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## Restriction map in a regular reduction of $\mathbf{SU}(n)^{2g}$

Sébastien Racanière

**Abstract.** The quasi-Hamiltonian reduction of  $\mathbf{SU}(n)^{2g}$  at a regular value, in the centre of  $\mathbf{SU}(n)$ , of the moment map is isomorphic to a moduli-space of semi-stable vector bundles over a Riemann surface. We describe the restriction map from the equivariant cohomology of  $\mathbf{SU}(n)^{2g}$  to the cohomology of the moduli space in terms of natural multiplicative generators of these cohomologies.

**Mathematics Subject Classification (2000).** 53D20, 14H60.

**Keywords.** Moduli spaces, symplectic reduction, quasi-Hamiltonian spaces,  $\mathbf{SU}(n)$ .

### Motivations

All cohomologies will be taken with coefficients in the field  $\mathbf{Q}$  of rational numbers. For a compact connected Lie group  $G$ , we denote  $EG \rightarrow BG$  the universal principal  $G$ -bundle. If  $G$  acts on a manifold  $M$ , we denote  $(M)_G$  the space  $M \times_G EG$ . The equivariant cohomology  $H_G^*(M)$  of  $M$  with respect to the action of  $G$  is by definition the Čech cohomology of  $(M)_G$ . For an account of equivariant cohomology see [6] and [14].

Let  $g$  be an integer bigger than 1. Let  $\pi$  be the group

$$\pi = \langle a_1, b_1, \dots, a_g, b_g, c; \prod_{k=1}^g [a_k, b_k] = c, c^n = 1 \rangle.$$

Let  $n$  and  $d$  be integers with  $n$  bigger than 1 and let  $\zeta$  be the  $n$ -th root of unity  $\zeta = e^{-2\pi i \frac{d}{n}}$ . Put  $\beta = \zeta \mathbf{I}$ , where  $\mathbf{I}$  is the identity matrix in the special unitary group  $\mathbf{SU}(n)$ . We define

$$S_\beta = \{\rho \in \text{Hom}(\pi, \mathbf{SU}(n)) \mid \rho(c) = \beta\}$$

the space of  $\mathbf{SU}(n)$ -representations of  $\pi$  such that  $\beta$  is the image of  $c$ . Because  $\beta$  is in the centre of  $\mathbf{SU}(n)$ , the group  $\mathbf{SU}(n)$  acts on  $S_\beta$  and its quotient

$$\mathfrak{m}_\beta = S_\beta / \mathbf{SU}(n)$$

is the moduli space of  $\mathbf{SU}(n)$ -representations of  $\pi$  that send  $c$  to  $\beta$ .

Narasimhan and Seshadri [17] have shown that  $\mathfrak{m}_\beta$  is isomorphic to the moduli space of holomorphic semi-stable vector bundles of rank  $n$ , degree  $d$ , and fixed determinant over a compact Riemann surface  $X$  of genus  $g$ . For  $d$  and  $n$  co prime,  $\mathfrak{m}_\beta$  is compact and smooth. In this case, Atiyah and Bott [1] showed that this space is symplectic, proposed a family of multiplicative generators of its cohomology and gave an inductive formula (on the rank  $n$ ) for the Betti numbers of  $\mathfrak{m}_\beta$ . Their method consists in studying an infinite dimensional Hamiltonian space. In 1993, Huebschmann [8] and Jeffrey [10] independently gave a group cohomology construction of the symplectic form on  $\mathfrak{m}_\beta$  (their results are summarised in a joint paper [9]). In 1998, Alekseev, Malkin and Meinrenken [2] showed that Huebschmann and Jeffrey's construction fits in a more general setting: one can get the moduli space  $\mathfrak{m}_\beta$  (and many others, moduli spaces of flat connections on a principal bundle) as the Marsden–Weinstein reduction of a quasi-Hamiltonian space. This space is  $\mathbf{SU}(n)^{2g}$ , it is relatively simple in contrast with the usual descriptions of  $\mathfrak{m}_\beta$  as a Hamiltonian reduction. A quasi-Hamiltonian (or q-Hamiltonian for short) space is a Hamiltonian space with a group valued moment map. Its 2-form is not symplectic in general but the Marsden–Weinstein reduction is well defined and the reduced space is symplectic.

An important result about Hamiltonian spaces is the

**Theorem 0.1** (Kirwan). *Let  $M$  be a symplectic manifold. Assume  $G$  is a compact Lie group acting symplectically on  $M$  and assume there exists a moment map  $\phi$  for this action. Let  $0$  be the null vector of the dual of the Lie algebra of  $G$ . The restriction map*

$$H_G^*(M) \longrightarrow H_G^*(\phi^{-1}(0))$$

*is surjective.*

It is a natural question to ask if this theorem is still true for q-Hamiltonian spaces. It is quite easy to see that the answer is no. For example, to get  $\mathfrak{m}_\beta$ , one considers  $\mathbf{SU}(n)^{2g}$  with moment map  $\mu$

$$\begin{aligned} \mathbf{SU}(n)^{2g} &\longrightarrow \mathbf{SU}(n) \\ (A_1, B_1, \dots, A_g, B_g) &\longmapsto \prod_{k=1}^g [A_k, B_k] \end{aligned}$$

the product of the commutators and a certain 2-form (see [2] for more details). Then the reduced space at  $\beta$  being symplectic and compact, its degree two cohomology (which is isomorphic to  $H_{\mathbf{SU}(n)}^2(\mu^{-1}(\beta))$ ) contains a non trivial class whereas  $H_{\mathbf{SU}(n)}^2(\mathbf{SU}(n)^{2g}) = \{0\}$ . Thus the map

$$r : H_{\mathbf{SU}(n)}^*(\mathbf{SU}(n)^{2g}) \longrightarrow H_{\mathbf{SU}(n)}^*(\mu^{-1}(\beta))$$

is not surjective.

Our aim is to give a description of this last map  $r$  (Theorem 5.1) when  $d$  and  $n$  are co prime. Note that in [3], a theorem of localisation in the context of quasi-

Hamiltonian spaces is given. It may be interesting to see how our theorem could be used to apply this localisation theorem to the reduction of  $\mathbf{SU}(n)^{2g}$  at  $\beta$ .

This paper is organised in the following way. Section 2 gives a (very short) review of the prerequisites on q-Hamiltonian spaces and semi-stable bundles. In particular, Narasimhan and Seshadri's theorem (see Theorems 2.9 and 2.13) is used throughout this article to identify  $\mathfrak{m}_\beta$  with  $\mu^{-1}(\beta)/\mathbf{SU}(n)$  and  $H^*(\mathfrak{m}_\beta)$  with  $H_{\mathbf{SU}(n)}^*(\mu^{-1}(\beta))$ . In Section 3, we give a construction of a universal bundle on  $\mathfrak{m}_\beta \times X$ , we then recall how Biswas and Raghavendra [4] use this bundle to define a set  $\{a_k, b_{k,j}, d_k, 2 \leq k \leq n, 1 \leq j \leq 2g\}$  of canonical multiplicative generators of the cohomology of  $\mathfrak{m}_\beta$  (Theorem 3.4). In the next section we define a bundle on  $\mathbf{SU}(n)^{2g} \times X - \{point\}$  and use it to get a set  $\{c_k, \sigma_{k,j}, 2 \leq k \leq n, 1 \leq j \leq g\}$  of multiplicative generators for the equivariant cohomology of  $\mathbf{SU}(n)^{2g}$  (Theorems 4.4 and 4.6). Finally in Section 5 we prove the

**Theorem 5.1.** *The restriction map*

$$r : H_{\mathbf{SU}(n)}^*(\mathbf{SU}(n)^{2g}) \longrightarrow H_{\mathbf{SU}(n)}^*(\mu^{-1}(\zeta\mathbf{I}))$$

*is given by*

$$\begin{aligned} r(c_k) &= a_k \text{ for } k = 2, \dots, n \\ r(\sigma_{k,j}) &= b_{k,j} \text{ for } k = 2, \dots, n, j = 1, \dots, 2g. \end{aligned}$$

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## 2. Prerequisites

In paragraph 2.1, we recall the definition of semi-stability for holomorphic vector bundles and state Narasimhan and Seshadri's theorem (Theorem 2.9). Then in §2.2, we give the definition of a q-Hamiltonian space and restate Narasimhan and Seshadri's result in the language of q-Hamiltonian spaces (Theorem 2.13).

### 2.1. Semi-stable bundle

The following constructions are due to Narasimhan and Seshadri [17]. Apart from the proof of Proposition 2.1, everything in this paragraph is from their article.



Let  $X$  be a Riemann surface of genus  $g$ ,  $g \geq 2$ . Fix a point  $x_0$  of  $X$ . We will first give a construction of a ramified covering  $Y \rightarrow X$  used in [17].

**Proposition 2.1.** *There exists a simply connected covering*

$$p : Y \rightarrow X$$

*with only one point of ramification  $x_0$  of order  $n$ . Outside of this point, the map*

$$Y - \{p^{-1}(x_0)\} \rightarrow X - \{x_0\}$$

*is a covering with group*

$$\pi = \langle a_1, b_1, \dots, a_g, b_g, c; c = \prod_{k=1}^g [a_k, b_k], c^n = 1 \rangle.$$

*Proof.* We start by constructing  $Y$ , then we show that it is simply connected. Let  $D$  be an open neighbourhood of  $x_0$  biholomorphic to an open disc in  $\mathbf{C}$  centred at zero. Let  $D' = D - \{x_0\}$ . The fundamental group of  $X' = X - \{x_0\}$  has a presentation

$$\pi_1(X') = \langle a_1, b_1, \dots, a_g, b_g \rangle,$$

such that the element  $\prod_{k=1}^g [a_k, b_k]$  is the class of a small circle  $\gamma$  included in  $D'$  and going counter clockwise around  $x_0$ . Let  $\pi$  be the group

$$\pi = \langle a_1, b_1, \dots, a_g, b_g, c; c = \prod_{k=1}^g [a_k, b_k], c^n = 1 \rangle.$$

The natural surjective map

$$\pi_1(X') \rightarrow \pi$$

defines a galoisian covering

$$p : Y' \rightarrow X'$$

with group  $\pi$ . Fix a point  $x_1$  of  $D'$ . We take it as the base point for the fundamental groups of  $X, X'$  and  $D'$ . Let us decompose  $p^{-1}(D')$  in its different connected components

$$p^{-1}(D') = \bigcup_{\alpha \in \Lambda} V_\alpha.$$

Each

$$V_\alpha \xrightarrow{p} D'$$

is a connected covering. As  $D'$  is a disc minus its centre, this last covering group is generated by the element in  $\pi$  corresponding to the loop  $\gamma$ . The connected covering of a disc with its centre removed is either the upper half complex plane with the exponential as a projective map or a disc with its centre removed and

a projection map of the type  $z \mapsto z^m$ , where  $m$  is a positive integer. Here, the class of  $\gamma$  acts as  $c$ , thus  $V_\alpha$  is a disc with its centre removed and, for each  $\alpha$ ,

$$V_\alpha \xrightarrow{p} D'$$

is the map  $z \mapsto z^n$ . Let

$$Y = (Y' \bigcup_{\alpha \in \Lambda} (D, \alpha)) / \sim$$

where for  $y \in Y'$  and  $(x, \alpha) \in (D, \alpha)$

$$y \sim (x, \alpha) \text{ if and only if } p(x) = y.$$

The natural projection

$$Y \longrightarrow X$$

is a covering with a unique ramification point at  $x_0$  with order  $n$ . We now have to check that  $Y$  is simply connected. We get  $Y'$  from  $Y$  by removing a discrete set of points. Hence the map

$$\pi_1(Y') \longrightarrow \pi_1(Y)$$

is surjective. The sequence

$$\{1\} \longrightarrow \pi_1(Y') \longrightarrow \pi_1(X') \longrightarrow \pi \longrightarrow \{1\}$$

is exact. The kernel of  $\pi_1(X') \longrightarrow \pi$  is the normal subgroup generated by  $c^n$ . Let  $a$  in  $\pi_1(X')$  be the class of a loop  $\eta : [0, 1] \longrightarrow X'$ . The class of  $\gamma^n$  is  $c^n$ . Let us lift  $\eta \cdot \gamma^n \cdot \eta^{-1}$  in  $Y'$ . To do so we have to take a lift  $\tilde{\eta}$  of  $\eta$  in  $Y'$  and then take a lift  $\tilde{\gamma}^n$  of  $\gamma^n$  satisfying  $\tilde{\gamma}^n(0) = \tilde{\eta}(1)$ . The loop we wanted is  $\tilde{\eta} \cdot \tilde{\gamma}^n \cdot \tilde{\eta}^{-1}$ . There exists  $\alpha$  such that

$$\tilde{\gamma}^n \subset V_\alpha,$$

thus  $\tilde{\gamma}^n$  is homotopic to the constant loop in  $Y$ . The image of  $a \cdot c^n \cdot a^{-1}$  by  $\pi_1(Y') \longrightarrow \pi_1(Y)$  is 1, hence the image of  $\pi_1(Y') \longrightarrow \pi_1(Y)$  is  $\{1\}$  and

$$\pi_1(Y) = \{1\}.$$

□

Choose a  $y_0$  in  $p^{-1}(x_0)$ . In the presentation

$$\pi = \langle a_1, b_1, \dots, a_g, b_g, c; \prod_{k=1}^g [a_k, b_k] = c, c^n = 1 \rangle$$

of  $\pi$ , we can assume that  $c$  is a generator of the isotropy group  $\pi_{y_0}$  of  $y_0$ .

For a representation  $\rho : \pi \rightarrow \mathbf{GL}(n)$  of  $\pi$ , we denote  $E_\pi(\rho)$  the vector bundle

$$Y \times \mathbf{C}^n \longrightarrow Y$$

with the action:

$$\begin{aligned} \pi \times (Y \times \mathbf{C}^n) &\longrightarrow Y \\ (\gamma, (y, v)) &\longmapsto (\gamma \cdot y, \rho(\gamma)v). \end{aligned}$$

Let  $\mathbf{E}$  be the sheaf of germs of holomorphic sections of  $E_\pi(\rho)$ . The group  $\pi$  acts on the image sheaf  $p_*(\mathbf{E})$ . Let  $p_*^\pi(\mathbf{E})$  be the subsheaf of  $\pi$ -invariant elements of  $p_*(\mathbf{E})$ . It is a rank  $n$  locally free sheaf of  $\mathbf{O}_X$ -modules. It defines a holomorphic vector bundle,  $p_*^\pi(E_\rho)$ , of rank  $n$  on  $X$ . A set of transition functions is obtained in the following way. Let  $\{U_i\}_{i=0}^m$  be a finite open covering of  $X$  satisfying:

- (1) all non-empty intersections of sets of the type  $U_i$  is contractible,
- (2)  $x_0 \in U_0$  and  $\bigcup_{i=1}^m U_i = X - \{x_0\}$ ,
- (3) there exist discs  $\{D_i\}_{i=0}^m$  in  $Y$  such that  $y_0 \in D_0$  and  $U_0$  is the quotient of  $D_0$  by  $\pi_{y_0}$ , the restriction  $p|_{D_i}$  is an homeomorphism of  $D_i$  with  $U_i$ , for all non zero  $i$ .

For each triplet  $i, j, k$ , choose a connected component  $W_{ij,k}$  of  $p^{-1}(U_i \cap U_j) \cap D_k$ . If  $U_i \cap U_j$  is not empty, we denote  $\gamma_{i,j}$  the element of  $\pi$  satisfying  $\gamma_{i,j} W_{ij,j} = W_{ji,i}$ . According to [17, p. 550] :

**Proposition 2.2.** *On each  $\{U_i\}_{i=0}^m$ , the bundle  $p_*^\pi(E_\rho)$  is trivial and a set of transition functions is given by:*

$$\begin{cases} g_{i,j} = \rho(\gamma_{i,j}) & \text{in } U_i \cap U_j, \text{ for } i, j \neq 0 \\ g_{0,i} = f_{0,i} \rho(\gamma_{0,i}) & \text{in } U_0 \cap U_i, \text{ for } i \neq 0 \end{cases}$$

where  $f_{0,i} : U_0 \cap U_i \rightarrow \mathbf{C}^*$  depends only on  $\tau$ .

**Definition 2.3.** Let  $W$  be a degree  $d(W)$  and rank  $r(W)$  holomorphic vector bundle on  $X$ . It is said to be stable, resp. semi-stable, if for each proper subbundle  $V$ , we have

$$\frac{d(V)}{r(V)} < \frac{d(W)}{r(W)}, \text{ resp. } \frac{d(V)}{r(V)} \leq \frac{d(W)}{r(W)}.$$

**Remark 2.4.** If  $d(W)$  and  $r(W)$  are co prime then the notions of stability and semi-stability are equivalent.

Recall that  $d$  is an integer,  $0 \leq d \leq n-1$ , and  $\zeta = e^{-2\pi i \frac{d}{n}}$  is an  $n$ -th root of unity. Let  $z$  be a coordinate in a neighbourhood of  $y_0$  such that  $\pi_{y_0}$  is the group of multiplications by  $\zeta^k$ . Up to a change of generator  $c$  of  $\pi_{y_0}$ , we can assume that  $c$  acts by multiplication by  $e^{\frac{2i\pi}{n}}$ . Let  $\tau$  be the character of  $\pi_{y_0}$  defined by  $\tau(c) = \zeta$ . A representation  $\rho : \pi \rightarrow \mathbf{U}(n)$  is said to be of type  $\tau$  if for all  $\gamma \in \pi_{y_0}$ , we have  $\rho(\gamma) = \tau(\gamma)\mathbf{I}$ . For any representation  $\rho$  of type  $\tau$  we have:

$$d(p_*^\pi(E_\rho)) = d - n \quad (\text{see [5, p. 13]}).$$

Again, according to [17]:

**Theorem 2.5.** *A holomorphic vector bundle of rank  $n$  and degree  $d - n$  on  $X$  is semi-stable if and only if it is isomorphic to a  $p_*^\pi(E_\rho)$ , where  $\rho : \pi \rightarrow \mathbf{U}(n)$  is a unitary representation of type  $\tau$ . This bundle is stable if and only if the*

representation  $\rho$  is irreducible. Moreover, two such bundles are isomorphic if and only if their corresponding unitary representations are isomorphic.

**Remark 2.6.** For  $d$  and  $n$  co prime, any representation  $\rho : \pi \rightarrow \mathbf{U}(n)$  of type  $\tau$  is irreducible [17, Prop. 9.3].

Let  $\mathbf{n}$  be the moduli space of rank  $n$ , degree  $d$ , stable holomorphic vector bundles over  $X$ .

**Remark 2.7.** Let  $M$  be a holomorphic line bundle of degree 1 over  $X$  (it always exists). The moduli space of rank  $n$  stable holomorphic vector bundles with fixed determinant of degree  $d - n$  over  $X$  is isomorphic to the moduli space of rank  $n$  stable holomorphic vector bundles with fixed determinant of degree  $d$  over  $X$ . The isomorphism is induced by the map which to a bundle  $E \rightarrow X$  associates  $E \otimes M$ .

We fix such a bundle  $M$  and use it to identify the two moduli spaces of Remark 2.7. Thus we have

**Theorem 2.8.** *The moduli space  $\mathbf{n}$  is isomorphic to the quotient of the space of unitary representations of type  $\tau$  of  $\pi$  by the action of  $\mathbf{U}(n)$ .*

The map which to a class of bundles in  $\mathbf{n}$  associates its determinant is a fibration over the moduli space of line bundles of degree  $d$ . Its fibre is called the moduli space of rank  $n$  stable holomorphic line bundles over  $X$  with fixed determinant (of degree  $d$ ). We get all such bundles by taking only representations  $\rho : \pi \rightarrow \mathbf{SU}(n)$  of type  $\tau$ . Let  $S$  be the set of such representations. We identify it with  $\{(A_1, B_1, \dots, A_g, B_g) \in \mathbf{SU}(n)^{2g} \mid \prod_{k=1}^g [A_k, B_k] = \zeta \mathbf{I}\}$  by:

$$\begin{aligned} S &\longrightarrow \{(A_1, B_1, \dots, A_g, B_g) \in \mathbf{SU}(n)^{2g} \mid \prod_{k=1}^g [A_k, B_k] = \zeta \mathbf{I}\} \\ \rho &\longmapsto (\rho(a_1), \rho(b_1), \dots, \rho(a_g), \rho(b_g)). \end{aligned}$$

The action of  $\mathbf{SU}(n)$  on the representations becomes, under this identification, the diagonal action by conjugation of  $\mathbf{SU}(n)$  on  $\mathbf{SU}(n)^{2g}$ . In this article we work with  $\mathbf{m}$  rather than  $\mathbf{n}$ . We have:

**Theorem 2.9.** *Let  $d$  be an integer,  $1 \leq d \leq n - 1$ , co prime with  $n$ . Let  $\mathbf{m}$  be the moduli space of rank  $n$  holomorphic stable vector bundles over  $X$  with fixed determinant (of degree  $d$ ). The map*

$$\begin{aligned} S &\longrightarrow \mathbf{m} \\ \rho &\longmapsto p_*^\pi(E_\rho) \end{aligned}$$

*is a  $\mathbf{PSU}(n)$ -principal bundle.*

*Proof.* The only thing that is left to check is that for a representation  $\rho = (A_1, B_1, \dots, A_g, B_g) \in S$  (recall we have identified  $S$  with a set of matrices), its stabiliser  $\text{Stab}(\rho)$  is the centre of  $\mathbf{SU}(n)$ . Let  $C$  be in the stabiliser of  $\rho$ . Let  $\lambda$  be an eigenvalue of  $C$  and let  $E_\lambda$  be its eigenspace. As  $C$  commutes with each of the  $A_i, B_i$ , the subspace  $E_\lambda$  is stable by the unitary representation  $\rho$ . As  $\rho$  is irreducible,  $E_\lambda = \mathbf{C}^n$  and  $C$  is in the centre of  $\mathbf{SU}(n)$ . On the other hand, any matrix in the centre of  $\mathbf{SU}(n)$  does leave  $\rho$  invariant. We have indeed a free action of  $\mathbf{PSU}(n)$  on  $S$ .  $\square$

## 2.2. Quasi-Hamiltonian spaces

The definition of a q-Hamiltonian space is due to Alekseev, Malkin and Meinrenken. Roughly speaking this is a Hamiltonian space with a group valued moment map. When the group is a torus, the definition reduces to the usual one of a Hamiltonian torus action whose moment map takes its values in the torus itself (see McDuff [15] and Weitsmann [19]).

Let  $G$  be a compact Lie group. Let  $\theta$  and  $\bar{\theta}$  be respectively the left invariant and right invariant Maurer-Cartan forms on  $G$ . Choose a  $G$ -invariant scalar product  $\langle, \rangle$  on the Lie algebra  $\mathfrak{g}$  of  $G$ . Define a 3-form  $\chi$  on  $G$  by

$$\chi = \frac{1}{12} \langle [\theta, \theta], \theta \rangle.$$

**Definition 2.10** ([2]). Let  $(M, G, \omega, \mu)$  be a 4-tuple where  $M$  is a manifold acted on by a compact Lie group  $G$ ,  $\omega$  is a  $G$ -invariant 2-form on  $M$  and  $\mu$  is an equivariant map from  $M$  to  $G$  (for the action by conjugation of  $G$  on itself). This 4-tuple (or simply  $M$  if there is no risk of confusion) is a q-Hamiltonian space if

- (B1)  $d\omega = -\mu^*\chi$
- (B2)  $\iota(v_\xi)\omega = \frac{1}{2}\mu^*\langle\theta + \bar{\theta}, \xi\rangle$
- (B3)  $\ker \omega_x = \{v_\xi(x) \mid \xi \in \ker \text{Ad}_{\mu(x)} + 1\}$ .

The map  $\mu$  is called the moment map.

This definition is a generalisation of the definition of a Hamiltonian space in the sense that any compact Hamiltonian space can be endowed with a q-Hamiltonian structure (this is an easy corollary of [2, Prop. 3.4.]).

A first example of a q-Hamiltonian space is a conjugacy class in a Lie group with moment map the inclusion of the conjugacy class in the group (see [2, §3]). The example that will be of interest to us is

**Theorem 2.11** ([2]). *Let  $G$  be a compact Lie group and  $g \geq 1$  an integer. There exists a 2-form  $\omega$  on  $G^{2g}$  such that the map*

$$\mu : \begin{array}{ccc} G^{2g} & \longrightarrow & G \\ (a_1, b_1, \dots, a_g, b_g) & \longmapsto & \prod_{k=1}^g [a_k, b_k] \end{array}$$

and the diagonal action of  $G$  on  $G^{2g}$  by conjugation makes  $(G^{2g}, G, \omega, \mu)$  into a  $q$ -Hamiltonian space.

In particular we will apply this theorem with  $G = \mathbf{SU}(n)$ . An important fact about  $q$ -Hamiltonian spaces is that one can take their Marsden–Weinstein reduction. More precisely:

**Theorem 2.12** ([2]). *Let  $(M, G, \omega, \mu)$  be a  $q$ -Hamiltonian space. Let  $h$  be in the centre of  $G$ . The moment map  $\mu$  is a submersion at  $x \in M$  if and only if the stabiliser of  $x$  in  $G$  is finite. If this is the case for any point of  $\mu^{-1}(h)$ , the reduced space  $\mu^{-1}(h)/G$  is an orbifold (a manifold if the action of  $G$  on  $\mu^{-1}(h)$  is principal) on which the restriction of  $\omega$  to  $\mu^{-1}(h)$  descends to define a symplectic form. We call this space the reduction of  $M$  at  $h$ .*

As a corollary of Theorems 2.2, 2.12 and 2.9 we have:

**Theorem 2.13.** *Let  $n, d$  be co prime integers,  $n \geq 2$  and  $0 \leq d \leq n - 1$ . Let  $\zeta = e^{-2\pi i \frac{d}{n}}$  be an  $n$ -th root of unity and  $\beta = \zeta \mathbf{I}$  in the centre of  $\mathbf{SU}(n)$ . The moduli space  $\mathfrak{m}_\beta$  of rank  $n$  stable holomorphic vector bundles with fixed determinant (and degree  $d$ ) over a Riemann surface  $X$  of genus  $g$  is isomorphic to the reduction of the  $q$ -Hamiltonian space  $\mathbf{SU}(n)^{2g}$  at  $\beta$ . It is a compact smooth symplectic manifold.*

### 2.3. Characteristic classes of principal bundles

Following Biswas and Raghavendra [4], we define in this section some characteristic classes of a projective bundle. We will see that when the projective bundle comes from a vector bundle of degree 0, these characteristic classes are the same as the Chern classes of the vector bundle.

Let  $\mathbf{Q}[X_1, \dots, X_n]$  be a polynomial ring in  $n$  variables. The cohomology of  $BU(n)$  is isomorphic to the subalgebra of invariant polynomials in the algebra  $\mathbf{Q}[X_1, \dots, X_n]$ , under the action of the symmetric group  $S_n$  on the variables. For  $k$  an integer in  $[1, n]$ , the Chern class  $c_k$  in  $H^*(BU(n))$  corresponds to the Schur polynomial

$$\sum_{1 \leq i_1 < \dots < i_k \leq n} X_{i_1} \dots X_{i_k}.$$

The projection from  $\mathbf{U}(n)$  to  $\mathbf{PU}(n)$  defines a fibration  $BU(n) \longrightarrow BPU(n)$  with fiber  $BU(1)$ . This fibration is cohomologically trivial and  $H^*(BPU(n))$  injects into  $H^*(BU(n))$ . Let us define

$$Y_k = X_k - \frac{1}{n} \sum_{k=1}^n X_k.$$

The image of  $H^*(BPU(n))$  in  $H^*(BU(n))$  is the ideal generated by the polyno-

mials

$$p_k = \sum_{1 \leq i_1 < \dots < i_k \leq n} Y_{i_1} \dots Y_{i_k}, \text{ for } 2 \leq k \leq n.$$

The  $k$ -th characteristic class of a projective bundle over a manifold  $M$  is the pull-back of  $p_k$  under the classifying map  $M \rightarrow \mathbf{BPU}(n)$ .

For a vector bundle  $F$  of degree 0, that is when the first Chern class vanishes, we have  $p_k(F) = c_k(F)$  for  $k$  in  $[2, n]$ . It will be the case in particular if the structure group of the vector bundle is  $\mathbf{SU}(n)$ . This corresponds to the fact that the projection  $\mathbf{BSU}(n) \rightarrow \mathbf{BPU}(n)$  defines an isomorphism in cohomology.

### 3. Construction of a universal bundle

In this section, we fix  $d$ ,  $n$ ,  $\zeta$  and  $\beta$  as in Theorem 2.13. We use the notations of that theorem and of Theorem 2.2 with  $G = \mathbf{SU}(n)$ . We construct a universal bundle on  $\mathfrak{m}_\beta$ , that is a vector bundle  $U$  over  $\mathfrak{m}_\beta \times X$ , holomorphic in the  $X$  direction, such that for any class  $[E]$  in  $\mathfrak{m}_\beta$ , the restriction of  $U$  to  $\{[E]\} \times X$  is in the class  $[E]$ . We then use this bundle to define natural multiplicative generators of the cohomology of  $\mathfrak{m}_\beta$ .

Recall that we defined page 399 an open covering of  $X$  by subsets  $\{U_i\}_{i=0}^m$ . Define a complex vector bundle  $T$  over  $S \times X$  (where we have identified  $S$  to  $\mu^{-1}(\zeta\mathbf{I})$ ) as being trivial over the  $S \times U_i$  and with transition functions:

$$\begin{aligned} (S \times U_i) \cap (S \times U_j) &\longrightarrow \mathbf{U}(n) \\ (\rho, x) &\longmapsto \begin{cases} \rho(\gamma_{i,j}) & x \in U_i \cap U_j, i, j \neq 0 \\ f_{0,i}(x)\rho(\gamma_{0,i}) & x \in U_0 \cap U_i, i \neq 0 \end{cases}. \end{aligned}$$

According to Proposition 2.2:

**Proposition 3.1.** *The bundle  $T$  satisfies: for all  $\rho$  in  $S$*

$$T|_{\{\rho\} \times X} \cong p_*^\pi(E_\rho).$$

Define an action of  $\mathbf{SU}(n)$  on  $T$  by defining it on each  $T|_{S \times U_i}$  by

$$\begin{aligned} \mathbf{SU}(n) \times (S \times U_i \times \mathbf{C}^n) &\longrightarrow S \times U_i \times \mathbf{C}^n \\ (g, (\rho, x, u)) &\longmapsto (g \cdot \rho, x, g(u)). \end{aligned}$$

This action is well defined because if  $x \in U_j \cap U_i$  and  $t = (\rho, x, u)$  is in  $S \times U_i \times \mathbf{C}^n$ , then in the trivialisation  $S \times U_j \times \mathbf{C}^n$ ,  $t$  is written  $t = (\rho, x, v(x)\rho(\gamma_{i,j})(u))$  where  $v(x)$  is a scalar and

$$\begin{aligned} g \cdot (\rho, x, v(x)\rho(\gamma_{i,j})(u)) &= (g \cdot \rho, x, g(v(x)\rho(\gamma_{i,j})(u))) \\ &= (g \cdot \rho, x, v(x)g\rho(\gamma_{i,j})g^{-1}g(u)). \end{aligned}$$

This last term is  $(g \cdot \rho, x, g(u))$  written in  $S \times U_j \times \mathbf{C}^n$ . This action is a lift for the action of  $\mathbf{SU}(n)$  on  $S \times X$ . Unfortunately it does not come from an action of

$\mathbf{PSU}(n)$  and the bundle  $T$  does not descend to a bundle on  $\mathfrak{m} \times X$ . Indeed the centre  $\mathbf{Z}/n\mathbf{Z}$  of  $\mathbf{SU}(n)$  acts trivially on  $S$  but the generator  $\zeta\mathbf{I}$  of  $\mathbf{Z}/n\mathbf{Z}$  acts by multiplication by  $\zeta$  in the fibres. To overcome this problem, we can construct a line bundle  $L$  on  $S$  with an action of  $\mathbf{SU}(n)$  lifting the action on  $S$  and such that  $\zeta\mathbf{I}$  also acts by multiplication by  $\zeta$  in the fibres. We will also denote  $L$  the induced bundle on  $S \times X$ . The bundle  $T \otimes L^*$  has the property of Proposition 3.1 but the action of  $\mathbf{SU}(n)$  reduces to an action of  $\mathbf{PSU}(n)$ . By taking the quotient we get

**Proposition 3.2.** *Let  $M$  be the line bundle of Remark 2.7. The bundle*

$$U = M \otimes (T \otimes L^*) / \mathbf{PSU}(n) \longrightarrow \mathfrak{m} \times X$$

*is a universal bundle for  $\mathfrak{m}_\beta$ . That is, if  $[E] \in \mathfrak{m}_\beta$  is the class of a bundle  $E \rightarrow X$  then  $U|_{[E] \times X}$  is isomorphic to  $E$ .*

We still have to prove the existence of the bundle  $L$ .

**Lemma 3.3.** *There exists a line bundle  $L$  over  $S$  with an action of  $\mathbf{SU}(n)$  lifting the one of  $\mathbf{SU}(n)$  on  $S$ . This action satisfies:  $\zeta\mathbf{I}$  acts by multiplication by  $\zeta$  in the fibres.*

*Proof.* The proof is inspired from [16].

The bundle  $M \otimes T$  is a family (parameterised by  $S$ ) of rank  $n$ , degree  $d$  stable holomorphic vector bundles. Let  $E$  be in this family and let  $k$  be an integer. By Serre duality,

$$H^1(E \otimes (\Omega_X^1)^k) = H^0(E^\vee \otimes (\Omega_X^1)^{1-k})^*$$

and this is the null vector space. Otherwise there would exist a non zero homomorphism  $(\Omega_X^1)^{k-1} \rightarrow E^\vee$  and thus a subbundle of  $E^\vee$  of degree bigger than or equal to  $2(g-1)(k-1) \geq 0$ . This is impossible because  $E$  is stable.

The  $H^0(E \otimes (\Omega_X^1)^k)$  form a holomorphic bundle (see [13])  $A_k$  over  $S$  of rank  $u_k$  the dimension of  $H^0(E \otimes (\Omega_X^1)^k)$ . By Riemann-Roch, we have

$$\begin{aligned} u_k &= d(E \otimes (\Omega_X^1)^k) + n(1-g) \\ &= d(E) + 2nk(g-1) + n(1-g) \\ &= d + n(g-1)(2k-1) \\ &= 2hk + d - h \quad (\text{where } h = n(g-1)). \end{aligned}$$

We have

$$\begin{aligned} (u_2, u_1) = 1 &\Leftrightarrow (d + 3h, d + h) = 1 \Leftrightarrow (2h, d + h) = 1 \\ &\Leftrightarrow d + h \text{ is odd and } (d, h) = 1. \end{aligned}$$

As  $d$  and  $n$  are co prime,  $d$  and  $h$  are co prime if and only if  $d$  and  $g-1$  are co prime. If in addition we assume  $g-1$  is odd then  $d+n(g-1)$  is odd ( $d$  and  $n$  have different parities). In this case, there exist integers  $a$  and  $b$  such that  $au_1 + bu_2 = 1$  and we can take

$$L = (\wedge^{u_1} A_1)^a \otimes (\wedge^{u_2} A_2)^b.$$



Otherwise, there exists  $g' \geq g$  such that  $g' - 1$  is odd and  $(d, g' - 1) = 1$ . The injection

$$\begin{array}{ccc} \mathbf{SU}(n)^{2g} & \longrightarrow & \mathbf{SU}(n)^{2g'} \\ (A_1, B_1, \dots, A_g, B_g) & \longmapsto & (A_1, B_1, \dots, A_g, B_g, 1, 1, \dots, 1, 1) \end{array}$$

restricts to an equivariant injection

$$S \rightarrow S'$$

where  $S'$  is the set of  $2g'$ -tuple of matrices

$$S' = \{(A_1, B_1, \dots, A_{g'}, B_{g'}), \prod_{k=1}^{g'} [A_k, B_k] = \zeta \mathbf{I}\}.$$

We have seen we can construct on  $S'$  a line bundle with the required properties. We take  $L$  to be the restriction of this bundle to  $S$ .  $\square$

Let us use the universal bundle to define classes in  $H^*(\mathfrak{m}_\beta)$ .

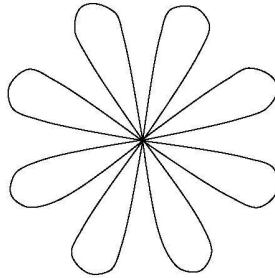


FIGURE 1. Bouquet of  $2g$  circles (with  $g = 4$ )

Let  $B$  be a bouquet of  $2g$  circles (Figure 1) embedded in  $X'$  in such a way that  $X'$  retracts on  $B$ . Each of the  $2g$  circles defines a class in  $H_1(X)$ . Let  $\alpha_1, \dots, \alpha_{2g}$  be their Poincaré duals. They form a basis of  $H^1(X)$ . Let  $\kappa$  be the class of the volume form on  $X$  of volume 1. Let us decompose the characteristic classes of the projective bundle  $P(U)$ . For  $k$  in  $[2, n]$ :

$$p_k(P(U)) = a_k \otimes \mathbf{1} + \sum_{j=1}^{2g} b_{k,j} \otimes \alpha_j + d_k \otimes \kappa.$$

Then, according to Biswas and Raghavendra [4], we have

**Theorem 3.4.** *The family*

$$\{a_k, b_{k,j}, d_k, 2 \leq k \leq n, 1 \leq j \leq 2g\}$$

*is a multiplicative system of generators of  $H^*(\mathfrak{m}_\beta) \simeq H_{\mathbf{SU}(n)}^*(\mu^{-1}(\beta))$ .*

#### 4. A bundle over $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times X'$ and its Chern classes

Let  $B$  be a bouquet of  $2g$  circles (Figure 1) embedded in  $X'$  in such a way that  $X'$  retracts on  $B$ . The theory of vector bundles with their Chern classes is the same on  $B$  and  $X'$ . We want to construct a complex vector bundle on  $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B$ . Denote  $B'$  the star with  $2g$  branches (see Figure 2), that is  $B' = (\cup_{i=1}^{2g} [0, 1]_i) / \sim$ ,

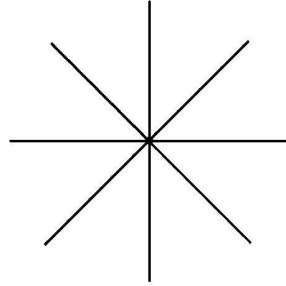


FIGURE 2. A star with  $2g$  branches (with again  $g = 4$ )

where  $\sim$  is the equivalence relation that identifies all the 0 to a point. There is a natural map

$$\eta : B' \longrightarrow B.$$

It is defined by means of the exponential  $\exp : [0, 1] \rightarrow S^1$ . Denote

$$D_n = (\mathbf{SU}(n)^{2g} \times E\mathbf{U}(n) \times B' \times \mathbf{C}^n) / \sim$$

where  $\sim$  is the relation:

$$\begin{aligned} ((\rho_1, \dots, \rho_{2g}), e, 0, v) &\sim ((\text{Ad}_A \rho_1, \dots, \text{Ad}_A \rho_{2g}), A \cdot e, 1_i, A \circ \rho_i(v)), \\ &\forall i \in [1, 2g], \forall A \in \mathbf{SU}(n). \end{aligned}$$

The projection

$$D_n \longrightarrow (\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B$$

makes  $D_n$  into a rank  $n$  complex vector bundle over  $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B$ . We wish to compute the characteristic classes of the projectivised bundle  $P(D_n)$  of  $D_n$ . Notice that as the structure group of  $D_n$  reduces to  $\mathbf{SU}(n)$ , the classes  $p_k(P(D))$  are equal to the Chern classes  $c_k(D)$ .

Let us describe the cohomology of  $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B$ . By the Künneth formula, we have

$$H^*((\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B) = H^*_{\mathbf{SU}(n)}(\mathbf{SU}(n)^{2g}) \otimes H^*(B).$$

**Proposition 4.1.** *Let  $G$  be a compact Lie group. Let  $k$  be an integer bigger than 0. Let  $G$  act on  $G^k$  diagonally by conjugation. The equivariant cohomology of  $G^k$  is isomorphic, as a graded algebra, to  $H^*(G^k) \otimes H^*(BG)$ .*

*Proof.* The fibration  $(G^k)_G \longrightarrow BG$  is cohomologically trivial (see [3]) so that we have an isomorphism of graded vector spaces between  $H^*_G(G^k)$  and  $H^*(G^k) \otimes H^*(BG)$ . The proposition then follows from the fact that for any compact Lie group, its cohomology is an exterior algebra on a finite number of elements and from the

**Lemma 4.2.** *Let  $q : N \longrightarrow M$  be a cohomologically trivial fibration with fiber  $F$ . Assume that the cohomology of  $F$  is an exterior algebra on a family  $\{\xi_1, \dots, \xi_r\}$ . Then the cohomology of  $N$  is isomorphic, as a graded algebra, to the tensor product of  $H^*(F)$  and  $H^*(M)$ .*

*Proof.* Let  $\mathfrak{J}$  be the set of strictly increasing sequences of integers  $I = (i_1, \dots, i_p)$  such that  $i_1 \geq 1$  and  $i_p \leq r$ . For  $I \in \mathfrak{J}$  with  $I = (i_1, \dots, i_p)$ , let

$$\xi_I = \xi_{i_1} \wedge \dots \wedge \xi_{i_p}.$$

The family  $\{\xi_I\}_{I \in \mathfrak{J}}$  forms a basis of  $H^*(F)$ .

To say that the fibration  $N \longrightarrow M$  is cohomologically trivial is equivalent (by the Leray-Hirsch Theorem) to saying that the inclusion of a fiber  $F$  into  $N$  induces a surjection  $H^*(N) \longrightarrow H^*(F)$ . For  $i \in [1, r]$ , let  $\zeta_i$ , in  $H^*(N)$ , be a pre-image of  $\xi_i$ . For  $I \in \mathfrak{J}$  with  $I = (i_1, \dots, i_p)$ , let

$$\zeta_I = \zeta_{i_1} \wedge \dots \wedge \zeta_{i_p}.$$

The map

$$\begin{aligned} H^*(F) &\longrightarrow H^*(N) \\ \sum \lambda_I \xi_I &\longmapsto \sum \lambda_I \zeta_I \end{aligned}$$

is a morphism of algebra and the map

$$\begin{aligned} H^*(F) \otimes H^*(M) &\longrightarrow H^*(N) \\ (\sum \lambda_I \xi_I) \otimes \chi &\longmapsto (\sum \lambda_I \zeta_I) \otimes q^*(\chi) \end{aligned}$$

is an isomorphism of graded algebra. □

□

According to the previous proposition, we have isomorphisms

$$\begin{aligned} H_{\mathbf{SU}(n)}^*(\mathbf{SU}(n)^{2g}) &\simeq H^*(\mathbf{SU}(n)^{2g}) \otimes H^*(BSU(n)) \\ &\simeq \otimes_{j=1}^{2g} H^*(\mathbf{SU}(n)) \otimes H^*(BSU(n)). \end{aligned} \quad (3.1)$$

For all  $k \geq 2$ , the fibration  $\mathbf{SU}(k) \longrightarrow S^{2k-1}$  with fiber  $\mathbf{SU}(k-1)$  is cohomologically trivial (see Hatcher [7]). Let  $\gamma_k$  be the volume form of volume 1 on  $S^{2k-1}$ . The cohomology of  $\mathbf{SU}(n)$  is the exterior algebra freely generated by the family  $\{\sigma_k, 2 \leq k \leq n\}$ , where  $\sigma_k$  is a class of degree  $2k-1$  which pulls-back under the restriction  $\mathbf{SU}(k) \longrightarrow \mathbf{SU}(n)$  to the image of  $\gamma_k$  by  $H^{2k-1}(S^{2k-1}) \longrightarrow H^{2k-1}(\mathbf{SU}(k))$ . Denote  $\sigma_{k,j}$  the image of  $\sigma_k \in H^{2k-1}(\mathbf{SU}(n))$  by the homomorphism  $H^*(\mathbf{SU}(n)) \rightarrow H^*(\mathbf{SU}(n)^{2g})$  induced by the projection on the  $j$ -th factor  $\mathbf{SU}(n)^{2g} \rightarrow \mathbf{SU}(n)$ . We have

**Lemma 4.3.** *The algebra  $H^*(\mathbf{SU}(n)^{2g})$  is the exterior algebra freely generated by the family  $\{\sigma_{k,j}, 2 \leq k \leq n, 1 \leq j \leq 2g, \deg \sigma_{k,j} = 2k-1\}$ .*

In addition, we know that  $H^*(BSU(n)) = \mathbf{Q}[c_2, \dots, c_n]$ . From the preceding lemma and Proposition 4.1, we deduce

**Theorem 4.4.** *Let  $\Lambda$  be the exterior algebra freely generated by the family  $\{\sigma_{k,j}, 2 \leq k \leq n, 1 \leq j \leq 2g, \deg \sigma_{k,j} = 2k-1\}$ . The  $\mathbf{SU}(n)$ -equivariant cohomology of  $\mathbf{SU}(n)^{2g}$  is isomorphic, as a graded algebra, to  $\Lambda \otimes \mathbf{Q}[c_2, \dots, c_n]$ .*

When there is no risk of confusion, we will write  $c_k$  and  $\sigma_{k,j}$  instead of respectively  $1 \otimes c_k$  and  $\sigma_{k,j} \otimes 1$ .

**Remark 4.5.** The injection  $\iota$  of  $\mathbf{SU}(n)$  into  $\mathbf{SU}(n+1)$  and the map  $BSU(n) \rightarrow BSU(n+1)$  induce isomorphisms

$$H^k(\mathbf{SU}(n+1)) \xrightarrow{\sim} H^k(\mathbf{SU}(n)) \text{ for } k \leq 2n \text{ and } k = 2n+2 \quad (4.1)$$

and

$$H^k(BSU(n+1)) \xrightarrow{\sim} H^k(BSU(n)) \text{ for } k \leq 2n. \quad (4.2)$$

With the notations of Theorem 4.4, we have

**Proposition 4.6.** *The Chern classes of  $D_n$  are:*

$$\begin{aligned} c_0(D_n) &= 1, \\ c_1(D_n) &= 0, \\ c_k(D_n) &= (1 \otimes c_k) \otimes 1 + \sum_{j=1}^{2g} (\sigma_{k,j} \otimes 1) \otimes \alpha_j \text{ for } k \geq 2. \end{aligned}$$

*Proof.* The classes  $c_0(D_n)$  and  $c_1(D_n)$  are trivially 1 and 0 (the structure group is  $\mathbf{SU}(n)$ ). Assume from now on that  $k \geq 2$ . Let us write the Chern classes of  $D_n$

in  $H^*((\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B)$  as

$$c_k(D_n) = \gamma_k^{(n)} \otimes 1 + \sum_{j=1}^{2g} \beta_{k,j}^{(n)} \otimes \alpha_j.$$

We will prove the proposition by induction on  $n$ . For  $n = 1$ ,  $\mathbf{SU}(1)$  is just a point, the bundle  $D_1$  is trivial and we are already done. Suppose the proposition to be true for a given  $n$ ,  $n \geq 1$  and let us prove it for  $n + 1$ . We need to prove that

$$\gamma_k^{(n+1)} = 1 \otimes c_k \text{ and } \beta_{k,j}^{(n+1)} = \sigma_{k,j} \otimes 1.$$

Let

$$m : (\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \longrightarrow (\mathbf{SU}(n+1)^{2g})_{\mathbf{SU}(n+1)},$$

the map induced by the inclusion  $\mathbf{SU}(n) \longrightarrow \mathbf{SU}(n+1)$  and

$$\ell = m \times \text{id}_B : (\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B \longrightarrow (\mathbf{SU}(n+1)^{2g})_{\mathbf{SU}(n+1)} \times B.$$

The bundle  $\ell^* D_{n+1}$  is isomorphic to  $D_n \oplus \mathbf{C}$ . Hence, for all  $k$ , we have  $c_k(\ell^* D_{n+1}) = c_k(D_n)$ . Thus

$$(m^* \gamma_k^{(n+1)}) \otimes 1 + \sum_{j=1}^{2g} (m^* \beta_{k,j}^{(n+1)}) \otimes \alpha_j = \gamma_k^{(n)} \otimes 1 + \sum_{j=1}^{2g} \beta_{k,j}^{(n)} \otimes \alpha_j.$$

From this we deduce

$$m^* \gamma_k^{(n+1)} = \gamma_k^{(n)} \text{ and } m^* \beta_{k,j}^{(n+1)} = \beta_{k,j}^{(n)}.$$

Because of the isomorphisms (4.1), (4.2) and the induction hypothesis, we have:

$$\begin{aligned} \gamma_k^{(n+1)} &= 1 \otimes c_k \text{ for } k \leq n, \\ \beta_{k,j}^{(n+1)} &= \sigma_{k,j} \otimes 1 \text{ for } k \leq n. \end{aligned}$$

There only remains to compute  $\gamma_{n+1}^{(n+1)}$  and the  $\beta_{n+1,j}^{(n+1)}$ . The class  $\gamma_{n+1}^{(n+1)}$  belongs to

$$H_{\mathbf{SU}(n+1)}^{2n+2}(\mathbf{SU}(n+1)^{2g}) = \bigoplus_{p+q=2n+2} H^p(\mathbf{SU}(n+1)^{2g}) \otimes H^q(B\mathbf{SU}(n+1)).$$

Let us decompose it

$$\gamma_{n+1}^{(n+1)} = \sum_{k=0}^{n+1} \varepsilon_k^{(n+1)} \otimes c_k$$

where  $\varepsilon_k^{(n+1)}$  is in  $H^{2n+2-2k}(\mathbf{SU}(n+1)^{2g})$  and where we have put  $c_0 = 1$  in  $H^0(B\mathbf{SU}(n))$ ,  $c_1 = 0$ . The classes  $\beta_{n+1,j}^{(n+1)}$  are in

$$H_{\mathbf{SU}(n+1)}^{2n+1}(\mathbf{SU}(n+1)^{2g}) = \bigoplus_{p+q=2n+1} H^p(\mathbf{SU}(n+1)^{2g}) \otimes H^q(B\mathbf{SU}(n+1)).$$

We decompose them in

$$\beta_{n+1,j}^{(n+1)} = \sum_{k=0}^n \delta_{k,j}^{(n+1)} \otimes c_k$$

where  $\delta_{k,j}^{(n+1)}$  belongs to  $H^{2n+1-2k}(\mathbf{SU}(n+1)^{2g})$ . The bundle  $\ell^* D_{n+1} = D_n \oplus \mathbf{C}$  has a nowhere vanishing section, hence its Euler class  $\ell^* c_{n+1}(D_{n+1})$  vanishes. Because of the isomorphisms (4.1) and (4.2), we deduce that the  $\{\varepsilon_k^{(n+1)}, 1 \leq k \leq n\}$  and the  $\{\delta_{k,j}^{(n+1)}, 1 \leq k \leq n, 1 \leq j \leq 2g\}$  vanish. Remark that the  $\{\delta_{n+1,j}^{(n+1)}, 1 \leq j \leq 2g\}$  are linear combinations of the  $\sigma_{2n+1,j}, 1 \leq j \leq 2g$ . Let us define a section

$$\begin{aligned} s : B\mathbf{SU}(n+1) &\longrightarrow (\mathbf{SU}(n+1)^{2g})_{\mathbf{SU}(n+1)} \times B \\ [e] &\longmapsto ([(\mathbf{I}, \dots, \mathbf{I}), e], 1) \end{aligned}$$

where, for  $e$  in  $E\mathbf{U}(n+1)$ , we denote by  $[e]$  its class in  $B\mathbf{SU}(n+1)$ . The Euler class of the bundle  $s^* D_{n+1}$  is  $\varepsilon X_{n+1}$ . Since  $s^* D_{n+1}$  is equal to  $E\mathbf{U}(n+1) \times_{\mathbf{SU}(n+1)} \mathbf{C}^{n+1}$  we have  $\varepsilon = 1$ . As a conclusion we have

$$\gamma_{n+1}^{(n+1)} = 1 \otimes X_{n+1}.$$

Let

$$h : \mathbf{SU}(n+1)^{2g} \longrightarrow (\mathbf{SU}(n+1)^{2g})_{\mathbf{SU}(n+1)}$$

be the inclusion of a fiber (we will always write  $h$  this application, omitting the subscript  $n$ ). The bundle

$$F_{n+1}^{2g} := (h \times \text{id}_B)^* D_{n+1}$$

is isomorphic to

$$F_{n+1}^{2g} \cong (\mathbf{SU}(n+1)^{2g} \times B' \times \mathbf{C}^{n+1}) / \sim,$$

where  $\sim$  is the relation:

$$((\rho_1, \dots, \rho_{2g}), 1_j, v) \sim ((\rho_1, \dots, \rho_{2g}), 0, \rho_j^{-1}(v)), \text{ for all } j \text{ in } [1, 2g].$$

The Euler class of  $F_{n+1}^{2g}$  is

$$c_{n+1}(F_{n+1}^{2g}) = \sum_{j=1}^{2g} \beta_{n+1,j}^{(n+1)} \otimes \alpha_j.$$

Let  $f_j : S^1 \rightarrow B$  (resp.  $g_j : \mathbf{SU}(n+1) \rightarrow \mathbf{SU}(n+1)^{2g}$ ) be the inclusion of the  $j$ -th circle (resp.  $\mathbf{SU}(n+1)$ ) in  $B$  (resp.  $\mathbf{SU}(n+1)^{2g}$ ). The  $\beta_{n+1,j}^{(n+1)}$  are characterised by:

$$c_{n+1}((\text{id}_{\mathbf{SU}(n+1)^{2g}} \times f_j)^* F_{2g}^{(n+1)}) = \beta_{n+1,j}^{(n+1)} \otimes f_j^* \alpha_j,$$

or

$$c_{n+1}((\text{id}_{\mathbf{SU}(n+1)^{2g}} \times f_j)^* F_{2g}^{(n+1)}) = \beta_{n+1,j}^{(n+1)} \otimes \frac{d\theta}{2\pi}. \quad (4.3)$$

Let us define a vector bundle  $E$  over  $\mathbf{SU}(n+1) \times S^1$  by

$$E = (\mathbf{SU}(n+1) \times [0, 1] \times \mathbf{C}) / \sim,$$

where  $\sim$  is the relation

$$(\rho, 1, v) \sim (\rho, 0, \rho^{-1}(v)).$$

The bundle  $(\mathrm{id}_{\mathbf{SU}(n+1)^{2g}} \times f_j)^* F_{2g}^{(n+1)}$  is isomorphic to  $(g_j \times \mathrm{id}_{S^1})^* E$ . Hence there exists a real  $\lambda$  such that

$$c_{n+1}(E) = \lambda \sigma_{2n+1} \otimes \frac{d\theta}{2\pi}.$$

If  $(\rho, t, v)$  belongs to  $\mathbf{SU}(n+1) \times [0, 1] \times \mathbf{C}^{n+1}$ , let us write  $[\rho, t, v]$  for its class in  $E$ . Let  $(e_1, \dots, e_{n+1})$  be the canonical basis, over the field  $\mathbf{C}$ , of  $\mathbf{C}^{n+1}$ . The family  $(e_1, ie_1, \dots, e_{n+1}, ie_{n+1})$  is then a basis of  $\mathbf{C}^{n+1}$  over  $\mathbf{R}$ . A section of  $E$  is given by:

$$\begin{aligned} s : \mathbf{SU}(n+1) \times S^1 &\longrightarrow E \\ (A, e^{2i\pi\theta}) &\longmapsto [A, \theta, (\theta A + (1-\theta)\mathrm{id})e_1]. \end{aligned}$$

Let us determine its zeros. The vector  $(\theta A + (1-\theta)\mathrm{id})e_1$  vanishes if  $\theta = \frac{1}{2}$  and  $A = \begin{bmatrix} -1 & 0 \\ 0 & \tilde{A} \end{bmatrix}$ ,  $\tilde{A} \in \mathbf{U}(n)$ ,  $\det \tilde{A} = -1$ . Fix  $\xi$  an  $n$ -th root of  $-1$ . The zero set  $Z$  of  $s$  is

$$Z = \left\{ \left( \begin{bmatrix} -1 & 0 \\ 0 & \xi \tilde{A} \end{bmatrix}, \frac{1}{2} \right), \tilde{A} \in \mathbf{SU}(n) \right\}.$$

**Lemma 4.7.** *The section  $s$  intersects the zero section  $s_0$  transversally.*

*Proof.* We want to prove that for all  $x$  of  $Z$

$$T_{s(x)}\mathrm{Im}s + T_{s(x)}\mathrm{Im}s_0 = T_{(x,0)}E.$$

We have

$$T_{(x,0)}E \simeq T_x(\mathbf{SU}(n+1) \times S^1) \oplus \mathbf{C}^{n+1} \simeq \mathfrak{su}(n+1) \oplus \mathbf{R} \oplus \mathbf{C}^{n+1}$$

and

$$T_{s(x)}\mathrm{Im}s_0 = \mathfrak{su}(n+1) \oplus \mathbf{R} \oplus \{0\},$$

$$T_{s(x)}\mathrm{Im}s = T_x s(T_x(\mathbf{SU}(n+1) \times S^1)).$$

Let  $x$  be the point  $(A = \begin{bmatrix} -1 & 0 \\ 0 & \xi \tilde{A} \end{bmatrix}, \frac{1}{2})$ ,

$$\begin{aligned} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} s(A, \tfrac{1}{2} + \varepsilon) &= [A, \tfrac{1}{2} + \varepsilon, ((\tfrac{1}{2} + \varepsilon)A + (\tfrac{1}{2} - \varepsilon)\mathrm{id})e_1] \\ &= [A, \tfrac{1}{2} + \varepsilon, -2\varepsilon e_1] \\ &= (0, 1, -2e_1). \end{aligned}$$

Let  $J$  be in  $\mathfrak{su}(n+1)$ ,

$$\begin{aligned} \frac{d}{d\varepsilon}|_{\varepsilon=0} s(\exp(\varepsilon J)A, \tfrac{1}{2}) &= \frac{d}{d\varepsilon}|_{\varepsilon=0} [\exp(\varepsilon J)A, \tfrac{1}{2}, \tfrac{1}{2}(\exp(\varepsilon J)A + \text{id})e_1] \\ &= \frac{d}{d\varepsilon}|_{\varepsilon=0} [\exp(\varepsilon J)A, \tfrac{1}{2}, \tfrac{1}{2}(\exp(\varepsilon J)(-e_1) + e_1)] \\ &= (J \cdot A, 0, \tfrac{1}{2}(-Je_1 + e_1)). \end{aligned}$$

We conclude the proof of Lemma 4.7 by noticing that, for any  $k$ , it is possible to find  $J$  in  $\mathfrak{su}(n+1)$  such that  $Je_1$  is equal to  $e_k$  or  $ie_k$ .  $\square$

**Lemma 4.8.** *The Euler class of the bundle  $E$  is*

$$c_{n+1}(E) = \sigma_{2n+1} \otimes \frac{d\theta}{2\pi}.$$

*Proof.* According to the preceding lemma, the Euler class of  $E$  is Poincaré dual of  $Z$ , that is it is characterised by

$$\forall \nu \in H^{n^2-1}(\mathbf{SU}(n+1) \times S^1), \int_Z \nu = \int_{\mathbf{SU}(n+1) \times S^1} \nu \wedge c_{n+1}(E)$$

where  $n^2 - 1 = \dim(\mathbf{SU}(n+1) \times S^1) - 2(n+1)$ . This Euler class is of the type

$$c_{n+1}(E) = \lambda \sigma_{2n+1} \otimes \frac{d\theta}{2\pi}$$

where  $\lambda$  is a real we are going to compute. The injection

$$\begin{aligned} \mathbf{SU}(n) &\longrightarrow \mathbf{SU}(n+1) \\ A &\longmapsto \begin{bmatrix} -1 & 0 \\ 0 & \xi A \end{bmatrix} \end{aligned}$$

identifies  $\mathbf{SU}(n)$  to the fibre above  $(-1, 0, \dots, 0)$  of the projection  $\mathbf{SU}(n+1) \rightarrow S^{2n+1}$ , that is  $Z$ . Let  $\gamma$  be the cohomology class of a volume form of volume 1 over  $\mathbf{SU}(n)$ . The decomposition  $H^*(\mathbf{SU}(n+1)) = H^*(\mathbf{SU}(n)) \otimes H^*(S^{2n+1})$  defines a class

$$\nu = \gamma \otimes 1.$$

As the integral of  $\nu$  on  $Z$  is 1, we have

$$\int_{\mathbf{SU}(n+1) \times S^1} \nu \wedge c_{n+1}(E) = 1,$$

that is

$$\lambda \int_{\mathbf{SU}(n+1) \times S^1} (\gamma \otimes 1) \wedge (\sigma_{2n+1} \otimes \frac{d\theta}{2\pi}) = 1.$$

The conclusion follows since the integral in the left-hand side of the equality is equal to 1.  $\square$

Proposition 4.6 follows from this lemma.  $\square$



## 5. Description of the restriction map

Using results of the previous sections, we wish to prove:

**Theorem 5.1.** *The restriction map  $r$  is described by*

$$\begin{aligned} r(c_k) &= a_k \text{ for } k = 2, \dots, n \\ r(\sigma_{k,j}) &= b_{k,j} \text{ for } k = 2, \dots, n, j = 1, \dots, 2g. \end{aligned}$$

*In particular,  $\text{Im}(r)$  is multiplicatively generated by*

$$\text{Im}(r) = \langle a_k, b_{k,j}, k = 2, \dots, n, j = 1, \dots, 2g \rangle.$$

Notice that for  $n$  equals 2, we get that  $r$  is surjective modulo the symplectic form on  $\mathfrak{m}_\beta$  (this result has been in [18]).

It is also very interesting to compare this theorem with [11, Theo. 7.1] where a group cohomological construction of multiplicative generators of  $H^*(\mathfrak{m}_\beta)$  is given.

*Proof.* The key point of the proof is to compare the bundles  $U$  of Section 3 and  $D_n$  of Section 4.

From now on, if  $g \in \mathbf{SU}(n)$ , we denote  $\bar{g}$  its class in  $\mathbf{PSU}(n)$ . Over each  $S \times U_i$ ,  $i = 0, \dots, m$ , the bundle  $M \otimes T \otimes L^*$  is trivial. In each of these sets, the action of  $\mathbf{PSU}(n)$  on  $M \otimes T \otimes L^*$  is

$$\begin{aligned} \mathbf{PSU}(n) \times M \otimes (S \times U_i \times \mathbf{C}^n) \otimes L^* &\longrightarrow M \otimes (S \times U_i \times \mathbf{C}^n) \otimes L^* \\ m \otimes (\bar{g}, (\rho, x, u) \otimes l) &\longmapsto m \otimes (g \cdot \rho, x, g(u)) \otimes (g \cdot l). \end{aligned}$$

**Lemma 5.2.** *We have*

$$P(U) = P(M \otimes (T \otimes L^*)/\mathbf{PSU}(n)) \cong P(T)/\mathbf{PSU}(n).$$

*Proof.* This time,  $\mathbf{PSU}(n)$  acts on  $P(T)$  by

$$\begin{aligned} \mathbf{PSU}(n) \times (S \times U_i \times \mathbf{CP}^n) &\longrightarrow (S \times U_i \times \mathbf{CP}^n) \\ (\bar{g}, (\rho, x, \bar{u})) &\longmapsto (g \cdot \rho, x, \overline{g(u)}) \end{aligned}$$

and the announced isomorphism is

$$\begin{aligned} P(U) &\xrightarrow{\cong} P(T)/\mathbf{PSU}(n) \\ \text{class of } m \otimes (\rho, x, u) \otimes l &\longmapsto \text{class of } (\rho, x, u). \end{aligned}$$

□

**Lemma 5.3.** *There exists an action of  $\pi \times \mathbf{PSU}(n)$  on  $S \times Y' \times \mathbf{CP}^{n-1}$  such that the quotient*

$$(S \times Y' \times \mathbf{CP}^{n-1})/(\pi \times \mathbf{PSU}(n))$$

*is isomorphic to*

$$P(U)|_{\mathfrak{m}_\beta \times X'}.$$

*Proof.* The bundle  $T$  restricted to  $S \times X'$  is trivial on each  $S \times U_i$ ,  $i \neq 0$  and transition functions are given by

$$\begin{aligned} (S \times U_i) \cap (S \times U_j) &\longrightarrow \mathbf{SU}(n) \\ (\rho, x) &\longmapsto \rho(\gamma_{i,j}). \end{aligned}$$

The group  $\pi$  acts freely on  $Y'$  and  $T|_{S \times X'}$  is  $(S \times Y' \times \mathbf{C}^n)/\pi$ , where the action of  $\pi$  is

$$\begin{aligned} \pi \times (S \times Y' \times \mathbf{C}^n) &\longrightarrow S \times Y' \times \mathbf{C}^n \\ (\gamma, (\rho, y, u)) &\longmapsto (\rho, \gamma \cdot y, \rho(\gamma)u). \end{aligned}$$

Let us consider the projective bundle  $P(T)|_{S \times X'}$ . It is isomorphic to  $(S \times Y' \times \mathbf{CP}^{n-1})/\pi$ . The subspace  $P(T)|_{S \times X'}$  is stable by  $\mathbf{PSU}(n)$  and the action comes from an action of  $\mathbf{PSU}(n)$  on  $S \times Y' \times \mathbf{CP}^{n-1}$ . That is

$$\begin{aligned} \mathbf{PSU}(n) \times (S \times Y' \times \mathbf{CP}^{n-1}) &\longrightarrow S \times Y' \times \mathbf{CP}^{n-1} \\ (\bar{g}, (\rho, y, \bar{u})) &\longmapsto (g \cdot \rho, y, \overline{g(u)}). \end{aligned}$$

This action commutes indeed with the one of  $\pi$ , the result follows.  $\square$

The pull-back of the bundle  $U \rightarrow (S/\mathbf{PSU}(n)) \times X'$  to  $(S)_{\mathbf{SU}(n)} \times X'$  by the natural map

$$f : (S)_{\mathbf{SU}(n)} \times X' \longrightarrow (S/\mathbf{PSU}(n)) \times X'$$

is a vector bundle, we will denote it  $F$ . Its projectivised bundle is

$$P(F) = (P(T))_{\mathbf{SU}(n)} \longrightarrow (S)_{\mathbf{SU}(n)} \times X'.$$

We will now state a proposition which will be our main tool in the study of the map  $r$ :

**Proposition 5.4.** *There is a projective bundle  $P(D)$  over  $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times X'$  whose restriction to  $(S)_{\mathbf{SU}(n)} \times X'$  is isomorphic to  $P(F)$ .*

*First proof.* The projection  $p : Y' \rightarrow X'$  is a covering. Its group is  $\pi$ . Let  $q : \tilde{Y}' \rightarrow Y'$  be the universal covering of  $Y'$ . The composed map  $\tilde{p} = p \circ q : \tilde{Y}' \rightarrow X'$  is the universal covering of  $X'$ . Its group is

$$\pi_1(X') = \langle a_1, b_1, \dots, a_g, b_g \rangle$$

and we have a projection  $\pi_1(X') \xrightarrow{\theta} \pi$  whose kernel is the group of the covering  $\tilde{Y}' \rightarrow Y'$ .

The open covering of  $X'$  by the  $\{U_i\}_{i=1}^m$  is such that any intersection of open sets of the type  $U_i$  is contractible. In particular, for all  $i$ , there exists a disc  $\tilde{D}_i$  in  $\tilde{Y}'$  such that  $\tilde{p} : \tilde{D}_i \rightarrow U_i$  is a diffeomorphism. Choose, for all  $i, j, k$ , a connected component  $W_{ij,k}$  of  $\tilde{p}^{-1}(U_i \cap U_j) \cap \tilde{D}_k$ . If  $U_i \cap U_j \neq \emptyset$ , let  $\tilde{\gamma}_{i,j}$  be the element of  $\pi_1(X')$  such that  $\tilde{\gamma}_{i,j} \tilde{W}_{ij,j} = \tilde{W}_{ji,i}$ . In Proposition 2.2, we can take the  $W_{ij,k}$  and  $\gamma_{i,j}$  such that

$$W_{ij,k} = \tilde{p}(\tilde{W}_{ij,k}), \gamma_{i,j} = \theta(\tilde{\gamma}_{i,j}).$$

Let us identify the set of representations  $\rho : \pi_1(X') \rightarrow \mathbf{SU}(n)$  to  $\mathbf{SU}(n)^{2g}$  by

$$\rho \mapsto (\rho(a_1), \rho(b_1), \dots, \rho(a_g), \rho(b_g)).$$

Let

$$T' \rightarrow \mathbf{SU}(n)^{2g} \times X'$$

be the rank  $n$  complex vector bundle defined by the following properties:

- (1)  $T'|_{\mathbf{SU}(n)^{2g} \times U_i}$  is trivial,
- (2) the transition functions are

$$g_{i,j} = \rho(\tilde{\gamma}_{i,j}) \text{ on } \mathbf{SU}(n)^{2g} \times (U_i \cap U_j).$$

The restriction of this bundle to  $S \times X'$  is  $T|_{S \times X'}$ . The action of  $\mathbf{SU}(n)$  on  $T|_{S \times X'}$  is then the restriction of the  $\mathbf{SU}(n)$  action on  $T'$  defined on each  $T'|_{\mathbf{SU}(n)^{2g} \times U_i}$  by

$$\begin{aligned} \mathbf{SU}(n) \times (\mathbf{SU}(n)^{2g} \times U_i \times \mathbf{C}^n) &\longrightarrow \mathbf{SU}(n)^{2g} \times U_i \times \mathbf{C}^n \\ (g, (\rho, x, u)) &\longmapsto (g \cdot \rho, x, g(u)). \end{aligned}$$

Notice that this action is a lift of the action of  $\mathbf{SU}(n)$  on  $\mathbf{SU}(n)^{2g} \times X'$ . Thus the bundle

$$P(F) = (P(T))_{\mathbf{SU}(n)} \rightarrow (S)_{\mathbf{SU}(n)} \times X'$$

is the restriction of the bundle

$$(P(T'))_{\mathbf{SU}(n)} \rightarrow (\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times X'.$$

*Second proof.* We have seen that

$$P(U)|_{\mathfrak{m}_\beta \times X'} \cong (S \times Y' \times \mathbf{CP}^{n-1})/(\pi \times \mathbf{PSU}(n)),$$

hence

$$P(F) \cong (S \times E\mathbf{U}(n) \times Y' \times \mathbf{CP}^{n-1})/(\pi \times \mathbf{SU}(n)).$$

Let us define, in a similar way as before, an action of  $\pi_1(X')$  on  $\mathbf{SU}(n)^{2g} \times E\mathbf{U}(n) \times \widetilde{Y}' \times \mathbf{C}^n$  and denote  $D$  the bundle we obtain when quotienting by  $\pi_1(X') \times \mathbf{SU}(n)$ . The projection  $S \times E\mathbf{U}(n) \times \widetilde{Y}' \times \mathbf{C}^n \rightarrow S \times E\mathbf{U}(n) \times Y' \times \mathbf{C}^n$  is equivariant for the respective actions of  $\pi_1(X')$  and  $\pi$ . It induces an action on the quotient and defines an isomorphism between

$$(S \times E\mathbf{U}(n) \times \widetilde{Y}' \times \mathbf{C}^n)/(\pi_1(X') \times \mathbf{SU}(n))$$

and

$$(S \times E\mathbf{U}(n) \times Y' \times \mathbf{C}^n)/(\pi \times \mathbf{SU}(n)).$$

We deduce that  $P(F)$  is isomorphic to  $P(D)|_{(S)_{\mathbf{SU}(n)} \times X'}$ . □

**Remark 5.5.** The bundle  $D \rightarrow (\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times X'$  is isomorphic to  $(T' \times E\mathbf{U}(n))/\mathbf{SU}(n)$ .

When restricted to  $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B$ , the bundle  $D$  is isomorphic to  $D_n$  (restricted to  $(\mu^{-1}(\zeta\mathbf{I}))_{\mathbf{SU}(n)} \times B$ ). Denote  $w$  the injection of  $(S)_{\mathbf{SU}(n)} \times X'$  in  $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times X'$ . The induced map  $w^*$  in cohomology is  $r \times \text{id}_{H^*(X')}$ . The restriction  $w^*D_n$  of  $D_n$  to  $(S)_{\mathbf{SU}(n)} \times X'$  has the same projectivisation as  $F$ . Thus, because of Proposition 4.6, we have for every  $k$

$$p_k(P(F)) = a_k \otimes 1 + \sum_{j=1}^{2g} b_k^j \otimes \alpha_j \quad (5.1)$$

$$\begin{aligned} &= p_k(P(w^*D_n)) \\ &= w^*p_k(P(D_n)) \\ &= r(1 \otimes p_k) \otimes 1 + \sum_{j=1}^{2g} r(\sigma_{k,j} \otimes 1) \otimes \alpha_j. \end{aligned} \quad (5.2)$$

Theorem 5.1 follows from the comparison of Line (5.1) and Line (5.2).  $\square$

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