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On the zero set of semi-invariants for tame quivers

Christine Riedtmann and Grzegorz Zwara

Abstract. Let **d** be a prehomogeneous dimension vector for a finite tame quiver Q. We show that the common zeros of all non-constant semi-invariants for the variety of representations of Q with dimension vector $N \cdot \mathbf{d}$, under the product of the general linear groups at all vertices, is a complete intersection for $N \geq 3$.

Mathematics Subject Classification (2000). 14L24, 16G20.

 ${\bf Keywords.}$ Semi-invariants, quivers, representations.

1. Introduction

Let k be an algebraically closed field, and let $Q = (Q_0, Q_1, t, h)$ be a finite quiver, i.e. a finite set $Q_0 = \{1, \ldots, n\}$ of vertices and a finite set Q_1 of arrows $\alpha : t\alpha \to h\alpha$, where $t\alpha$ and $h\alpha$ denote the tail and the head of α , respectively.

A representation of Q over k is a collection $(X(i); i \in Q_0)$ of finite dimensional k-vector spaces together with a collection $(X(\alpha): X(t\alpha) \to X(h\alpha); \alpha \in Q_1)$ of k-linear maps. A morphism $f: X \to Y$ between two representations is a collection $(f(i): X(i) \to Y(i))$ of k-linear maps such that

$$f(h\alpha) \circ X(\alpha) = Y(\alpha) \circ f(t\alpha)$$
 for all $\alpha \in Q_1$.

By $\sigma(X)$ we denote the number of pairwise non-isomorphic indecomposable direct summands occurring in a decomposition of X into indecomposables. According to the theorem of Krull–Schmidt, $\sigma(X)$ is well-defined. The dimension vector of a representation X of Q is the vector

$$\operatorname{\mathbf{dim}} X = (\operatorname{\mathrm{dim}} X(1), \dots, \operatorname{\mathrm{dim}} X(n)) \in \mathbb{N}^{Q_0}.$$

We denote the category of representations of Q by $\operatorname{rep}(Q)$, and for any vector $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^{Q_0}$

$$\operatorname{rep}(Q, \operatorname{\mathbf{d}}) = \prod_{\alpha \in Q_1} \operatorname{Mat}(d_{h\alpha} \times d_{t\alpha}, k)$$

is the vector space of representations X of Q with $X(i) = k^{d_i}$, $i \in Q_0$. The group

$$\operatorname{Gl}(\mathbf{d}) = \prod_{i=1}^{n} \operatorname{Gl}(d_i, k)$$

acts on $rep(Q, \mathbf{d})$ by

$$((g_1,\ldots,g_n)\star X)(\alpha)=g_{h\alpha}\cdot X(\alpha)\cdot g_{t\alpha}^{-1}.$$

Note that the $Gl(\mathbf{d})$ -orbit of X consists of the representations Y in $rep(Q, \mathbf{d})$ which are isomorphic to X.

We call \mathbf{d} a prehomogeneous dimension vector if $\mathrm{Gl}(\mathbf{d}) \star T$ is an open orbit for some T in $\mathrm{rep}(Q,\mathbf{d})$. Such a representation T is characterized by $\mathrm{Ext}_Q^1(T,T)=0$ [9]. If Q admits only finitely many indecomposable representations, or equivalently if the underlying graph of Q is a disjoint union of Dynkin diagrams of type \mathbb{A} , \mathbb{D} or \mathbb{E} [6], every vector \mathbf{d} is prehomogeneous. Indeed, any representation is a direct sum of indecomposables and therefore $\mathrm{rep}(Q,\mathbf{d})$ contains finitely many orbits, one of which must be open.

Let **d** be prehomogeneous, and let $f_1, \ldots, f_s \in k[\operatorname{rep}(Q, \mathbf{d})]$ be the irreducible monic polynomials whose zeros $Z(f_1), \ldots, Z(f_s)$ are the irreducible components of codimension 1 of $\operatorname{rep}(Q, \mathbf{d}) \setminus \operatorname{Gl}(\mathbf{d}) \star T$, where $\operatorname{Gl}(\mathbf{d}) \star T$ is the open orbit. It is easy to see that

$$g \cdot f_i = \chi_i(g) \cdot f_i$$

for $g \in \mathrm{Gl}(\mathbf{d})$, where $\chi_i : \mathrm{Gl}(\mathbf{d}) \to k^*$ is a character. A regular function with this property is called a semi-invariant. By [11], any semi-invariant is a scalar multiple of a monomial in f_1, \ldots, f_s , and f_1, \ldots, f_s are algebraically independent. We denote by

$$\mathcal{Z}_{Q,\mathbf{d}} = \{ X \in \text{rep}(Q,\mathbf{d}); \ f_i(X) = 0, \ i = 1,\dots, s \}$$

the closed subscheme of $\operatorname{rep}(Q,\operatorname{\mathbf{d}})$ of common zeros of all non-constant semi-invariants. Obviously we have $\operatorname{codim} \mathcal{Z}_{Q,\operatorname{\mathbf{d}}} \leq s$, and equality means that $\mathcal{Z}_{Q,\operatorname{\mathbf{d}}}$ is a complete intersection.

Let T_1, \ldots, T_r be pairwise non-isomorphic indecomposable representations of Q such that $\operatorname{Ext}^1_Q(T_i, T_j) = 0$ for any $i, j \leq r$. In [8] we showed that there is a positive integer N such that $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection and irreducible for any dimension vector $\mathbf{d} = \sum_{i=1}^r \lambda_i \dim T_i$ with $\lambda_i \geq N$, $i = 1, 2, \ldots, r$. Now our goal is to prove that N is quite small in case Q is tame; i.e., every connected component Δ of Q is either a Dynkin quiver or an extended Dynkin quiver. Our methods are completely different.

Assume that Q is tame, and set

$$N(Q) = \max N(\Delta),$$

352

where Δ ranges over the connected components of Q and where

$$N(\Delta) = \begin{cases} 1 & \text{if } |\Delta| = \mathbb{A}_m \text{ or } \widetilde{\mathbb{A}}_m, \\ 2 & \text{if } |\Delta| = \mathbb{D}_m, \mathbb{E}_6, \mathbb{E}_7 \text{ or } \mathbb{E}_8, \\ 3 & \text{if } |\Delta| = \widetilde{\mathbb{D}}_m, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7 \text{ or } \widetilde{\mathbb{E}}_8, \end{cases}$$

and $|\Delta|$ denotes the underlying non-oriented graph of the quiver Δ . Note that $N(K) \leq N(Q)$ for any subquiver K of Q.

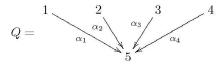
Theorem 1.1. Suppose Q is tame. Let T_1, \ldots, T_r be pairwise non-isomorphic indecomposable representations of Q such that $\operatorname{Ext}_Q^1(T_i, T_j) = 0$ for any $i, j \leq r$. Choose positive integers $\lambda_1, \ldots, \lambda_r$ and set $\lambda = \min \lambda_i$, $\mathbf{d} = \sum_{i=1}^r \lambda_i \operatorname{\mathbf{dim}} T_i$. Then

- (i) a complete intersection provided $\lambda \geq N(Q)$,
- (ii) irreducible provided $\lambda \geq N(Q) + 1$.

Note that the case of a Dynkin quiver of type \mathbb{A}_n has been treated by Chang and Weyman in [5]

In case k is the field \mathbb{C} of complex numbers, the fact that $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection implies that $rep(Q, \mathbf{d})$ is cofree as a representation of the subgroup $Sl(\mathbf{d}) = \prod_{i=1}^{n} Sl(d_i)$ of $Gl(\mathbf{d})$; i.e., $\mathbb{C}[rep(Q, \mathbf{d})]$ is a free module over the ring $\mathbb{C}[rep(Q, \mathbf{d})]^{Sl(\mathbf{d})}$ of $Sl(\mathbf{d})$ -invariant polynomials [13, §17], [8].

Example. Let us consider the quiver



with the dimension vector $\mathbf{d} = \lambda \cdot \mathbf{e}$, $\lambda \in \mathbb{N}$ and $\mathbf{e} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ as an example.

There is an indecomposable representation T_1 in $\operatorname{rep}(Q,\mathbf{e})$, whose orbit is open. The complement of the open orbit of $T = T_1^{\lambda}$ in rep (Q, \mathbf{d}) has 4 components of codimension 1, defined by

$$\det\left(X(\alpha_1)\cdots\widehat{X(\alpha_j)}\cdots X(\alpha_4)\right)=0,$$

j=1,2,3,4, where the hat means "omit $X(\alpha_j)$ ". Using the results developed later, we know that X belongs to $\mathcal{Z}_{Q,\mathbf{d}}$ if and only if X either contains the simple projective P_5 or else the direct sum $\bigoplus_{j=1}^4 P_j$ of the two-dimensional projectives associated to the vertices $1, \ldots, 4$ as a direct summand. It is easy to check that

- $\mathcal{Z}_{Q,\mathbf{e}}$ is irreducible of codimension 2,
- $\mathcal{Z}_{Q,2\mathbf{e}}$ has two irreducible components of codimension 3 and 4, respectively,
- Z_{Q,3e} has two irreducible components of codimension 4,
 Z_{Q,λ·e} is irreducible of codimension 4 for λ ≥ 4.

2. Notations and preliminaries

The varieties considered in this paper are locally closed subsets of a k-vector space. If $\mathcal{A} \subseteq \mathcal{B}$ are two such varieties and \mathcal{B} is irreducible, we denote by $\operatorname{codim}_{\mathcal{B}} \mathcal{A}$ the codimension of \mathcal{A} in \mathcal{B} . In case $\mathcal{B} = \operatorname{rep}(Q, \mathbf{d})$, we omit the subscript \mathcal{B} .

We will assume throughout that the representation $T = \bigoplus_{i=1}^r T_i^{\lambda_i}$ is sincere, i.e., $T(l) \neq 0$ for any $l \in Q_0$. As the full subquiver K of Q which supports T is still tame with $N(K) \leq N(Q)$, this is no restriction. The assumption excludes oriented cycles as subquivers of Q. Indeed, a sincere representation of an oriented cycle cannot have an open orbit.

The Euler form of Q is the \mathbb{Z} -bilinear form on \mathbb{Z}^{Q_0} defined by

$$\langle \mathbf{d}, \mathbf{e} \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{lpha \in Q_1} d_{tlpha} e_{hlpha}.$$

For $X \in \operatorname{rep}(Q, \mathbf{d})$, $Y \in \operatorname{rep}(Q, \mathbf{e})$ it can be computed as

$$\langle \mathbf{d}, \mathbf{e} \rangle = [X, Y] - {}^{1}[X, Y],$$

where

$$[X,Y] = \dim_k \operatorname{Hom}_{\mathcal{Q}}(X,Y)$$
 and ${}^1[X,Y] = \dim_k \operatorname{Ext}_{\mathcal{Q}}^1(X,Y)$.

The quadratic form

$$q(\mathbf{d}) = \langle \mathbf{d}, \mathbf{d} \rangle$$

associated with the Euler form is the Tits form of Q. It is positive semi-definite as Q is tame and positive definite if Q does not contain extended Dynkin diagrams.

We follow Schofield [12] in order to describe the semi-invariants of rep(Q, \mathbf{d}): For a representation U of Q, the right perpendicular category U^{\perp} is the full subcategory of rep(Q) whose objects are

$${Y;\ [U,Y]={}^{1}[U,Y]=0}.$$

Dually, $^{\perp}U$ has as objects

$${Z; [Z, U] = {}^{1}[Z, U] = 0}.$$

Note that $U^{\perp} = {}^{\perp}(\tau U)$, where τ is the Auslander–Reiten translation for all non-projective indecomposable direct summands of U and $\tau(P_l) = I_l$, where P_l and I_l are the projective and injective indecomposable representations associated to the vertex $l \in Q_0$, respectively. If ${}^1[U,U] = 0$, the category U^{\perp} is equivalent to the category of representations of a quiver with $n - \sigma(U)$ vertices.

category of representations of a quiver with $n - \sigma(U)$ vertices. Thus T^{\perp} contains n - r simple objects if $T = \bigoplus_{i=1}^{r} T_i^{\lambda_i}$ is a representation of Q as in the statement of the theorem. If S is one of them, the set

$${X \in \text{rep}(Q, \mathbf{d}); [X, S] \neq 0}$$

is a component of codimension 1 of the complement

$$rep(Q, \mathbf{d}) \setminus Gl(\mathbf{d}) \star T$$
.

Non-isomorphic simple objects lead to distinct components, and all components of codimension 1 are obtained in this way. Thus $\mathcal{Z}_{Q,\mathbf{d}}$ is the zero set of n-r (algebraically independent) polynomials. From now on, we will denote the underlying reduced variety of $\mathcal{Z}_{Q,\mathbf{d}}$ by the same symbol. This will cause no confusion since we are only interested in the irreducibility and the dimension of $\mathcal{Z}_{Q,\mathbf{d}}$. We have the following descriptions:

$$\mathcal{Z}_{Q,\mathbf{d}} = \{X \in \operatorname{rep}(Q,\mathbf{d}); \ [X,S] \neq 0 \text{ for all simple objects } S \in T^{\perp}\}\$$

= $\{X \in \operatorname{rep}(Q,\mathbf{d}); \ [S',X] \neq 0 \text{ for all simple objects } S' \in {}^{\perp}T\}.$

The material presented here can be found in [12]; compare also [8]. In order to obtain part (i) of our theorem it suffices to prove codim $\mathcal{Z}_{Q,\mathbf{d}} \geq n-r$.

Fix a sink $z \in Q_0$; i.e., a vertex z which is the head of some arrows $\alpha_j: y_j \to z$, $j=1,\ldots,s$, but the tail of none. The vertices y_1,\ldots,y_s need not be distinct. Let E be the simple projective supported at z. By \overline{Q} we denote the full subquiver of Q with $\overline{Q}_0 = Q_0 \setminus \{z\}$ and by $\overline{\mathbf{d}}$ the restriction of \mathbf{d} to \overline{Q}_0 . Note that the orbit of the restriction $\overline{T} = \bigoplus_{i=1}^r \overline{T}_i^{\lambda_i}$ to \overline{Q} is open in $\operatorname{rep}(\overline{Q}, \overline{\mathbf{d}})$. As E is the simple projective supported at z, we have

$$E^{\perp} = \{ X \in \text{rep}(Q); \ X(z) = 0 \},$$

which we identify with $rep(\overline{Q})$. There is a short exact sequence

$$0 \to E^{d_z} \to T \to \overline{T} \to 0.$$

Considering the long exact sequence of Hom's and Ext^1 's from it, we find that $E^\perp \cap T^\perp = E^\perp \cap \overline{T}^\perp = \overline{T}^{\perp_{\overline{Q}}}$.

We decompose $\mathcal{Z}_{Q,\mathbf{d}}$ as a disjoint union

$$\mathcal{Z}_{Q,\mathbf{d}} = \mathcal{Z}'_{Q,\mathbf{d}} \cup \mathcal{Z}''_{Q,\mathbf{d}},$$

where

$$\mathcal{Z}'_{Q,\mathbf{d}} = \{X \in \mathcal{Z}_{Q,\mathbf{d}}; \ [X,E] = 0\} \qquad \text{and} \qquad \mathcal{Z}''_{Q,\mathbf{d}} = \{X \in \mathcal{Z}_{Q,\mathbf{d}}; \ [X,E] \neq 0\}.$$

We will estimate the codimensions of $\mathcal{Z}'_{Q,\mathbf{d}}$ and $\mathcal{Z}''_{Q,\mathbf{d}}$ in $\operatorname{rep}(Q,\mathbf{d})$ separately.

Throughout the article, $T = \bigoplus_{i=1}^r T_i^{\lambda_i}$ will denote a sincere representation of a tame quiver Q, and we set $\lambda = \min \lambda_i \geq 1$ and $\dim T = \mathbf{d}$.

3. The variety $\mathcal{Z}''_{Q,\mathbf{d}}$

Proposition 3.1. A representation X in $\mathcal{Z}_{Q,\mathbf{d}}$ belongs to $\mathcal{Z}''_{Q,\mathbf{d}}$ if and only if

- (i) the restriction \overline{X} to \overline{Q} lies in $\mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}}$ and
 - (ii) $\operatorname{rank}(X(\alpha_1) \cdots X(\alpha_s)) < d_z$.

In particular,

$$\operatorname{codim} \mathcal{Z}_{Q,\mathbf{d}}'' = \operatorname{codim}_{\operatorname{rep}(\overline{Q},\overline{\mathbf{d}})} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} + \max \bigg(0, \bigg(\sum_{j=1}^s d_{y_j} \bigg) - d_z + 1 \bigg).$$

Proof. The second condition just says that E is a direct summand of X, or equivalently that $[X, E] \neq 0$. A representation $X = X' \oplus E$ belongs to $\mathcal{Z}_{Q,\mathbf{d}}$ if and only if

$$[X,S] = [X',S] + [E,S] > 0$$

for any simple object $S \in T^{\perp}$. Equivalently,

holds for any simple representation $S \in T^{\perp}$ with $[E,S] = \dim S(z) = 0$. These are precisely the simple objects of $\overline{T}^{\perp_{\overline{Q}}}$, and moreover we have

$$[X',S]=[\overline{X'},S]=[\overline{X},S]>0$$

since S(z) = 0.

As for the statement about codim $\mathcal{Z}''_{Q,\mathbf{d}}$, observe that, in case $d_z > \sum_{j=1}^s d_{y_j}$, any $d_z \times \sum_{j=1}^s d_{y_j}$ -matrix has rank less than d_z , whereas for $d_z \leq \sum_{j=1}^s d_{y_j}$, the subvariety

$$\mathcal{N}_{\mathbf{d}} = \left\{ A \in \operatorname{Mat}\left(d_z \times \sum_{j=1}^s d_{y_j}\right); \operatorname{rank} A < d_z \right\}$$

is of codimension $\left(\sum_{j=1}^{s} d_{y_j}\right) - d_z + 1$.

Corollary 3.2. Suppose that $\lambda \geq N(Q)$ and that E is not a direct summand of T.

(i) We have

$$\operatorname{codim} \mathcal{Z}_{Q,\mathbf{d}}'' - n + \sigma(T) \geq \operatorname{codim} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} - (n-1) + \sigma(\overline{T}).$$

(ii) If moreover $d_z < \sum_{j=1}^s d_{y_j}$, we have

$$\operatorname{codim} \mathcal{Z}''_{Q,\operatorname{\mathbf{d}}} - n + \sigma(T) \geq \operatorname{codim} \mathcal{Z}_{\overline{Q},\overline{\operatorname{\mathbf{d}}}} - (n-1) + \sigma(\overline{T}) + \lambda - N(Q).$$

In order to prove this result, we need some information about the number $\sigma(\overline{T})$ of pairwise non-isomorphic indecomposables occurring as direct summands of \overline{T} . We start by estimating $\sigma(\overline{U})$ for an indecomposable representation U:

Lemma 3.3. For an indecomposable representation $U \neq E$ of Q, we have

$$\sigma(\overline{U}) \le 1 + N(Q) \cdot \left(\left(\sum_{j=1}^{s} \dim U(y_j) \right) - \dim U(z) \right).$$

 ${\it Proof.}$ As U is indecomposable, we may assume Q to be connected. We use the following abbreviations:

$$\dim U(z)=u, \quad \dim U(y_j)=u_j, j=1,\ldots,s, \quad u'=\left(\sum_{i=1}^s u_j
ight)-u.$$

Note that $u' \geq 0$ since U is indecomposable and $U \neq E$. If u = 0, $\overline{U} = U$ is indecomposable and $\sigma(\overline{U}) = 1$. In case u' = 0, the map

$$[U(\alpha_1), \cdots, U(\alpha_s)] : \bigoplus_{j=1}^s U(y_j) \to U(z)$$

is an isomorphism, and again \overline{U} is indecomposable. Thus we may suppose u>0 and u'>0.

Recall that the value of the Tits form $q(\operatorname{\mathbf{dim}} U)$ equals 0 or 1, as Q is tame. We compute:

$$q(\operatorname{\mathbf{dim}} U - \operatorname{\mathbf{dim}} E) = q(\operatorname{\mathbf{dim}} U) + q(\operatorname{\mathbf{dim}} E) - \langle \operatorname{\mathbf{dim}} U, \operatorname{\mathbf{dim}} E \rangle - \langle \operatorname{\mathbf{dim}} E, \operatorname{\mathbf{dim}} U \rangle$$
$$= q(\operatorname{\mathbf{dim}} U) + q(\operatorname{\mathbf{dim}} E) + u' - u \le 2 + u' - u.$$

As q is positive definite or positive semi-definite in case Q is a Dynkin quiver or an extended Dynkin quiver, respectively, we obtain:

$$u \leq \begin{cases} u' + 2 \leq 2u' + 1 & \text{if } Q \text{ is an extended Dynkin quiver,} \\ u' + 1 & \text{if } Q \text{ is a Dynkin quiver.} \end{cases}$$

Now clearly \overline{U} has at most $\sum_{j=1}^{s} u_j$ indecomposable direct summands, and thus

$$\sigma(\overline{U}) \leq \sum_{i=1}^{s} u_{j} = u + u' \leq \begin{cases} 1 + 3u' & \text{if } Q \text{ is an extended Dynkin quiver,} \\ 1 + 2u' & \text{if } Q \text{ is a Dynkin quiver,} \end{cases}$$

which proves the lemma except in case $|Q| = \mathbb{A}_n$ or $|Q| = \widetilde{\mathbb{A}}_{n-1}$.

If $|Q| = \mathbb{A}_n$, we have $u \leq 1$ and hence $\sigma(\overline{U}) \leq 1 + u'$. In case $|Q| = \mathbb{A}_{n-1}$, the number of indecomposable (possible isomorphic) direct summands in a decomposition of \overline{U} is at most 1 + u'. This can be seen by inspecting the list of indecomposable representations of Q. Such representations are string or band representations, and they are described by words (non-oriented paths) in Q (see [4] for details).

Proof of Corollary 3.2. We set

$$t_i' = \left(\sum_{j=1}^s \dim T_i(y_j)\right) - \dim T_i(z), \ i = 1, \dots, r$$

and

$$t' = \sum_{i=1}^r t_i'.$$

Note that, by definition,

$$\sum_{i=1}^r \lambda_i t_i' = \left(\sum_{j=1}^s d_{y_j}\right) - d_z.$$

Our lemma implies:

$$\begin{split} \sigma(\overline{T}) &\leq \sum_{i=1}^r \sigma(\overline{T_i}) \leq r + N(Q) \cdot t' \leq r + \left(\sum_{i=1}^r \lambda_i t_i'\right) - (\lambda - N(Q)) \cdot t' \\ &= \sigma(T) + \left(\sum_{j=1}^s d_{y_j}\right) - d_z - (\lambda - N(Q)) \cdot t'. \end{split}$$

Combining this with Proposition 3.1 we find that

$$\begin{aligned} \operatorname{codim} \mathcal{Z}_{Q,\mathbf{d}}'' - n + \sigma(T) &= \operatorname{codim}_{\operatorname{rep}(\overline{Q},\overline{\mathbf{d}})} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} + \left(\sum_{j=1}^s d_{y_j}\right) - d_z + 1 - n + \sigma(T) \\ &\geq \operatorname{codim}_{\operatorname{rep}(\overline{Q},\overline{\mathbf{d}})} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} - (n-1) + \sigma(\overline{T}) + (\lambda - N(Q)) \cdot t'. \end{aligned}$$

As $t_i' \ge 0$ for all i, this yields part (i) of Corollary 3.2. Part (ii) follows from the fact that $\sum_{i=1}^r \lambda_i t_i' = \left(\sum_{j=1}^s d_{y_j}\right) - d_z > 0$ implies $t_i' > 0$ for some i and hence t' > 0.

4. Reflection functors

We define two new quivers \widetilde{Q} and Q': \widetilde{Q} is obtained from Q by adding a vertex z' and arrows $\beta_j: z' \to y_j, j = 1, \ldots, s$. Deleting z and $\alpha_1, \ldots, \alpha_s$ in \widetilde{Q} yields Q'. Note that Q' is tame as well. We denote by E' the simple injective representation of Q' supported at z'.

We consider the reflection functor

$$\mathcal{F}: \operatorname{rep}(Q) \to \operatorname{rep}(Q')$$

associated with z. Recall that

$$(\mathcal{F}X)(i) = \begin{cases} X(i) & i \neq z' \\ \ker\left(\bigoplus X(y_j) \xrightarrow{[X(\alpha_1), \dots, X(\alpha_s)]} X(z)\right) & i = z', \end{cases}$$

and that

$$(\mathcal{F}X)(\beta_l):(\mathcal{F}X)(z')\to (\mathcal{F}X)(y_l)=X(y_l)$$

is the inclusion of $(\mathcal{F}X)(z')$ into $\bigoplus_{j=1}^s X(y_j)$ followed by the projection to $X(y_l)$ (see [1], [6]). The functor \mathcal{F} restricts to an equivalence

$$\mathcal{F}: (\operatorname{rep}(Q))' \to (\operatorname{rep}(Q'))'$$

from the full subcategory $(\operatorname{rep}(Q))'$ of $\operatorname{rep}(Q)$ whose objects do not contain E as a direct summand, or equivalently have no non-trivial morphisms to E, to the full subcategory $(\operatorname{rep}(Q'))'$ of $\operatorname{rep}(Q')$ whose objects do not contain E' as a direct summand.

Suppose that E is neither a direct summand of T nor an element of T^{\perp} . This implies that [T, E] = 0 and ${}^{1}[T, E] > 0$ and thus the vector $\mathbf{d}' \in \mathbb{Z}^{Q'_{0}}$, where Q'_{0} denotes the set of vertices of Q', defined by

$$d_x' = \begin{cases} d_x, & x \neq z' \\ \left(\sum_{j=1}^s d_{y_j}\right) - d_z, & x = z' \end{cases}$$

has positive entries. Indeed, we have

$$d'_{z'} = \left(\sum_{j=1}^{s} d_{y_j}\right) - d_z = -\langle \mathbf{d}, \dim E \rangle = -[T, E] + {}^{1}[T, E] > 0.$$
 (4.1)

Note that in fact we have $d'_{z'} \geq \lambda$ as ${}^{1}[T_{i}, E] > 0$ for some i implies ${}^{1}[T, E] \geq \lambda_{i} \geq \lambda$. We let $\widetilde{\mathbf{d}}$ be the dimension vector for \widetilde{Q} which coincides with \mathbf{d} on Q_{0} and with \mathbf{d}' on Q'_{0} .

As E is not a direct summand of T, the latter belongs to $(\operatorname{rep} Q)'$. Therefore $\mathcal{F}T$ lies in $(\operatorname{rep} Q')'$, and we have $\dim \mathcal{F}T = \mathbf{d}'$, ${}^1[\mathcal{F}T,\mathcal{F}T] = {}^1[T,T] = 0$, and thus \mathbf{d}' is prehomogeneous. Choose T' in the open orbit of $\operatorname{rep}(Q',\mathbf{d}')$. As T' is isomorphic to $\mathcal{F}T$, we have $T' = \bigoplus_{i=1}^r (T_i')^{\lambda_i}$ with T_i' indecomposable, pairwise non-isomorphic and ${}^1[T_i',T_j'] = 0$ for all i,j. Moreover, we know $T^\perp \subseteq (\operatorname{rep} Q)'$, as E does not belong to T^\perp , and $(T')^\perp \subseteq (\operatorname{rep} Q')'$, as $d'_{z'} = [T',E'] > 0$. We conclude that $(T')^\perp$ is equivalent to $\mathcal{F}(T^\perp)$, the category of representations of a quiver with n-r vertices. Hence $\mathcal{Z}_{Q',\mathbf{d}'}$ is given by n-r equations as well. We decompose $\mathcal{Z}_{Q',\mathbf{d}'}$ as a disjoint union $\mathcal{Z}_{Q',\mathbf{d}'} = \mathcal{W}'_{Q',\mathbf{d}'} \cup \mathcal{W}''_{Q',\mathbf{d}'}$, where

$$\mathcal{W}'_{Q',\mathbf{d}'} = \{ X' \in \mathcal{Z}_{Q',\mathbf{d}'}; [E',X'] = 0 \}$$

and

$$\mathcal{W}''_{Q',\mathbf{d}'} = \{ X' \in \mathcal{Z}_{Q',\mathbf{d}'}; \ [E',X'] \neq 0 \}.$$

Proposition 4.1. Suppose E is neither a direct summand of T nor an element of T^{\perp} . Then we have

- (i) $\operatorname{codim} \mathcal{Z}'_{Q,\mathbf{d}} = \operatorname{codim}_{\operatorname{rep}(Q',\mathbf{d}')} \mathcal{W}'_{Q',\mathbf{d}'}$ and
- (ii) $\mathcal{Z}'_{Q,\mathbf{d}}$ is irreducible if $\mathcal{W}'_{Q',\mathbf{d}'}$ has this property.

Proof. By construction, X belongs to $\mathcal{Z}'_{Q,\mathbf{d}}$ if and only if $\mathcal{F}X$ is isomorphic to some $X' \in \mathcal{W}'_{Q',\mathbf{d}'}$, but unfortunately the functor \mathcal{F} cannot be made into a regular map from

$$rep(Q, \mathbf{d})' = \{X \in rep(Q, \mathbf{d}); [X, E] = 0\}$$

to

$$\operatorname{rep}(Q', \mathbf{d}')' = \{X' \in \operatorname{rep}(Q', \mathbf{d}'); [E', X'] = 0\}.$$

We use the following détour (compare [7] and Section 4.2 in [3]): The set

$$\left\{X \in \operatorname{rep}(\widetilde{Q}, \widetilde{\mathbf{d}}); \sum_{j=1}^{s} X(\alpha_j) X(\beta_j) = 0, [X(\beta_1), \dots, X(\beta_s)]^t \text{ injective}, \\ [X(\alpha_1), \dots, X(\alpha_s)] \text{ surjective} \right\}$$

is a principal $\mathrm{Gl}(d'_{z'})$ -bundle over $\mathrm{rep}(Q,\mathbf{d})'$ and a principal $\mathrm{Gl}(d_z)$ -bundle over $\mathrm{rep}(Q',\mathbf{d}')'$ via the projections π and π' deleting z' and z, respectively. Hence the claim follows from $\pi^{-1}(\mathcal{Z}'_{Q,\mathbf{d}}) = (\pi')^{-1}(\mathcal{W}'_{Q',\mathbf{d}'})$.

5. Proof of Theorem 1.1

We proceed by induction on the number n of vertices of Q. We may assume the theorem to be true for $\mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}}$. First we treat the cases that

- (i) E is a direct summand of T and
- (ii) E belongs to T^{\perp} .

In both cases, we have that E is a direct summand of X for all $X \in \mathcal{Z}_{Q,\mathbf{d}}$; i.e., $\mathcal{Z}''_{Q,\mathbf{d}} = \mathcal{Z}_{Q,\mathbf{d}}$. Indeed, in case (i) this follows from the fact that $\operatorname{Hom}_Q(E,T) \neq 0$, which is a closed condition. In case (ii), E is a simple object in T^{\perp} .

As any direct summand $T_i \not\simeq E$ of T belongs to $^{\perp}E$, we have

$$\dim T_i(z) - \sum_{j=1}^s \dim T_i(y_j) = \langle \operatorname{\mathbf{dim}} T_i, \operatorname{\mathbf{dim}} E \rangle = [T_i, E] - {}^1[T_i, E] = 0.$$

By Lemma 3.3, $\overline{T_i}$ is indecomposable, and therefore

$$\sigma(\overline{T}) = \begin{cases} r - 1 & \text{in case (i),} \\ r & \text{in case (ii).} \end{cases}$$

The induction hypothesis together with Corollary 3.2 implies the first part of our theorem. We conclude from Proposition 3.1 that $\mathcal{Z}_{Q,\mathbf{d}} \simeq \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} \times \mathcal{N}_{\mathbf{d}}$, where

$$\mathcal{N}_{\mathbf{d}} = \{ A \in \operatorname{Mat}\left(d_z \times \sum_{j=1}^s d_{y_j}\right); \text{ rank } A < d_z \}.$$

The second part follows from the fact that the set $\mathcal{N}_{\mathbf{d}}$ is irreducible in case $d_z \geq \sum_{j=1}^{s} dy_j$.

(iii) Finally, suppose that E is neither a direct summand of T nor does it belong to T^{\perp} , or equivalently that $d_z < \sum_{j=1}^s d_{y_j}$. Using Corollary 3.2 and its dual, Proposition 4.1 and remembering that the codimension of any irreducible

component of $\mathcal{Z}_{Q,\mathbf{d}}$ is at most n-r, we see that the theorem is true for $\mathcal{Z}_{Q,\mathbf{d}}$ if and only if it holds for $\mathcal{Z}_{Q',\mathbf{d}'}$.

In case either T contains a preprojective direct summand or T^{\perp} a preprojective representation, we may apply a series of reflection functors until we reach the situation that a simple projective either is a direct summand of T or else belongs to T^{\perp} , and we can reduce by (i) or (ii). This finishes the proof in case Q is of finite representation type as any indecomposable representation is preprojective.

If Q is not representation finite, we are left with the situation that no preprojective representation is a direct summand of T nor an element of T^{\perp} . Dually, we may assume T does not contain a preinjective direct summand either. Indeed, suppose a simple injective representation E' is a direct summand of T or belongs to $^{\perp}T$, a situation we will reach after a series of (inverse) reflection functors. Then apply the dual of the first or the second reduction step above; recall that $\mathcal{Z}_{Q,\mathbf{d}}$ has a dual description as

$$\mathcal{Z}_{Q,\mathbf{d}} = \{X \in \operatorname{rep}(Q,\mathbf{d}); [S',X] \neq 0 \text{ for all simple objects } S' \in {}^{\perp}T\}.$$

The following lemma finishes the proof of Theorem 1.1.

Lemma 5.1. Let Q be an extended Dynkin quiver. Suppose T is a regular representation with an open orbit. Then T^{\perp} contains a non-zero preprojective representation.

Proof. Consider a Bongartz completion \widetilde{T} for T [2]; i.e., an exact sequence

$$0 \to kQ \to \widetilde{T} \to \bigoplus_{i=1}^r T_i^{\nu_i} \to 0$$

for which the induced map

$$\operatorname{Hom}_Q\left(T_l, \bigoplus_{i=1}^r T_i^{\nu_i}\right) \to \operatorname{Ext}_Q^1(T_l, kQ)$$

is surjective for $l=1,\ldots,r$. There is a \mathbb{Z} -linear map $\partial:\mathbb{Z}^{Q_0}\to\mathbb{Z}$, called defect, such that any indecomposable representation Y of Q is preprojective, regular and preinjective if and only if $\partial(\dim Y)$ is negative, zero and positive, respectively (see for instance [10]). As T is regular, $\partial \widetilde{T}=\partial kQ<0$ and therefore \widetilde{T} contains an indecomposable preprojective direct summand Y, and $Y\in T^{\perp}$. Indeed, we have ${}^1[T,Y]=0$ for all direct summands of \widetilde{T} and [T,Y]=0 since Y is preprojective and T is regular [10, Theorem 3.6.5].

Example. Working out the following example, one can see that if Q is not tame, it may happen that both T and T^{\perp} belong to the set of regular representations:

As $q(\mathbf{d}) = 1$, there exists an irreducible $T \in \operatorname{rep}(Q, \mathbf{d})$ having an open orbit. The simple objects in T^{\perp} have dimension vectors

It is easy to check that these simple objects are regular representations of Q.

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References

- I. M. Bernstein, I. N. Gelfand and V. A. Ponomarev, Coxeter functors and Gabriel theorem, Russ. Math. Surveys 28 (1973), 17–32.
- [2] K. Bongartz, Tilted algebras, in: Representations of Algebras, Lecture Notes in Math. 903 (1981), 26–38.
- [3] K. Bongartz, Minimal singularities for representations of Dynkin quivers, *Comment. Math. Helv.* **63** (1994), 575–611.
- [4] M. Butler and C. M. Ringel, Auslander–Reiten sequences with few middle terms and applications to string algebras, Comm. Algebra 15 (1987), 145–179.
- [5] C. Chang and J. Weyman, Representations of quivers with free module of covariants, Preprint.
- [6] P. Gabriel, Représentations indécomposables, Séminaire Bourbaki 1973/74, Lecture Notes in Math. 431 (1975), 143–169.
- [7] H. Kraft and Ch. Riedtmann, Geometry of representations of quivers, LMS lecture notes 116 (1985), 109–147.
- [8] Ch. Riedtmann and G. Zwara, On the zero set of semi-invariants for quivers, Preprint 2002, http://www.mat.uni.torun.pl/~gzwara/semii1.ps.
- [9] C. M. Ringel, The rational invariants of tame quivers, Inv. Math. 58 (1980), 217–239.
- [10] C. M. Ringel, Tame algebras and integral quadratic forms, Lecture Notes in Math. 1099, Springer Verlag, 1984.
- [11] M. Sato and T. Kimura, A classification of irreducible prehomogeneous vector spaces and their relative invariants, Nagoya J. Math. 65 (1977), 1–155.
- [12] A. Schofield, Semi-invariants of quivers, J. London Math. Soc. 43 (1991), 385–395.
- [13] G. W. Schwarz, Lifting smooth homotopies of orbit spaces, Inst. Hautes Études Sci. Publ. Math. 51 (1980), 37–135.

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