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Some geometric properties of the Bakry–Émery–Ricci tensor

John Lott

Abstract. The Bakry–Émery tensor gives an analog of the Ricci tensor for a Riemannian manifold with a smooth measure. We show that some of the topological consequences of having a positive or nonnegative Ricci tensor are also valid for the Bakry–Émery tensor. We show that the Bakry–Émery tensor is nondecreasing under a Riemannian submersion whose fiber transport preserves measures up to constants. We give some relations between the Bakry–Émery tensor and measured Gromov–Hausdorff limits.

Mathematics Subject Classification (2000). Primary 53C21; Secondary 58G25.

Keywords. Ricci tensor, metric-measure space, Riemannian submersion.

1. Introduction

When considering the metric structure of manifolds with lower Ricci curvature bounds, it is natural to carry along the extra structure of a measure and consider metric-measure spaces. This is especially relevant for collapsing, and has been discussed by Cheeger–Colding [8, 9, 10], Fukaya [15] and Gromov [17, Chapter 3 $\frac{1}{2}$].

In this paper we consider smooth metric-measure spaces. Let M be an n -dimensional Riemannian manifold, with metric g . Let $d\text{vol}_M$ denote the Riemannian density on M . Let ϕ be a smooth positive function on M . Then $(M, \phi d\text{vol}_M)$ is a smooth metric-measure space. For reasons coming from the study of diffusion processes, Bakry and Émery [4] defined a generalization of the Ricci tensor of M by

$$\widetilde{\text{Ric}}_\infty = \text{Ric} - \text{Hess}(\ln \phi). \quad (1.1)$$

In terms of indices, $\left(\widetilde{\text{Ric}}_\infty\right)_{\alpha\beta} = \text{Ric}_{\alpha\beta} - (\ln \phi)_{;\alpha\beta}$.

It turns out that the Bakry–Émery tensor (1.1) has interesting connections to logarithmic Sobolev inequalities, isoperimetric inequalities and heat semigroups. We refer to [2] and [19] for information on these connections. (In fact, Bakry and Émery defined their tensor in a more abstract setting than what we consider.)

We are interested in the geometric implications of bounds on the Bakry–Émery tensor. As in [20], let us define a related tensor $\widetilde{\text{Ric}}_q$. Given $q \in (0, \infty)$, put

$$\begin{aligned}\widetilde{\text{Ric}}_q &= \text{Ric} - \text{Hess}(\ln \phi) - \frac{1}{q} d \ln \phi \otimes d \ln \phi \\ &= \text{Ric} - \frac{\text{Hess}(\phi)}{\phi} + \left(1 - \frac{1}{q}\right) \frac{d\phi}{\phi} \otimes \frac{d\phi}{\phi} \\ &= \text{Ric} - q \frac{\text{Hess}\left(\phi^{\frac{1}{q}}\right)}{\phi^{\frac{1}{q}}}.\end{aligned}\tag{1.2}$$

Clearly, if $\widetilde{\text{Ric}}_q \geq rg$ then $\widetilde{\text{Ric}}_\infty \geq rg$. In the terminology of [3], a condition of the form $\widetilde{\text{Ric}}_q \geq rg$ implies a curvature-dimension inequality $\text{CD}(r, n+q)$.

Our first result extends some classical topological results about the Ricci tensor (i.e. when ϕ is constant) to the setting of the Bakry–Émery tensor.

Theorem 1. *Suppose that M is connected and closed.*

1. *If $\widetilde{\text{Ric}}_\infty > 0$ then $\pi_1(M)$ is finite.*
2. *If $\widetilde{\text{Ric}}_q \geq 0$ and $q \in (0, \infty)$ then $\pi_1(M)$ has a finite-index free abelian subgroup of rank at most n .*
3. *If $\widetilde{\text{Ric}}_\infty \geq 0$ then $H^1(M; \mathbb{R})$ is isomorphic to the linear space of parallel 1-forms on M whose pairing with $\text{grad}(\phi)$ vanishes identically. In particular, if $\widetilde{\text{Ric}}_\infty \geq 0$ then $b_1(M) \leq n$. If $\widetilde{\text{Ric}}_\infty \geq 0$ and $b_1(M) = n$ then M is a flat torus and ϕ is constant.*
4. *If $\widetilde{\text{Ric}}_\infty < 0$ then the isometry group of (M, g) is finite.*
5. *If $\widetilde{\text{Ric}}_\infty \leq 0$ then any Killing vector field on (M, g) is parallel and annihilates ϕ .*

Remark. Theorem 1.2 is a strengthening of [20, Theorem 6], which says that if $\widetilde{\text{Ric}}_q \geq 0$ and $q \in (0, \infty)$ then $\pi_1(M)$ has polynomial growth of order at most $n+q$.

The proofs of parts 3–5 of Theorem 1 use a Bochner-type identity. If the pair (g, ϕ) is only $C^0 \cap H^1$ -regular then one can use this identity to still make sense of the notion $\widetilde{\text{Ric}}_\infty \geq rg$ or $\widetilde{\text{Ric}}_q \geq rg$ (see Definition 1 of Section 2).

An important result in the study of manifolds of nonnegative sectional curvature is O’Neill’s theorem, which says that sectional curvature is nondecreasing under a Riemannian submersion [7, Chapter 9]. We show that there is a Ricci analog of O’Neill’s theorem, provided that one uses the Bakry–Émery tensor and assumes that the fiber transport of the Riemannian submersion preserves measures up to multiplicative constants.

Suppose that a Riemannian submersion $p : M \rightarrow B$ has compact fiber F . Put $F_b = p^{-1}(b)$. Given a smooth curve $\gamma : [0, 1] \rightarrow B$ and a point $m \in F_{\gamma(0)}$, let $\rho(m)$ be the endpoint $\bar{\gamma}(1)$ of the horizontal lift $\bar{\gamma}$ of γ that starts at $\bar{\gamma}(0)$. Then ρ is the fiber transport diffeomorphism from $F_{\gamma(0)}$ to $F_{\gamma(1)}$.

Given the positive function ϕ^M on M , define ϕ^B , a smooth positive function on B , by

$$p_*(\phi^M d\text{vol}_M) = \phi^B d\text{vol}_B. \quad (1.3)$$

Let $\widetilde{\text{Ric}}_\infty^M$ and $\widetilde{\text{Ric}}_\infty^B$ denote the corresponding Bakry–Émery tensors. Let $d\text{vol}_F$ denote the fiberwise Riemannian density.

Theorem 2. *Suppose that fiber transport preserves the fiberwise measure $\phi_M d\text{vol}_F$ up to a multiplicative constant, i.e. for any smooth curve $\gamma : [0, 1] \rightarrow B$, there is a constant $c_\gamma > 0$ such that $\rho^* \left(\phi^M \Big|_{F_{\gamma(1)}} d\text{vol}_{F_{\gamma(1)}} \right) = c_\gamma \phi^M \Big|_{F_{\gamma(0)}} d\text{vol}_{F_{\gamma(0)}}$.*

1. *For any $r \in \mathbb{R}$, if $\widetilde{\text{Ric}}_\infty^M \geq rg^M$ then $\widetilde{\text{Ric}}_\infty^B \geq rg^B$.*
2. *Suppose in addition that $\phi^M = 1$. Put $q = \dim(F)$. For any $r \in \mathbb{R}$, if $\widetilde{\text{Ric}}^M \geq rg^M$ then $\widetilde{\text{Ric}}_q^B \geq rg^B$.*

Using Theorem 2, we show a relationship between $\widetilde{\text{Ric}}_q$ and collapsing.

Theorem 3. 1. *Given $r \in \mathbb{R}$ and an integer $q \geq 2$, let (B, ϕ) be a smooth closed measured Riemannian manifold with $\widetilde{\text{Ric}}_q^B \geq rg^B$. Then (B, ϕ) is the measured Gromov–Hausdorff limit of a sequence of $(n + q)$ -dimensional closed Riemannian manifolds (M_i, g_i) with $\text{Ric}(M_i, g_i) \geq rg_i$.*

2. *Let $\{(M_i, g_i)\}_{i=1}^\infty$ be a sequence of N -dimensional connected closed Riemannian manifolds with sectional curvatures bounded above in absolute value by Λ and diameters bounded above by D , for some $D, \Lambda \in \mathbb{R}^+$. Let (X, μ) be a limit point for $\{(M_i, g_i)\}_{i=1}^\infty$ in the measured Gromov–Hausdorff topology. Suppose that for some $r \in \mathbb{R}$ and all $i \in \mathbb{Z}^+$, $\text{Ric}(M_i, g_i) \geq rg_i$. Suppose that X is an n -dimensional closed manifold. Put $q = N - n$.*

- a. *If $q = 0$ then X has $\text{Ric} \geq rg$ in the generalized sense of Definition 1 below.*
- b. *If $q > 0$ then X has $\widetilde{\text{Ric}}_q \geq rg$ in the generalized sense of Definition 1 below.*

Finally, we give a condition in terms of distances and masses that is equivalent to having Bakry–Émery tensor bounded below by r . If \mathcal{O} is a measurable subset of M , put

$$\text{vol}_\phi(\mathcal{O}) = \int_{\mathcal{O}} \phi d\text{vol}_M. \quad (1.4)$$

Following [17, Section 5.45], we define the notion of a distance tube in M . Let T_0 be a closed subset of M . A subset $T \subset M$ containing T_0 is a distance tube with base T_0 if for all $t \in T$, there is a segment $s \subset T$ from some $t_0 \in T_0$ to t with length $l(s) = d(t, T_0)$. For $0 < u_1 < u_2$, define the distance annulus

$$A(u_1, u_2) = \{t \in T : u_1 \leq d(t, T_0) \leq u_2\}. \quad (1.5)$$

Given $c \in \mathbb{R}$, put

$$\widehat{v}(u_1, u_2, c) = \int_{u_1}^{u_2} e^{-\frac{r}{2}x^2 + cx} dx. \quad (1.6)$$

Theorem 4. *Suppose that $\widetilde{\text{Ric}}_\infty(M, g, \phi) \geq rg$ for some $r \in \mathbb{R}$. Given numbers $0 < u_1 < u_2 < u_3$, we assume that the tube T is a disjoint union of segments s , starting at T_0 , of length at least u_3 . We also assume that $\text{vol}_\phi(A(u_2, u_3)) > 0$. Suppose that for some $c \in \mathbb{R}$,*

$$\frac{\text{vol}_\phi(A(u_2, u_3))}{\text{vol}_\phi(A(u_1, u_2))} \leq \frac{\widehat{v}(u_2, u_3, c)}{\widehat{v}(u_1, u_2, c)}. \quad (1.7)$$

Then there is a subtube $T' \subset T$ consisting of a union of segments s from T_0 , such that

1.

$$\frac{\text{vol}_\phi(T' \cap A(u_1, u_2))}{\text{vol}_\phi(A(u_1, u_2))} \geq 1 - \frac{\text{vol}_\phi(A(u_2, u_3))}{\text{vol}_\phi(A(u_1, u_2))} \left(\frac{\widehat{v}(u_2, u_3, c)}{\widehat{v}(u_1, u_2, c)} \right)^{-1}, \quad (1.8)$$

2. *If a segment $s \subset T$, starting from T_0 , intersects $T' \cap A(u_2, u_3)$ then $s \subset T'$, and*

3. *For all $u_4 > u_3$,*

$$\frac{\text{vol}_\phi(T' \cap A(u_3, u_4))}{\text{vol}_\phi(T' \cap A(u_2, u_3))} \leq \frac{\widehat{v}(u_3, u_4, c)}{\widehat{v}(u_2, u_3, c)}. \quad (1.9)$$

Conversely, suppose that there is a number $r \in \mathbb{R}$ so that for each tube T and $c \in \mathbb{R}$ satisfying (1.7), there is a subtube T' with the above properties. Then $\widetilde{\text{Ric}}_\infty(M, g, \phi) \geq rg$.

In Sections 2–5 we prove Theorems 1–4, respectively. In Section 6 we make some remarks.

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2. Proof of Theorem 1

We first prove parts 1 and 2 of the theorem. If $\widetilde{\text{Ric}}_\infty > 0$ then $\widetilde{\text{Ric}}_q > 0$ for some $q \in (0, \infty)$. Increasing q if necessary, we may assume without loss of generality that q is an integer greater than one. Thus for parts 1 and 2, it is enough to consider the case when $\widetilde{\text{Ric}}_q > 0$ or $\widetilde{\text{Ric}}_q \geq 0$, for some integer q greater than one.

Given $i \in \mathbb{Z}^+$, consider $S^q \times M$ with the warped product metric $g^{S^q \times M} = g^M + i^{-2} \phi^{\frac{2}{q}} g^{S^q}$. Let $p : S^q \times M \rightarrow M$ be the projection. Let \overline{X} be the horizontal lift to $S^q \times M$ of a vector field X on M and let \overline{U} be a vertical vector field on

$S^q \times M$. From [7, Proposition 9.106],

$$\begin{aligned} \operatorname{Ric}^{S^q \times M}(\overline{X}, \overline{X}) &= p^* \left(\operatorname{Ric}^M(X, X) - q \frac{\operatorname{Hess}(\phi^{\frac{1}{q}})(X, X)}{\phi^{\frac{1}{q}}} \right), \\ \operatorname{Ric}^{S^q \times M}(\overline{X}, \overline{U}) &= 0 \\ \operatorname{Ric}^{S^q \times M}(\overline{U}, \overline{U}) &= \operatorname{Ric}^{S^q}(\overline{U}, \overline{U}) + (\overline{U}, \overline{U}) p^* \left(-\frac{\nabla^2 \phi^{\frac{1}{q}}}{\phi^{\frac{1}{q}}} - (q-1) \frac{|\nabla \phi^{\frac{1}{q}}|^2}{\phi^{\frac{2}{q}}} \right). \end{aligned} \quad (2.1)$$

Taking $i \rightarrow \infty$, we see that if $\widetilde{\operatorname{Ric}}_q(M, g, \phi) \geq rg$ then (M, g^M, ϕ) is the limit of a sequence of $(n+q)$ -dimensional manifolds with Ricci curvature bounded below by r . If r is positive then from Myers' theorem, $\pi_1(S^q \times M) \cong \pi_1(M)$ is finite. This proves part 1 of the theorem.

Now suppose that $r \geq 0$. For i large, the warped product metric on $S^q \times M$ has nonnegative Ricci curvature. There is a $k \geq 0$ so that $\pi_1(S^q \times M) \cong \pi_1(M)$ has a finite-index free abelian subgroup of rank k and the universal cover $S^q \times \widetilde{M}$ has an isometric splitting as $\mathbb{R}^k \times Y^{n+q-k}$, where Y is closed and simply-connected [12]. Considering the cohomology groups of $S^q \times \widetilde{M} \cong \mathbb{R}^k \times Y^{n+q-k}$, it follows that

$$q + \max\{j : H^j(\widetilde{M}; \mathbb{Z}) \neq 0\} = n + q - k. \quad (2.2)$$

Then $k = n - \max\{j : H^j(\widetilde{M}; \mathbb{Z}) \neq 0\} \leq n$, which proves part 2 of the theorem.

To prove the rest of the theorem, if V is a vector field on M , let V^\sharp denote the dual 1-form. If ω is a 1-form on M , let ω_\sharp denote the dual vector field. Let i_V denote interior multiplication with respect to V and let \mathcal{L}_V denote Lie differentiation with respect to V .

If T is a tensor field on M , let $(T, T) \in C^\infty(M)$ be the inner product coming from the Riemannian metric g . Put

$$\langle T, T \rangle = \int_M (T, T)(m) \phi(m) d\operatorname{vol}_M(m). \quad (2.3)$$

Let $(\Omega^*(M), d)$ denote the de Rham complex of M . Let δ be the formal adjoint of d with respect to the Riemannian metric g , i.e. in the case $\phi = 1$, and let $\widetilde{\delta}$ be the formal adjoint of d with respect to $\langle \cdot, \cdot \rangle$. Then

$$\widetilde{\delta} = \delta - i_{(d \ln \phi)_\sharp}. \quad (2.4)$$

Put $\Delta = d\delta + \delta d$ and $\widetilde{\Delta} = d\widetilde{\delta} + \widetilde{\delta}d$. Then

$$\widetilde{\Delta} = \Delta - di_{(d \ln \phi)_\sharp} - i_{(d \ln \phi)_\sharp}d = \Delta - \mathcal{L}_{(d \ln \phi)_\sharp}. \quad (2.5)$$

The Bochner identity says that if ω is a 1-form then there is an equality of functions on M :

$$\frac{1}{2} \delta d(\omega, \omega) = (\omega, \Delta \omega) - (\nabla \omega, \nabla \omega) - (\omega, \operatorname{Ric} \omega). \quad (2.6)$$

On the other hand,

$$\frac{1}{2}i_{(d \ln \phi)^\sharp} d(\omega, \omega) = \frac{1}{2} \mathcal{L}_{(d \ln \phi)^\sharp}(\omega, \omega). \quad (2.7)$$

We have

$$\mathcal{L}_{(d \ln \phi)^\sharp} g = 2 \operatorname{Hess}(\ln \phi). \quad (2.8)$$

Then

$$\frac{1}{2}i_{(d \ln \phi)^\sharp} d(\omega, \omega) = (\omega, \mathcal{L}_{(d \ln \phi)^\sharp} \omega) - (\omega, \operatorname{Hess}(\ln \phi) \omega). \quad (2.9)$$

(The minus sign in (2.9) comes from the fact that the pairing is on 1-forms instead of vector fields.) Equations (2.4), (2.5), (2.6) and (2.9) give

$$\frac{1}{2} \widetilde{\delta} d(\omega, \omega) = (\omega, \widetilde{\Delta} \omega) - \langle \nabla \omega, \nabla \omega \rangle - (\omega, \widetilde{\operatorname{Ric}}_\infty \omega). \quad (2.10)$$

Multiplying (2.10) by ϕ and integrating over M , we obtain

$$0 = \langle \omega, \widetilde{\Delta} \omega \rangle - \langle \nabla \omega, \nabla \omega \rangle - \langle \omega, \widetilde{\operatorname{Ric}}_\infty \omega \rangle, \quad (2.11)$$

or

$$\langle d\omega, d\omega \rangle + \langle \widetilde{\delta} \omega, \widetilde{\delta} \omega \rangle - \langle \nabla \omega, \nabla \omega \rangle = \langle \omega, \widetilde{\operatorname{Ric}}_\infty \omega \rangle. \quad (2.12)$$

We can apply usual elliptic theory to the de Rham complex, with the inner product $\langle \cdot, \cdot \rangle$, to obtain an isomorphism

$$H^*(M; \mathbb{R}) \cong \{\omega \in \Omega^*(M) : d\omega = \widetilde{\delta} \omega = 0\}. \quad (2.13)$$

If $\widetilde{\operatorname{Ric}}_\infty \geq 0$ and a 1-form ω satisfies $d\omega = \widetilde{\delta} \omega = 0$ then (2.12) implies that $\nabla \omega = 0$. Hence $\delta \omega = 0$. Along with $\widetilde{\delta} \omega = 0$, (2.4) now implies that $\omega(\operatorname{grad}(\phi)) = 0$. Conversely, if $\nabla \omega = \omega(\operatorname{grad}(\phi)) = 0$ then $d\omega = \widetilde{\delta} \omega = 0$. This proves the isomorphism in part 3 of the theorem.

If $b_1(M) = n$ then there are n linearly-independent parallel 1-forms on M that annihilate $\operatorname{grad}(\phi)$. The usual argument shows that M is a flat torus. As the parallel 1-forms on M annihilate $\operatorname{grad}(\phi)$, ϕ must be constant. This proves part 3 of the theorem.

A pointwise algebraic computation shows that

$$(d\omega, d\omega) + (\mathcal{L}_{\omega^\sharp} g, \mathcal{L}_{\omega^\sharp} g) = 2\langle \nabla \omega, \nabla \omega \rangle. \quad (2.14)$$

Then (2.12) becomes

$$\langle \nabla \omega, \nabla \omega \rangle + \langle \widetilde{\delta} \omega, \widetilde{\delta} \omega \rangle - \langle \omega, \widetilde{\operatorname{Ric}}_\infty \omega \rangle = \langle \mathcal{L}_{\omega^\sharp} g, \mathcal{L}_{\omega^\sharp} g \rangle. \quad (2.15)$$

If $\widetilde{\operatorname{Ric}}_\infty < 0$ and $\mathcal{L}_V g = 0$ then taking $\omega = V^\sharp$, (2.15) implies that $V = 0$. Hence the isometry group of (M, g) is discrete and, being compact, must be finite. This proves part 4 of the theorem.

If $\widetilde{\text{Ric}}_\infty \leq 0$ and $\mathcal{L}_V g = 0$ then (2.15) implies that $\nabla V^\sharp = \widetilde{\delta} V^\sharp = 0$. As before, we obtain that $V\phi = 0$. This proves part 5 of the theorem.

Remarks. 1. If we put $\omega = df$ in (2.10) then we recover the definition of $\widetilde{\text{Ric}}_\infty$ from [4].

2. Jianguo Cao pointed out to me that a formula related to (2.12) has been used to study the $\bar{\partial}$ -operator on complete Kähler manifolds [14, Théorème 5.1].

3. The operator $\widetilde{\Delta}$ is related to the Witten Laplacian of [22], but the two operators are distinct. To see the relation, note that $\widetilde{\delta} = \phi^{-1} \delta \phi$. Put $D = \phi^{\frac{1}{2}} d \phi^{-\frac{1}{2}}$ and $D^* = \phi^{-\frac{1}{2}} \delta \phi^{\frac{1}{2}}$. Then the Witten Laplacian $DD^* + D^*D$ is related to $\widetilde{\Delta}$ by

$$DD^* + D^*D = \phi^{\frac{1}{2}} \widetilde{\Delta} \phi^{-\frac{1}{2}}. \quad (2.16)$$

The Bochner-type identity (2.12), when translated to a statement about $DD^* + D^*D$, becomes

$$DD^* + D^*D = \left(\phi^{\frac{1}{2}} \nabla \phi^{-\frac{1}{2}} \right)^* \left(\phi^{\frac{1}{2}} \nabla \phi^{-\frac{1}{2}} \right) + \widetilde{\text{Ric}}_\infty, \quad (2.17)$$

where the adjoints are with respect to the unweighted L^2 -inner product. In contrast, in Morse–Witten theory one collects the terms differently, by writing $DD^* + D^*D = \nabla^* \nabla + \dots$

4. The equality (2.12) gives a way of defining the notion of $\widetilde{\text{Ric}}_\infty \geq rg$ for a class of nonsmooth measured manifolds (M, g, ϕ) . Namely, suppose that M is a manifold whose transition maps are $C^{1,1}$ -regular. Let g be a Riemannian metric on M whose components, in local charts, are in $C^0 \cap H^1$, where H^1 denotes the Sobolev space. Let $\phi \in C^0(M) \cap H^1_{loc}(M)$ be a positive function. (There are a smooth manifold M' and a $C^{1,1}$ -diffeomorphism $M' \rightarrow M$. Hence after pulling back, if one wants then one can assume that g and ϕ are defined on a smooth manifold.)

Definition 1. We say that $\text{Ric}(M, g) \geq rg$ if for all compactly-supported Lipschitz-regular 1-forms ω on M ,

$$\int_M (d\omega, d\omega) d\text{vol}_M + \int_M (\delta\omega, \delta\omega) d\text{vol}_M - \int_M (\nabla\omega, \nabla\omega) d\text{vol}_M \geq r \int_M (\omega, \omega) d\text{vol}_M. \quad (2.18)$$

We say that $\widetilde{\text{Ric}}_\infty(M, g, \phi) \geq rg$ if for all compactly-supported Lipschitz-regular 1-forms ω on M ,

$$\langle d\omega, d\omega \rangle + \langle \widetilde{\delta}\omega, \widetilde{\delta}\omega \rangle - \langle \nabla\omega, \nabla\omega \rangle \geq r \langle \omega, \omega \rangle. \quad (2.19)$$

We say that $\widetilde{\text{Ric}}_q(M, g, \phi) \geq rg$ if for all compactly-supported Lipschitz-regular 1-forms ω on M ,

$$\langle d\omega, d\omega \rangle + \langle \widetilde{\delta}\omega, \widetilde{\delta}\omega \rangle - \langle \nabla\omega, \nabla\omega \rangle - \frac{1}{q} \int_M (\omega(\nabla \ln \phi))^2 \phi d\text{vol}_M \geq r \langle \omega, \omega \rangle. \quad (2.20)$$

An immediate consequence of the definition is the following lemma.

Lemma 1. *Let M be a smooth closed manifold.*

1. *If $\{g_i\}_{i=1}^\infty$ is a sequence of smooth Riemannian metrics on M with $\text{Ric}(M, g_i) \geq rg_i$, and $g_i \xrightarrow{C^0 \cap H^1} g$ for some $C^0 \cap H^1$ -regular metric g , then $\text{Ric}(M, g) \geq rg$.*
2. *If $\{(g_i, \phi_i)\}_{i=1}^\infty$ is a sequence of smooth Riemannian metrics and smooth positive functions on M with $\widetilde{\text{Ric}}_\infty(M, g_i, \phi_i) \geq rg_i$, and $(g_i, \phi_i) \xrightarrow{C^0 \cap H^1} (g, \phi)$ for some $C^0 \cap H^1$ -regular pair (g, ϕ) , then $\widetilde{\text{Ric}}_\infty(M, g, \phi) \geq rg$.*
3. *If $\{(g_i, \phi_i)\}_{i=1}^\infty$ is a sequence of smooth Riemannian metrics and smooth positive functions on M with $\widetilde{\text{Ric}}_q(M, g_i, \phi_i) \geq rg_i$, and $(g_i, \phi_i) \xrightarrow{C^0 \cap H^1} (g, \phi)$ for some $C^0 \cap H^1$ -regular pair (g, ϕ) , then $\widetilde{\text{Ric}}_q(M, g, \phi) \geq rg$.*

For example, let $\{(M_i, g_i)\}_{i=1}^\infty$ be a sequence of n -dimensional closed Riemannian manifolds with Ricci curvatures bounded below by $r \in \mathbb{R}$, injectivity radii bounded below by $i_0 \in \mathbb{R}^+$ and diameters bounded above by $D \in \mathbb{R}^+$. Then $\{(M_i, g_i)\}_{i=1}^\infty$ has a limit point X in the Gromov–Hausdorff topology. From [1], X is an n -dimensional closed manifold with a Riemannian metric g that is $W^{1,p}$ -regular for all $p \in [1, \infty)$. From the Sobolev embedding theorem, g is also $C^{0,\alpha}$ -regular for all $\alpha \in (0, 1)$. After applying diffeomorphisms one has $W^{1,p}$ -convergence of a subsequence of $\{(M_i, g_i)\}_{i=1}^\infty$ to (X, g) , and so $\text{Ric}(X, g) \geq rg$ in the sense of Definition 1.

For another example, suppose that M is a compact Kähler manifold with local complex coordinates $\{z^\alpha\}$ and metric $g_{\alpha\bar{\beta}}$. Its Ricci form, in local coordinates, is the $(1, 1)$ -form $-\frac{1}{2}\partial\bar{\partial}\ln\det(g)$. Now suppose that the $g_{\alpha\bar{\beta}}$ are only $C^0 \cap H^1$ -regular. The Kähler condition still makes sense distributionally, and the Ricci form makes sense as a closed $(1, 1)$ -current. Then $\text{Ric}(M, g) \geq 0$ in the sense of Definition 1 if and only if $-\frac{1}{2}\partial\bar{\partial}\ln\det(g)$ is a positive current. (This last condition makes sense for a much larger class of g .)

3. Proof of Theorem 2

We (mostly) use the notation of [7, Chapter 9]. If X is a vector field on B , let \overline{X} be its horizontal lift to M . Let N be the mean curvature vector field to the fibers F . Let A be the curvature of the horizontal distribution. Let T be the second fundamental form tensor of the fibers F . Let ∇^M be the covariant derivative operator on M and let ∇^B be the covariant derivative operator on B . From [7, (9.36c)], there is an identity of functions on M :

$$\text{Ric}^M(\overline{X}, \overline{X}) = \text{Ric}^B(X, X) - 2(A_{\overline{X}}, A_{\overline{X}}) - (T\overline{X}, T\overline{X}) + (\overline{X}, \nabla_{\overline{X}}^M N). \quad (3.1)$$

Given $b \in B$, let $\{\theta_t\}_{t \in (-\epsilon, \epsilon)}$ be the flow of X as defined in a neighborhood of b and for t in some interval $(-\epsilon, \epsilon)$. Let $\{\bar{\theta}_t\}_{t \in (-\epsilon, \epsilon)}$ be the flow of \overline{X} that covers θ_t .

It sends fibers to fibers diffeomorphically. Hence it makes sense to define $\mathcal{L}_{\overline{X}} d\text{vol}_F$ by

$$(\mathcal{L}_{\overline{X}} d\text{vol}_F)|_{F_b} = \frac{d}{dt} \Big|_{t=0} (\bar{\theta}_t^* d\text{vol}_F)|_{F_b}. \quad (3.2)$$

With our conventions,

$$\mathcal{L}_{\overline{X}} d\text{vol}_F = -(\overline{X}, N) d\text{vol}_F. \quad (3.3)$$

We have

$$\phi^B = \int_F \phi^M d\text{vol}_F. \quad (3.4)$$

Then

$$\begin{aligned} X\phi^B &= \mathcal{L}_X \phi^B = \mathcal{L}_X \int_F \phi^M d\text{vol}_F = \int_F \mathcal{L}_{\overline{X}}(\phi^M d\text{vol}_F) \\ &= \int_F (\overline{X}\phi^M - (\overline{X}, N)\phi^M) d\text{vol}_F \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} XX\phi^B &= \int_F [\overline{X}(\overline{X}\phi^M - (\overline{X}, N)\phi^M) - (\overline{X}, N)(\overline{X}\phi^M - (\overline{X}, N)\phi^M)] d\text{vol}_F \\ &= \int_F [\overline{X}\overline{X}\phi^M - \overline{X}(\overline{X}, N)\phi^M - 2(\overline{X}, N)\overline{X}\phi^M + (\overline{X}, N)^2\phi^M] d\text{vol}_F \\ &= \int_F \left[\frac{\overline{X}\overline{X}\phi^M}{\phi^M} - (\nabla_{\overline{X}}^M \overline{X}, N) - (\overline{X}, \nabla_{\overline{X}}^M N) - \left(\frac{\overline{X}\phi^M}{\phi^M} \right)^2 \right. \\ &\quad \left. + \left(\frac{\overline{X}\phi^M}{\phi^M} - (\overline{X}, N) \right)^2 \right] \phi^M d\text{vol}_F. \end{aligned} \quad (3.6)$$

Using the fact that $\nabla_{\overline{X}}^M \overline{X} = \overline{\nabla_X^B X}$ [7, (9.25d)], it follows that

$$\begin{aligned} \text{Hess}(\phi_B)(X, X) &= XX\phi^B - (\nabla_X^B X)\phi^B \\ &= \int_F \left[\frac{\text{Hess}(\phi^M)(\overline{X}, \overline{X})}{\phi^M} - (\overline{X}, \nabla_{\overline{X}}^M N) - \left(\frac{\overline{X}\phi^M}{\phi^M} \right)^2 \right. \\ &\quad \left. + \left(\frac{\overline{X}\phi^M}{\phi^M} - (\overline{X}, N) \right)^2 \right] \phi^M d\text{vol}_F \\ &= \int_F [\text{Hess}(\ln \phi^M)(\overline{X}, \overline{X}) - (\overline{X}, \nabla_{\overline{X}}^M N) \\ &\quad + \left(\frac{\overline{X}\phi^M}{\phi^M} - (\overline{X}, N) \right)^2] \phi^M d\text{vol}_F. \end{aligned} \quad (3.7)$$

Substituting $(\bar{X}, \nabla_{\bar{X}}^M N)$ from (3.1) gives

$$\begin{aligned} \text{Ric}^B(X, X)\phi^B - \text{Hess}(\phi^B)(X, X) &= \int_F \left[\widetilde{\text{Ric}}_\infty^M(\bar{X}, \bar{X}) + 2(A_{\bar{X}}, A_{\bar{X}}) + (T\bar{X}, T\bar{X}) \right. \\ &\quad \left. - \left(\frac{\bar{X}\phi^M}{\phi^M} - (\bar{X}, N) \right)^2 \right] \phi^M d\text{vol}_F \end{aligned} \quad (3.8)$$

Using (3.5),

$$\begin{aligned} \widetilde{\text{Ric}}_\infty^B(X, X)\phi^B &= \left[\text{Ric}^B(X, X) - \frac{\text{Hess}(\phi^B)(X, X)}{\phi^B} + \frac{(X\phi^B)^2}{(\phi^B)^2} \right] \phi^B \\ &= \int_F \left[\widetilde{\text{Ric}}_\infty^M(\bar{X}, \bar{X}) + 2(A_{\bar{X}}, A_{\bar{X}}) + (T\bar{X}, T\bar{X}) \right. \\ &\quad \left. - \left(\frac{\bar{X}\phi^M}{\phi^M} - (\bar{X}, N) \right)^2 \right] \phi^M d\text{vol}_F \\ &\quad + \left(\int_F \left(\frac{\bar{X}\phi^M}{\phi^M} - (\bar{X}, N) \right) \phi^M d\text{vol}_F \right)^2 (\phi^B)^{-1}. \end{aligned} \quad (3.9)$$

We have

$$\mathcal{L}_{\bar{X}}(\phi^M d\text{vol}_F) = \left(\frac{\bar{X}\phi^M}{\phi^M} - (\bar{X}, N) \right) \phi^M d\text{vol}_F. \quad (3.10)$$

By assumption, $\frac{\bar{X}\phi^M}{\phi^M} - (\bar{X}, N)$ is constant on a fiber F . Then

$$\begin{aligned} \widetilde{\text{Ric}}_\infty^B(X, X)\phi^B &= \int_F \left[\widetilde{\text{Ric}}_\infty^M(\bar{X}, \bar{X}) + 2(A_{\bar{X}}, A_{\bar{X}}) + (T\bar{X}, T\bar{X}) \right] \phi^M d\text{vol}_F \\ &\geq \int_F \widetilde{\text{Ric}}_\infty^M(\bar{X}, \bar{X}) \phi^M d\text{vol}_F. \end{aligned} \quad (3.11)$$

If $\widetilde{\text{Ric}}_\infty^M(\bar{X}, \bar{X}) \geq rg^M(\bar{X}, \bar{X})$ then (3.11) implies that $\widetilde{\text{Ric}}_\infty^B(X, X) \geq rg^B(X, X)$. This proves Theorem 2.1.

Now suppose that $\phi^M = 1$. Equations (1.2) and (3.9) imply that

$$\begin{aligned} \widetilde{\text{Ric}}_q^B(X, X)\phi^B &= \int_F \left[\text{Ric}^M(\bar{X}, \bar{X}) + 2(A_{\bar{X}}, A_{\bar{X}}) + (T\bar{X}, T\bar{X}) - \frac{1}{q}(\bar{X}, N)^2 \right] d\text{vol}_F \\ &\quad + \left(1 - \frac{1}{q} \right) \left(- \int_F (\bar{X}, N)^2 d\text{vol}_F + \left(\int_F (\bar{X}, N) d\text{vol}_F \right)^2 \right) (\phi^B)^{-1}. \end{aligned} \quad (3.12)$$

As $(\overline{X}, N) = -\operatorname{Tr}(T\overline{X})$, we know that $(T\overline{X}, T\overline{X}) - \frac{1}{q}(\overline{X}, N)^2 \geq 0$. By assumption, (\overline{X}, N) is constant on a fiber F . Then

$$\begin{aligned} & \widetilde{\operatorname{Ric}}_q^B(X, X) \phi^B \\ &= \int_F \left[\operatorname{Ric}^M(\overline{X}, \overline{X}) + 2(A_{\overline{X}}, A_{\overline{X}}) + (T\overline{X}, T\overline{X}) - \frac{1}{q}(\overline{X}, N)^2 \right] d\operatorname{vol}_F \quad (3.13) \\ &\geq \int_F \operatorname{Ric}^M(\overline{X}, \overline{X}) d\operatorname{vol}_F. \end{aligned}$$

If $\widetilde{\operatorname{Ric}}_\infty^M(\overline{X}, \overline{X}) \geq rg^M(\overline{X}, \overline{X})$ then

$$\widetilde{\operatorname{Ric}}_q^B(X, X) \phi^B \geq r \int_F g^M(\overline{X}, \overline{X}) d\operatorname{vol}_F = rg^B(X, X) \phi^B. \quad (3.14)$$

This proves Theorem 2.2.

Example. Let $p : M \rightarrow B$ be a Riemannian submersion, with M compact, whose fiber transport preserves the fiberwise metric up to multiplicative constants. Equivalently, the Riemannian metric g on M comes from starting with a submersion metric g' with totally geodesic fibers, along with a positive function $f \in C^\infty(B)$, and then multiplying the fiberwise metric of g' on F_b by $f^2(b)$. One can think of g as a generalized warped product metric.

Suppose that the fibers F have nonnegative Ricci curvature. For $\epsilon > 0$, let g_ϵ be the Riemannian metric on M which comes from multiplying the fiberwise Riemannian metrics by ϵ^2 . Then as $\epsilon \rightarrow 0$, the metrics g_ϵ have Ricci curvatures that are uniformly bounded below. Explicitly, let \overline{X} be the horizontal lift of a vector field X on B and let \overline{U} be a vertical vector field. Then as $\epsilon \rightarrow 0$, with the notation of [7, Chapter 9],

$$\begin{aligned} \operatorname{Ric}_\epsilon^M(\overline{X}, \overline{X}) &\sim p^* \operatorname{Ric}^B(X, X) - (T\overline{X}, T\overline{X}) + (\overline{X}, \nabla_{\overline{X}}^M N), \quad (3.15) \\ \operatorname{Ric}_\epsilon^M(\overline{X}, \overline{U}) &\sim 0 \\ \operatorname{Ric}_\epsilon^M(\overline{U}, \overline{U}) &\sim \operatorname{Ric}^F(\overline{U}, \overline{U}) + \epsilon^2 \left((\tilde{\delta}T)(\overline{U}, \overline{U}) - (N, T_{\overline{U}}\overline{U}) \right). \end{aligned}$$

(The terms on the right-hand side of (3.15) are evaluated with respect to the metric g_1 .) This is an example of a collapse with Ricci curvature bounded below, to which Theorem 2.2 applies.

For another example, let M be a compact Riemannian manifold on which a Lie group G acts isometrically and effectively. Suppose that the G -action on M has a single orbit type and put $B = G \backslash M$. Then there is a natural Riemannian submersion $p : M \rightarrow B$. As the orbits of the G -action on M are all G -diffeomorphic to a homogeneous space G/H , and G/H has a unique G -invariant volume form up to constants, it follows that the fiber transport of the Riemannian submersion preserves measures up to constants. Hence Theorem 2.2 applies.

4. Proof of Theorem 3

We refer to [15] for the definition of the measured Gromov–Hausdorff topology.

To prove Theorem 3.1, we just apply the warped product construction of the proof of Theorem 1.1 to $S^q \times B$.

Let $\{M_i, g_i\}_{i=1}^\infty$ be a sequence as in the statement of Theorem 3.2. We may assume that $\lim_{i \rightarrow \infty} (M_i, g_i, d\text{vol}_i) = (X, \mu)$ in the measured Gromov–Hausdorff topology. If $q = 0$ then X is a smooth manifold with a $C^{1,\alpha}$ -regular metric g^X and after taking a subsequence and applying diffeomorphisms, we may assume that (M_i, g_i) converges to (X, g^X) in the $C^{1,\alpha}$ -topology (see, for example, [18]). In this case, the theorem follows from Lemma 1.1.

Suppose that $q > 0$. By saying that X is a manifold, we mean that in the construction of X as a quotient space $\hat{X}/O(N)$ [16], the action of $O(N)$ on the manifold \hat{X} has a single orbit type. Then X has the structure of a smooth manifold with a $C^{1,\alpha}$ -regular pair (g^X, ϕ^X) .

For any $\epsilon > 0$, we can apply smoothing results of Abresch and others [11, Theorem 1.12] to obtain new metrics $g_i(\epsilon)$ with

$$\begin{aligned} e^{-\epsilon} g_i &\leq g_i(\epsilon) \leq e^\epsilon g_i, \\ |\nabla_{g_i} - \nabla_{g_i(\epsilon)}| &\leq \epsilon, \\ |\nabla_{g_i(\epsilon)}^k \text{Riem}(M_i, g_i(\epsilon))| &\leq C_k(N, \epsilon, \Lambda), \end{aligned} \tag{4.1}$$

where the constants are uniform. We can also assume that $\text{Ric}(M_i, g_i(\epsilon)) \geq (r - \epsilon)g_i(\epsilon)$ [13, Remark 2, p. 51]. (See [21, Theorem 2.1] for a similar statement about sectional curvature.) For small ϵ , let $B(\epsilon)$ be a Gromov–Hausdorff limit of a subsequence of $\{(M_i, g_i(\epsilon))\}_{i=1}^\infty$. We relabel the subsequence as $\{(M_i, g_i(\epsilon))\}_{i=1}^\infty$. From [11, Proposition 4.9], for large i , there is a small C^2 -perturbation $g'_i(\epsilon)$ of $g_i(\epsilon)$ which is invariant with respect to a Nil -structure. In particular, we may assume that $\text{Ric}(M_i, g'_i(\epsilon)) \geq (r - 2\epsilon)g'_i(\epsilon)$. Now $(M_i, g'_i(\epsilon))$ is the total space of a Riemannian submersion $M_i \rightarrow B(\epsilon)$ with infranil fibers and affine holonomy. Let $(g_i^{B(\epsilon)}, \phi_i^{B(\epsilon)})$ denote the induced metric and measure on $B(\epsilon)$. As the fiber transport of the Riemannian submersion preserves the affine-parallel volume forms of the fibers, up to constants, Theorem 2.2 implies that $\widehat{\text{Ric}}_q(B(\epsilon), g_i^{B(\epsilon)}, \phi_i^{B(\epsilon)}) \geq (r - 2\epsilon)g_i^{B(\epsilon)}$. Varying i and ϵ , we can extract a subsequence of $\{(B(\epsilon), g_i^{B(\epsilon)}, \phi_i^{B(\epsilon)})\}$ with $i \rightarrow \infty$ and $\epsilon \rightarrow 0$ that converges in the $C^{1,\alpha}$ -topology to (X, g^X, ϕ^X) . The theorem now follows from Lemma 1.3.

5. Proof of Theorem 4

Let s be a segment from $t_0 \in T_0$ to $t \in T$, with length $l(s) > u_3$ and arc-length parameter u . By definition, s is length-minimizing. We can decompose the measure $\phi d\text{vol}_M$ on $A(u_1, u_4)$ as $\phi \text{area}_s(u) du \mu(s)$, where μ is a measure on the

space \mathcal{S} of distinct segments s that make up $A(u_1, u_4)$, du is the length measure along a segment s and $\text{area}_s(u)$ is the relative size of the transverse Riemannian area density along s , as measured with respect to the fan of segments. Let h denote the trace of the second fundamental form Π of a level set of constant distance from T_0 . (With our conventions, the boundary of the unit ball in \mathbb{R}^n has positive mean curvature.) Differentiating along s , with respect to u , gives

$$\partial_u \ln(\phi(u)\text{area}_s(u)) \equiv \frac{\partial_u(\phi(u)\text{area}_s(u))}{\phi(u)\text{area}_s(u)} = h(u) + \partial_u \ln \phi(u) \quad (5.1)$$

and

$$\partial_u^2 \ln(\phi(u)\text{area}_s(u)) = \partial_u h(u) + \partial_u^2 \ln \phi(u). \quad (5.2)$$

From the Riccati equation for Π ,

$$\partial_u h(u) = -\text{Tr}(\Pi^2) - \text{Ric}(\partial_u, \partial_u) \leq -\text{Ric}(\partial_u, \partial_u). \quad (5.3)$$

Then

$$\partial_u^2 \ln(\phi(u)\text{area}_s(u)) \leq -\widetilde{\text{Ric}}_\infty(\partial_u, \partial_u) \leq -r. \quad (5.4)$$

Hence for any $c \in \mathbb{R}$,

$$\partial_u^2 \left(\ln(\phi(u)\text{area}_s(u)) + \frac{r}{2}u^2 - cu \right) \leq 0. \quad (5.5)$$

Fix s and put

$$a(u) = \phi(u)\text{area}_s(u), \quad (5.6)$$

$$\widehat{a}(u) = e^{-\frac{r}{2}u^2 + cu}, \quad (5.7)$$

$$v(u_1, u_2) = \int_{u_1}^{u_2} a(u) du \quad (5.8)$$

and

$$\widehat{v}(u_1, u_2) = \int_{u_1}^{u_2} \widehat{a}(u) du. \quad (5.9)$$

Then (5.5) says that

$$\frac{d^2}{du^2} \ln \left(\frac{a}{\widehat{a}} \right) \leq 0, \quad (5.10)$$

i.e. that $\ln \left(\frac{a}{\widehat{a}} \right)$ is concave in u .

Lemma 2. *If $\frac{v(u_2, u_3)}{\widehat{v}(u_2, u_3)} \leq \frac{v(u_1, u_2)}{\widehat{v}(u_1, u_2)}$ then $\frac{a(u_3)}{\widehat{a}(u_3)} \leq \frac{v(u_2, u_3)}{\widehat{v}(u_2, u_3)}$.*

Proof. Suppose that

$$\frac{a(u_3)}{\widehat{a}(u_3)} > \frac{v(u_2, u_3)}{\widehat{v}(u_2, u_3)} = \frac{\int_{u_2}^{u_3} \frac{a(u)}{\widehat{a}(u)} \widehat{a}(u) du}{\int_{u_2}^{u_3} \widehat{a}(u) du}. \quad (5.11)$$

If $\frac{a(u_2)}{\widehat{a}(u_2)} \geq \frac{a(u_3)}{\widehat{a}(u_3)}$ then the concavity of $\ln\left(\frac{a}{\widehat{a}}\right)$ implies that

$$\frac{a(u_3)}{\widehat{a}(u_3)} \leq \frac{\int_{u_2}^{u_3} \frac{a(u)}{\widehat{a}(u)} \widehat{a}(u) du}{\int_{u_2}^{u_3} \widehat{a}(u) du}, \quad (5.12)$$

which is a contradiction. Thus

$$\frac{a(u_2)}{\widehat{a}(u_2)} < \frac{a(u_3)}{\widehat{a}(u_3)}. \quad (5.13)$$

With the concavity of $\ln\left(\frac{a}{\widehat{a}}\right)$, (5.13) implies that $\frac{a(u)}{\widehat{a}(u)}$ is decreasing in u for $u < u_2$ and so

$$\frac{\int_{u_1}^{u_2} \frac{a(u)}{\widehat{a}(u)} \widehat{a}(u) du}{\int_{u_1}^{u_2} \widehat{a}(u) du} < \frac{a(u_2)}{\widehat{a}(u_2)}. \quad (5.14)$$

The concavity of $\ln\left(\frac{a}{\widehat{a}}\right)$ and (5.13) also imply that

$$\frac{a(u_2)}{\widehat{a}(u_2)} < \frac{\int_{u_2}^{u_3} \frac{a(u)}{\widehat{a}(u)} \widehat{a}(u) du}{\int_{u_2}^{u_3} \widehat{a}(u) du}. \quad (5.15)$$

Thus we have

$$\frac{\int_{u_1}^{u_2} \frac{a(u)}{\widehat{a}(u)} \widehat{a}(u) du}{\int_{u_1}^{u_2} \widehat{a}(u) du} < \frac{a(u_2)}{\widehat{a}(u_2)} < \frac{\int_{u_2}^{u_3} \frac{a(u)}{\widehat{a}(u)} \widehat{a}(u) du}{\int_{u_2}^{u_3} \widehat{a}(u) du}, \quad (5.16)$$

which contradicts the assumption. \square

Lemma 3. If $\frac{v(u_2, u_3)}{\widehat{v}(u_2, u_3)} \leq \frac{v(u_1, u_2)}{\widehat{v}(u_1, u_2)}$ then for $u_4 \in (u_3, l(s))$, $\frac{v(u_3, u_4)}{\widehat{v}(u_3, u_4)} \leq \frac{v(u_2, u_3)}{\widehat{v}(u_2, u_3)}$.

Proof. For $u \in (u_3, l(s))$, put

$$F(u) = \ln \left(\frac{v(u_3, u)}{\widehat{v}(u_3, u)} / \frac{v(u_2, u_3)}{\widehat{v}(u_2, u_3)} \right). \quad (5.17)$$

Then

$$F'(u) = \frac{a(u)}{v(u_3, u)} - \frac{\widehat{a}(u)}{\widehat{v}(u_3, u)} = \frac{\widehat{a}(u)}{v(u_3, u)} \left[\frac{a(u)}{\widehat{a}(u)} - \frac{v(u_3, u)}{\widehat{v}(u_3, u)} \right]. \quad (5.18)$$

Lemma 2 implies that if $F(u) \leq 0$ then $F'(u) \leq 0$. We can extend $F(u)$ smoothly to $u = u_3$, with

$$F(u_3) = \ln \left(\frac{a(u_3)}{\widehat{a}(u_3)} / \frac{v(u_2, u_3)}{\widehat{v}(u_2, u_3)} \right). \quad (5.19)$$

By Lemma 2, $F(u_3) \leq 0$. It follows that $F(u) \leq 0$ for all $u \in (u_3, l(s))$, which proves the lemma. \square

We have

$$\frac{\text{vol}_\phi(A(u_2, u_3))}{\text{vol}_\phi(A(u_1, u_2))} = \frac{\int_{\mathcal{S}} \frac{v_s(u_2, u_3)}{v_s(u_1, u_2)} v_s(u_1, u_2) d\mu(s)}{\int_{\mathcal{S}} v_s(u_1, u_2) d\mu(s)}. \quad (5.20)$$

Put

$$S' = \left\{ s \in S : \frac{v_s(u_2, u_3)}{v_s(u_1, u_2)} < \frac{\widehat{v}(u_2, u_3)}{\widehat{v}(u_1, u_2)} \right\} \quad (5.21)$$

and

$$T' = \bigcup_{s \in S'} s. \quad (5.22)$$

We claim that (1.8) is satisfied. If it is not satisfied, put $S'' = S - S'$ and $T'' = T - T'$. Then

$$\frac{\text{vol}_\phi(T'' \cap A(u_1, u_2))}{\text{vol}_\phi(A(u_1, u_2))} > \frac{\text{vol}_\phi(A(u_2, u_3))}{\text{vol}_\phi(A(u_1, u_2))} \left(\frac{\widehat{v}(u_2, u_3)}{\widehat{v}(u_1, u_2)} \right)^{-1}. \quad (5.23)$$

However, from the definition of T'' ,

$$\begin{aligned} \text{vol}_\phi(A(u_2, u_3)) &\geq \text{vol}_\phi(T'' \cap A(u_2, u_3)) = \int_{S''} \frac{v_s(u_2, u_3)}{v_s(u_1, u_2)} v_s(u_1, u_2) d\mu(s) \\ &\geq \int_{S''} \frac{\widehat{v}(u_2, u_3)}{\widehat{v}(u_1, u_2)} v_s(u_1, u_2) d\mu(s) = \frac{\widehat{v}(u_2, u_3)}{\widehat{v}(u_1, u_2)} \text{vol}_\phi(T'' \cap A(u_1, u_2)), \end{aligned} \quad (5.24)$$

which contradicts (5.23).

If there is a cutpoint along s , with respect to its basepoint in T_0 , at $u_c \in (u_3, u_4)$ then we put $v_s(u_3, u_4) = \int_{u_3}^{u_c} a_s(u) du$, and otherwise we put $v_s(u_3, u_4) = \int_{u_3}^{u_4} a_s(u) du$. Using Lemma 3,

$$\frac{\text{vol}_\phi(T' \cap A(u_3, u_4))}{\text{vol}_\phi(T' \cap A(u_2, u_3))} = \frac{\int_{S'} \frac{v_s(u_3, u_4)}{v_s(u_2, u_3)} v_s(u_2, u_3) d\mu(s)}{\int_{S'} v_s(u_2, u_3) d\mu(s)} \leq \frac{\widehat{v}_s(u_3, u_4)}{\widehat{v}_s(u_2, u_3)}. \quad (5.25)$$

This proves the first part of the theorem.

Suppose that there is a number $r \in \mathbb{R}$ so that for each tube T and $c \in \mathbb{R}$ satisfying (1.7), there is a subtube T' satisfying the properties of the theorem. Given $m \in M$ and a unit vector $v \in T_m M$, let T_0 be a hypersurface passing through m such that $T_m(T_0) = v^\perp$ and the second fundamental form of T_0 at m vanishes. Let s be a minimizing segment with $s(0) = m$ and $s'(0) = v$. From (5.1),

$$\left. \frac{d}{du} \right|_{u=0} (\ln(\phi(u) \text{area}(u))) = v(\ln \phi). \quad (5.26)$$

From (5.2) and the Riccati equation,

$$\left. \frac{d^2}{du^2} \right|_{u=0} (\ln(\phi(u) \text{area}(u))) = -\widetilde{\text{Ric}}_\infty(v, v). \quad (5.27)$$

Put $c_0 = v(\ln \phi)$ and $r_0 = \widetilde{\text{Ric}}_\infty(v, v)$. Then for small u ,

$$\ln(\phi(u) \text{area}(u)) \sim \text{const.} + c_0 u - \frac{r_0}{2} u^2. \quad (5.28)$$

For small $u_1 < u_2 < u_3 < u_4$, we have

$$\frac{v(u_2, u_3)}{v(u_1, u_2)} \sim \frac{\int_{u_2}^{u_3} e^{-\frac{r_0}{2} u^2 + c_0 u} du}{\int_{u_1}^{u_2} e^{-\frac{r_0}{2} u^2 + c_0 u} du} \quad (5.29)$$

and

$$\frac{v(u_3, u_4)}{v(u_2, u_3)} \sim \frac{\int_{u_3}^{u_4} e^{-\frac{r_0}{2}u^2 + c_0u} du}{\int_{u_2}^{u_3} e^{-\frac{r_0}{2}u^2 + c_0u} du}. \quad (5.30)$$

Take T to be a small tube around s (with small base T_0), take u_3 small relative to u_4 and take $c = c_0 + \epsilon$ with $\epsilon > 0$ small so that (1.7) holds. If there is to be a subtube T' such that (1.9) holds, for all such choices, then we must have $r_0 \geq r$. This proves the theorem.

6. Remarks

1. If M^n is compact and $\widetilde{\text{Ric}}_q \geq rg$, with q an integer greater than one, then Theorem 3.1 says that (M, g, ϕ) is the limit of a sequence of $(n+q)$ -dimensional manifolds with Ricci curvature bounded below by r . As in the proof of Theorem 1.2, we can then apply standard results about manifolds with Ricci curvature bounded below, in order to obtain conclusions about (M, g, ϕ) . For example, applying the Bishop–Gromov inequality to the $(n+q)$ -dimensional manifolds and taking the limit, we obtain a Bishop–Gromov-type inequality for the measures of the distance balls in M . Namely, let vol_ϕ denote the weighted measure. Then for $0 < u_1 < u_2$, $\frac{\text{vol}_\phi(B_{u_2})}{\text{vol}_\phi(B_{u_1})}$ is less than or equal to the corresponding quantity in the $(n+q)$ -dimensional space form of Ricci curvature r . If $r > 0$ then applying Myers' theorem to the $(n+q)$ -dimensional manifolds and taking the limit, we obtain that $\text{diam}(M) \leq \pi \sqrt{\frac{n+q-1}{r}}$. This gives alternative proofs of some results of Qian [20, Corollary 2 and Theorem 5] in the special case when q is an integer greater than one. (The results of [20] are valid for all positive q .) One can also show that if $\widetilde{\text{Ric}}_q \geq rg$ with $q \in (0, \infty)$ then (M, g, ϕ) satisfies the directional Bishop–Gromov inequality of [8, (A.2.2)] with respect to a model space of formal dimension $n+q$.

2. Similarly, if q is an integer greater than one then there are Sobolev inequalities for the $(n+q)$ -dimensional collapsing manifolds [6, Theorem 3, p. 397]. Applying these inequalities to functions that pullback from M , we obtain weighted Sobolev inequalities for M . Namely, put $V = \int_M \phi d\text{vol}_M$. Given $\alpha, \beta \in [1, \infty)$ such that $\alpha \leq \frac{(n+q)\beta}{n+q-\beta}$, let $\Sigma(n+q; \alpha, \beta)$ be the Sobolev constant of the standard $(n+q)$ -sphere S^{n+q} , defined by

$$\Sigma(n+q; \alpha, \beta) = \sup \left\{ \frac{\|f\|_\alpha}{\|df\|_\beta} : f \in W^{1,\beta}(S^{n+q}), f \neq 0, \int_{S^{n+q}} f = 0 \right\}. \quad (6.1)$$

Then if $\widetilde{\text{Ric}}_q(M, g, \phi) \geq \frac{n+q-1}{R^2}g$, we have

$$\begin{aligned} \left(\int_M f^\alpha \phi \, d\text{vol}_M \right)^{\frac{1}{\alpha}} &\leq \Sigma(n+q; \alpha, \beta) R \left(\frac{V}{\text{vol}(S^{n+q})} \right)^{\frac{1}{\alpha}-\frac{1}{\beta}} \left(\int_M |\nabla f|^\beta \phi \, d\text{vol}_M \right)^{\frac{1}{\beta}} \\ &\quad + V^{\frac{1}{\alpha}-\frac{1}{\beta}} \left(\int_M f^\beta \phi \, d\text{vol}_M \right)^{\frac{1}{\beta}} \end{aligned} \quad (6.2)$$

for $f \in W^{1,\beta}(M)$. In the case $\beta = 2$, these inequalities appeared in [3].

3. From the Bishop–Gromov-type inequalities, one can easily show that for any $q, D \in \mathbb{R}^+$ and $r \in \mathbb{R}$, the space of Riemannian manifolds (M, g) with a smooth positive probability measure $\phi \, d\text{vol}_M$ satisfying $\widetilde{\text{Ric}}_q(M, g, \phi) \geq rg$ and $\text{diam}(M, g) \leq D$, taken modulo diffeomorphisms, is precompact in the measured Gromov–Hausdorff topology.

Since the relative volume in \mathbb{R}^{n+q} of B_{u_2} and B_{u_1} is $\left(\frac{u_2}{u_1}\right)^{n+q}$, we cannot expect any Bishop–Gromov-type comparison theorem for the masses of balls in spaces with $\widetilde{\text{Ric}}_\infty$ bounded below, i.e. when $q \rightarrow \infty$ in $\widetilde{\text{Ric}}_q$. However, it is interesting that spaces with $\widetilde{\text{Ric}}_\infty \geq rg$ for $r > 0$ do admit isoperimetric inequalities [5].

4. It is an interesting question whether there is a good synthetic notion of a metric-measure space with Ricci curvature bounded below, in analogy to the notion of an Alexandrov space with curvature bounded below. See [8, Appendix 2] for discussion. It is clear from Theorem 3.1 that triples (M, g, ϕ) with $\widetilde{\text{Ric}}_q \geq rg$ are examples of metric-measure spaces with generalized Ricci curvature bounded below by r , at least if q is an integer greater than one.

There are various ways that one could try to extend the notion of Ricci curvature bounded below, from smooth metric-measure spaces to more general metric-measure spaces. One could fix $q \in (0, \infty)$ and try to extend the notion of having $\widetilde{\text{Ric}}_q \geq rg$. Or one could consider all q simultaneously, and say in particular that a triple (M, g, ϕ) has generalized Ricci curvature bounded below by r if $\widetilde{\text{Ric}}_q \geq rg$ for some $q \in (0, \infty)$. Or one could consider a triple (M, g, ϕ) to have generalized Ricci curvature bounded below by r if $\widetilde{\text{Ric}}_\infty \geq rg$.

We note that there is a difference between having $\widetilde{\text{Ric}}_q \geq rg$ for some $q \in (0, \infty)$ and having $\widetilde{\text{Ric}}_\infty \geq rg$. For example, if $r > 0$ and $\widetilde{\text{Ric}}_q \geq rg$ for some $q \in (0, \infty)$ then M is compact [20, Theorem 5], whereas if $\widetilde{\text{Ric}}_\infty \geq rg$ then M can be noncompact (as in the case of \mathbb{R} with $\phi(x) = e^{-\frac{r}{2}x^2}$.) It is also easy to see that triples (M, g, ϕ) with $\widetilde{\text{Ric}}_\infty \geq 0$ generally do not satisfy the splitting principle.

If one does consider a triple (M, g, ϕ) with $\widetilde{\text{Ric}}_\infty \geq rg$ to be an admissible space with generalized Ricci curvature bounded below by r then one has a large class of examples. For instance, from this viewpoint it would be reasonable to say that flat \mathbb{R}^n with the measure $e^{-V} dx_1 \dots dx_n$ has nonnegative generalized Ricci curvature if V is any convex function on \mathbb{R}^n .

References

- [1] M. Anderson and J. Cheeger, C^α -Compactness for Manifolds with Ricci Curvature and Injectivity Radius Bounded Below, *J. Diff. Geom.* **35** (1992), 265–281.
- [2] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto and G. Scheffer, *Sur les Inégalités de Sobolev Logarithmiques*, Panoramas et Synthèses 10, Soc. Math. de France, Paris, 2000.
- [3] D. Bakry, L'Hypercontractivité et son Utilisation en Théorie des Semigroupes, in: *Lectures on Probability Theory (Saint-Flour, 1992)*, 1–114, Lecture Notes in Math. 1581, Springer, Berlin, 1994.
- [4] D. Bakry and M. Émery, Diffusions Hypercontractives, in: *Séminaire de probabilités XIX, 1983/84*, 177–206, Lecture Notes in Math. 1123, Springer, Berlin, 1985.
- [5] D. Bakry and M. Ledoux, Lévy–Gromov's Isoperimetric Inequality for an Infinite-Dimensional Diffusion Generator, *Inv. Math.* **123** (1996), 259–281.
- [6] P. Bérard, From Vanishing Theorems to Estimating Theorems: the Bochner Technique Revisited, *Bull. Amer. Math. Soc.* **19** (1988), 371–406.
- [7] A. Besse, *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 10, Springer-Verlag, Berlin, 1987.
- [8] J. Cheeger and T. Colding, On the Structure of Spaces with Ricci Curvature Bounded Below I, *J. Diff. Geom.* **46** (1997), 406–480.
- [9] J. Cheeger and T. Colding, On the Structure of Spaces with Ricci Curvature Bounded Below II, *J. Diff. Geom.* **54** (2000), 13–35.
- [10] J. Cheeger and T. Colding, On the Structure of Spaces with Ricci Curvature Bounded Below III, *J. Diff. Geom.* **54** (2000), 37–74.
- [11] J. Cheeger, K. Fukaya and M. Gromov, Nilpotent Structures and Invariant Metrics on Collapsed Manifolds, *J. Amer. Math. Soc.* **5** (1992), 327–372.
- [12] J. Cheeger and D. Gromoll, The Splitting Theorem for Manifolds of Nonnegative Ricci Curvature, *J. Diff. Geom.* **6** (1971), 119–128.
- [13] X. Dai, G. Wei and R. Ye, Smoothing Riemannian Metrics with Ricci Curvature Bounds, *Manuscripta Math.* **90** (1996), 49–61.
- [14] J.-P. Demailly, Estimations L^2 pour l'Opérateur $\bar{\partial}$ d'un Fibré Vectoriel Holomorphe Semi-Positif Au-Dessus d'une Variété Kählérienne Complète, *Ann. Sci. École Norm. Sup.* (4) **15** (1982), 457–511.
- [15] K. Fukaya, Collapsing of Riemannian Manifolds and Eigenvalues of Laplace Operator, *Inv. Math.* **87** (1987), 517–547.
- [16] K. Fukaya, A Boundary of the Set of the Riemannian Manifolds with Bounded Curvatures and Diameters, *J. Diff. Geom.* **28** (1988), 1–21.
- [17] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Birkhäuser, Boston, 1999.
- [18] A. Kasue, A Convergence Theorem for Riemannian Manifolds and Some Applications, *Nagoya Math. J.* **114** (1989), 21–51.
- [19] M. Ledoux, The Geometry of Markov Diffusion Generators, *Ann. Fac. Sci. Toulouse Math.* (6) **9** (2000), 305–366.
- [20] Z. Qian, Estimates for Weighted Volumes and Applications, *Quart. J. Math. Oxford Ser.* (2) **48** (1997), 235–242.
- [21] X. Rong, On the Fundamental Groups of Manifolds of Positive Sectional Curvature, *Ann. of Math.* **143** (1996), 397–411.
- [22] E. Witten, Supersymmetry and Morse Theory, *J. Diff. Geom.* **17** (1982), 661–692.

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