor, we obtain that $\text{div}(\alpha^*\omega) + mL^W \geq B$, i.e., $\alpha^*\omega \in H^0(W, \mathcal{O}_W(mL^W - B))$. \square

Let $f: M \rightarrow X$ be a semigood resolution and E the exceptional divisor of f . Since $\pi|_{X \setminus X_0}$ admits a weak simultaneous resolution, there exists a positive integer r such that rN_t is a cycle for any $t \in T$. Assume that $r(K_Y + F)$ is a Cartier divisor. Let $\psi_m: X_m \rightarrow X$ be the blowing-up of X with respect to the sheaf $f_*\mathcal{O}_M(K_M + mr(K_M + E))$ for $m \geq 0$. Note that these sheaves are independent of the choice of the semigood resolution.

In the following, an RDP good resolution of X_t means a partial resolution which is the RDP good resolution of a non-log-canonical singularity (X_t, x_t) and an isomorphism over $X_t \setminus \{x_t\}$.

Theorem 3.6 (see the proof of [15, Theorem 4.2]). *Suppose that $-P_t \cdot P_t$ is constant. Let γ be as in Lemma 3.4 and $\beta(X)$ the maximum of $\{\beta(X_t, \gamma(t)) \mid t \in T\}$ (see Theorem 2.8). Then for any $m \geq \beta(X)$, there exists a neighborhood T_m of $0 \in T$ such that each $(\psi_m)_t: (X_m)_t \rightarrow X_t$ is the RDP good resolution for $t \in T_m$.*

To simplify the notation, we write T (resp. π) instead of T_m (resp. $\pi|_{\pi^{-1}(T_m)}$).

Proposition 3.7. *Suppose that $-P_t \cdot P_t$ is constant. Then the natural rational map $\varphi_m: X_m \rightarrow Y$ is a morphism for $m \gg 0$. If $m \geq \beta(X)$ and φ_m is a morphism, then φ_m is the canonical model of Y .*

Proof. Assume that $m \geq \beta(X)$. Let A' be the exceptional set of $(\psi_m)_0: (X_m)_0 \rightarrow X_0$. Then φ_m is a morphism on $X_m \setminus A'$, since $\pi|_{X \setminus X_0}$ admits a weak simultaneous resolution. There exists an effective divisor Z on Y such that $K_Y \sim -Z$ and $\text{Supp}(Z) = F$. Let $g': Y' \rightarrow Y$ be the normalization of the blowing-up of Y with respect to the sheaf of ideals $\mathcal{O}_Y(-Z)$. We take a semigood resolution $f_m: M_m \rightarrow X$ of X such that the following diagram of morphisms is commutative:

$$\begin{array}{ccccc} M_m & \xrightarrow{\tilde{\psi}_m} & X_m & & \\ h_m \downarrow & & \downarrow \psi_m & & \\ Y' & \xrightarrow{g'} & Y & \xrightarrow{g} & X \end{array}$$

where $f_m = \psi_m \circ \tilde{\psi}_m$. Let G' be a Cartier divisor on Y' such that $\mathcal{O}_{Y'}(G') = g'^*\mathcal{O}_Y(-Z)/\text{torsion}$ and $G_m = h_m^*G'$. Let E_m be the exceptional divisor of f_m .

We put $L_m^M = mr(K_{M_m} + E_m)$, $L_m^Y = mr(K_Y + F)$ and $P_m = (g' \circ h_m)^* L_m^Y$. Let D_m be a Cartier divisor on M_m such that

$$\mathcal{O}_{M_m}(D_m) = f_m^* f_{m*} \mathcal{O}_{M_m}(K_{M_m} + L_m^M) / \text{torsion}.$$

Then D_m and P_m are f_m -nef.

Now let us show the claim: $D_m \sim G_m + P_m$ for $m \gg 0$. Since L_1^Y is a g -ample Cartier divisor, the natural homomorphism

$$g^* g_* \mathcal{O}_Y(K_Y + L_m^Y) \rightarrow \mathcal{O}_Y(K_Y + L_m^Y)$$

is surjective for $m \gg 0$. Then we have the surjection

$$(g' \circ h_m)^* g^* g_* \mathcal{O}_Y(K_Y + L_m^Y) \rightarrow \mathcal{O}_{M_m}(G_m + P_m).$$

By Lemma 3.5, the left hand side is equal to $f_m^* f_{m*} \mathcal{O}_{M_m}(K_{M_m} + L_m^M)$. Hence we have $\mathcal{O}_{M_m}(D_m) \cong \mathcal{O}_{M_m}(G_m + P_m)$.

To show that φ_m is a morphism, it suffices to prove that if $D_m \cdot \ell = 0$ for an irreducible curve $\ell \subset \tilde{\psi}_m^{-1}(A')$, then $P_m \cdot \ell = 0$. Let Λ be the set of irreducible curves on Y'_0 which are $g \circ g'$ -exceptional but not g' -exceptional. Since g' is isomorphic over the non-singular locus of Y , each curve in Λ is the strict transform of an irreducible component of F_0 . We take m such that $D_m \sim G_m + P_m$ and $-m < \min\{G' \cdot \ell' \mid \ell' \in \Lambda\}$. Suppose that $D_m \cdot \ell = 0$ and $P_m \cdot \ell > 0$ for a curve $\ell \subset \tilde{\psi}_m^{-1}(A')$. Then $h_m(\ell) \in \Lambda$. Let d be the degree of the finite morphism $\ell \rightarrow h_m(\ell)$. Since L_1^Y is Cartier, $P_m \cdot \ell \geq dm$. Then we have $dG' \cdot h_m(\ell) = G_m \cdot \ell \leq -dm$: however it contradicts the choice of m .

Assume that φ_m is a morphism on X_m . By Lemma 2.7, the divisor $K_{X_m}|_{(X_m)_t}$ is $(\varphi_m)_t$ -ample for any $t \in T$. Hence K_{X_m} is φ_m -ample. By Theorem 3.6 and [16, Theorem 2.6], X_m has only canonical singularities. Hence φ_m is the canonical model of Y . □

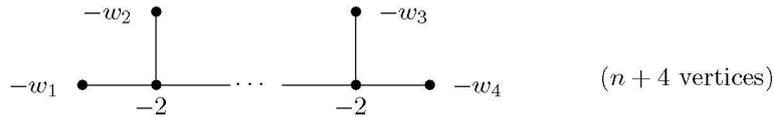
4. The main result

Let (X_0, x_0) be a normal Gorenstein surface singularity and $\pi: X \rightarrow T$ a deformation of $X_0 = \pi^{-1}(0)$. We always assume that T is sufficiently small; so $\pi|_{X \setminus X_0}$ admits a weak simultaneous resolution. We shall prove that the constancy of $-P_t \cdot P_t$ implies the existence of a simultaneous resolution $f: M \rightarrow X$ and a section $\gamma: T \rightarrow X$ which satisfy the following

Condition 4.1. Let E denote the reduced exceptional divisor on M such that $f(E) = \gamma(T)$.

- (1) For each $t \in T$, $f_t: M_t \rightarrow X_t$ is the minimal semigood resolution and E_t is the reduced divisor supported on $f_t^{-1}(\gamma(t))$.
- (2) There exists a divisor $S \leq E$ such that S_t is the sum of all maximal strings at the ends of E_t for each $t \in T$ and that $\pi \circ f|_S: S \rightarrow T$ is a locally trivial deformation.

Example 4.2. Let (X_0, x_0) be a minimally elliptic singularity which has the following weighted dual graph (we denote it by $A_n(w_1, w_2, w_3, w_4)$):



By using [4, Corollary 3.9], for any positive integer $k < n$, we can construct a deformation $\pi: X \rightarrow T$ of X_0 , a section $\gamma: T \rightarrow X$ and a simultaneous resolution $f: M \rightarrow X$ which satisfy Condition 4.1 such that the weighted dual graph of $(X_t, \gamma(t))$ is $A_k(w_1, w_2, w_3, w_4)$ for $t \neq 0$.

In general, some rational double points of type A_q arise on X_t . There is a concrete example. According to Table 1 in [8, V], the weighted dual graph of the singularity $(\{z^2 - (y + x^3)(y^2 + x^{n+5}) = 0\}, o) \subset (\mathbb{C}^3, o)$ is $A_n(w_1, w_2, w_3, w_4)$. Assume that $n - k \geq 2$. Let us consider a family $X_t = \{z^2 - (y + x^3)(y^2 + x^{k+5}(x - t)^{n-k}) = 0\}$. If $t \neq 0$, then the points $(0, 0, 0)$ and $(t, 0, 0)$ are singularities of X_t ; the singularity $(0, 0, 0)$ is an equisingular deformation of $(\{z^2 - (y + x^3)(y^2 + x^{k+5}) = 0\}, o)$, and $(t, 0, 0)$ is a rational double point of type A_{n-k-1} .

Theorem 4.3. *Assume that $-P_t \cdot P_t$ is constant. Then, after a finite base change, there exists a section $\gamma: T \rightarrow X$ such that each $(X_t, \gamma(t))$ is a non-log-canonical singularity and a simultaneous resolution which satisfy the conditions in Condition 4.1; furthermore $X_t \setminus \{\gamma(t)\}$ has only rational double points of type A_n .*

Proof. By Theorem 3.6, there exists a simultaneous RDP good resolution of π . It follows from [1] that there exists a finite base change $T' \rightarrow T$ and a resolution $f': M' \rightarrow X' = X \times_T T'$ such that each $f'_t: M'_t \rightarrow X'_t, t \in T'$, is the minimal semi-good resolution; M' is obtained by resolving the singularities of the simultaneous RDP good resolution of $X' \rightarrow T'$ simultaneously. To simplify, we write $f: M \rightarrow X$ (resp. T) instead of $f': M' \rightarrow X'$ (resp. T'). By Theorem 3.3, there exists the simultaneous log-canonical model $g: Y \rightarrow X$. By Proposition 3.7, we may assume that there exists a morphism $h: M \rightarrow Y$ such that $f = g \circ h$. Let $\gamma: T \rightarrow X$ be the section in Lemma 3.4. We will show that $f: M \rightarrow X$ and $\gamma: X \rightarrow T$ satisfy the conditions in Condition 4.1.

Let F (resp. E) be the reduced exceptional divisor on Y (resp. on M over $\gamma(T)$). We define the \mathbb{Q} -divisors \mathcal{P} and \mathcal{N}' on M by $\mathcal{P} = h^*(K_Y + F)$ and $\mathcal{N}' = K_M + E - \mathcal{P}$, respectively. Since $K_Y + F$ is log-canonical, \mathcal{N}' is an effective exceptional divisor. Let $\mathcal{N}' = \sum n_i E^i$, where $\{E^i\}$ is the set of the exceptional prime divisors on M . Let $\mathcal{N} = \sum_{E^i \subset E} n_i E^i$. For each $t \in T$, we put $K_t = (K_M)_t$; in fact, $(K_M)_t$ is a canonical divisor on M_t . Now suppose $t \in T \setminus \{0\}$. Since $\pi|_{X \setminus X_0}$ admits a weak simultaneous resolution, E_t is the reduced exceptional divisor on M_t and $\mathcal{P}_t + \mathcal{N}_t$ is the Zariski decomposition of $K_t + E_t$ (by using the notation

in the previous section, we can write $\mathcal{P}_t = P_{t,\gamma(t)}$ and $\mathcal{N}_t = N_{t,\gamma(t)}$. Let A be the exceptional set on M_0 and $P + N$ the Zariski decomposition of $K_0 + A$. Then $P = h_0^*((K_Y + F)|_{Y_0}) = \mathcal{P}_0$. Since $K_0 + E_0 = \mathcal{P}_0 + \mathcal{N}_0$, we have $\mathcal{N}_0 - N = E_0 - A$. These divisors are effective since $[\mathcal{N}_0] = E_0 - A$ by Lemma 2.4. Thus $(\mathcal{N}_0)_{red} \geq N_{red}$ and $(E_0)_{red} = A$. If $\mathcal{N} = 0$, then $N = 0$ and $E_0 = A$; hence the conditions in Condition 4.1 are satisfied. Assume that $\mathcal{N} \neq 0$ and let $S = \mathcal{N}_{red}$.

Let C be the cycle supported in A defined in Preliminaries and $C = \bigcup_{j=1}^n C^j$ the decomposition into connected components. Since $P \cdot \mathcal{N}_0 = 0$, we have $(\mathcal{N}_0)_{red} \leq C$. Let $H = A - C$. Each C^j is one of the following three types (see [6, Theorem 9.6]):

- (1) Type A: C^j is a maximal string at an end of A .
- (2) Type \tilde{A} : C^j has the following dual graph



where symbols \bullet and \blacksquare represent a component of C^j and H , respectively.

- (3) Type D: C^j has the following dual graph



We write $S = \sum S^i$, where $\{S^i\}$ is a set of reduced divisors such that $\{(S^i)_t\}$ is the set of all maximal strings at the ends of E_t . Let S_t^i denote $(S^i)_t$. Note that $S_t^i \cdot S_t^j = 0$ if $i \neq j$. By [10, Lemma 3.1, Theorem 3.17], S_0^i is connected and reduced for any i . Hence each S_0^i is contained in a unique C^j . Let $A = \cup A_i$ be the decomposition into irreducible components.

Suppose that C^1 is a cycle of type \tilde{A} . Let $\sigma = \{i \mid S_0^i \leq C^1\}$. Assume that $\sigma \neq \emptyset$. Let A_k be a component at an end of $(\sum_{i \in \sigma} S_0^i)_{red}$. Assume that $A_k \leq S_0^i - \sum_{j \neq i} S_0^j$. Then the coefficient of A_k in S_0 is 1. Since A_k is not a component of N and $[\mathcal{N}] = 0$ by Lemma 2.4, it follows from Proposition 2.3 that the coefficient of A_k in $\mathcal{N}_0 - N$ is a positive number less than 1; however it contradicts that $\mathcal{N}_0 - N = E_0 - A$. If $A_k \subset S_0^i \cap S_0^j$, then $S_0^i \cdot S_0^j < 0$. Hence $\sigma = \emptyset$.

Next suppose that C^1 is a cycle of type D and that A_1 and A_2 are the maximal strings at ends of A in C^1 . Let $C' = C^1 - A_1 - A_2$ and

$$\tau = \{i \mid S_0^i \text{ and } C' \text{ have a common component}\}.$$

Suppose that $\tau \neq \emptyset$ and A_k is the component of $\sum_{i \in \tau} S_0^i$ nearest to H . Assume that $A_k \subset S_0^i \cap S_0^j$ with $i \neq j$. Then the condition $S_t^i \cdot S_t^j = 0$ implies that any component of $S_0^i + S_0^j$ is a (-2) -curve. Thus there exists an open set in M containing $S^i \cup S^j$ which is a simultaneous resolution space of a deformation of a rational double point (see [11, p.12]); however S_0^i and S_0^j can have no common component by virtue of [10, Theorem 3.9] or [7, §4.3]. Hence $\tau = \emptyset$.

Now we obtain that $(\mathcal{N}_0)_{red} = N_{red}$. By arguments similar to above, we see that S_0 is a disjoint union of S_0^j 's. Since $[\mathcal{N}] = 0$, we have $[\mathcal{N}_0] = 0$. It follows from $\mathcal{N}_0 - N = E_0 - A \geq 0$ that $\mathcal{N}_0 = N$ and $E_0 = A$. So (1) in Condition 4.1

follows. Let $S = \bigcup_{i=1}^{\alpha} E^i$ be the decomposition into irreducible components. By Lemma 2.2, each $(E^i)_0$ is irreducible. Hence (2) in Condition 4.1 holds.

Next we will show a rational double point $p \in X_t \setminus \{\gamma(t)\}$ is of type A_n . Let D be a reduced exceptional divisor on M such that $D_t = f_t^{-1}(p)$. Then D_0 is reduced, connected and contained in C . By the minimality of the semigood resolution, any component of D_0 is a (-2) -curve. Let D'_0 be the sum of the components $A_i \leq D_0$ such that $(D_0 - A_i) \cdot A_i = 2$. Note that if $A_i \leq D_0$ and $D_0 \cdot A_i = 0$ then $A_i \leq D'_0$. Since $A_i \cdot D_0 = 0$ for any $A_i \subset S_0^j$, we have $S_0^j \leq D'_0$ or $\text{Supp}(S_0^j) \cap \text{Supp}(D_0) = \emptyset$. Since S_0^j is a maximal string at an end of A , the first case does not occur. Hence D_0 is a chain and so is D_t . \square

We use the notation of the proof of Theorem 4.3 in the following two remarks.

Remark 4.4. The converse of the theorem is true. In fact, the following conditions are equivalent:

- (1) π admits a section and a simultaneous resolution as in Theorem 4.3 after a finite base change;
- (2) $\delta_m(X_t) = \sum_{p \in \text{Sing}(X_t)} \delta_m(X_t, p)$ is constant for any $m \in \mathbb{N}$;
- (3) $-P_t \cdot P_t$ is constant.

We show a sketch of the proof. Suppose that (1) holds. Then we see that $\mathcal{P}_t \cdot \mathcal{P}_t$ and $K_t \cdot \mathcal{P}_t$ are constant. The existence of the simultaneous resolution implies that $p_g(X_t, \gamma(t))$ is constant too (see [11, Theorem 5.3]). Hence $\delta_m(X_t, \gamma(t))$ is constant by Theorem 2.5. Now (2) follows from the fact that $\delta_m = 0$ for any quotient singularity and $m \in \mathbb{N}$ ([22, Theorem 1.5]).

Remark 4.5. A component A_i is called a node unless it is a nonsingular rational curve with at most two intersections with other curves. Suppose that $-P_t \cdot P_t$ is constant. From the proof of the theorem, we see that X_t ($t \neq 0$) has only one singular point $\gamma(t)$ if any chain in A connecting two nodes contains no (-2) -curves.

Corollary 4.6. *Suppose that $-P_t \cdot P_t$ is constant and that the weighted dual graph of (X_0, x_0) is a star-shaped graph. Then π admits a weak simultaneous resolution.*

Proof. If the weighted dual graph of (X_0, x_0) is a star-shaped graph, then X_t has only one singular point by Remark 4.5 and a simultaneous resolution with the conditions in Condition 4.1 is just a weak simultaneous resolution. Thus we need no finite base changes. \square

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