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Kazhdan's property (T), L^2-spectrum and isoperimetric inequalities for locally symmetric spaces

Enrico Leuzinger

Abstract. Let $V = \Gamma\backslash G/K$ be a Riemannian locally symmetric space of nonpositive sectional curvature and such that the isometry group G of its universal covering space has Kazhdan's property (T). We establish strong dichotomies between the finite and infinite volume case. In particular, we characterize lattices (or, equivalently, arithmetic groups) among discrete subgroups $\Gamma \subset G$ in various ways (e.g., in terms of critical exponents, the bottom of the spectrum of the Laplacian and the behaviour of the Brownian motion on V).

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1. Introduction

In the study of discrete subgroups of Lie groups, lattices (i.e., discrete subgroups of finite covolume) and their remarkable properties have attracted most attention. Among the deepest results are the strong rigidity theorem of G. D. Mostow and the superrigidity and arithmeticity theorems of G. A. Margulis for lattices in semisimple Lie groups of higher rank (see e.g. [14], [23] and [25]). In contrast only very little is known about general discrete subgroups (except, of course, the elaborated theory of Fuchsian and Kleinian groups). We refer to [4], [13], [18] and [24] for interesting examples of discrete subgroups of Lie groups of higher rank, which have infinite covolume and which are not subgroups of lattices and in particular not arithmetic.

It is a general idea that lattices are isolated (in various ways) in the set of all discrete subgroups of a given semisimple Lie group. Moreover this isolation should become stronger if the complexity of the ambient group is increasing. The present paper supports that philosophy.

A basic measure for the size of a discrete group acting on a metric space is the exponent of growth or critical exponent. We next define that notion in the case of discrete subgroups of semisimple Lie groups acting isometrically on symmetric spaces. Consider a connected, noncompact, semisimple Lie group G with finite center. Associated to G is the (globally) symmetric space $X = G/K$, where K is a chosen maximal compact subgroup of G. Equipped with a left-invariant Riemannian metric X is a Hadamard manifold (i.e., a complete, simply connected Riemannian manifold of nonpositive sectional curvature). Let Γ be a torsion-free, discrete subgroup of G. Then Γ acts isometrically and properly discontinuously on X and the resulting quotient space $V = \Gamma\backslash X$ is locally symmetric (i.e., the geodesic symmetry at each of its points is a local isometry). For $x \in X$ let $B_R(x)$ be the closed ball of radius R in $X = G/K$ centered at x. We denote by $N(x, y; R) := |B_R(x)) \cap \Gamma \cdot y|$ the number of orbit points of $y \in X$ under Γ contained in $B_R(x)$. The *critical exponent of* Γ is then defined as

$$\delta(\Gamma) := \limsup_{R\to\infty} \frac{1}{R} \log N(x, y; R).$$

This number is independent of the chosen points $x, y \in X$. We remark that if Γ is a lattice, i.e., $\mathrm{vol}(\Gamma\backslash X) < \infty$, then the critical exponent $\delta(\Gamma)$ is equal to the volume entropy of V (or of its universal covering space X). The latter is defined as $\limsup_{R\to\infty} R^{-1} \log \mathrm{vol}(B_R(x))$, where $B_R(x)$ is as above with arbitrary $x \in X$.

The present paper was motivated by the following remarkable *rigidity phenomenon* discovered by K. Corlette for discrete subgroups of the isometry groups of quaternionic hyperbolic spaces $H^n\mathbb{H}, n \geq 2$, and the Cayley hyperbolic plane $H^2\mathbb{C}a$ respectively.

Theorem (K. Corlette, [10]). (i) *If* Γ *is a discrete group of isometries of* $H^n\mathbb{H}$, $n \geq 2$, *with the metric normalized such that the sectional curvature is pinched between* -1 *and* -4, *then* $\delta(\Gamma) = 4n + 2$ *or* $\delta(\Gamma) \leq 4n$. *If* $\delta(\Gamma) = 4n + 2$, *then* Γ *is a lattice.*

(ii) *If* Γ *is a discrete group of isometries of* $H^2\mathbb{C}a$ *with the metric normalized such that the sectional curvature is pinched between* -1 *and* -4, *then* $\delta(\Gamma) = 22$ *or* $\delta(\Gamma) \leq 16$. *If* $\delta(\Gamma) = 22$, *then* Γ *is a lattice.*

The isometry groups of $H^n\mathbb{H}, n \geq 2$, and of $H^2\mathbb{C}a$ are precisely the noncompact simple Lie groups of real *rank one* which have Kazhdan's property (T). Corlette's result roughly says that property (T) has the effect of creating a gap in the possible sizes of discrete subgroups measured by the critical exponent. Lattices are exactly the discrete subgroups on one side of the gap. Corlette uses property (T) to show first that there is a similar gap in the L^2-spectrum of the Laplace–Beltrami operator of the locally symmetric space corresponding to a discrete group Γ: either zero belongs to the spectrum in which case Γ is a lattice, or the spectrum has a strictly positive lower bound $c > 0$ (independent of Γ). Generalizing a formula of Patterson–Sullivan he then relates the bottom of the spectrum to the critical exponent of Γ and also to the Hausdorff dimension of the limit set (see [10]).

For a lattice in the isometry group of a quaternionic or Cayley hyperbolic space the above number $c > 0$ turns out to be also a lower bound for the non-zero spectrum of the Laplacian. This qualitatively generalizes the celebrated estimate of A. Selberg which asserts that the bottom λ_1 of the non-zero spectrum of the Laplacian Δ on the Riemann surface $\Gamma(n)\backslash SL(2,\mathbb{R})/SO(2)$ satisfies $\lambda_1 \geq \frac{3}{16}$ for any congruence subgroup $\Gamma(n) \subset SL(2,\mathbb{Z})$. It is conjectured that actually $\lambda_1 \geq \frac{1}{4}$ (see [31] for a recent discussion). Qualitative estimates of λ_1 of Δ for finite volume locally symmetric spaces corresponding to lattices with property (T) have been obtained by R. Brooks, A. Lubotzky, R. Zimmer and others (see e.g. [7], [20], [21] and [22]).

A main goal of the present paper is to extend the above mentioned results to arbitrary discrete subgroups (in particular to those of *infinite* covolume) of a semisimple Lie group G, such that G has property (T). It is well-known that G has property (T) if and only if G has no simple factors locally isomorphic to $SO(n,1)$ or $SU(n,1)$ (see [16], Ch. 2, 9). Notice that a lattice $\Gamma \subset G$ also has property (T) if the ambient group G has (T), but that this need not be the case for an arbitrary discrete subgroup of G. For instance a generalized Schottky group (see [4]) is a free group and hence does not have property (T).

In the main theorem below we list several dichotomies between locally symmetric spaces of finite resp. infinite volume. We emphasize that all these dichotomies actually characterize lattices among arbitrary discrete subgroups. In order to state them we introduce some additional notation. Consider a discrete, torsion-free subgroup Γ of a semisimple Lie group G as above and let $X = G/K$ resp. $V = \Gamma\backslash G/K$ be the associated globally resp. locally symmetric spaces. By $\lambda_0(V) := \inf \mathrm{Spec}\Delta \subset [0,\infty)$ we denote the bottom of the L^2-spectrum of the Laplace–Beltrami operator Δ of V. If $\mathrm{vol}(V) < \infty$, the constants are in $L^2(V)$ and $\lambda_0(V) = 0$. We denote by $\lambda_1(V) := \inf\{\mathrm{Spec}\Delta \setminus \{0\}\}$ the bottom of the non-zero spectrum of Δ on V.

The *Poincaré series* of Γ with exponent s is defined as

$$P_s(x,y) := \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)} \qquad (x,y \in X),$$

where $d = d_X$ is the distance function associated to the Riemannian metric on X. The critical exponent of Γ defined above satisfies

$$\delta(\Gamma) = \inf\{s \in \mathbb{R} \mid P_s(x,y) < \infty\}.$$

Finally, if 2ρ denotes the sum of all positive restricted roots of G counted with multiplicity, then its norm $2\|\rho\|$ is equal to the volume-entropy of X (see section 6 for the details).

Main Theorem (Dichotomy). *Let G be a connected, semisimple Lie group with finite center, without compact factors and with* Kazhdan's property (T). *Let Γ be a discrete, torsion-free subgroup of G and let $V = \Gamma\backslash G/K$ be the associated locally symmetric space. Then there exist constants $c(G) > 0$ and $c^*(G) > 0$ depending only on G but not on Γ such that:*

(a) *The following assertions are equivalent:*

(1) $\mathrm{vol}(V) < \infty$, *i.e.*, Γ *is a lattice.*

(2) $\lambda_0(V) = 0 < c(G) \leq \lambda_1(V)$.

(3) $\delta(\Gamma) = 2\|\rho\|$.

(4) *The Poincaré series of* Γ *diverges at exponent* $2\|\rho\|$.

(5) *There does not exist a Green function on* V.

(6) *The Brownian motion on* V *is recurrent.*

(b) *The following assertions are equivalent:*

(1) $\mathrm{vol}(V) = \infty$.

(2) $0 < c(G) \leq \lambda_0(V)$.

(3) $\delta(\Gamma) \leq 2\|\rho\| - c^*(G)$.

(4) *The Poincaré series of* Γ *converges at exponent* $2\|\rho\|$.

(5) *There exists a Green function on* V.

(6) *The Brownian motion on* V *is transient.*

For semisimple groups of $\mathbb{R}$-*rank one* with property (T) the equivalence of (a.1) with (a.3) and of (b.1) with (b.3) of the theorem was previously proved by K. Corlette (see [10], Theorem 4.4.) as we mentioned above. That (a.1) implies (a.2) in the rank one case has been shown by M. Burger and V. Schroeder (see [8], p. 280 (2), (3)).

We outline the plan of the paper. We equip G with a left invariant Riemannian metric and then first work in the space $\Gamma\backslash G$ instead of $\Gamma\backslash G/K$. The reason to do this is that G acts transitively and measure preserving (though not isometrically) on $\Gamma\backslash G$ *from the right* (section 2). In section 3 we use property (T) to derive a positive lower bound of Cheeger's isoperimetric constant $h(\Gamma\backslash G)$ which is universal (i.e., independent of Γ). An inequality of J. Cheeger then yields a universal lower bound for the bottom of the (non-zero) L^2-spectrum first of $\Gamma\backslash G$ and then also for $\Gamma\backslash G/K$ (section 4). In section 5 we use an inequality of P. Buser to derive *isoperimetric inequalities* for the various locally symmetric spaces V and to estimate their volume growth. We show that locally symmetric spaces of finite volume are *expanding* while those with infinite volume are *open at infinity.* In a previous paper [19] we estimated $\lambda_0(V)$ from above and below in terms of quadratic polynomials in the critical exponent of Γ. In section 6 we use that result to deduce a new rigidity property for lattices which (qualitatively) extends the above quoted result of Corlette to higher rank. In section 7 we estimate the growth of orbital counting functions for arbitrary discrete groups $\Gamma \subset G$. Finally in section 8 we put everything together to prove the main theorem.

It is possible to deduce the spectral gap also more directly from property (T) using Casimir operators and the methods of [5]. But the present proof via Cheeger's constant is more adapted to possible generalizations. In fact, parts of the main theorem probably hold for a more general class of groups and spaces. For instance a result of R. S. Phillips, P. Sarnak and P. Doyle asserts the following: For $n \geq 3$ there is a number $d_n < n-1$ such that for any (classical) Schottky group $\Gamma \subset SO(n,1)$, the isometry group of the n-dimensional real hyperbolic space, one has $\delta(\Gamma) \leq d_n$ (see [26], [11]). Note that in this case $2\|\rho\| = n-1$.

In contrast it is well-known that assertion (a.2) is not true for (cocompact) lattices in $SO(n,1)$. In fact, by a result of B. Randol there exist compact hyperbolic manifolds in all dimensions with arbitrarily numerous small eigenvalues as close to zero as one wishes (see [29]).

2. Some elementary properties of $\Gamma\backslash G$

Let G be a connected, noncompact, semisimple Lie group without compact factors and with finite center and let K be a maximal compact subgroup of G. Let B be the Killing form on the Lie algebra $\mathfrak{g}$ of G, let θ be the Cartan involution of $\mathfrak{g}$ with respect to $\mathfrak{k}$ and set

$$g_0(X,Y) := -B(X,\theta Y) \qquad (X, Y \in \mathfrak{g}).$$

Then g_0 is a positive definite scalar product on $\mathfrak{g}$ invariant under $\mathrm{Ad}(K)$. We equip G with the *left-invariant* Riemannian metric, say g, which is equal to g_0 on $\mathfrak{g} = T_eG$. Let d_G be the associated left-invariant distance function on G and set $|x| := d_G(e,x)$ for $x \in G$.

Let Γ be a torsion-free, discrete subgroup of G. Then Γ acts freely and properly discontinuously from the left on G by isometries so that $\Gamma\backslash G$ is a manifold. We endow it with the Riemannian metric which makes $\pi : G \to \Gamma\backslash G$ into a Riemannian covering.

Lemma 1. *For any Γ as above the infimum of the Ricci curvature of $\Gamma\backslash G$ is non-positive:* $\inf \mathrm{Ric} = -\kappa^2(n-1)g$ *where $\kappa \geq 0$, $n = \dim G$ and g is the Riemannian metric.*

Proof. Since $\Gamma\backslash G$ is locally isometric to G we have $\mathrm{Ric}(\Gamma\backslash G) = \mathrm{Ric}(G)$. Since G is homogeneous $-\infty < ag \leq \mathrm{Ric} \leq bg < \infty$ and since G is not compact $a \leq 0$ by Myer's theorem (see [30], IV, 3.1). □

The distance function on $\Gamma\backslash G$ is given by

$$d_{\Gamma\backslash G}(\pi(g_1), \pi(g_2)) := \inf_{\gamma\in\Gamma} d_G(\gamma g_1, g_2).$$

We also use the action of G on $\Gamma\backslash G$ from the *right*:

$$\Gamma\backslash G \times G \longrightarrow \Gamma\backslash G; \quad (\Gamma g, h) \mapsto r_h(\Gamma g) := \Gamma g h.$$

Lemma 2. *For any $\Gamma \subset G$ and any $h \in G$ holds:*

(i) *r_h is measure preserving for the Riemannian measure on $\Gamma\backslash G$.*

(ii) *$d_{\Gamma\backslash G}(x, r_h(x)) \leq d_G(e,h) = |h| = |h^{-1}|$ for all $x \in \Gamma\backslash G$.*

Proof. (i) The Riemannian measure induced by the left-invariant Riemannian metric is a left-invariant Haar measure on G. As a semisimple Lie group G is unimodular, so that this measure is also *right-invariant* and descends to a right-invariant measure on $\Gamma\backslash G$.

(ii) For any $h \in G$ we have

$$d_{\Gamma\backslash G}(\pi(g), r_h(\pi(g))) = d_{\Gamma\backslash G}(\pi(g), \pi(gh)) = \inf_{\gamma\in\Gamma} d_G(\gamma g, gh) \leq$$
$$\leq d_G(eg, gh) = d_G(e,h) = |h|.$$

□

3. Kazhdan's property (T) and Cheeger's isoperimetric constant

Certain group-theoretic properties of locally compact groups are reflected by the so-called property (T) introduced by D. Kazhdan. Suppose that $\rho : G \longrightarrow \mathrm{Aut}\mathcal{H}$ is a unitary representation of a locally compact topological group G in a separable Hilbert space $\mathcal{H}$. For any compact subset H of G and any positive number $\varepsilon > 0$, the vector $v \in \mathcal{H}$ is said to be *(ε, H)-invariant*, if $\|\rho(h)v - v\| < \varepsilon\|v\|$ for all $h \in H$. One says that *the representation ρ contains almost invariant vectors*, if it contains (ε, H)-invariant vectors for arbitrary ε and H. A vector $v \in \mathcal{H}$ is *G-invariant* if $\rho(g)(v) = v$ for all $g \in G$. A locally compact group G has *Kazdan's property (T)*, if any one of its unitary representations, containing almost invariant vectors, contains non-zero G-invariant vectors.

It is well-known that a connected, semisimple Lie group has property (T) if and only if it has no simple factors locally isomorphic to the rank one groups $SO(n,1)$ or $SU(n,1)$ (see [16], [23], [34]).

The following proposition is reformulation of [16], 1.15 and will be most convenient for our purposes.

Proposition 1. *Let G be a connected, semisimple Lie group without compact factors and with finite center and let H be a compact neighbourhood of the identity of G (which in particular generates G). If G has property (T), then there exists a number $\varepsilon = \varepsilon(G,H) > 0$ which depends only on G and the choice of H such that the following assertion holds: If ρ is a unitary representation of G on a Hilbert space $\mathcal{H}$ which does not have nontrivial invariant vectors, then, given any $v \in \mathcal{H}$, there is an $h \in H$ such that $\|\rho(h)v - v\| \geq \varepsilon\|v\|$.*

Next consider a complete, connected Riemannian manifold M. For an open submanifold Ω of M with smooth boundary $\partial\Omega$ we denote by $V(\Omega)$ (resp. by $A(\partial\Omega)$) its n-dimensional (resp. $(n-1)$-dimensional) Riemann-Lebesgue measure. The *Cheeger isoperimetric constant of M* is defined as

$$h(M) := \inf_{\Omega} \frac{A(\partial\Omega)}{V(\Omega)}$$

where the infimum is taken over all open submanifolds Ω of M with compact closure and smooth boundary $\partial\Omega$ such that $V(\Omega) \leq \frac{1}{2}\mathrm{vol}(M)$. Notice that the last condition is automatically satisfied if M has infinite volume. A tubular neighbourhood of some subset $S \subset M$ will be denoted by $\mathcal{U}_r(S) := \{p \in M \mid d_M(p, S) \leq r\}$, $r > 0$.

The following technical result due to P. Buser is crucial for what follows.

Proposition 2. *Let M be a complete, n-dimensional Riemannian manifold whose Ricci curvature satisfies* $\mathrm{Ric} \geq -(n-1)\kappa^2$ $(\kappa \geq 0)$. *Let $\Omega \subset M$ be a relatively compact domain with $\frac{A(\partial\Omega)}{V(\Omega)} \geq h(M)$ and let C be a sufficiently large ball containing Ω (if M is compact let $C = M$). Then there are constants $c', r > 0$, depending only on $n = \dim M$, and a domain $\tilde{\Omega} \subseteq \mathcal{U}_r(\Omega)$ with boundary $\partial\tilde{\Omega}$ such that*

$$(a) \qquad V(\tilde{\Omega}) \geq \frac{1}{2}V(\Omega) \quad \text{and} \quad V(C \setminus \tilde{\Omega}) \geq \frac{1}{2}V(C \setminus \Omega)$$

$$(b) \qquad \frac{V(\mathcal{U}_t(\partial\tilde{\Omega}))}{V(\tilde{\Omega})} \leq c' e^{(n-1)\kappa t} \cdot \frac{A(\partial\Omega)}{V(\Omega)} \quad \text{for all } t \geq r.$$

Proof. Proposition 2 is just a slightly modified version of Lemma 7.2 in [9]. Since we are interested in $\inf \frac{A(\partial\Omega)}{V(\Omega)}$ we can assume without loss of generality that $\frac{A(\partial\Omega)}{V(\Omega)} \leq 1$. This in particular implies that the parameter r in Buser's proof can be chosen $\leq \frac{1}{2}c'$ (and independent of Ω). That observation yields (b). As for (a) we remark that if M is noncompact one choses a sufficiently large ball C (not necessarily homeomorphic to a euclidean ball) which contains the given Ω and for which one may assume that $\mathrm{vol}(C) \geq \frac{3}{4}\mathrm{vol}(M)$ if $\mathrm{vol}(M) < \infty$ (see [9], section 7). Assertion (a) is then a consequence of formula (4.9) and its proof in [9]. □

Theorem 1. *Let G be a semisimple Lie group without compact factors and with finite center. Assume that G has property (T). Then there exists a universal constant $c_1(G) > 0$ which depends only on G such that for any torsion-free discrete subgroup Γ the isoperimetric constant of $\Gamma\backslash G$ satisfies*

$$h(\Gamma\backslash G) \geq c_1(G) > 0.$$

3.1. Proof of Theorem 1. Part (a): the case of infinite volume

We choose a compact neighbourhood of the identity $H \subset G$ as in Proposition 1. We consider a relatively compact domain Ω in $\Gamma\backslash G$. By Lemma 1 the Ricci curvature of $\Gamma\backslash G$ has a non-positive lower bound and we can apply Proposition 2 in order to replace Ω by a domain $\tilde{\Omega}$ whose tubular neighbourhood can be estimated.

Using the (non-isometric) *right* action of G on $\Gamma\backslash G$ we define the *right regular representation* $\mathcal{R}$ of G on $L^2(\Gamma\backslash G)$ by $\mathcal{R}_g(f)(x) = f(r_g(x))$ for $f \in L^2(\Gamma\backslash G)$, $x \in \Gamma\backslash G$ and $g \in G$. The representation $\mathcal{R}$ is unitary by Lemma 2(i). Since G acts transitively on $\Gamma\backslash G$ the invariant vectors of $\mathcal{R}$ are precisely the constant L^2-functions on $\Gamma\backslash G$. Since $\mathrm{vol}(\Gamma\backslash G) = \infty$ we conclude that $\mathcal{R}$ does not have non-zero invariant vectors.

Let $\chi_{\tilde{\Omega}} \in L^2(\Gamma\backslash G)$ be the characteristic function of $\tilde{\Omega}$. By Proposition 1 there is $h \in H$ and $\varepsilon > 0$ depending only on G and H (but not on $\tilde{\Omega}$ and Γ) such that

$$\begin{aligned} &\varepsilon^2 V(\tilde{\Omega}) \\ &= \varepsilon^2 \|\chi_{\tilde{\Omega}}\|^2 \leq \|\mathcal{R}_h(\chi_{\tilde{\Omega}}) - \chi_{\tilde{\Omega}}\|^2 = \int_{\Gamma\backslash G} |\chi_{\tilde{\Omega}}(r_h(x)) - \chi_{\tilde{\Omega}}(x)|^2 d\mu(x) = V(E \cup F) \end{aligned}$$

where $E := \{x \in \Gamma\backslash G \mid x \in \tilde{\Omega}, r_h(x) \notin \tilde{\Omega}\}$ and $F := \{x \in \Gamma\backslash G \mid r_h(x) \in \tilde{\Omega}, x \notin \tilde{\Omega}\}$.

We claim that the sets E and F are contained in the $|h|$-neighbourhood $\mathcal{U}_{|h|}(\partial\tilde{\Omega})$ of the boundary $\partial\tilde{\Omega}$. To see this pick $x \in E$, i.e., $x \in \tilde{\Omega}$ and $r_h(x) \notin \tilde{\Omega}$. Assume that $d_{\Gamma\backslash G}(x, \partial\tilde{\Omega}) \geq |h| + 2a$ for some $a > 0$. Then the ball $B_{|h|+a}(x)$ is contained in $\tilde{\Omega}$. Since $d_{\Gamma\backslash G}(x, r_h(x)) \leq |h|$ by Lemma 2(ii) we have $r_h(x) \in \tilde{\Omega}$, which is a contradiction. The proof for F is similar.

So far we have

$$\varepsilon^2 V(\tilde{\Omega}) \leq V(\mathcal{U}_{|h|}(\partial\tilde{\Omega})). \tag{1}$$

From Proposition 2 (b) we get

$$V(\mathcal{U}_{|h|}(\partial\tilde{\Omega})) \leq c' e^{(n-1)\kappa|h|} V(\tilde{\Omega}) \frac{A(\partial\Omega)}{V(\Omega)}$$

where the constant $c' > 0$ depends only on $\dim \Gamma\backslash G = \dim G$.

If we set $|H| := \max_{h\in H} |h| = \max_{h\in H} d_G(e, h)$ for the compact subset H then we have

$$V(\mathcal{U}_{|h|}(\partial\tilde{\Omega})) \leq \mathrm{const} \cdot V(\tilde{\Omega}) \frac{A(\partial\Omega)}{V(\Omega)}. \tag{2}$$

where the positive constant $\mathrm{const} = \mathrm{const}(G, |H|)$ depends only on G and the size of the chosen compact subset $H \subset G$. Combining (1) and (2) we obtain

$$0 < \frac{\varepsilon^2}{\mathrm{const}} \leq \frac{A(\partial\Omega)}{V(\Omega)}$$

which eventually yields

$$0 < \frac{\varepsilon^2(G, |H|)}{\mathrm{const}(G, |H|)} \leq \inf \frac{A(\partial\Omega)}{V(\Omega)} = h(\Gamma\backslash G).$$

This concludes the proof of theorem 1 in case that $\mathrm{vol}(\Gamma\backslash G) = \infty$.

3.2. Proof of Theorem 1: (b) the case of lattices

The idea of the proof is the same as in (a) with the following two modifications. Firstly, the constants are the (only) invariant vectors in $L^2(\Gamma\backslash G)$ for the right regular representation. We thus have to consider the subspace $L_0^2(\Gamma\backslash G)$ orthogonal to the constant functions. Secondly, we recall that in the case at hand the Cheeger constant is

$$h(\Gamma\backslash G) = \inf_{\Omega} \frac{A(\partial\Omega)}{V(\Omega)}$$

where the infimum is taken over all relatively compact domains Ω of $\Gamma\backslash G$ with $V(\Omega) \leq \frac{1}{2}\mathrm{vol}(\Gamma\backslash G) =: v < \infty$.

We now proceed as in (a). Using Proposition 2 we first replace Ω with $\frac{A(\partial\Omega)}{V(\Omega)}$ close to $h(\Gamma\backslash G)$ by a relatively compact domain $\tilde{\Omega}$ whose tubular neighbourhoods can be estimated. We then set

$$a := V(\tilde{\Omega}) \qquad b := v - a \quad (v = \mathrm{vol}(\Gamma\backslash G) < \infty)$$

and define

$$f : \Gamma\backslash G \longrightarrow \mathbb{R}; \quad f(x) := \begin{cases} b & \text{if } x \in \tilde{\Omega} \\ -a & \text{if } x \notin \tilde{\Omega}. \end{cases}$$

We compute

$$\int_{\Gamma\backslash G} f\,d\mu = bV(\tilde{\Omega}) + (-a)(v - V(\tilde{\Omega})) = 0 \quad \text{and} \quad \int_{\Gamma\backslash G} f^2 d\mu = (a+b)ab = vab < \infty,$$

i.e., $f \in L_0^2(\Gamma\backslash G)$. Since the right regular representation $\mathcal{R}$ of G on $L_0^2(\Gamma\backslash G)$ has no nontrivial invariant vectors, Proposition 1 asserts that there is $h \in H$ such that

$$\varepsilon^2 \|f\|^2 \leq \|\mathcal{R}_h f - f\|^2.$$

But $\|\mathcal{R}_h f - f\|^2 = (a+b)^2 V(E \cup F)$ where as in (a) $E := \{x \in \Gamma\backslash G \mid x \in \tilde{\Omega}, r_h(x) \notin \tilde{\Omega}\}$ and $F := \{x \in \Gamma\backslash G \mid r_h(x) \in \tilde{\Omega}, x \notin \tilde{\Omega}\}$ and $E \cup F \subset \mathcal{U}_{|h|}(\partial\tilde{\Omega})$. Proposition 2 (b) and the compactness of H yield a constant const $=$ const$(G, |H|) > 0$ depending only on G and H such that

$$\varepsilon^2 abv \leq (a+b)^2 \mathrm{const} \cdot V(\tilde{\Omega}) \frac{A(\partial\Omega)}{V(\Omega)}$$

or, equivalently,

$$\frac{\varepsilon^2}{\mathrm{const}} \cdot \frac{v - V(\tilde{\Omega})}{v} \leq \frac{A(\partial\Omega)}{V(\Omega)}.$$

In the proof of Proposition 2 one choses a large ball C which contains the given Ω (if $\Gamma\backslash G$ is compact, set $C = \Gamma\backslash G$) and for which one can assume that $V(C) \geq \frac{3}{4}v$ (where $v = \mathrm{vol}(\Gamma\backslash G)$). On the other hand by Proposition 2 (a) for such a C holds $V(C \setminus \tilde{\Omega}) \geq \frac{1}{2}V(C \setminus \Omega)$ and and since we also have $V(\Omega) \leq \frac{1}{2}v$ we find

$$v - V(\tilde{\Omega}) \geq V(C) - V(\tilde{\Omega}) \geq \frac{1}{2}(V(C) - V(\Omega)) \geq \frac{1}{2}\left(\frac{3}{4}v - \frac{1}{2}v\right) = \frac{1}{8}v.$$

In conclusion we obtain

$$0 < \frac{\varepsilon^2}{8 \cdot \mathrm{const}} \leq \frac{A(\partial\Omega)}{V(\Omega)}.$$

This completes the proof also in the case of lattices.

4. A lower bound for the bottom of the spectrum

For a complete Riemannian manifold M we define the *Laplace–Beltrami operator* on C^∞-functions with compact support on M by $\Delta_M := -\mathrm{div}\ \mathrm{grad}$. This operator has a unique selfadjoint extension to $L^2(M)$ which we also denote by Δ_M (see e.g. [32] for details). Let $\mathrm{Spec}\Delta_M$ denote the L^2-spectrum of Δ_M and define the *bottom of the spectrum* $\lambda_0(M) := \inf \mathrm{Spec}\Delta_M \geq 0$. Finally denote by $\lambda_1(M) := \inf\{\mathrm{Spec}\Delta_M \setminus \{0\}\}$ the bottom of the non-zero spectrum of Δ on M.

Theorem 2. *Let G be a semisimple Lie group G without compact factors, with finite center and with property (T). Then there exists a constant $c_2(G) > 0$, which depends only on G, such that for any torsion-free, discrete subgroup Γ of G the following holds:*

(a) *If* $\mathrm{vol}(\Gamma\backslash G) = \infty$, *then the bottom of the L^2-spectrum of the Laplace–Beltrami operator on $\Gamma\backslash G$ satisfies $\lambda_0(\Gamma\backslash G) \geq c_2(G) > 0$.*

(b) *If* $\mathrm{vol}(\Gamma\backslash G) < \infty$, *then the bottom of the non-zero spectrum of the Laplace–Beltrami operator on $\Gamma\backslash G$ satisfies $\lambda_1(\Gamma\backslash G) \geq c_2(G) > 0$.*

Proof. According to Cheeger's inequality (see e.g. [9]) we have $\frac{1}{4}h(\Gamma\backslash G)^2 \leq \lambda_i(\Gamma\backslash G)$, where $i = 0$ if $\mathrm{vol}(\Gamma\backslash G) = \infty$ and $i = 1$ if $\mathrm{vol}(\Gamma\backslash G) < \infty$. Theorem 2 is thus a direct consequence of these inequalities and Theorem 1. □

We can now state a main result in which we pass from $\Gamma\backslash G$ to the locally symmetric space $V = \Gamma\backslash G/K$. Note that the Riemannian structure on G/K is given by the restriction $B\mid_{\mathfrak{p}} = g_0\mid_{\mathfrak{p}}$ of the Killing form B of the Lie algebra $\mathfrak{g}$ of G to $\mathfrak{p} = \mathfrak{g}/\mathfrak{k} \cong T_{eK}G/K$ (see section 2). In particular, the canonical projection $G \longrightarrow G/K$ is a Riemannian submersion.

Theorem 3. *Let G be a semisimple Lie group G without compact factors, with finite center and with property (T). Then there exists a constant $c(G) > 0$, which depends only on G, such that for any torsion-free, discrete subgroup Γ of G with asssociated locally symmetric space $V = \Gamma\backslash G/K$ one has:*

(a) *If* $\mathrm{vol}(V) = \infty$, *then the bottom of the L^2-spectrum of the Laplace–Beltrami operator on V satisfies $\lambda_0(V) \geq c(G) > 0$.*

(b) *If* $\mathrm{vol}(V) < \infty$, *then the bottom of the non-zero spectrum of the Laplace–Beltrami operator on V satisfies $\lambda_1(V) \geq c(G) > 0 = \lambda_0(V)$.*

Proof. It is well-known that $L^2(\Gamma\backslash G/K) = L^2(V)$ is isomorphic to the K-fixed part $L^2(\Gamma\backslash G)^K$ of $L^2(\Gamma\backslash G)$ by averaging over K-orbits:

$$f \in C_0^\infty(\Gamma\backslash G) \mapsto f^*(\Gamma g K) := \int_K f(\Gamma g k)dk$$

(for more details see e.g. [33]). Moreover by Fubini's theorem for Riemannian submersions (see [30], Theorem 5.6) we have for any $f \in C_0^\infty(\Gamma\backslash G)^K$

$$\|f\|_2^{\Gamma\backslash G} = \|f^*\|_2^V \cdot \int_K dk.$$

The variational definition of $\lambda_0(V)$ (resp. $\lambda_1(V)$) via Rayleigh quotients together with the above observations about the L^2-norms on $\Gamma\backslash G$ and V respectively then yields

$$\lambda_0(V) = \inf_{f^*\in C_0^\infty(V), f^*\neq 0} \frac{\|\nabla f^*\|_2}{\|f^*\|_2} \geq \lambda_0(\Gamma\backslash G) = \inf_{f\in C_0^\infty(\Gamma\backslash G), f\neq 0} \frac{\|\nabla f\|_2}{\|f\|_2}.$$

By Theorem 1 the last term is bounded from below by $c_2(G) > 0$ which completes the proof. □

Theorem 3 yields (universal) estimates for the volume growth of locally symmetric spaces $V = \Gamma\backslash G/K$. For $v \in V$ let $B_R(v)$ be the ball of radius R in V centered at v. We set

$$\mu(V) := \limsup_{R\to\infty} \frac{1}{R} \log \operatorname{vol}(B_R(v)).$$

This number μ is independent of v and called the exponential growth of V.

Corollary 1. *There is a positive constant $\hat{c}(G) > 0$ depending only on G such that for any $V = \Gamma\backslash G/K$ as above with* $\operatorname{vol}(V) = \infty$ *holds*

$$0 < \hat{c}(G) \leq \mu(V) \leq 2\|\rho\|.$$

Proof. By a result of R. Brooks $\lambda_0(V) \leq \frac{1}{4}\mu(V)^2$ (see [6], Thm. 1). The lower bound thus follows from Theorem 3. The upper bound follows from Proposition 4 below. □

5. Isoperimetric inequalities for locally symmetric spaces

Let M be a complete Riemannian manifold. If the Cheeger isoperimetric constant satsfies $h(M) \geq c > 0$ for some positive constant c and if $\operatorname{vol}(M) = \infty$ (resp. $\operatorname{vol}(M) < \infty$), then M is called *open at infinity* (resp. *expanding*).

Theorem 4. *Let $V = \Gamma\backslash G/K$ be a locally symmetric space with nonpositive sectional curvature and such that G has Kazdan's property (T). Then there is a positive constant $c^*(G) > 0$ depending only on G such that the Cheeger isoperimetric* *constant satisfies*

$$0 < c^*(G) \leq h(V);$$

and consequently:

(a) *If* $\mathrm{vol}(V) = \infty$*, then* V *is open at infinity.*

(b) *If* $\mathrm{vol}(V) < \infty$*, then* V *is expanding.*

Proof. It is well-known that the Ricci curvature of V (which is equal to the Ricci curvature of G/K) is bounded from below by a strictly negative constant, say $-(\kappa^*)^2(n-1) < 0$, $n = \dim V$ (see e.g. [12], 2.14). Inequalities of Buser (see [9], Theorem 1.2, Theorem 7.1) then assert that there are constants α and β depending only on $n = \dim V$ such that $\lambda_0(V) \leq \alpha\kappa^* h(V)$ if V is noncompact of infinite volume, that $\lambda_1(V) \leq \alpha\kappa^* h(V)$ if V is noncompact of finite volume and that $\lambda_1(V) \leq \beta(\kappa^* h(V) + h(V)^2)$ if V is compact.

From Theorem 3 we know that $\lambda_0(V) > c > 0$ if $\mathrm{vol}(V) = \infty$ resp. $\lambda_1(V) > c > 0$ if $\mathrm{vol}(V) < \infty$. Hence by the above inequalities there is a constant $c^*(G)$ such that

$$0 < c^*(G) \leq h(V)$$

and assertions (a) and (b) follow. □

6. Critical exponents and a rigidity result

In section 1 we defined the critical exponent of a discrete subgroup Γ of a semisimple Lie group G as

$$\delta(\Gamma) := \limsup_{R\to\infty} \frac{1}{R} \log N(x, y; R)$$

where $N(x, y; R) := |B_R(x)) \cap \Gamma \cdot y|$ is the number of orbit points of y contained in the closed ball $B_R(x))$ in the globally symmetric space $X = G/K$. This number $\delta(\Gamma)$ does not depend on $x, y \in X$ but only on Γ (see [19]). The *Poincaré series* of a discrete subgroup Γ of G is defined as

$$P_s(x, y) := \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)}, \qquad x, y \in X,$$

where $d = d_X$ is the distance function associated to the Riemannian metric on $X = G/K$. One can show (see e.g. [19]) that

$$\delta(\Gamma) = \inf\{s \in \mathbb{R} \mid P_s(x, y) < \infty\}.$$

The next proposition is proved in [19]; it relates $\delta(\Gamma)$ to the bottom $\lambda_0(V)$ of the L^2-spectrum of the Laplace–Beltrami operator of $V = \Gamma\backslash G/K$. In order to state it we need some additional notations.

Let $\mathfrak{g}$ be the Lie algebra of the semisimple group G, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition and let $\mathfrak{a}$ be a maximal abelian subalgebra in $\mathfrak{p}$. Let further Σ^+

denote the set of positive restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$ with respect to some Weyl chamber $\mathfrak{a}^+$ (see e.g. [15], Chapter VI, for definitions). We also denote by $m_\alpha = \dim \mathfrak{g}_\alpha$ the multiplicity of $\alpha \in \Sigma^+$ and set $\rho := \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha \in \mathfrak{a}^*$. Since $\mathfrak{g}$ is semisimple the Killing form $\langle\ ,\ \rangle$ is non-degenerate and defines an isomorphism $\mathfrak{a}^* \simeq \mathfrak{a}; \alpha \mapsto \vec{\alpha}$ by $\langle \vec{\alpha}, H\rangle := \alpha(H)$ for all $H \in \mathfrak{a}$; finally we set $\langle \alpha, \beta\rangle := \langle \vec{\alpha}, \vec{\beta}\rangle$. Let $\overline{\mathfrak{a}^+}(1)$ be the set of all unit vectors in the closed Weyl chamber $\overline{\mathfrak{a}^+}$ and set $\rho_{\min} := \min_{H\in\overline{\mathfrak{a}^+}(1)} \rho(H)$. Note that $\max_{H\in\overline{\mathfrak{a}^+}(1)} \rho(H) = \|\rho\|$ since $\vec{\rho} \in \mathfrak{a}^+$ (see [19]).

Proposition 3. *Let Γ be a torsion-free, discrete subgroup of a semisimple Lie group G without compact factors and with finite center and $V = \Gamma\backslash G/K$ the associated locally symmetric space. Then*

$$\begin{array}{rll} \lambda_0(V) = \|\rho\|^2 & & \text{if } \delta(\Gamma) \in [0, \rho_{\min}] \\ \|\rho\|^2 - (\delta(\Gamma) - \rho_{\min})^2 \leq \lambda_0(V) \leq \|\rho\|^2 & & \text{if } \delta(\Gamma) \in [\rho_{\min}, \|\rho\|] \\ \max\{0; \|\rho\|^2 - (\delta(\Gamma) - \rho_{\min})^2\} \leq \lambda_0(V) \leq \|\rho\|^2 - (\delta(\Gamma) - \|\rho\|)^2 & & \text{if } \delta(\Gamma) \in [\|\rho\|, 2\|\rho\|]. \end{array}$$

For the proof of Proposition 3 we refer to [19]. The idea can be roughly described as follows. In order to determine the bottom of the spectrum of the laplacian Δ_V on $V = \Gamma\backslash G/K$ one looks at the resolvent $(\lambda \mathrm{Id} - \Delta_V)^{-1}$. Its kernel is given by the λ-Green function of V. The latter can be written as a series, $\sum_{\gamma\in\Gamma} G_\lambda(x, \gamma y)$, where G_λ is the λ-Green function of X. This series is then compared with the Poincaré series of Γ using the estimates on G_λ given in [2].

A basic idea is that lattices are isolated in various ways from other discrete subgroups. The following rigidity result is a quantitative manifestation of this philosophy.

Theorem 5. *Let G be a connected semisimple Lie group without compact factors, with finite center and with property (T). Let $X = G/K$ be the associated symmetric space. Let Γ be a discrete, torsion-free subgroup of G and $\Gamma\backslash X$ the associated locally symmetric space, then the following holds:*

(a) *If Γ is a lattice, i.e., $\mathrm{vol}(\Gamma\backslash X) < \infty$, then $\delta(\Gamma) = 2\|\rho\|$.*

(b) *If $\mathrm{vol}(\Gamma\backslash X) = \infty$, then there is a positive constant $c^*(G) > 0$ depending only on G but not on Γ such that $\delta(\Gamma) \leq 2\|\rho\| - c^*(G)$.*

Proof. (a) is well-known (see e.g. [1] or [19]).
In order to prove (b) we may assume that $\delta(\Gamma) > \|\rho\|$ (otherwise we may set $c^* := \|\rho\| > 0$). By Proposition 3 and Theorem 3 (a) we have

$$0 < c(G) \leq \lambda_0(\Gamma\backslash X) \leq \|\rho\|^2 - (\delta(\Gamma) - \|\rho\|)^2$$

and therefore

$$0 < (\delta(\Gamma) - \|\rho\|)^2 \leq \|\rho\|^2 - c(G),$$

which gives

$$\delta(\Gamma) \leq \sqrt{\|\rho\|^2 - c(G)} + \|\rho\| =: 2\|\rho\| - c^*(G).$$

$\square$

Remark 1. The above rigidity phenomenon was discovered by K. Corlette for quotients of quaternionic (resp. Cayley) hyperbolic spaces. He also related the critical exponent to the Hausdorff dimension of the limit set (see [10] and section 1). The fact that Corlette's results are much more precise than those of Theorem 5 is a consequence of explicit information about the unitary dual which is due to B. Kostant.

Remark 2. It is known that $\lambda_0(G/K) = \|\rho\|^2$ (see e.g. [19]). Together with Theorem 3 and its proof this fact may be used to obtain upper bounds for Kazhdan constants: For $\Gamma = \{\mathrm{id}\}$ we have

$$\frac{1}{4}\left(\frac{\varepsilon^2}{\mathrm{const}(G, |H|)}\right)^2 \leq \frac{1}{4}h(G)^2 \leq \lambda_0(G) \leq \lambda_0(G/K) = \|\rho\|^2$$

(for the last equality see e.g. [19]) and hence $\varepsilon^2 \leq \mathrm{const}(G, |H|)2\|\rho\|$.

7. Estimates of orbital counting functions

In this section we generalize certain results which are well-known for hyperbolic manifolds of constant negative curvature (see e.g. [28]).

The following proposition is crucial for this section. For a proof see e.g. [17].

Proposition 4. *The volume of a ball $B_R(x)$ of radius R and center x in the globally symmetric space $X = G/K$ satisfies*

$$\mathrm{const}_1 \cdot R^{\frac{r-1}{2}} e^{2\|\rho\|R} \leq \mathrm{vol}(B_R(x)) \leq \mathrm{const}_2 \cdot R^{\frac{r-1}{2}} e^{2\|\rho\|R},$$

where $r = \mathrm{rank}\, X$ and the constants $\mathrm{const}_1, \mathrm{const}_2$ depend only on X.

Recall that, for a discrete subgroup Γ of G, $N(x, y; R) := |B_R(x)) \cap \Gamma \cdot y|$ denotes the number of orbit points of y which are contained in the ball $B_R(x) \subset X$.

Lemma 3. *Let Γ be a discrete, torsion-free subgroup of G acting isometrically on the symmetric space $X = G/K$ of rank r. Then there is a constant A_1 depending only on Γ such that for any $x, y \in X$*

$$N(x, y; R) < A_1 R^{\frac{r-1}{2}} e^{2\|\rho\|R}.$$

Proof. Let $B_\varepsilon(y)$ be a ball centered at y of radius $\varepsilon > 0$ so small that no two

Γ-images overlap. Then by Proposition 4 we have

$$\begin{aligned}\operatorname{vol}(B_\varepsilon(y))N(x,y;R) < \operatorname{vol}(B_{R+\varepsilon}(x)) \le \\ \le \operatorname{const}_2 \cdot (R+\varepsilon)^{\frac{r-1}{2}} e^{2\|\rho\|(R+\varepsilon)} \le \operatorname{const}' \cdot R^{\frac{r-1}{2}} e^{2\|\rho\|R}\end{aligned}$$

and the claim follows. $\square$

Lemma 4. *Let X be as above and let Γ be a lattice, i.e., $\operatorname{vol}(\Gamma\backslash X) < \infty$. Then for given $x, y \in X$ there is a constant A_2 depending only on x, y and on Γ and a positive real number R_0 such that for all $R > R_0$*

$$N(x,y;R) > A_2 R^{\frac{r-1}{2}} e^{2\|\rho\|R}.$$

Proof. Consider the ball $B_R(x)$. Let D be the Dirichlet-region of Γ centered at x and set $D(R) := D \cap B_R(x)$. Since $\operatorname{vol}(D) < \infty$, for $\varepsilon > 0$ we may find R_1 such that for all $R > R_1$

$$\operatorname{vol}(D \setminus D(R)) < \varepsilon. \tag{3}$$

Then for $R > R_1$

$$\operatorname{vol}(B_R(x)) = \operatorname{vol}(B_R(x) \cap \Gamma D(R_1)) + \operatorname{vol}(B_R(x) \cap \Gamma(D \setminus D(R_1))) \tag{4}$$

and the second term on the right is less or equal to

$$\int_{D\setminus D(R_1)} N(x,z;R)\,dv(z) < \varepsilon A_1 R^{\frac{r-1}{2}} e^{2\|\rho\|R} \tag{5}$$

where the estimate follows from Lemma 3 and (3). Using (5) and Proposition 4 we get from (4) that

$$\operatorname{vol}(B_R(x) \cap \Gamma D(R_1)) > \operatorname{const}_1 R^{\frac{r-1}{2}} e^{2\|\rho\|R} - \varepsilon A_1 R^{\frac{r-1}{2}} e^{2\|\rho\|R} = A_1' R^{\frac{r-1}{2}} e^{2\|\rho\|R}$$

for some positive A_1' provided ε was chosen small enough.

By choosing R_1 large enough we can assume that $y \in D(R_1)$. The Γ-images of $D(R_1)$ are disjoint and if one of them meets $B_R(x)$ then the corresponding Γ-image of y must lie in $B_{R+2R_1}(x)$ and thus we have

$$\operatorname{vol}(D(R_1))N(x,y;R+2R_1) > A_1'(R+2R_1)^{\frac{r-1}{2}} e^{2\|\rho\|(R+2R_1)}.$$

Hence there is a positive constant A_2 such that

$$N(x,y;\hat{R}) > A_2 \hat{R}^{\frac{r-1}{2}} e^{2\|\rho\|\hat{R}}$$

for $\hat{R}$ sufficently large. $\square$

Lemma 5. *For Γ as in Lemma 3 and $s > 2\|\rho\|$ we have*

$$\sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x)} < \infty \qquad (x \in X).$$

Proof. We consider the partial sum

$$\sum_{\substack{\gamma\in\Gamma \\ d(x,\gamma x)<R}} e^{-sd(x,\gamma x)} = \int_0^R e^{-st} dN(x,x;t)$$

$$= N(x,x;R)e^{-sR} + s\int_0^R N(x,x;R)e^{-st}dt.$$

The claim then follows immediately from Lemma 3. □

Theorem 6. *Let G be a semisimple Lie group without compact factors, with finite center and with property (T). Let $X = G/K$ be the associated symmetric space (of rank r) and let Γ be a torsion-free, discrete subgroup of G.*

(a) *If* $\mathrm{vol}(\Gamma\backslash X) = \infty$*, then the Poincaré series of Γ converges at the exponent $2\|\rho\|$, i.e.,*

$$P_{2\|\rho\|}(x_0,x_0) = \sum_{\gamma\in\Gamma} e^{-2\|\rho\| d(x_0,\gamma x_0)} < \infty.$$

(b) *If Γ is a lattice, i.e.,* $\mathrm{vol}(\Gamma\backslash X) < \infty$*, then the Poincaré series of Γ diverges at the exponent* $2\|\rho\|$*, i.e.,*

$$P_{2\|\rho\|}(x_0,x_0) = \sum_{\gamma\in\Gamma} e^{-2\|\rho\| d(x_0,\gamma x_0)} = \infty.$$

Proof. (a) follows from Theorem 5 (b) and the very definition of the critical exponent $\delta(\Gamma)$.

(b) is a consequence of Lemma 4:

$$\sum_{\gamma\in\Gamma} e^{-2\|\rho\| d(x_0,\gamma x_0)} = \int_0^R e^{-2\|\rho\| t} dN(x_0,x_0;t) =$$

$$= N(x_0,x_0;R)e^{-2\|\rho\| R} + 2\|\rho\| \int_0^R e^{-2\|\rho\| t} N(x_0,x_0;t)dt >$$

$$> 2\|\rho\| A_2 \int_0^R t^{\frac{r-1}{2}} dt \longrightarrow \infty \quad (R\longrightarrow\infty).$$

□

8. Proof of the main theorem

In this section we complete the proof of the main theorem (MT) in section 1, which summarizes the various dichotomies between locally symmetric spaces of finite and infinite volume.

The equivalences MT: (a.1) $\Longleftrightarrow$ (a.2) resp. MT: (b.1) $\Longleftrightarrow$ (b.2) hold by Theorem 3.

The equivalences MT: (a.1) $\iff$ (a.3) resp. MT: (b.1) $\iff$ (b.3) follow directly from Theorem 5.

The equivalences MT: (a.1) $\iff$ (a.4) resp. MT: (b.1) $\iff$ (b.4) hold by Theorem 6.

The equivalences MT: (a.5) $\iff$ (a.6) resp. MT: (b.5) $\iff$ (b.6) are well-known (see e.g. [3]).

It remains to prove MT: (a.1) $\iff$ (a.6) resp. MT: (b.1) $\iff$ (b.6).

Assume first that (b.1) holds, i.e., $\mathrm{vol}(V) = \infty$. By the above we then have (b.2): $\lambda_0(V) > c > 0$. This implies that the Brownian motion on V is transient (see e.g. [27]) and hence there exists a Green function on $V = \Gamma\backslash X$. The latter can be written as a (convergent) series $\sum_{\gamma\in\Gamma} G_0(x, \gamma y)$ where G_0 is the Green function of X. From the explicit form of the latter (see [2]) one deduces that the Green function on V dominates the Poincaré series of Γ at exponent $2\|\rho\|$ (see [19] 3.3 estimate $(**)$ for the details). Hence (b.4) holds. By the equivalences already shown above this implies (b.1): $\mathrm{vol}(V) = \infty$. Thus we have shown that (b.1) $\iff$ (b.6). Since (a.1) $\iff$ (a.6) if and only if (b.1) $\iff$ (b.6), the proof of the main theorem is complete.

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