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## Some geometric properties of the Bakry–Émery–Ricci tensor

John Lott

**Abstract.** The Bakry–Émery tensor gives an analog of the Ricci tensor for a Riemannian manifold with a smooth measure. We show that some of the topological consequences of having a positive or nonnegative Ricci tensor are also valid for the Bakry–Émery tensor. We show that the Bakry–Émery tensor is nondecreasing under a Riemannian submersion whose fiber transport preserves measures up to constants. We give some relations between the Bakry–Émery tensor and measured Gromov–Hausdorff limits.

**Mathematics Subject Classification (2000).** Primary 53C21; Secondary 58G25.

**Keywords.** Ricci tensor, metric-measure space, Riemannian submersion.

### 1. Introduction

When considering the metric structure of manifolds with lower Ricci curvature bounds, it is natural to carry along the extra structure of a measure and consider metric-measure spaces. This is especially relevant for collapsing, and has been discussed by Cheeger–Colding [8, 9, 10], Fukaya [15] and Gromov [17, Chapter 3 $\frac{1}{2}$ ].

In this paper we consider smooth metric-measure spaces. Let  $M$  be an  $n$ -dimensional Riemannian manifold, with metric  $g$ . Let  $d\text{vol}_M$  denote the Riemannian density on  $M$ . Let  $\phi$  be a smooth positive function on  $M$ . Then  $(M, \phi d\text{vol}_M)$  is a smooth metric-measure space. For reasons coming from the study of diffusion processes, Bakry and Émery [4] defined a generalization of the Ricci tensor of  $M$  by

$$\widetilde{\text{Ric}}_\infty = \text{Ric} - \text{Hess}(\ln \phi). \quad (1.1)$$

In terms of indices,  $\left(\widetilde{\text{Ric}}_\infty\right)_{\alpha\beta} = \text{Ric}_{\alpha\beta} - (\ln \phi)_{;\alpha\beta}$ .

It turns out that the Bakry–Émery tensor (1.1) has interesting connections to logarithmic Sobolev inequalities, isoperimetric inequalities and heat semigroups. We refer to [2] and [19] for information on these connections. (In fact, Bakry and Émery defined their tensor in a more abstract setting than what we consider.)

We are interested in the geometric implications of bounds on the Bakry–Émery tensor. As in [20], let us define a related tensor  $\widetilde{\text{Ric}}_q$ . Given  $q \in (0, \infty)$ , put

$$\begin{aligned}\widetilde{\text{Ric}}_q &= \text{Ric} - \text{Hess}(\ln \phi) - \frac{1}{q} d \ln \phi \otimes d \ln \phi \\ &= \text{Ric} - \frac{\text{Hess}(\phi)}{\phi} + \left(1 - \frac{1}{q}\right) \frac{d\phi}{\phi} \otimes \frac{d\phi}{\phi} \\ &= \text{Ric} - q \frac{\text{Hess}\left(\phi^{\frac{1}{q}}\right)}{\phi^{\frac{1}{q}}}.\end{aligned}\tag{1.2}$$

Clearly, if  $\widetilde{\text{Ric}}_q \geq rg$  then  $\widetilde{\text{Ric}}_\infty \geq rg$ . In the terminology of [3], a condition of the form  $\widetilde{\text{Ric}}_q \geq rg$  implies a curvature-dimension inequality  $\text{CD}(r, n+q)$ .

Our first result extends some classical topological results about the Ricci tensor (i.e. when  $\phi$  is constant) to the setting of the Bakry–Émery tensor.

**Theorem 1.** *Suppose that  $M$  is connected and closed.*

1. *If  $\widetilde{\text{Ric}}_\infty > 0$  then  $\pi_1(M)$  is finite.*
2. *If  $\widetilde{\text{Ric}}_q \geq 0$  and  $q \in (0, \infty)$  then  $\pi_1(M)$  has a finite-index free abelian subgroup of rank at most  $n$ .*
3. *If  $\widetilde{\text{Ric}}_\infty \geq 0$  then  $H^1(M; \mathbb{R})$  is isomorphic to the linear space of parallel 1-forms on  $M$  whose pairing with  $\text{grad}(\phi)$  vanishes identically. In particular, if  $\widetilde{\text{Ric}}_\infty \geq 0$  then  $b_1(M) \leq n$ . If  $\widetilde{\text{Ric}}_\infty \geq 0$  and  $b_1(M) = n$  then  $M$  is a flat torus and  $\phi$  is constant.*
4. *If  $\widetilde{\text{Ric}}_\infty < 0$  then the isometry group of  $(M, g)$  is finite.*
5. *If  $\widetilde{\text{Ric}}_\infty \leq 0$  then any Killing vector field on  $(M, g)$  is parallel and annihilates  $\phi$ .*

**Remark.** Theorem 1.2 is a strengthening of [20, Theorem 6], which says that if  $\widetilde{\text{Ric}}_q \geq 0$  and  $q \in (0, \infty)$  then  $\pi_1(M)$  has polynomial growth of order at most  $n+q$ .

The proofs of parts 3–5 of Theorem 1 use a Bochner-type identity. If the pair  $(g, \phi)$  is only  $C^0 \cap H^1$ -regular then one can use this identity to still make sense of the notion  $\widetilde{\text{Ric}}_\infty \geq rg$  or  $\widetilde{\text{Ric}}_q \geq rg$  (see Definition 1 of Section 2).

An important result in the study of manifolds of nonnegative sectional curvature is O’Neill’s theorem, which says that sectional curvature is nondecreasing under a Riemannian submersion [7, Chapter 9]. We show that there is a Ricci analog of O’Neill’s theorem, provided that one uses the Bakry–Émery tensor and assumes that the fiber transport of the Riemannian submersion preserves measures up to multiplicative constants.

Suppose that a Riemannian submersion  $p : M \rightarrow B$  has compact fiber  $F$ . Put  $F_b = p^{-1}(b)$ . Given a smooth curve  $\gamma : [0, 1] \rightarrow B$  and a point  $m \in F_{\gamma(0)}$ , let  $\rho(m)$  be the endpoint  $\bar{\gamma}(1)$  of the horizontal lift  $\bar{\gamma}$  of  $\gamma$  that starts at  $\bar{\gamma}(0)$ . Then  $\rho$  is the fiber transport diffeomorphism from  $F_{\gamma(0)}$  to  $F_{\gamma(1)}$ .

Given the positive function  $\phi^M$  on  $M$ , define  $\phi^B$ , a smooth positive function on  $B$ , by

$$p_*(\phi^M d\text{vol}_M) = \phi^B d\text{vol}_B. \quad (1.3)$$

Let  $\widetilde{\text{Ric}}_\infty^M$  and  $\widetilde{\text{Ric}}_\infty^B$  denote the corresponding Bakry–Émery tensors. Let  $d\text{vol}_F$  denote the fiberwise Riemannian density.

**Theorem 2.** *Suppose that fiber transport preserves the fiberwise measure  $\phi_M d\text{vol}_F$  up to a multiplicative constant, i.e. for any smooth curve  $\gamma : [0, 1] \rightarrow B$ , there is a constant  $c_\gamma > 0$  such that  $\rho^* \left( \phi^M \Big|_{F_{\gamma(1)}} d\text{vol}_{F_{\gamma(1)}} \right) = c_\gamma \phi^M \Big|_{F_{\gamma(0)}} d\text{vol}_{F_{\gamma(0)}}$ .*

1. *For any  $r \in \mathbb{R}$ , if  $\widetilde{\text{Ric}}_\infty^M \geq rg^M$  then  $\widetilde{\text{Ric}}_\infty^B \geq rg^B$ .*
2. *Suppose in addition that  $\phi^M = 1$ . Put  $q = \dim(F)$ . For any  $r \in \mathbb{R}$ , if  $\widetilde{\text{Ric}}^M \geq rg^M$  then  $\widetilde{\text{Ric}}_q^B \geq rg^B$ .*

Using Theorem 2, we show a relationship between  $\widetilde{\text{Ric}}_q$  and collapsing.

**Theorem 3.** 1. *Given  $r \in \mathbb{R}$  and an integer  $q \geq 2$ , let  $(B, \phi)$  be a smooth closed measured Riemannian manifold with  $\widetilde{\text{Ric}}_q^B \geq rg^B$ . Then  $(B, \phi)$  is the measured Gromov–Hausdorff limit of a sequence of  $(n + q)$ -dimensional closed Riemannian manifolds  $(M_i, g_i)$  with  $\text{Ric}(M_i, g_i) \geq rg_i$ .*

2. *Let  $\{(M_i, g_i)\}_{i=1}^\infty$  be a sequence of  $N$ -dimensional connected closed Riemannian manifolds with sectional curvatures bounded above in absolute value by  $\Lambda$  and diameters bounded above by  $D$ , for some  $D, \Lambda \in \mathbb{R}^+$ . Let  $(X, \mu)$  be a limit point for  $\{(M_i, g_i)\}_{i=1}^\infty$  in the measured Gromov–Hausdorff topology. Suppose that for some  $r \in \mathbb{R}$  and all  $i \in \mathbb{Z}^+$ ,  $\text{Ric}(M_i, g_i) \geq rg_i$ . Suppose that  $X$  is an  $n$ -dimensional closed manifold. Put  $q = N - n$ .*

- a. *If  $q = 0$  then  $X$  has  $\text{Ric} \geq rg$  in the generalized sense of Definition 1 below.*
- b. *If  $q > 0$  then  $X$  has  $\widetilde{\text{Ric}}_q \geq rg$  in the generalized sense of Definition 1 below.*

Finally, we give a condition in terms of distances and masses that is equivalent to having Bakry–Émery tensor bounded below by  $r$ . If  $\mathcal{O}$  is a measurable subset of  $M$ , put

$$\text{vol}_\phi(\mathcal{O}) = \int_{\mathcal{O}} \phi d\text{vol}_M. \quad (1.4)$$

Following [17, Section 5.45], we define the notion of a distance tube in  $M$ . Let  $T_0$  be a closed subset of  $M$ . A subset  $T \subset M$  containing  $T_0$  is a distance tube with base  $T_0$  if for all  $t \in T$ , there is a segment  $s \subset T$  from some  $t_0 \in T_0$  to  $t$  with length  $l(s) = d(t, T_0)$ . For  $0 < u_1 < u_2$ , define the distance annulus

$$A(u_1, u_2) = \{t \in T : u_1 \leq d(t, T_0) \leq u_2\}. \quad (1.5)$$



Given  $c \in \mathbb{R}$ , put

$$\widehat{v}(u_1, u_2, c) = \int_{u_1}^{u_2} e^{-\frac{r}{2}x^2 + cx} dx. \quad (1.6)$$

**Theorem 4.** *Suppose that  $\widetilde{\text{Ric}}_\infty(M, g, \phi) \geq rg$  for some  $r \in \mathbb{R}$ . Given numbers  $0 < u_1 < u_2 < u_3$ , we assume that the tube  $T$  is a disjoint union of segments  $s$ , starting at  $T_0$ , of length at least  $u_3$ . We also assume that  $\text{vol}_\phi(A(u_2, u_3)) > 0$ . Suppose that for some  $c \in \mathbb{R}$ ,*

$$\frac{\text{vol}_\phi(A(u_2, u_3))}{\text{vol}_\phi(A(u_1, u_2))} \leq \frac{\widehat{v}(u_2, u_3, c)}{\widehat{v}(u_1, u_2, c)}. \quad (1.7)$$

*Then there is a subtube  $T' \subset T$  consisting of a union of segments  $s$  from  $T_0$ , such that*

1.

$$\frac{\text{vol}_\phi(T' \cap A(u_1, u_2))}{\text{vol}_\phi(A(u_1, u_2))} \geq 1 - \frac{\text{vol}_\phi(A(u_2, u_3))}{\text{vol}_\phi(A(u_1, u_2))} \left( \frac{\widehat{v}(u_2, u_3, c)}{\widehat{v}(u_1, u_2, c)} \right)^{-1}, \quad (1.8)$$

2. *If a segment  $s \subset T$ , starting from  $T_0$ , intersects  $T' \cap A(u_2, u_3)$  then  $s \subset T'$ , and*

3. *For all  $u_4 > u_3$ ,*

$$\frac{\text{vol}_\phi(T' \cap A(u_3, u_4))}{\text{vol}_\phi(T' \cap A(u_2, u_3))} \leq \frac{\widehat{v}(u_3, u_4, c)}{\widehat{v}(u_2, u_3, c)}. \quad (1.9)$$

*Conversely, suppose that there is a number  $r \in \mathbb{R}$  so that for each tube  $T$  and  $c \in \mathbb{R}$  satisfying (1.7), there is a subtube  $T'$  with the above properties. Then  $\widetilde{\text{Ric}}_\infty(M, g, \phi) \geq rg$ .*

In Sections 2–5 we prove Theorems 1–4, respectively. In Section 6 we make some remarks.

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## 2. Proof of Theorem 1

We first prove parts 1 and 2 of the theorem. If  $\widetilde{\text{Ric}}_\infty > 0$  then  $\widetilde{\text{Ric}}_q > 0$  for some  $q \in (0, \infty)$ . Increasing  $q$  if necessary, we may assume without loss of generality that  $q$  is an integer greater than one. Thus for parts 1 and 2, it is enough to consider the case when  $\widetilde{\text{Ric}}_q > 0$  or  $\widetilde{\text{Ric}}_q \geq 0$ , for some integer  $q$  greater than one.

Given  $i \in \mathbb{Z}^+$ , consider  $S^q \times M$  with the warped product metric  $g^{S^q \times M} = g^M + i^{-2} \phi^{\frac{2}{q}} g^{S^q}$ . Let  $p : S^q \times M \rightarrow M$  be the projection. Let  $\overline{X}$  be the horizontal lift to  $S^q \times M$  of a vector field  $X$  on  $M$  and let  $\overline{U}$  be a vertical vector field on

$S^q \times M$ . From [7, Proposition 9.106],

$$\begin{aligned} \operatorname{Ric}^{S^q \times M}(\overline{X}, \overline{X}) &= p^* \left( \operatorname{Ric}^M(X, X) - q \frac{\operatorname{Hess}(\phi^{\frac{1}{q}})(X, X)}{\phi^{\frac{1}{q}}} \right), \\ \operatorname{Ric}^{S^q \times M}(\overline{X}, \overline{U}) &= 0 \\ \operatorname{Ric}^{S^q \times M}(\overline{U}, \overline{U}) &= \operatorname{Ric}^{S^q}(\overline{U}, \overline{U}) + (\overline{U}, \overline{U}) p^* \left( -\frac{\nabla^2 \phi^{\frac{1}{q}}}{\phi^{\frac{1}{q}}} - (q-1) \frac{|\nabla \phi^{\frac{1}{q}}|^2}{\phi^{\frac{2}{q}}} \right). \end{aligned} \quad (2.1)$$

Taking  $i \rightarrow \infty$ , we see that if  $\widetilde{\operatorname{Ric}}_q(M, g, \phi) \geq rg$  then  $(M, g^M, \phi)$  is the limit of a sequence of  $(n+q)$ -dimensional manifolds with Ricci curvature bounded below by  $r$ . If  $r$  is positive then from Myers' theorem,  $\pi_1(S^q \times M) \cong \pi_1(M)$  is finite. This proves part 1 of the theorem.

Now suppose that  $r \geq 0$ . For  $i$  large, the warped product metric on  $S^q \times M$  has nonnegative Ricci curvature. There is a  $k \geq 0$  so that  $\pi_1(S^q \times M) \cong \pi_1(M)$  has a finite-index free abelian subgroup of rank  $k$  and the universal cover  $S^q \times \widetilde{M}$  has an isometric splitting as  $\mathbb{R}^k \times Y^{n+q-k}$ , where  $Y$  is closed and simply-connected [12]. Considering the cohomology groups of  $S^q \times \widetilde{M} \cong \mathbb{R}^k \times Y^{n+q-k}$ , it follows that

$$q + \max\{j : H^j(\widetilde{M}; \mathbb{Z}) \neq 0\} = n + q - k. \quad (2.2)$$

Then  $k = n - \max\{j : H^j(\widetilde{M}; \mathbb{Z}) \neq 0\} \leq n$ , which proves part 2 of the theorem.

To prove the rest of the theorem, if  $V$  is a vector field on  $M$ , let  $V^\sharp$  denote the dual 1-form. If  $\omega$  is a 1-form on  $M$ , let  $\omega_\sharp$  denote the dual vector field. Let  $i_V$  denote interior multiplication with respect to  $V$  and let  $\mathcal{L}_V$  denote Lie differentiation with respect to  $V$ .

If  $T$  is a tensor field on  $M$ , let  $(T, T) \in C^\infty(M)$  be the inner product coming from the Riemannian metric  $g$ . Put

$$\langle T, T \rangle = \int_M (T, T)(m) \phi(m) d\operatorname{vol}_M(m). \quad (2.3)$$

Let  $(\Omega^*(M), d)$  denote the de Rham complex of  $M$ . Let  $\delta$  be the formal adjoint of  $d$  with respect to the Riemannian metric  $g$ , i.e. in the case  $\phi = 1$ , and let  $\widetilde{\delta}$  be the formal adjoint of  $d$  with respect to  $\langle \cdot, \cdot \rangle$ . Then

$$\widetilde{\delta} = \delta - i_{(d \ln \phi)_\sharp}. \quad (2.4)$$

Put  $\Delta = d\delta + \delta d$  and  $\widetilde{\Delta} = d\widetilde{\delta} + \widetilde{\delta}d$ . Then

$$\widetilde{\Delta} = \Delta - di_{(d \ln \phi)_\sharp} - i_{(d \ln \phi)_\sharp}d = \Delta - \mathcal{L}_{(d \ln \phi)_\sharp}. \quad (2.5)$$

The Bochner identity says that if  $\omega$  is a 1-form then there is an equality of functions on  $M$ :

$$\frac{1}{2} \delta d(\omega, \omega) = (\omega, \Delta \omega) - (\nabla \omega, \nabla \omega) - (\omega, \operatorname{Ric} \omega). \quad (2.6)$$

On the other hand,

$$\frac{1}{2}i_{(d \ln \phi)^\sharp} d(\omega, \omega) = \frac{1}{2} \mathcal{L}_{(d \ln \phi)^\sharp}(\omega, \omega). \quad (2.7)$$

We have

$$\mathcal{L}_{(d \ln \phi)^\sharp} g = 2 \operatorname{Hess}(\ln \phi). \quad (2.8)$$

Then

$$\frac{1}{2}i_{(d \ln \phi)^\sharp} d(\omega, \omega) = (\omega, \mathcal{L}_{(d \ln \phi)^\sharp} \omega) - (\omega, \operatorname{Hess}(\ln \phi) \omega). \quad (2.9)$$

(The minus sign in (2.9) comes from the fact that the pairing is on 1-forms instead of vector fields.) Equations (2.4), (2.5), (2.6) and (2.9) give

$$\frac{1}{2} \widetilde{\delta} d(\omega, \omega) = (\omega, \widetilde{\Delta} \omega) - \langle \nabla \omega, \nabla \omega \rangle - (\omega, \widetilde{\operatorname{Ric}}_\infty \omega). \quad (2.10)$$

Multiplying (2.10) by  $\phi$  and integrating over  $M$ , we obtain

$$0 = \langle \omega, \widetilde{\Delta} \omega \rangle - \langle \nabla \omega, \nabla \omega \rangle - \langle \omega, \widetilde{\operatorname{Ric}}_\infty \omega \rangle, \quad (2.11)$$

or

$$\langle d\omega, d\omega \rangle + \langle \widetilde{\delta} \omega, \widetilde{\delta} \omega \rangle - \langle \nabla \omega, \nabla \omega \rangle = \langle \omega, \widetilde{\operatorname{Ric}}_\infty \omega \rangle. \quad (2.12)$$

We can apply usual elliptic theory to the de Rham complex, with the inner product  $\langle \cdot, \cdot \rangle$ , to obtain an isomorphism

$$H^*(M; \mathbb{R}) \cong \{\omega \in \Omega^*(M) : d\omega = \widetilde{\delta} \omega = 0\}. \quad (2.13)$$

If  $\widetilde{\operatorname{Ric}}_\infty \geq 0$  and a 1-form  $\omega$  satisfies  $d\omega = \widetilde{\delta} \omega = 0$  then (2.12) implies that  $\nabla \omega = 0$ . Hence  $\delta \omega = 0$ . Along with  $\widetilde{\delta} \omega = 0$ , (2.4) now implies that  $\omega(\operatorname{grad}(\phi)) = 0$ . Conversely, if  $\nabla \omega = \omega(\operatorname{grad}(\phi)) = 0$  then  $d\omega = \widetilde{\delta} \omega = 0$ . This proves the isomorphism in part 3 of the theorem.

If  $b_1(M) = n$  then there are  $n$  linearly-independent parallel 1-forms on  $M$  that annihilate  $\operatorname{grad}(\phi)$ . The usual argument shows that  $M$  is a flat torus. As the parallel 1-forms on  $M$  annihilate  $\operatorname{grad}(\phi)$ ,  $\phi$  must be constant. This proves part 3 of the theorem.

A pointwise algebraic computation shows that

$$(d\omega, d\omega) + (\mathcal{L}_{\omega^\sharp} g, \mathcal{L}_{\omega^\sharp} g) = 2\langle \nabla \omega, \nabla \omega \rangle. \quad (2.14)$$

Then (2.12) becomes

$$\langle \nabla \omega, \nabla \omega \rangle + \langle \widetilde{\delta} \omega, \widetilde{\delta} \omega \rangle - \langle \omega, \widetilde{\operatorname{Ric}}_\infty \omega \rangle = \langle \mathcal{L}_{\omega^\sharp} g, \mathcal{L}_{\omega^\sharp} g \rangle. \quad (2.15)$$

If  $\widetilde{\operatorname{Ric}}_\infty < 0$  and  $\mathcal{L}_V g = 0$  then taking  $\omega = V^\sharp$ , (2.15) implies that  $V = 0$ . Hence the isometry group of  $(M, g)$  is discrete and, being compact, must be finite. This proves part 4 of the theorem.

If  $\widetilde{\text{Ric}}_\infty \leq 0$  and  $\mathcal{L}_V g = 0$  then (2.15) implies that  $\nabla V^\sharp = \widetilde{\delta} V^\sharp = 0$ . As before, we obtain that  $V\phi = 0$ . This proves part 5 of the theorem.

**Remarks.** 1. If we put  $\omega = df$  in (2.10) then we recover the definition of  $\widetilde{\text{Ric}}_\infty$  from [4].

2. Jianguo Cao pointed out to me that a formula related to (2.12) has been used to study the  $\bar{\partial}$ -operator on complete Kähler manifolds [14, Théorème 5.1].

3. The operator  $\widetilde{\Delta}$  is related to the Witten Laplacian of [22], but the two operators are distinct. To see the relation, note that  $\widetilde{\delta} = \phi^{-1} \delta \phi$ . Put  $D = \phi^{\frac{1}{2}} d \phi^{-\frac{1}{2}}$  and  $D^* = \phi^{-\frac{1}{2}} \delta \phi^{\frac{1}{2}}$ . Then the Witten Laplacian  $DD^* + D^*D$  is related to  $\widetilde{\Delta}$  by

$$DD^* + D^*D = \phi^{\frac{1}{2}} \widetilde{\Delta} \phi^{-\frac{1}{2}}. \quad (2.16)$$

The Bochner-type identity (2.12), when translated to a statement about  $DD^* + D^*D$ , becomes

$$DD^* + D^*D = \left( \phi^{\frac{1}{2}} \nabla \phi^{-\frac{1}{2}} \right)^* \left( \phi^{\frac{1}{2}} \nabla \phi^{-\frac{1}{2}} \right) + \widetilde{\text{Ric}}_\infty, \quad (2.17)$$

where the adjoints are with respect to the unweighted  $L^2$ -inner product. In contrast, in Morse–Witten theory one collects the terms differently, by writing  $DD^* + D^*D = \nabla^* \nabla + \dots$

4. The equality (2.12) gives a way of defining the notion of  $\widetilde{\text{Ric}}_\infty \geq rg$  for a class of nonsmooth measured manifolds  $(M, g, \phi)$ . Namely, suppose that  $M$  is a manifold whose transition maps are  $C^{1,1}$ -regular. Let  $g$  be a Riemannian metric on  $M$  whose components, in local charts, are in  $C^0 \cap H^1$ , where  $H^1$  denotes the Sobolev space. Let  $\phi \in C^0(M) \cap H^1_{loc}(M)$  be a positive function. (There are a smooth manifold  $M'$  and a  $C^{1,1}$ -diffeomorphism  $M' \rightarrow M$ . Hence after pulling back, if one wants then one can assume that  $g$  and  $\phi$  are defined on a smooth manifold.)

**Definition 1.** We say that  $\text{Ric}(M, g) \geq rg$  if for all compactly-supported Lipschitz-regular 1-forms  $\omega$  on  $M$ ,

$$\int_M (d\omega, d\omega) d\text{vol}_M + \int_M (\delta\omega, \delta\omega) d\text{vol}_M - \int_M (\nabla\omega, \nabla\omega) d\text{vol}_M \geq r \int_M (\omega, \omega) d\text{vol}_M. \quad (2.18)$$

We say that  $\widetilde{\text{Ric}}_\infty(M, g, \phi) \geq rg$  if for all compactly-supported Lipschitz-regular 1-forms  $\omega$  on  $M$ ,

$$\langle d\omega, d\omega \rangle + \langle \widetilde{\delta}\omega, \widetilde{\delta}\omega \rangle - \langle \nabla\omega, \nabla\omega \rangle \geq r \langle \omega, \omega \rangle. \quad (2.19)$$

We say that  $\widetilde{\text{Ric}}_q(M, g, \phi) \geq rg$  if for all compactly-supported Lipschitz-regular 1-forms  $\omega$  on  $M$ ,

$$\langle d\omega, d\omega \rangle + \langle \widetilde{\delta}\omega, \widetilde{\delta}\omega \rangle - \langle \nabla\omega, \nabla\omega \rangle - \frac{1}{q} \int_M (\omega(\nabla \ln \phi))^2 \phi d\text{vol}_M \geq r \langle \omega, \omega \rangle. \quad (2.20)$$

An immediate consequence of the definition is the following lemma.

**Lemma 1.** *Let  $M$  be a smooth closed manifold.*

1. *If  $\{g_i\}_{i=1}^\infty$  is a sequence of smooth Riemannian metrics on  $M$  with  $\text{Ric}(M, g_i) \geq rg_i$ , and  $g_i \xrightarrow{C^0 \cap H^1} g$  for some  $C^0 \cap H^1$ -regular metric  $g$ , then  $\text{Ric}(M, g) \geq rg$ .*
2. *If  $\{(g_i, \phi_i)\}_{i=1}^\infty$  is a sequence of smooth Riemannian metrics and smooth positive functions on  $M$  with  $\widetilde{\text{Ric}}_\infty(M, g_i, \phi_i) \geq rg_i$ , and  $(g_i, \phi_i) \xrightarrow{C^0 \cap H^1} (g, \phi)$  for some  $C^0 \cap H^1$ -regular pair  $(g, \phi)$ , then  $\widetilde{\text{Ric}}_\infty(M, g, \phi) \geq rg$ .*
3. *If  $\{(g_i, \phi_i)\}_{i=1}^\infty$  is a sequence of smooth Riemannian metrics and smooth positive functions on  $M$  with  $\widetilde{\text{Ric}}_q(M, g_i, \phi_i) \geq rg_i$ , and  $(g_i, \phi_i) \xrightarrow{C^0 \cap H^1} (g, \phi)$  for some  $C^0 \cap H^1$ -regular pair  $(g, \phi)$ , then  $\widetilde{\text{Ric}}_q(M, g, \phi) \geq rg$ .*

For example, let  $\{(M_i, g_i)\}_{i=1}^\infty$  be a sequence of  $n$ -dimensional closed Riemannian manifolds with Ricci curvatures bounded below by  $r \in \mathbb{R}$ , injectivity radii bounded below by  $i_0 \in \mathbb{R}^+$  and diameters bounded above by  $D \in \mathbb{R}^+$ . Then  $\{(M_i, g_i)\}_{i=1}^\infty$  has a limit point  $X$  in the Gromov–Hausdorff topology. From [1],  $X$  is an  $n$ -dimensional closed manifold with a Riemannian metric  $g$  that is  $W^{1,p}$ -regular for all  $p \in [1, \infty)$ . From the Sobolev embedding theorem,  $g$  is also  $C^{0,\alpha}$ -regular for all  $\alpha \in (0, 1)$ . After applying diffeomorphisms one has  $W^{1,p}$ -convergence of a subsequence of  $\{(M_i, g_i)\}_{i=1}^\infty$  to  $(X, g)$ , and so  $\text{Ric}(X, g) \geq rg$  in the sense of Definition 1.

For another example, suppose that  $M$  is a compact Kähler manifold with local complex coordinates  $\{z^\alpha\}$  and metric  $g_{\alpha\bar{\beta}}$ . Its Ricci form, in local coordinates, is the  $(1, 1)$ -form  $-\frac{1}{2}\partial\bar{\partial}\ln\det(g)$ . Now suppose that the  $g_{\alpha\bar{\beta}}$  are only  $C^0 \cap H^1$ -regular. The Kähler condition still makes sense distributionally, and the Ricci form makes sense as a closed  $(1, 1)$ -current. Then  $\text{Ric}(M, g) \geq 0$  in the sense of Definition 1 if and only if  $-\frac{1}{2}\partial\bar{\partial}\ln\det(g)$  is a positive current. (This last condition makes sense for a much larger class of  $g$ .)

### 3. Proof of Theorem 2

We (mostly) use the notation of [7, Chapter 9]. If  $X$  is a vector field on  $B$ , let  $\overline{X}$  be its horizontal lift to  $M$ . Let  $N$  be the mean curvature vector field to the fibers  $F$ . Let  $A$  be the curvature of the horizontal distribution. Let  $T$  be the second fundamental form tensor of the fibers  $F$ . Let  $\nabla^M$  be the covariant derivative operator on  $M$  and let  $\nabla^B$  be the covariant derivative operator on  $B$ . From [7, (9.36c)], there is an identity of functions on  $M$ :

$$\text{Ric}^M(\overline{X}, \overline{X}) = \text{Ric}^B(X, X) - 2(A_{\overline{X}}, A_{\overline{X}}) - (T\overline{X}, T\overline{X}) + (\overline{X}, \nabla_{\overline{X}}^M N). \quad (3.1)$$

Given  $b \in B$ , let  $\{\theta_t\}_{t \in (-\epsilon, \epsilon)}$  be the flow of  $X$  as defined in a neighborhood of  $b$  and for  $t$  in some interval  $(-\epsilon, \epsilon)$ . Let  $\{\bar{\theta}_t\}_{t \in (-\epsilon, \epsilon)}$  be the flow of  $\overline{X}$  that covers  $\theta_t$ .

It sends fibers to fibers diffeomorphically. Hence it makes sense to define  $\mathcal{L}_{\overline{X}} d\text{vol}_F$  by

$$(\mathcal{L}_{\overline{X}} d\text{vol}_F)|_{F_b} = \frac{d}{dt} \Big|_{t=0} (\bar{\theta}_t^* d\text{vol}_F)|_{F_b}. \quad (3.2)$$

With our conventions,

$$\mathcal{L}_{\overline{X}} d\text{vol}_F = -(\overline{X}, N) d\text{vol}_F. \quad (3.3)$$

We have

$$\phi^B = \int_F \phi^M d\text{vol}_F. \quad (3.4)$$

Then

$$\begin{aligned} X\phi^B &= \mathcal{L}_X \phi^B = \mathcal{L}_X \int_F \phi^M d\text{vol}_F = \int_F \mathcal{L}_{\overline{X}}(\phi^M d\text{vol}_F) \\ &= \int_F (\overline{X}\phi^M - (\overline{X}, N)\phi^M) d\text{vol}_F \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} XX\phi^B &= \int_F [\overline{X}(\overline{X}\phi^M - (\overline{X}, N)\phi^M) - (\overline{X}, N)(\overline{X}\phi^M - (\overline{X}, N)\phi^M)] d\text{vol}_F \\ &= \int_F [\overline{X}\overline{X}\phi^M - \overline{X}(\overline{X}, N)\phi^M - 2(\overline{X}, N)\overline{X}\phi^M + (\overline{X}, N)^2\phi^M] d\text{vol}_F \\ &= \int_F \left[ \frac{\overline{X}\overline{X}\phi^M}{\phi^M} - (\nabla_{\overline{X}}^M \overline{X}, N) - (\overline{X}, \nabla_{\overline{X}}^M N) - \left( \frac{\overline{X}\phi^M}{\phi^M} \right)^2 \right. \\ &\quad \left. + \left( \frac{\overline{X}\phi^M}{\phi^M} - (\overline{X}, N) \right)^2 \right] \phi^M d\text{vol}_F. \end{aligned} \quad (3.6)$$

Using the fact that  $\nabla_{\overline{X}}^M \overline{X} = \overline{\nabla_X^B X}$  [7, (9.25d)], it follows that

$$\begin{aligned} \text{Hess}(\phi_B)(X, X) &= XX\phi^B - (\nabla_X^B X)\phi^B \\ &= \int_F \left[ \frac{\text{Hess}(\phi^M)(\overline{X}, \overline{X})}{\phi^M} - (\overline{X}, \nabla_{\overline{X}}^M N) - \left( \frac{\overline{X}\phi^M}{\phi^M} \right)^2 \right. \\ &\quad \left. + \left( \frac{\overline{X}\phi^M}{\phi^M} - (\overline{X}, N) \right)^2 \right] \phi^M d\text{vol}_F \\ &= \int_F [\text{Hess}(\ln \phi^M)(\overline{X}, \overline{X}) - (\overline{X}, \nabla_{\overline{X}}^M N) \\ &\quad + \left( \frac{\overline{X}\phi^M}{\phi^M} - (\overline{X}, N) \right)^2] \phi^M d\text{vol}_F. \end{aligned} \quad (3.7)$$

Substituting  $(\bar{X}, \nabla_{\bar{X}}^M N)$  from (3.1) gives

$$\begin{aligned} \text{Ric}^B(X, X)\phi^B - \text{Hess}(\phi^B)(X, X) &= \int_F \left[ \widetilde{\text{Ric}}_\infty^M(\bar{X}, \bar{X}) + 2(A_{\bar{X}}, A_{\bar{X}}) + (T\bar{X}, T\bar{X}) \right. \\ &\quad \left. - \left( \frac{\bar{X}\phi^M}{\phi^M} - (\bar{X}, N) \right)^2 \right] \phi^M d\text{vol}_F \end{aligned} \quad (3.8)$$

Using (3.5),

$$\begin{aligned} \widetilde{\text{Ric}}_\infty^B(X, X)\phi^B &= \left[ \text{Ric}^B(X, X) - \frac{\text{Hess}(\phi^B)(X, X)}{\phi^B} + \frac{(X\phi^B)^2}{(\phi^B)^2} \right] \phi^B \\ &= \int_F \left[ \widetilde{\text{Ric}}_\infty^M(\bar{X}, \bar{X}) + 2(A_{\bar{X}}, A_{\bar{X}}) + (T\bar{X}, T\bar{X}) \right. \\ &\quad \left. - \left( \frac{\bar{X}\phi^M}{\phi^M} - (\bar{X}, N) \right)^2 \right] \phi^M d\text{vol}_F \\ &\quad + \left( \int_F \left( \frac{\bar{X}\phi^M}{\phi^M} - (\bar{X}, N) \right) \phi^M d\text{vol}_F \right)^2 (\phi^B)^{-1}. \end{aligned} \quad (3.9)$$

We have

$$\mathcal{L}_{\bar{X}}(\phi^M d\text{vol}_F) = \left( \frac{\bar{X}\phi^M}{\phi^M} - (\bar{X}, N) \right) \phi^M d\text{vol}_F. \quad (3.10)$$

By assumption,  $\frac{\bar{X}\phi^M}{\phi^M} - (\bar{X}, N)$  is constant on a fiber  $F$ . Then

$$\begin{aligned} \widetilde{\text{Ric}}_\infty^B(X, X)\phi^B &= \int_F \left[ \widetilde{\text{Ric}}_\infty^M(\bar{X}, \bar{X}) + 2(A_{\bar{X}}, A_{\bar{X}}) + (T\bar{X}, T\bar{X}) \right] \phi^M d\text{vol}_F \\ &\geq \int_F \widetilde{\text{Ric}}_\infty^M(\bar{X}, \bar{X}) \phi^M d\text{vol}_F. \end{aligned} \quad (3.11)$$

If  $\widetilde{\text{Ric}}_\infty^M(\bar{X}, \bar{X}) \geq rg^M(\bar{X}, \bar{X})$  then (3.11) implies that  $\widetilde{\text{Ric}}_\infty^B(X, X) \geq rg^B(X, X)$ . This proves Theorem 2.1.

Now suppose that  $\phi^M = 1$ . Equations (1.2) and (3.9) imply that

$$\begin{aligned} \widetilde{\text{Ric}}_q^B(X, X)\phi^B &= \int_F \left[ \text{Ric}^M(\bar{X}, \bar{X}) + 2(A_{\bar{X}}, A_{\bar{X}}) + (T\bar{X}, T\bar{X}) - \frac{1}{q}(\bar{X}, N)^2 \right] d\text{vol}_F \\ &\quad + \left( 1 - \frac{1}{q} \right) \left( - \int_F (\bar{X}, N)^2 d\text{vol}_F + \left( \int_F (\bar{X}, N) d\text{vol}_F \right)^2 \right) (\phi^B)^{-1}. \end{aligned} \quad (3.12)$$

As  $(\overline{X}, N) = -\operatorname{Tr}(T\overline{X})$ , we know that  $(T\overline{X}, T\overline{X}) - \frac{1}{q}(\overline{X}, N)^2 \geq 0$ . By assumption,  $(\overline{X}, N)$  is constant on a fiber  $F$ . Then

$$\begin{aligned} & \widetilde{\operatorname{Ric}}_q^B(X, X) \phi^B \\ &= \int_F \left[ \operatorname{Ric}^M(\overline{X}, \overline{X}) + 2(A_{\overline{X}}, A_{\overline{X}}) + (T\overline{X}, T\overline{X}) - \frac{1}{q}(\overline{X}, N)^2 \right] d\operatorname{vol}_F \\ &\geq \int_F \operatorname{Ric}^M(\overline{X}, \overline{X}) d\operatorname{vol}_F. \end{aligned} \quad (3.13)$$

If  $\widetilde{\operatorname{Ric}}_\infty^M(\overline{X}, \overline{X}) \geq rg^M(\overline{X}, \overline{X})$  then

$$\widetilde{\operatorname{Ric}}_q^B(X, X) \phi^B \geq r \int_F g^M(\overline{X}, \overline{X}) d\operatorname{vol}_F = rg^B(X, X) \phi^B. \quad (3.14)$$

This proves Theorem 2.2.

**Example.** Let  $p : M \rightarrow B$  be a Riemannian submersion, with  $M$  compact, whose fiber transport preserves the fiberwise metric up to multiplicative constants. Equivalently, the Riemannian metric  $g$  on  $M$  comes from starting with a submersion metric  $g'$  with totally geodesic fibers, along with a positive function  $f \in C^\infty(B)$ , and then multiplying the fiberwise metric of  $g'$  on  $F_b$  by  $f^2(b)$ . One can think of  $g$  as a generalized warped product metric.

Suppose that the fibers  $F$  have nonnegative Ricci curvature. For  $\epsilon > 0$ , let  $g_\epsilon$  be the Riemannian metric on  $M$  which comes from multiplying the fiberwise Riemannian metrics by  $\epsilon^2$ . Then as  $\epsilon \rightarrow 0$ , the metrics  $g_\epsilon$  have Ricci curvatures that are uniformly bounded below. Explicitly, let  $\overline{X}$  be the horizontal lift of a vector field  $X$  on  $B$  and let  $\overline{U}$  be a vertical vector field. Then as  $\epsilon \rightarrow 0$ , with the notation of [7, Chapter 9],

$$\begin{aligned} \operatorname{Ric}_\epsilon^M(\overline{X}, \overline{X}) &\sim p^* \operatorname{Ric}^B(X, X) - (T\overline{X}, T\overline{X}) + (\overline{X}, \nabla_{\overline{X}}^M N), \\ \operatorname{Ric}_\epsilon^M(\overline{X}, \overline{U}) &\sim 0 \\ \operatorname{Ric}_\epsilon^M(\overline{U}, \overline{U}) &\sim \operatorname{Ric}^F(\overline{U}, \overline{U}) + \epsilon^2 \left( (\tilde{\delta}T)(\overline{U}, \overline{U}) - (N, T_{\overline{U}}\overline{U}) \right). \end{aligned} \quad (3.15)$$

(The terms on the right-hand side of (3.15) are evaluated with respect to the metric  $g_1$ .) This is an example of a collapse with Ricci curvature bounded below, to which Theorem 2.2 applies.

For another example, let  $M$  be a compact Riemannian manifold on which a Lie group  $G$  acts isometrically and effectively. Suppose that the  $G$ -action on  $M$  has a single orbit type and put  $B = G \backslash M$ . Then there is a natural Riemannian submersion  $p : M \rightarrow B$ . As the orbits of the  $G$ -action on  $M$  are all  $G$ -diffeomorphic to a homogeneous space  $G/H$ , and  $G/H$  has a unique  $G$ -invariant volume form up to constants, it follows that the fiber transport of the Riemannian submersion preserves measures up to constants. Hence Theorem 2.2 applies.



#### 4. Proof of Theorem 3

We refer to [15] for the definition of the measured Gromov–Hausdorff topology.

To prove Theorem 3.1, we just apply the warped product construction of the proof of Theorem 1.1 to  $S^q \times B$ .

Let  $\{M_i, g_i\}_{i=1}^\infty$  be a sequence as in the statement of Theorem 3.2. We may assume that  $\lim_{i \rightarrow \infty} (M_i, g_i, d\text{vol}_i) = (X, \mu)$  in the measured Gromov–Hausdorff topology. If  $q = 0$  then  $X$  is a smooth manifold with a  $C^{1,\alpha}$ -regular metric  $g^X$  and after taking a subsequence and applying diffeomorphisms, we may assume that  $(M_i, g_i)$  converges to  $(X, g^X)$  in the  $C^{1,\alpha}$ -topology (see, for example, [18]). In this case, the theorem follows from Lemma 1.1.

Suppose that  $q > 0$ . By saying that  $X$  is a manifold, we mean that in the construction of  $X$  as a quotient space  $\hat{X}/O(N)$  [16], the action of  $O(N)$  on the manifold  $\hat{X}$  has a single orbit type. Then  $X$  has the structure of a smooth manifold with a  $C^{1,\alpha}$ -regular pair  $(g^X, \phi^X)$ .

For any  $\epsilon > 0$ , we can apply smoothing results of Abresch and others [11, Theorem 1.12] to obtain new metrics  $g_i(\epsilon)$  with

$$\begin{aligned} e^{-\epsilon} g_i &\leq g_i(\epsilon) \leq e^\epsilon g_i, \\ |\nabla_{g_i} - \nabla_{g_i(\epsilon)}| &\leq \epsilon, \\ |\nabla_{g_i(\epsilon)}^k \text{Riem}(M_i, g_i(\epsilon))| &\leq C_k(N, \epsilon, \Lambda), \end{aligned} \tag{4.1}$$

where the constants are uniform. We can also assume that  $\text{Ric}(M_i, g_i(\epsilon)) \geq (r - \epsilon)g_i(\epsilon)$  [13, Remark 2, p. 51]. (See [21, Theorem 2.1] for a similar statement about sectional curvature.) For small  $\epsilon$ , let  $B(\epsilon)$  be a Gromov–Hausdorff limit of a subsequence of  $\{(M_i, g_i(\epsilon))\}_{i=1}^\infty$ . We relabel the subsequence as  $\{(M_i, g_i(\epsilon))\}_{i=1}^\infty$ . From [11, Proposition 4.9], for large  $i$ , there is a small  $C^2$ -perturbation  $g'_i(\epsilon)$  of  $g_i(\epsilon)$  which is invariant with respect to a *Nil*-structure. In particular, we may assume that  $\text{Ric}(M_i, g'_i(\epsilon)) \geq (r - 2\epsilon)g'_i(\epsilon)$ . Now  $(M_i, g'_i(\epsilon))$  is the total space of a Riemannian submersion  $M_i \rightarrow B(\epsilon)$  with infranil fibers and affine holonomy. Let  $(g_i^{B(\epsilon)}, \phi_i^{B(\epsilon)})$  denote the induced metric and measure on  $B(\epsilon)$ . As the fiber transport of the Riemannian submersion preserves the affine-parallel volume forms of the fibers, up to constants, Theorem 2.2 implies that  $\widehat{\text{Ric}}_q(B(\epsilon), g_i^{B(\epsilon)}, \phi_i^{B(\epsilon)}) \geq (r - 2\epsilon)g_i^{B(\epsilon)}$ . Varying  $i$  and  $\epsilon$ , we can extract a subsequence of  $\{(B(\epsilon), g_i^{B(\epsilon)}, \phi_i^{B(\epsilon)})\}$  with  $i \rightarrow \infty$  and  $\epsilon \rightarrow 0$  that converges in the  $C^{1,\alpha}$ -topology to  $(X, g^X, \phi^X)$ . The theorem now follows from Lemma 1.3.

#### 5. Proof of Theorem 4

Let  $s$  be a segment from  $t_0 \in T_0$  to  $t \in T$ , with length  $l(s) > u_3$  and arc-length parameter  $u$ . By definition,  $s$  is length-minimizing. We can decompose the measure  $\phi d\text{vol}_M$  on  $A(u_1, u_4)$  as  $\phi \text{area}_s(u) du \mu(s)$ , where  $\mu$  is a measure on the

space  $\mathcal{S}$  of distinct segments  $s$  that make up  $A(u_1, u_4)$ ,  $du$  is the length measure along a segment  $s$  and  $\text{area}_s(u)$  is the relative size of the transverse Riemannian area density along  $s$ , as measured with respect to the fan of segments. Let  $h$  denote the trace of the second fundamental form  $\Pi$  of a level set of constant distance from  $T_0$ . (With our conventions, the boundary of the unit ball in  $\mathbb{R}^n$  has positive mean curvature.) Differentiating along  $s$ , with respect to  $u$ , gives

$$\partial_u \ln(\phi(u)\text{area}_s(u)) \equiv \frac{\partial_u(\phi(u)\text{area}_s(u))}{\phi(u)\text{area}_s(u)} = h(u) + \partial_u \ln \phi(u) \quad (5.1)$$

and

$$\partial_u^2 \ln(\phi(u)\text{area}_s(u)) = \partial_u h(u) + \partial_u^2 \ln \phi(u). \quad (5.2)$$

From the Riccati equation for  $\Pi$ ,

$$\partial_u h(u) = -\text{Tr}(\Pi^2) - \text{Ric}(\partial_u, \partial_u) \leq -\text{Ric}(\partial_u, \partial_u). \quad (5.3)$$

Then

$$\partial_u^2 \ln(\phi(u)\text{area}_s(u)) \leq -\widetilde{\text{Ric}}_\infty(\partial_u, \partial_u) \leq -r. \quad (5.4)$$

Hence for any  $c \in \mathbb{R}$ ,

$$\partial_u^2 \left( \ln(\phi(u)\text{area}_s(u)) + \frac{r}{2}u^2 - cu \right) \leq 0. \quad (5.5)$$

Fix  $s$  and put

$$a(u) = \phi(u)\text{area}_s(u), \quad (5.6)$$

$$\widehat{a}(u) = e^{-\frac{r}{2}u^2 + cu}, \quad (5.7)$$

$$v(u_1, u_2) = \int_{u_1}^{u_2} a(u) du \quad (5.8)$$

and

$$\widehat{v}(u_1, u_2) = \int_{u_1}^{u_2} \widehat{a}(u) du. \quad (5.9)$$

Then (5.5) says that

$$\frac{d^2}{du^2} \ln \left( \frac{a}{\widehat{a}} \right) \leq 0, \quad (5.10)$$

i.e. that  $\ln \left( \frac{a}{\widehat{a}} \right)$  is concave in  $u$ .

**Lemma 2.** *If  $\frac{v(u_2, u_3)}{\widehat{v}(u_2, u_3)} \leq \frac{v(u_1, u_2)}{\widehat{v}(u_1, u_2)}$  then  $\frac{a(u_3)}{\widehat{a}(u_3)} \leq \frac{v(u_2, u_3)}{\widehat{v}(u_2, u_3)}$ .*

*Proof.* Suppose that

$$\frac{a(u_3)}{\widehat{a}(u_3)} > \frac{v(u_2, u_3)}{\widehat{v}(u_2, u_3)} = \frac{\int_{u_2}^{u_3} \frac{a(u)}{\widehat{a}(u)} \widehat{a}(u) du}{\int_{u_2}^{u_3} \widehat{a}(u) du}. \quad (5.11)$$

If  $\frac{a(u_2)}{\widehat{a}(u_2)} \geq \frac{a(u_3)}{\widehat{a}(u_3)}$  then the concavity of  $\ln\left(\frac{a}{\widehat{a}}\right)$  implies that

$$\frac{a(u_3)}{\widehat{a}(u_3)} \leq \frac{\int_{u_2}^{u_3} \frac{a(u)}{\widehat{a}(u)} \widehat{a}(u) du}{\int_{u_2}^{u_3} \widehat{a}(u) du}, \quad (5.12)$$

which is a contradiction. Thus

$$\frac{a(u_2)}{\widehat{a}(u_2)} < \frac{a(u_3)}{\widehat{a}(u_3)}. \quad (5.13)$$

With the concavity of  $\ln\left(\frac{a}{\widehat{a}}\right)$ , (5.13) implies that  $\frac{a(u)}{\widehat{a}(u)}$  is decreasing in  $u$  for  $u < u_2$  and so

$$\frac{\int_{u_1}^{u_2} \frac{a(u)}{\widehat{a}(u)} \widehat{a}(u) du}{\int_{u_1}^{u_2} \widehat{a}(u) du} < \frac{a(u_2)}{\widehat{a}(u_2)}. \quad (5.14)$$

The concavity of  $\ln\left(\frac{a}{\widehat{a}}\right)$  and (5.13) also imply that

$$\frac{a(u_2)}{\widehat{a}(u_2)} < \frac{\int_{u_2}^{u_3} \frac{a(u)}{\widehat{a}(u)} \widehat{a}(u) du}{\int_{u_2}^{u_3} \widehat{a}(u) du}. \quad (5.15)$$

Thus we have

$$\frac{\int_{u_1}^{u_2} \frac{a(u)}{\widehat{a}(u)} \widehat{a}(u) du}{\int_{u_1}^{u_2} \widehat{a}(u) du} < \frac{a(u_2)}{\widehat{a}(u_2)} < \frac{\int_{u_2}^{u_3} \frac{a(u)}{\widehat{a}(u)} \widehat{a}(u) du}{\int_{u_2}^{u_3} \widehat{a}(u) du}, \quad (5.16)$$

which contradicts the assumption.  $\square$

**Lemma 3.** If  $\frac{v(u_2, u_3)}{\widehat{v}(u_2, u_3)} \leq \frac{v(u_1, u_2)}{\widehat{v}(u_1, u_2)}$  then for  $u_4 \in (u_3, l(s))$ ,  $\frac{v(u_3, u_4)}{\widehat{v}(u_3, u_4)} \leq \frac{v(u_2, u_3)}{\widehat{v}(u_2, u_3)}$ .

*Proof.* For  $u \in (u_3, l(s))$ , put

$$F(u) = \ln \left( \frac{v(u_3, u)}{\widehat{v}(u_3, u)} / \frac{v(u_2, u_3)}{\widehat{v}(u_2, u_3)} \right). \quad (5.17)$$

Then

$$F'(u) = \frac{a(u)}{v(u_3, u)} - \frac{\widehat{a}(u)}{\widehat{v}(u_3, u)} = \frac{\widehat{a}(u)}{v(u_3, u)} \left[ \frac{a(u)}{\widehat{a}(u)} - \frac{v(u_3, u)}{\widehat{v}(u_3, u)} \right]. \quad (5.18)$$

Lemma 2 implies that if  $F(u) \leq 0$  then  $F'(u) \leq 0$ . We can extend  $F(u)$  smoothly to  $u = u_3$ , with

$$F(u_3) = \ln \left( \frac{a(u_3)}{\widehat{a}(u_3)} / \frac{v(u_2, u_3)}{\widehat{v}(u_2, u_3)} \right). \quad (5.19)$$

By Lemma 2,  $F(u_3) \leq 0$ . It follows that  $F(u) \leq 0$  for all  $u \in (u_3, l(s))$ , which proves the lemma.  $\square$

We have

$$\frac{\text{vol}_\phi(A(u_2, u_3))}{\text{vol}_\phi(A(u_1, u_2))} = \frac{\int_{\mathcal{S}} \frac{v_s(u_2, u_3)}{v_s(u_1, u_2)} v_s(u_1, u_2) d\mu(s)}{\int_{\mathcal{S}} v_s(u_1, u_2) d\mu(s)}. \quad (5.20)$$

Put

$$S' = \left\{ s \in S : \frac{v_s(u_2, u_3)}{v_s(u_1, u_2)} < \frac{\widehat{v}(u_2, u_3)}{\widehat{v}(u_1, u_2)} \right\} \quad (5.21)$$

and

$$T' = \bigcup_{s \in S'} s. \quad (5.22)$$

We claim that (1.8) is satisfied. If it is not satisfied, put  $S'' = S - S'$  and  $T'' = T - T'$ . Then

$$\frac{\text{vol}_\phi(T'' \cap A(u_1, u_2))}{\text{vol}_\phi(A(u_1, u_2))} > \frac{\text{vol}_\phi(A(u_2, u_3))}{\text{vol}_\phi(A(u_1, u_2))} \left( \frac{\widehat{v}(u_2, u_3)}{\widehat{v}(u_1, u_2)} \right)^{-1}. \quad (5.23)$$

However, from the definition of  $T''$ ,

$$\begin{aligned} \text{vol}_\phi(A(u_2, u_3)) &\geq \text{vol}_\phi(T'' \cap A(u_2, u_3)) = \int_{S''} \frac{v_s(u_2, u_3)}{v_s(u_1, u_2)} v_s(u_1, u_2) d\mu(s) \\ &\geq \int_{S''} \frac{\widehat{v}(u_2, u_3)}{\widehat{v}(u_1, u_2)} v_s(u_1, u_2) d\mu(s) = \frac{\widehat{v}(u_2, u_3)}{\widehat{v}(u_1, u_2)} \text{vol}_\phi(T'' \cap A(u_1, u_2)), \end{aligned} \quad (5.24)$$

which contradicts (5.23).

If there is a cutpoint along  $s$ , with respect to its basepoint in  $T_0$ , at  $u_c \in (u_3, u_4)$  then we put  $v_s(u_3, u_4) = \int_{u_3}^{u_c} a_s(u) du$ , and otherwise we put  $v_s(u_3, u_4) = \int_{u_3}^{u_4} a_s(u) du$ . Using Lemma 3,

$$\frac{\text{vol}_\phi(T' \cap A(u_3, u_4))}{\text{vol}_\phi(T' \cap A(u_2, u_3))} = \frac{\int_{S'} \frac{v_s(u_3, u_4)}{v_s(u_2, u_3)} v_s(u_2, u_3) d\mu(s)}{\int_{S'} v_s(u_2, u_3) d\mu(s)} \leq \frac{\widehat{v}_s(u_3, u_4)}{\widehat{v}_s(u_2, u_3)}. \quad (5.25)$$

This proves the first part of the theorem.

Suppose that there is a number  $r \in \mathbb{R}$  so that for each tube  $T$  and  $c \in \mathbb{R}$  satisfying (1.7), there is a subtube  $T'$  satisfying the properties of the theorem. Given  $m \in M$  and a unit vector  $v \in T_m M$ , let  $T_0$  be a hypersurface passing through  $m$  such that  $T_m(T_0) = v^\perp$  and the second fundamental form of  $T_0$  at  $m$  vanishes. Let  $s$  be a minimizing segment with  $s(0) = m$  and  $s'(0) = v$ . From (5.1),

$$\left. \frac{d}{du} \right|_{u=0} (\ln(\phi(u) \text{area}(u))) = v(\ln \phi). \quad (5.26)$$

From (5.2) and the Riccati equation,

$$\left. \frac{d^2}{du^2} \right|_{u=0} (\ln(\phi(u) \text{area}(u))) = -\widetilde{\text{Ric}}_\infty(v, v). \quad (5.27)$$

Put  $c_0 = v(\ln \phi)$  and  $r_0 = \widetilde{\text{Ric}}_\infty(v, v)$ . Then for small  $u$ ,

$$\ln(\phi(u) \text{area}(u)) \sim \text{const.} + c_0 u - \frac{r_0}{2} u^2. \quad (5.28)$$

For small  $u_1 < u_2 < u_3 < u_4$ , we have

$$\frac{v(u_2, u_3)}{v(u_1, u_2)} \sim \frac{\int_{u_2}^{u_3} e^{-\frac{r_0}{2} u^2 + c_0 u} du}{\int_{u_1}^{u_2} e^{-\frac{r_0}{2} u^2 + c_0 u} du} \quad (5.29)$$

and

$$\frac{v(u_3, u_4)}{v(u_2, u_3)} \sim \frac{\int_{u_3}^{u_4} e^{-\frac{r_0}{2}u^2 + c_0 u} du}{\int_{u_2}^{u_3} e^{-\frac{r_0}{2}u^2 + c_0 u} du}. \quad (5.30)$$

Take  $T$  to be a small tube around  $s$  (with small base  $T_0$ ), take  $u_3$  small relative to  $u_4$  and take  $c = c_0 + \epsilon$  with  $\epsilon > 0$  small so that (1.7) holds. If there is to be a subtube  $T'$  such that (1.9) holds, for all such choices, then we must have  $r_0 \geq r$ . This proves the theorem.

## 6. Remarks

1. If  $M^n$  is compact and  $\widetilde{\text{Ric}}_q \geq rg$ , with  $q$  an integer greater than one, then Theorem 3.1 says that  $(M, g, \phi)$  is the limit of a sequence of  $(n+q)$ -dimensional manifolds with Ricci curvature bounded below by  $r$ . As in the proof of Theorem 1.2, we can then apply standard results about manifolds with Ricci curvature bounded below, in order to obtain conclusions about  $(M, g, \phi)$ . For example, applying the Bishop–Gromov inequality to the  $(n+q)$ -dimensional manifolds and taking the limit, we obtain a Bishop–Gromov-type inequality for the measures of the distance balls in  $M$ . Namely, let  $\text{vol}_\phi$  denote the weighted measure. Then for  $0 < u_1 < u_2$ ,  $\frac{\text{vol}_\phi(B_{u_2})}{\text{vol}_\phi(B_{u_1})}$  is less than or equal to the corresponding quantity in the  $(n+q)$ -dimensional space form of Ricci curvature  $r$ . If  $r > 0$  then applying Myers' theorem to the  $(n+q)$ -dimensional manifolds and taking the limit, we obtain that  $\text{diam}(M) \leq \pi \sqrt{\frac{n+q-1}{r}}$ . This gives alternative proofs of some results of Qian [20, Corollary 2 and Theorem 5] in the special case when  $q$  is an integer greater than one. (The results of [20] are valid for all positive  $q$ .) One can also show that if  $\widetilde{\text{Ric}}_q \geq rg$  with  $q \in (0, \infty)$  then  $(M, g, \phi)$  satisfies the directional Bishop–Gromov inequality of [8, (A.2.2)] with respect to a model space of formal dimension  $n+q$ .

2. Similarly, if  $q$  is an integer greater than one then there are Sobolev inequalities for the  $(n+q)$ -dimensional collapsing manifolds [6, Theorem 3, p. 397]. Applying these inequalities to functions that pullback from  $M$ , we obtain weighted Sobolev inequalities for  $M$ . Namely, put  $V = \int_M \phi d\text{vol}_M$ . Given  $\alpha, \beta \in [1, \infty)$  such that  $\alpha \leq \frac{(n+q)\beta}{n+q-\beta}$ , let  $\Sigma(n+q; \alpha, \beta)$  be the Sobolev constant of the standard  $(n+q)$ -sphere  $S^{n+q}$ , defined by

$$\Sigma(n+q; \alpha, \beta) = \sup \left\{ \frac{\|f\|_\alpha}{\|df\|_\beta} : f \in W^{1,\beta}(S^{n+q}), f \neq 0, \int_{S^{n+q}} f = 0 \right\}. \quad (6.1)$$

Then if  $\widetilde{\text{Ric}}_q(M, g, \phi) \geq \frac{n+q-1}{R^2}g$ , we have

$$\begin{aligned} \left( \int_M f^\alpha \phi \, d\text{vol}_M \right)^{\frac{1}{\alpha}} &\leq \Sigma(n+q; \alpha, \beta) R \left( \frac{V}{\text{vol}(S^{n+q})} \right)^{\frac{1}{\alpha}-\frac{1}{\beta}} \left( \int_M |\nabla f|^\beta \phi \, d\text{vol}_M \right)^{\frac{1}{\beta}} \\ &\quad + V^{\frac{1}{\alpha}-\frac{1}{\beta}} \left( \int_M f^\beta \phi \, d\text{vol}_M \right)^{\frac{1}{\beta}} \end{aligned} \quad (6.2)$$

for  $f \in W^{1,\beta}(M)$ . In the case  $\beta = 2$ , these inequalities appeared in [3].

3. From the Bishop–Gromov-type inequalities, one can easily show that for any  $q, D \in \mathbb{R}^+$  and  $r \in \mathbb{R}$ , the space of Riemannian manifolds  $(M, g)$  with a smooth positive probability measure  $\phi \, d\text{vol}_M$  satisfying  $\widetilde{\text{Ric}}_q(M, g, \phi) \geq rg$  and  $\text{diam}(M, g) \leq D$ , taken modulo diffeomorphisms, is precompact in the measured Gromov–Hausdorff topology.

Since the relative volume in  $\mathbb{R}^{n+q}$  of  $B_{u_2}$  and  $B_{u_1}$  is  $\left(\frac{u_2}{u_1}\right)^{n+q}$ , we cannot expect any Bishop–Gromov-type comparison theorem for the masses of balls in spaces with  $\widetilde{\text{Ric}}_\infty$  bounded below, i.e. when  $q \rightarrow \infty$  in  $\widetilde{\text{Ric}}_q$ . However, it is interesting that spaces with  $\widetilde{\text{Ric}}_\infty \geq rg$  for  $r > 0$  do admit isoperimetric inequalities [5].

4. It is an interesting question whether there is a good synthetic notion of a metric-measure space with Ricci curvature bounded below, in analogy to the notion of an Alexandrov space with curvature bounded below. See [8, Appendix 2] for discussion. It is clear from Theorem 3.1 that triples  $(M, g, \phi)$  with  $\widetilde{\text{Ric}}_q \geq rg$  are examples of metric-measure spaces with generalized Ricci curvature bounded below by  $r$ , at least if  $q$  is an integer greater than one.

There are various ways that one could try to extend the notion of Ricci curvature bounded below, from smooth metric-measure spaces to more general metric-measure spaces. One could fix  $q \in (0, \infty)$  and try to extend the notion of having  $\widetilde{\text{Ric}}_q \geq rg$ . Or one could consider all  $q$  simultaneously, and say in particular that a triple  $(M, g, \phi)$  has generalized Ricci curvature bounded below by  $r$  if  $\widetilde{\text{Ric}}_q \geq rg$  for some  $q \in (0, \infty)$ . Or one could consider a triple  $(M, g, \phi)$  to have generalized Ricci curvature bounded below by  $r$  if  $\widetilde{\text{Ric}}_\infty \geq rg$ .

We note that there is a difference between having  $\widetilde{\text{Ric}}_q \geq rg$  for some  $q \in (0, \infty)$  and having  $\widetilde{\text{Ric}}_\infty \geq rg$ . For example, if  $r > 0$  and  $\widetilde{\text{Ric}}_q \geq rg$  for some  $q \in (0, \infty)$  then  $M$  is compact [20, Theorem 5], whereas if  $\widetilde{\text{Ric}}_\infty \geq rg$  then  $M$  can be noncompact (as in the case of  $\mathbb{R}$  with  $\phi(x) = e^{-\frac{r}{2}x^2}$ .) It is also easy to see that triples  $(M, g, \phi)$  with  $\widetilde{\text{Ric}}_\infty \geq 0$  generally do not satisfy the splitting principle.

If one does consider a triple  $(M, g, \phi)$  with  $\widetilde{\text{Ric}}_\infty \geq rg$  to be an admissible space with generalized Ricci curvature bounded below by  $r$  then one has a large class of examples. For instance, from this viewpoint it would be reasonable to say that flat  $\mathbb{R}^n$  with the measure  $e^{-V} dx_1 \dots dx_n$  has nonnegative generalized Ricci curvature if  $V$  is any convex function on  $\mathbb{R}^n$ .

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