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## Logarithmic cohomology of the complement of a plane curve

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and Francisco J. Castro Jiménez<sup>1</sup>

**Abstract.** Let  $D, x$  be a plane curve germ. We prove that the complex  $\Omega^\bullet(\log D)_x$  computes the cohomology of the complement of  $D, x$  *only if*  $D$  is quasihomogeneous. This is a partial converse to a theorem of [5], which asserts that this complex does compute the cohomology of the complement, whenever  $D$  is a locally weighted homogeneous free divisor (and so in particular when  $D$  is a quasihomogeneous plane curve germ). We also give an example of a free divisor  $D \subset \mathbb{C}^3$  which is not locally weighted homogeneous, but for which this (second) assertion continues to hold.

**Mathematics Subject Classification (2000).** Primary 32S20; Secondary 32S40, 14F40.

**Keywords.** Free divisor, logarithmic de Rham complex, plane curve, local quasi-homogeneity.

### 1. Introduction

In [5] the last three authors showed that if  $D$  is a locally quasi-homogeneous free divisor in the complex manifold  $X$  then locally the complex  $\Omega^\bullet(\log D)$  of holomorphic differential forms with logarithmic poles along  $D$  calculates the cohomology of the complement of  $D$  in  $X$ . More precisely, the following two equivalent statements hold:

**Theorem 1.1.** *With  $D$  as above,*

1. *If  $V \subset X$  is a Stein open set then the de Rham map (integration of forms over cycles) gives rise to an isomorphism*

$$h^k(\Gamma(V, \Omega^\bullet(\log D))) \xrightarrow{\sim} H^k(V \setminus D; \mathbb{C}).$$

2. *Denoting by  $U$  the complement of  $D$  in  $X$  and by  $j : U \hookrightarrow X$  the inclusion, the de Rham morphism gives rise to an isomorphism*

$$\Omega^\bullet(\log D) \xrightarrow{\sim} \mathbf{R}j_*(\mathbb{C}_U). \quad \square$$

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By analogy with Grothendieck's Comparison Theorem [8], in which the complex  $\Omega^\bullet(\log D)$  is replaced in these two statements by  $\Omega^\bullet(*D)$ , but which holds for an arbitrary divisor, we summarise this with a slogan: if  $D \hookrightarrow X$  is a locally quasi-homogeneous free divisor then the *logarithmic comparison theorem* holds.

The definition of local quasi-homogeneity, (called *strong* quasi-homogeneity in [5]), is as follows:

**Definition 1.2.**

1. The polynomial  $h(z_1, \dots, z_n) = \sum a_{i_1, \dots, i_n} z_1^{i_1} \dots z_n^{i_n} \in \mathcal{O}_{\mathbb{C}^n}$  is *weighted homogeneous* if there exist positive integer weights  $w_1, \dots, w_n$  such that  $h(z_1^{w_1}, \dots, z_n^{w_n})$  is homogeneous.
2. The divisor  $D \subset X$  is *locally quasi-homogeneous* if for all  $x \in D$  there are local coordinates on  $X$ , centered at  $x$ , with respect to which  $D$  has a weighted homogeneous defining equation.

Every plane curve is a free divisor, since the module of logarithmic vector fields  $\text{Der}(\log D)$  is reflexive and thus has depth at least 2. In [4, Cor. 4.2.2] the first author showed that if  $D$  is a plane curve then the logarithmic de Rham complex  $\Omega^\bullet(\log D)$  is perverse, a necessary condition for the logarithmic comparison theorem.

In [6] the logarithmic comparison theorem has been tested for the following non locally quasi-homogeneous plane curve (cf. [9]):  $D = \{f = x_1^4 + x_2^5 + x_2^4 x_1 = 0\} \subset X = \mathbb{C}^2$ . A basis for  $\text{Der}(\log D)$  is given by:

$$\begin{aligned} \delta_1 &= (16x_1^2 + 20x_1x_2) \frac{\partial}{\partial x_1} + (12x_1x_2 + 16x_2^2) \frac{\partial}{\partial x_2} \\ \delta_2 &= (16x_1x_2^2 + 4x_2^3 - 125x_1x_2) \frac{\partial}{\partial x_1} + (12x_2^3 - 4x_1^2 + 5x_1x_2 - 100x_2^2) \frac{\partial}{\partial x_2}. \end{aligned}$$

Let  $\mathcal{D}_X$  be the sheaf of linear differential operators with holomorphic coefficients on  $X$  and  $I$  the left  $\mathcal{D}_X$ -ideal generated by  $\delta_1, \delta_2$ . By [4, Th. 4.2.1], we have a (canonical) isomorphism (in the derived category)

$$\Omega^\bullet(\log D) \simeq \mathbf{R}\text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/I, \mathcal{O}_X),$$

and so we can compute the characteristic cycle  $CC(\Omega^\bullet(\log D))$  as the cycle  $Z$  in  $T^*X$  determined by the ideal  $J = \sigma(I)$  generated by the principal symbols of elements in  $I$ . The symbols  $\sigma_1 = \sigma(\delta_1), \sigma_2 = \sigma(\delta_2)$  form a regular sequence in  $\mathcal{O}_{T^*X}$  and so, by [4, Prop. 4.1.2], the ideal  $J$  is generated by  $\sigma_1, \sigma_2$ . An easy computation shows that the multiplicity of the conormal at 0 in  $Z$  is 4. On the other hand, the multiplicity of the conormal at 0 in  $CC(\mathbf{R}j_*(\mathbb{C}_U))$  is equal to  $\text{mult}_0(D) - 1 = 3$  (cf. [3]), and so the logarithmic comparison theorem does not hold for  $D$ .

For the family of non locally quasi-homogeneous plane curves (cf. [9])

$$x_1^q + x_2^p + x_2^{p-1}x_1 = 0, \quad p \geq q + 1 \geq 5,$$

the multiplicities of the conormal at 0 in  $CC(\Omega^\bullet(\log D))$  and in  $CC(\mathbf{R}j_*(\mathbb{C}_U))$  are  $2(q-2)$  and  $q-1$  respectively, and so these curves also do not satisfy the logarithmic comparison theorem.

A natural question is therefore whether or not the logarithmic comparison theorem holds for a given free divisor.

The purpose of this paper is to prove a partial converse to Theorem 1.1. We prove:

**Theorem 1.3.** *Let  $D$  be a reduced plane curve. If the logarithmic comparison theorem holds for  $D$ , then  $D$  is locally quasi-homogeneous.*

Our proof shows that if  $h$  is a local equation of  $D$ , and the logarithmic comparison theorem holds, then there is a vector field germ  $\chi$  such that  $\chi \cdot h = h$ . As a reduced curve has isolated singularities, we can then apply the theorem of K. Saito [10]: if  $h \in \mathcal{O}_{\mathbb{C}^n,0}$  has isolated singularity and  $h$  belongs to its Jacobian ideal  $J_h$  then in suitable coordinates  $h$  is weighted homogeneous.

We conjecture that in higher dimensions the following version of our Theorem 1.3 holds:

**Conjecture 1.4.** *If  $D \hookrightarrow X$  is a free divisor and if the logarithmic comparison theorem holds, then for all  $x \in D$  there is a local equation  $h$  for  $D$  around  $x$ , and a germ of vector field  $\chi$  vanishing at  $x$  such that  $\chi \cdot h = h$ .*

A singular free divisor of dimension greater than 1 has non-isolated singularities, so even if this conjecture is true, Saito's theorem cannot be used to deduce local quasi-homogeneity. Indeed, it is *not* true in higher dimensions that if the logarithmic comparison theorem holds for a free divisor  $D$  then  $D$  is necessarily locally quasi-homogeneous. This is shown by an example in Section 4 below: the logarithmic comparison theorem holds for the free divisor

$$D = \{(x, y, z) : xy(x+y)(zx+y) = 0\}$$

(the total space of a family of four lines in the plane with varying cross-ratio, cf. [4]), in the neighbourhood of  $(0, 0, \lambda)$ , with  $\lambda \in \mathbb{C} \setminus \{0, 1\}$ ; however it is well known that this divisor is not locally quasi-homogeneous. On the other hand, it does satisfy Conjecture 1.4.

Adrian Langer has indicated to us that he has subsequently found a shorter proof of Theorem 1.3, using globalisation and a comparison of Chern classes<sup>1</sup>.

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<sup>1</sup>Added on November 2001.

## 2. Preliminary results

In this section we recall the spectral sequence argument used in [5] to compare the cohomology of the logarithmic complex  $\Omega^\bullet(\log D)$  with the cohomology of  $X \setminus D$ . Except for referring to “local” rather than “strong” quasi-homogeneity, we will use the same notation as [5].

Without loss of generality we assume  $X = \mathbb{C}^n$  with coordinates  $z_i$  and  $x_0 = 0$ . Let  $V$  be a Stein neighbourhood (sufficiently small) of 0, let  $\mathcal{U}$  be the open cover of  $V \setminus \{0\}$  consisting of the sets  $U_i = V \cap \{z_i \neq 0\}$ , and let  $\mathcal{U}'$  be the open cover of  $V \setminus D$  consisting of the open sets  $U'_i = (V \setminus D) \cap \{z_i \neq 0\} = U_i \setminus D$ .

We consider the two double complexes

$$K^{p,q} = \check{C}^q(\mathcal{U}, \Omega^p(\log D))$$

and

$$\tilde{K}^{p,q} = \check{C}^q(\mathcal{U}', \Omega^p),$$

equipped with the exterior derivative  $d$  (the horizontal differential) and the Čech differential  $\delta$  (the vertical differential). There is an obvious restriction morphism  $\rho_{p,q} : K^{p,q} \rightarrow \tilde{K}^{p,q}$  which commutes with both differentials, and thus gives rise to morphisms of the two spectral sequences arising from each double complex. These spectral sequences have  $E_1$  terms

$$\begin{aligned} {}''E_1^{p,q} &= \check{H}^q(\mathcal{U}, \Omega^p(\log D)) \\ {}''\tilde{E}_1^{p,q} &= \check{H}^q(\mathcal{U}', \Omega^p) \\ {}'E_1^{p,q} &= \oplus_{1 \leq i_1 < \dots < i_{q+1} \leq n} h^p \left( \Gamma \left( \bigcap_j U_{i_j}, \Omega^\bullet(\log D) \right) \right) \\ {}'\tilde{E}_1^{p,q} &= \oplus_{1 \leq i_1 < \dots < i_{q+1} \leq n} h^p \left( \Gamma \left( \bigcap_j U'_{i_j}, \Omega^\bullet \right) \right). \end{aligned}$$

As both  $\mathcal{U}$  and  $\mathcal{U}'$  are Stein covers,

$$\check{H}^q(\mathcal{U}, \Omega^p(\log D)) = \check{H}^q(V \setminus \{0\}, \Omega^p(\log D))$$

and

$$\check{H}^q(\mathcal{U}', \Omega^p) = \check{H}^q(V \setminus D, \Omega^p).$$

As  $V \setminus D$  is Stein,  $\check{H}^q(V \setminus D, \Omega^p) = 0$  if  $q > 0$ . It follows that

$${}''\tilde{E}_2^{p,q} = \begin{cases} H^p(V \setminus D; \mathbb{C}) & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases},$$

and in particular the spectral sequence  ${}''\tilde{E}$  converges to the cohomology of  $V \setminus D$ .

Now assume that outside 0,  $D$  is locally quasi-homogeneous, so that by 1.1  $\mathbf{R}j_*(\mathbb{C}_U) \simeq \Omega^\bullet(\log D)$ , again outside 0. As  $\mathcal{U}$  and  $\mathcal{U}'$  are Stein covers, by 1.1 the quotient of the restriction  $\rho_{p,q}$  defines an isomorphism  ${}'\rho_{p,q} : {}'E_1^{p,q} \rightarrow {}'\tilde{E}_1^{p,q}$  for all  $p, q$ . This isomorphism persists to give an isomorphism of the cohomology of the

total complexes  $K^{\text{tot}}$  and  $\tilde{K}^{\text{tot}}$  as calculated by the spectral sequences. It follows that the spectral sequence  ${}''E$ , like  ${}''\tilde{E}$ , also converges to the cohomology of  $V \setminus D$ :

$$H^k(V \setminus D; \mathbb{C}) \simeq \oplus_{p+q=k} {}''E_{\infty}^{p,q}.$$

As  $D$  is a free divisor,  $\check{H}^q(V \setminus \{0\}, \Omega^p(\log D)) = 0$  for  $q \neq 0, n-1$ , so  ${}''E_1$  has only two non-null rows; writing for the moment  $\Omega^p(D)$  and  $V^*$  in place of  $\Omega^p(\log D)$  and  $V \setminus \{0\}$ ,  ${}''E_1$  thus looks like

$$\begin{array}{ccccccc} \check{H}^{n-1}(V^*, \Omega^0(D)) & \xrightarrow{d_1} & \dots & \xrightarrow{d_1} & \check{H}^{n-1}(V^*, \Omega^p(D)) & \xrightarrow{d_1} & \dots & \xrightarrow{d_1} & \check{H}^{n-1}(V^*, \Omega^n(D)) \\ 0 & & \dots & & 0 & & \dots & & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & & \dots & & 0 & & \dots & & 0 \\ \Gamma(V, \Omega^0(D)) & \xrightarrow{d_1} & \dots & \xrightarrow{d_1} & \Gamma(V, \Omega^p(\log D)) & \xrightarrow{d_1} & \dots & \xrightarrow{d_1} & \Gamma(V, \Omega^n(\log D)). \end{array}$$

(Note that as  $n \geq 2$  and as the  $\Omega^p(\log D)$  are free modules, we have  $\Gamma(V^*, \Omega^p(D)) = \Gamma(V, \Omega^p(D))$ .)

As this spectral sequence converges to the cohomology of  $V \setminus D$ , we have

$$H^{n-1}(V \setminus D; \mathbb{C}) \simeq E_{\infty}^{0,n-1} \oplus \dots \oplus E_{\infty}^{n-1,0} = E_{n+2}^{0,n-1} \oplus h^{n-1}(\Gamma(V, \Omega^{\bullet}(\log D)))$$

$$H^n(V \setminus D; \mathbb{C}) = E_{\infty}^{0,n} \oplus \dots \oplus E_{\infty}^{0,n} = E_{n+2}^{1,n-1} \oplus \frac{h^n(\Gamma(V, \Omega^{\bullet}(\log D)))}{d_{n+1}(E_{n+2}^{0,n-1})},$$

where

$$E_{n+2}^{0,n-1} = \text{Ker } d_1 : \check{H}^{n-1}(V^*, \Omega^0(D)) \rightarrow \check{H}^{n-1}(V^*, \Omega^1(D)).$$

In [5], the main theorem was proved by showing that if  $D$  is locally quasi-homogeneous then the complex

$$(\check{H}^{n-1}(V \setminus \{0\}, \Omega^{\bullet}(\log D)), d_1)$$

is exact.

### 3. Proof of the Theorem

We continue with the discussion of the last paragraph. If the natural morphism  $\Omega^{\bullet}(\log D) \rightarrow \mathbf{R}j_*(\mathbb{C}_U)$  is a quasi-isomorphism (i.e. if the logarithmic comparison theorem holds for  $D$ ) then by the formulae of the last section,  $d_1 : \check{H}^{n-1}(V \setminus \{0\}, \Omega^0(\log D)) \rightarrow \check{H}^{n-1}(V \setminus \{0\}, \Omega^1(\log D))$  is injective.

Let  $\{\omega_1, \dots, \omega_n\}$  be a free basis of  $\Omega^1(\log D)$  as  $\mathcal{O}_V$ -module, and let  $\delta_1, \dots, \delta_n$  be the dual basis of  $\mathcal{D}er(\log D)$ . Then  $\check{H}^{n-1}(V \setminus \{0\}, \Omega^0(\log D)) = \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n})$  and  $\check{H}^{n-1}(V \setminus \{0\}, \Omega^1(\log D)) \simeq \oplus_1^n \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n})$ . The morphism  $d_1 : \check{H}^{n-1}(V \setminus \{0\}, \Omega^0(\log D)) \rightarrow \check{H}^{n-1}(V \setminus \{0\}, \Omega^1(\log D))$  now becomes

$$\begin{array}{ccc} \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n}) & \xrightarrow{d_1} & \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n})^n \\ [g] & \mapsto & ([\delta_1 \cdot g], \dots, [\delta_n \cdot g]). \end{array}$$

where  $g \in \Gamma(V \setminus \cup_i \{z_i = 0\}, \mathcal{O}_{\mathbb{C}^n}) = \Gamma(\mathbb{C}^n \setminus \cup_i \{z_i = 0\}, \mathcal{O}_{\mathbb{C}^n})$  represents the class  $[g]$  in  $\check{H}^{n-1}(\mathbb{C}^n \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n})$ .

For  $\delta \in \text{Der}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n})$ , we denote by  $d_\delta$  the homomorphism

$$d_\delta : \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n}) \rightarrow \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n}), \quad d_\delta([g]) = [\delta \cdot g].$$

**Proposition 3.1.** *Let  $\mathfrak{m}_{\mathbb{C}^n,0}$  be the maximal ideal of  $\mathcal{O}_{\mathbb{C}^n,0}$  and let  $\delta \in \mathfrak{m}_{\mathbb{C}^n,0} \text{Der}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n})$ ,*

$$\delta = (x_1, \dots, x_n) \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} \partial/\partial x_1 \\ \vdots \\ \partial/\partial x_n \end{pmatrix} + \delta_{\geq 1}$$

*with the  $a_{i,j} \in \mathbb{C}$  and  $\delta_{\geq 1} \in \mathfrak{m}_{\mathbb{C}^n,0}^2 \text{Der}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n})$ . If  $d_\delta$  is injective, then the eigenvalues of  $A$  do not satisfy any relation with positive integer coefficients (in this case, we will say that  $\delta$  satisfies condition (I)).*

*Proof.* By a coordinate change we can make  $A$  lower triangular. Its eigenvalues  $a_1, \dots, a_n$  are then the elements of the diagonal. The group  $\check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n})$  is isomorphic to the space of Laurent series, convergent for all  $\underline{x} = (x_1, \dots, x_n)$  with  $\underline{x} \neq 0$ , whose non-zero coefficients are those with strictly negative indices in all variables, i.e.

$$\sum_{i_1, \dots, i_n < 0} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n}.$$

For  $p \geq n$ , we set

$$G^p = \left\{ \sum_{\substack{i_1, \dots, i_n < 0 \\ i_1 + \dots + i_n = -p}} c_{\mathbf{i}} x_1^{i_1} \cdots x_n^{i_n} \right\},$$

$$F^p = \left\{ \sum_{\substack{i_1, \dots, i_n < 0 \\ i_1 + \dots + i_n \geq -p}} c_{\mathbf{i}} x_1^{i_1} \cdots x_n^{i_n} \right\}.$$

Then  $F^p = G^p \oplus G^{p-1} \oplus \cdots \oplus G^n$ . Each  $G^p$  is a finite-dimensional  $\mathbb{C}$ -vector space, whose dimension we denote by  $r_p$ , and  $d_\delta$  restricts to morphisms of vector spaces

$$d_\delta|_{F^p} : F^p \rightarrow F^p$$

and

$$d_\delta|_{G^p} : G^p \rightarrow F^p.$$

Let us denote by  $d_{\delta,p}^p$  the component of this second restriction lying in  $G^p$ . Then  $d_{\delta,p}^p$  depends only on the weight 0 part  $\delta_0$  of  $\delta$ . We claim that with respect to a suitable ordered basis of  $G^p$ , its matrix  $[d_{\delta,p}^p]$  is lower triangular.

As basis for  $G^p$  we take the monomials

$$\frac{1}{x_1^{i_1} \cdots x_n^{i_n}}$$

with  $i_1 + \cdots + i_n = p$ .

We have

$$d_\delta(x_1^{-i_1} \cdots x_n^{-i_n}) = - \sum_{j,k} i_k a_{j,k} x_1^{-i_1} \cdots x_k^{-(i_k-1)} \cdots x_j^{-(i_j+1)} \cdots x_n^{-i_n}. \quad (1)$$

Thus, if we give our basis of  $G^p$  the lexicographic order corresponding to the order of the coordinates  $x_1, \dots, x_n$ , then since  $a_{j,k} = 0$  if  $j < k$  (recall that we have chosen our coordinates so that  $A$  is lower triangular), the matrix  $[d_{\delta,p}^p]$  is lower triangular.

Let  $q \leq p$ . Then  $d_\delta(G^q) \subset G^q + G^{q-1} + \cdots + G^n$ . Thus, it follows from the above that if we give  $F^p$  the ordered basis consisting of the ordered bases for each  $G^q$ ,  $n \leq q \leq p$  that we have chosen, and order these by descending value of  $q$ , then the matrix of  $d_\delta|_{F^p}$  is also lower triangular.

What are its diagonal elements? In the right-hand side of equation (1), the coefficient of  $x_1^{-i_1} \cdots x_n^{-i_n}$  is equal to

$$i_1 a_{1,1} + \cdots + i_n a_{n,n};$$

this is the diagonal element in the matrix of  $d_\delta|_{F^p}$  in the row and column corresponding to the basis element  $x_1^{-i_1} \cdots x_n^{-i_n}$ . Note that the diagonal elements of  $A$  are its eigenvalues; thus, the diagonal elements in the matrix of  $d_\delta|_{F^p}$  with respect to the chosen basis are all linear combinations  $i_1 \lambda_1 + \cdots + i_n \lambda_n$  of the eigenvalues  $\lambda_1, \dots, \lambda_n$  of  $A$ , with the  $i_j$  positive integers and  $i_1 + \cdots + i_n \leq p$ . As this matrix is lower triangular,  $d_\delta|_{F^p}$  is injective only if the product of these diagonal elements is non-zero.  $\square$

**Remark 3.2.** We have used in the proof of this lemma the fact that if  $d_\delta$  is injective then so is its restriction to each  $F^p$ . We do not know if the opposite implication holds. It seems likely that an argument involving faithful flatness would prove it. However, we do not need it in what follows.

Let  $D$  be a plane curve. We suppose as above that 0 is the singular point of  $D$ . In this case the upper non-zero row in the  $E_2$  page of the spectral sequence  $'\tilde{E}$  begins

$$d_1 : \check{H}^1(\mathbb{C}^2 \setminus \{0\}, \mathcal{O}_{\mathbb{C}^2}) \rightarrow \oplus_1^2 \check{H}^1(\mathbb{C}^2 \setminus \{0\}, \mathcal{O}_{\mathbb{C}^2}).$$



**Theorem 3.3.** *Let  $D$  be a plane curve, singular at 0. If  $d_1$  is injective, then there is a local equation  $h$  for  $D$  around 0, and a germ of vector field  $\chi$  at 0 such that  $\chi \cdot h = h$ .*

*Proof.* Any reduced plane curve whose equation has non-zero quadratic part is quasihomogeneous, by the classification of singularities of functions of two variables: such a curve is equivalent to  $A_k$ ,  $x^2 + y^{k+1} = 0$ , for some  $k$ . For a quasihomogeneous curve, the conclusion of the theorem of course holds. Thus, we may assume that the equation  $h$  of  $D$  lies in  $\mathbf{m}_{\mathbb{C}^2, 0}^3$ . As the determinant of the coefficients of a free basis of  $\text{Der}(\log D)$  is a local defining equation for  $D$  ([11]), we may therefore choose a free basis  $\delta, \gamma$  for  $\text{Der}(\log D)$  such that  $\gamma$  has zero linear part. In fact the supposition that  $d_1$  is injective implies that at least one member of the basis has non-zero linear part, as otherwise  $d_1([1/xy]) = ([\delta \cdot 1/xy], [\gamma \cdot 1/xy]) = 0$ .

We may thus take

$$\delta = \delta_0 + \delta_1 + \delta_2 + \cdots = \sum_{k \geq 0} \sum_{i+j=k+1} \left( \alpha_{ij} x^i y^j \frac{\partial}{\partial x} + \beta_{ij} x^i y^j \frac{\partial}{\partial y} \right)$$

where  $\delta_0 = \underline{x} A \underline{\partial_x}^t$ , with  $A \neq 0$  and in Jordan normal form, i.e.

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix}.$$

Let  $h$  be the reduced equation of  $D$ :

$$h = h_n + h_{n+1} + h_{n+2} + \cdots = \sum_{k \geq n} h_k = \sum_{k \geq n} \sum_{i+j=k} a_{ij} x^i y^j,$$

where the polynomials  $h_i$  are homogeneous of degree  $i$ .

Let us now suppose that  $\delta$  is not an Euler vector field for  $h$ , we will see that (up to multiplication by a non-zero constant) the only possibility for  $h$  and  $\delta$  is

$$h_1 = \cdots = h_{n-1} = 0, h_n = x^a y^b \quad \text{and} \quad \delta_0 = qx \frac{\partial}{\partial x} - py \frac{\partial}{\partial y}.$$

First case:  $h_n = \sum_{i+j=n} a_{ij} x^i y^j$  and  $\delta_0 = \lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y}$ . Then

$$0 = \delta_0(h_n) = \sum_{i+j=n} (i\lambda_1 + j\lambda_2) a_{ij} x^i y^j.$$

So,  $a_{ij} = 0$  if  $i\lambda_1 + j\lambda_2 \neq 0$ ; thus, since by assumption  $h_n \neq 0$ , we have  $q\lambda_1 = -p\lambda_2$  and  $p + q = n$  ( $p, q \in \mathbb{N}$ ). In this case,

$$h_n = x^p y^q, \quad \delta_0 = qx \frac{\partial}{\partial x} - py \frac{\partial}{\partial y}.$$

Second case:  $h_n = \sum_{i+j=n} a_{ij} x^i y^j$  and  $\delta_0 = (\lambda_1 x + y) \frac{\partial}{\partial x} + \lambda_1 y \frac{\partial}{\partial y}$ . Then

$$0 = \delta_0(h_n) = n\lambda_1 a_{n0} x^n + \sum_{i+j=n, j \geq 1} (n\lambda_1 a_{ij} + i a_{i+1, j-1}) x^i y^j.$$

So, if  $\lambda_1 \neq 0$ , then we must have  $a_{n0} = 0$ , then  $a_{n-1,1} = 0, \dots, a_{1,n-1} = 0, a_{0n} = 0$ , so that  $h_n = 0$ . This is absurd, by hypothesis.

If  $\lambda_1 = 0$ , then  $d_1$  is not injective, because

$$d_1([1/xy]) = (d_\delta([1/xy]), d_\gamma([1/xy])) = (0, 0).$$

Then, we have

$$h = x^p y^q + h_{n+1} + h_{n+2} + \dots, \quad \delta_0 = qx \frac{\partial}{\partial x} - py \frac{\partial}{\partial y}.$$

We will prove that, in this case, after a coordinate change  $h$  can be reduced to  $h = x^p y^q$  with  $p + q = n \geq 3$ . This contradicts our supposition that  $h$  is reduced. Then our initial supposition about  $\delta$  is false, and  $\delta$  is an Euler vector field for  $h$ .

Inductively, for all  $k \geq 0$ , we construct coordinates  $(x_{(k)}, y_{(k)})$  and functions  $h^{(k)}$  such that

$$h(x, y) = h^{(k)}(x_{(k)}, y_{(k)}) = x_{(k)}^p y_{(k)}^q + \sum_{s \geq n+k} h_s^{(k)}(x_{(k)}, y_{(k)}) \equiv x_{(k)}^p y_{(k)}^q (\mathbf{m}_{\mathbb{C}^2, 0}^{n+k}),$$

where  $h_i^{(k)}$  is homogeneous of degree  $i$ . Then, by Artin approximation [1, Theorem 1.2], there exist coordinates  $z_1, z_2$  solving the equation

$$h(x, y) - z_1^p z_2^q = 0.$$

Let us construct the  $x_{(k)}, y_{(k)}, h^{(k)}$ . We suppose that we have  $x_{(k)}, y_{(k)}$  and  $h^{(k)} \in \mathbb{C}\{x_{(k)}, y_{(k)}\}$ , such that

$$h(x, y) = h^{(k)}(x_{(k)}, y_{(k)}) = x_{(k)}^p y_{(k)}^q + \sum_{s \geq n+k} h_s^{(k)},$$

$$\delta_0^{(k)} = qx_{(k)} \frac{\partial}{\partial x_{(k)}} - py_{(k)} \frac{\partial}{\partial y_{(k)}}.$$

We define  $x_{(k+1)}, y_{(k+1)}$  and  $h^{(k+1)} \in \mathbb{C}\{x_{(k+1)}, y_{(k+1)}\}$ , such that

$$h(x, y) = h^{(k+1)}(x_{(k+1)}, y_{(k+1)}) = x_{(k+1)}^p y_{(k+1)}^q + \sum_{s \geq n+k+1} h_s^{(k+1)},$$

$$\delta_0^{(k+1)} = qx_{(k+1)} \frac{\partial}{\partial x_{(k+1)}} - py_{(k+1)} \frac{\partial}{\partial y_{(k+1)}}.$$

Let  $h_{n+k}^{(k)} = \sum_{i+j=n+k} a_{i,j}^{(k)} x_{(k)}^i y_{(k)}^j$ , then

$$\delta_0^{(k)}(h_{n+k}^{(k)}) = \sum_{i+j=n+k} (iq - jp) a_{i,j}^{(k)} x_{(k)}^i y_{(k)}^j.$$

As the part of  $h^{(k)}$  of degree less than  $n+k$  is  $x_{(k)}^p y_{(k)}^q$ , it follows that the part of degree  $n+k$  of  $\delta^{(k)}(h^{(k)}) \in \mathbf{m}_{\mathbb{C}^2, 0} h^{(k)}$  belongs to  $(x_{(k)}^p y_{(k)}^q)$ :

$$[\delta^{(k)}(h^{(k)})]_{n+k} = \delta_0^{(k)}(h_{n+k}^{(k)}) + \delta_k^{(k)}(x_{(k)}^p y_{(k)}^q) \in (x_{(k)}^p y_{(k)}^q),$$

but

$$\delta_k^{(k)}(x_{(k)}^p y_{(k)}^q) \in (x_{(k)}^{p-1} y_{(k)}^q, x_{(k)}^p y_{(k)}^{q-1}),$$

then

$$\delta_0^{(k)}(h_{n+k}^{(k)}) \in (x_{(k)}^{p-1} y_{(k)}^q, x_{(k)}^p y_{(k)}^{q-1}),$$

so

$$(iq - jp)a_{i,j}^{(k)} = 0 \quad (i + j = n + k) \text{ if } i < p - 1 \text{ or } j < q - 1,$$

but if  $iq - jp = 0$ , then  $(i, j) = \frac{n+k}{n}(p, q)$ , and  $i > p$ ,  $j > q$ . So  $h_{n+k}^{(k)} \in (x_{(k)}^{p-1} y_{(k)}^q, x_{(k)}^p y_{(k)}^{q-1})$ :

$$h_{n+k}^{(k)} = x_{(k)}^{p-1} y_{(k)}^q f_{k+1}(x_{(k)}, y_{(k)}) + x_{(k)}^p y_{(k)}^{q-1} g_{k+1}(x_{(k)}, y_{(k)}).$$

Let

$$x_{(k+1)} = x_{(k)} + \frac{1}{p} f_{k+1}(x_{(k)}, y_{(k)}) \quad y_{(k+1)} = y_{(k)} + \frac{1}{q} g_{k+1}(x_{(k)}, y_{(k)}).$$

We have

$$h(x, y) = x_{(k+1)}^p y_{(k+1)}^q + \sum_{r \geq k+1} \sum_{i+j=n+r} a_{i,j}^{(k+1)} x_{(k+1)}^i y_{(k+1)}^j.$$

We define  $h^{(k+1)}$  by the equation  $h(x, y) = h^{(k+1)}(x_{(k+1)}, y_{(k+1)})$ , where

$$h^{(k+1)} = x_{(k+1)}^p y_{(k+1)}^q + \sum_{s \geq n+k+1} h_s^{(k+1)},$$

with  $h_s^{(k+1)} = \sum_{i+j=s} a_{i,j}^{(k+1)} x_{(k+1)}^i y_{(k+1)}^j$  homogeneous polynomials of degree  $s \geq n + k + 1$ . Moreover, as

$$x_{(k+1)} = x_{(k)}; \quad y_{(k+1)} = y_{(k)} \pmod{\mathbf{m}_{\mathbb{C}^2, 0}^2},$$

we have  $\delta = \sum_{q \geq 0} \delta_q^{(k+1)}$ , where each  $\delta_q^{(k+1)}$  is homogeneous of degree  $q$ , and

$$\delta_0^{(k+1)} = qx_{(k+1)} \frac{\partial}{\partial x_{(k+1)}} - py_{(k+1)} \frac{\partial}{\partial y_{(k+1)}}. \quad \square$$

**Proposition 3.4.** *Let  $D$  a plane curve, singular at 0. If there exists  $\delta \in \text{Der}(\log D)$  satisfying condition (I), then there exists a unit  $\alpha$  such that  $\alpha\delta \cdot h = h$ , and so  $D$  is Euler homogeneous.*

*Proof.* The proof is similar to the proof of Theorem 3.3. There, we consider the case where  $h_n = x^p y^q$  and  $\delta_0 = qx\partial/\partial x - py\partial/\partial y$ , with  $p, q \in \mathbb{N}$ . Condition (I) forces one of  $p$  and  $q$  to be 0. The proof now proceeds as before, with this additional hypothesis.

**Theorem 3.5.** *Let  $(D, 0) \subset (\mathbb{C}^2, 0)$  be a plane curve. The following conditions are equivalent:*

- a) *There exists  $\delta \in \text{Der}(\log D)_0$  such that  $d_\delta$  is injective.*
- b) *There exists  $\delta \in \text{Der}(\log D)_0$  satisfying condition (I).*
- c)  *$d_1$  is injective.*
- d)  *$(D, 0)$  is Euler homogeneous.*
- e)  *$(D, 0)$  is quasi-homogeneous.*
- f) *The logarithmic comparison theorem holds for  $(D, 0)$  on a neighbourhood of 0.*

*Proof.* By Theorem 3.3, if  $d_1$  is injective, then  $(D, 0)$  is Euler homogeneous. By Saito's theorem [10] (for a function  $h$  with isolated singularity,  $h \in J_h$  is equivalent to the quasihomogeneity of  $h$ ) to be Euler homogeneous or quasi-homogeneous is the same. Theorem 1.1 proves that if  $(D, 0)$  is quasi-homogeneous, the logarithmic comparison theorem holds for  $(D, 0)$  on a neighbourhood of 0. From the results of section 2 we can easily deduce that logarithmic comparison theorem implies the injectivity of  $d_1$ . Then, the last four conditions are equivalent. If  $\chi = w_1 \frac{\partial}{\partial x} + w_2 \frac{\partial}{\partial y}$  is the Euler vector field then  $d_\chi$  is injective. Proposition 3.1 shows that if  $d_\delta$  is injective, then  $\delta$  satisfies (I) and, finally, by proposition 3.4,  $\delta \in \text{Der}(\log D)$  implies that  $D$  is Euler homogeneous.

#### 4. Example

In this section we give an example of a free divisor  $D \subset \mathbb{C}^3$  which is Euler homogeneous but not locally quasi-homogeneous, and for which the logarithmic comparison theorem does hold. This example is studied in [4], where the perversity of  $\Omega^\bullet(\log D)$  is proved. We remark that  $D$  is the total space of an equisingular one-parameter deformation of a plane curve singularity. In [7], Damon shows that under mild additional hypotheses, all surfaces obtained in this way are free divisors.

$D$  is defined by the equation

$$h(x, y, z) = xy(x + y)((z - \lambda)x + y) = h_1 h_2 h_3 h_4, \quad \lambda \in \mathbb{C} \setminus \{0, 1\}.$$

$\text{Der}(\log D)$  has free basis  $\{\delta_1, \delta_2, \delta_3\}$

$$\begin{aligned} \delta_1 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \\ \delta_2 &= \frac{\partial}{\partial z} + ((z - \lambda)x + y) \frac{\partial}{\partial z} \\ \delta_3 &= x^2 \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y} - (z - \lambda)(x + y) \frac{\partial}{\partial z}. \end{aligned}$$

Note that  $\delta_1 \cdot h = 4h$ , so that  $h$  is Euler homogeneous. Note also that it is easy to check that each of these vector fields is logarithmic, and that the determinant of their coefficients is a reduced equation for  $D$ . From this it follows by a theorem

of K. Saito ([11]) that they really do form a basis for  $\mathcal{D}er(\log D)$ ; as no linear combination of them has non-singular linear part, it follows that  $D$  cannot be quasihomogeneous.

This example of free divisor is interesting also as it provides a counterexample to the “logarithmic Sard’s theorem”: every point of  $\mathbb{C} = z$ -axis is a logarithmic critical value with respect to the projection  $(x, y, z) \mapsto z$ .

The basis of  $\Omega^1(\log D)$  dual to  $\{\delta_1, \delta_2, \delta_3\}$  is

$$\begin{aligned}\omega_1 &= \frac{y^2 dx + x^2 dy}{xy(x+y)} \\ \omega_2 &= \frac{y(z-\lambda) dx - x(z-\lambda) dy + xy dz}{xy(x(z-\lambda) + y)} \\ \omega_3 &= \frac{y dx - x dy}{xy(x+y)}.\end{aligned}$$

We have to calculate homology groups of the stalk at 0 of the logarithmic de Rham complex

$$0 \rightarrow \Omega^0(\log D) \xrightarrow{d_0} \Omega^1(\log D) \xrightarrow{d_1} \Omega^2(\log D) \xrightarrow{d_2} \Omega^3(\log D) \xrightarrow{d_3} 0.$$

Although  $D$  is not weighted homogeneous in the strict sense, it is homogeneous if we assign weights 1, 1, 0 to the variables  $x, y, z$ . The Lie derivative with respect to the vector field  $\delta_1$ ,

$$L_{\delta_1}(\omega) = \iota_{\delta_1}(d\omega) + d(\iota_{\delta_1}(\omega)),$$

then defines a contracting homotopy from  $\Omega^\bullet(\log D)$  to its weight-zero part  $\Omega_0^\bullet(\log D)$ . For if  $\omega \in \Omega^k(\log D)$  is a sum of homogeneous parts  $\omega_i$ , and if  $d\omega = 0$ , then  $d\omega_i = 0$  for all  $i$ . Since  $L_{\delta_1}(\omega_i) = i\omega_i$ , each  $\omega_i$ , for  $i \neq 0$ , is then exact, and  $\omega$  is cohomologous to  $\omega - \iota_{\delta_1}(\sum_{i \neq 0} (1/i)\omega_i)$ .

Thus we consider only the weight 0 subcomplex

$$0 \rightarrow \Omega_0^0(\log D) \xrightarrow{d_0^0} \Omega_0^1(\log D) \xrightarrow{d_1^0} \Omega_0^2(\log D) \xrightarrow{d_2^0} \Omega_0^3(\log D) \xrightarrow{d_3^0} 0.$$

- We have  $\Omega_0^0(\log D) = \mathbb{C}\{z\}$ , and  $d_0(z^k) = kz^{k-1}[(z-\lambda)x + y]\omega_2 - (z-\lambda)(x+y)\omega_3$  ( $k \geq 0$ ), so

$$\text{Im}(d_0^0) = \mathbb{C}\{z\}dz = \mathbb{C}\{z\} \langle ((z-\lambda)x + y)\omega_2 - (z-\lambda)(x+y)\omega_3 \rangle.$$

- $\Omega_0^1(\log D) = \mathbb{C}\{z\} \langle \omega_1, x\omega_2, y\omega_2, x\omega_3, y\omega_3 \rangle$ , and we find

$$\begin{aligned}d_1(\omega_1) &= d_1(x\omega_2) = d_1(x\omega_3) = d_1(y\omega_3) = 0 \\ d_1(z^k\omega_1) &= kz^{k-1}((x(\lambda-z) - y)\omega_1 \wedge \omega_2 + (z-\lambda)(x+y)\omega_1 \wedge \omega_3) \\ d_1(y\omega_2) &= (xy + y^2)\omega_2 \wedge \omega_3 \\ d_1(z^kx\omega_2) &= kz^{k-1}((z-\lambda)(x+y)x\omega_2 \wedge \omega_3) \\ d_1(z^ky\omega_2) &= ((k+1)z^k - k\lambda z^{k-1})(x+y)y\omega_2 \wedge \omega_3 \\ d_1(z^kx\omega_3) &= kz^{k-1}x(x(z-\lambda) + y)\omega_2 \wedge \omega_3 \\ d_1(z^ky\omega_3) &= kz^{k-1}y(x(z-\lambda) + y)\omega_2 \wedge \omega_3.\end{aligned}$$

It follows that  $\text{Ker}(d_1^0) = \mathbb{C} \langle \omega_1, x\omega_2, x\omega_3, y\omega_3 \rangle \oplus \text{Im}(d_0^0)$ , so

$$h^1(\Omega^\bullet(\log D)_0) = \mathbb{C} \langle \omega_1, x\omega_2, x\omega_3, y\omega_3 \rangle$$

is 4-dimensional. Also we have

$$\begin{aligned} \text{Im}(d_1^0) &= \mathbb{C}\{z\} \langle ((\lambda - z)x - y)\omega_1 \wedge \omega_2 + (z - \lambda)(x + y)\omega_1 \wedge \omega_3 \rangle \oplus \\ &\quad \mathbb{C}\{z\} \langle x^2, xy, y^2 \rangle \omega_2 \wedge \omega_3. \end{aligned}$$

- $\Omega_0^2(\log D)$  is generated over  $\mathbb{C}\{z\}$  by

$$x\omega_1 \wedge \omega_2, y\omega_1 \wedge \omega_2, x\omega_3 \wedge \omega_1, y\omega_3 \wedge \omega_1, x^2\omega_2 \wedge \omega_3, xy\omega_2 \wedge \omega_3, y^2\omega_2 \wedge \omega_3.$$

We find

$$\begin{aligned} d_2(x\omega_1 \wedge \omega_2) &= d_2(x\omega_1 \wedge \omega_3) = d_2(y\omega_1 \wedge \omega_3) = 0 \\ d_2(z^k x^2 \omega_2 \wedge \omega_3) &= d_2(z^k xy \omega_2 \wedge \omega_3) = d_2(z^k y^2 \omega_2 \wedge \omega_3) = 0 \\ d_2(z^k x \omega_1 \wedge \omega_2) &= kz^{k-1}(\lambda - z)(x + y)\omega_1 \wedge \omega_2 \wedge \omega_3 \\ d_2(y \omega_1 \wedge \omega_2) &= (xy + y^2)\omega_1 \wedge \omega_2 \wedge \omega_3 \\ d_2(z^k y \omega_1 \wedge \omega_2) &= z^{k-1}(x + y)(ky(\lambda - z) - zy)\omega_1 \wedge \omega_2 \wedge \omega_3 \\ d_2(z^k x \omega_1 \wedge \omega_3) &= -kz^{k-1}x((z - \lambda)x + y)\omega_1 \wedge \omega_2 \wedge \omega_3 \\ d_2(z^k y \omega_1 \wedge \omega_3) &= -kz^{k-1}y((z - \lambda)x + y)\omega_1 \wedge \omega_2 \wedge \omega_3. \end{aligned}$$

We deduce that  $\text{Ker}(d_2^0) = \mathbb{C} \langle x\omega_1 \wedge \omega_2, x\omega_1 \wedge \omega_3, y\omega_1 \wedge \omega_3 \rangle \oplus \text{Im}(d_1^0)$ , and thus that

$$h^2(\Omega^\bullet(\log D)_0) = \mathbb{C} \langle x\omega_1 \wedge \omega_2, x\omega_1 \wedge \omega_3, y\omega_1 \wedge \omega_3 \rangle$$

is 3-dimensional.

- Finally,

$$\text{Im}(d_2^0) = \mathbb{C}\{z\} \langle x^2, xy, y^2 \rangle \omega_1 \wedge \omega_2 \wedge \omega_3 = \Omega_0^3(\log D),$$

and, consequently,

$$h^3(\Omega^\bullet(\log D)_0) = 0.$$

Now consider the intersection  $D_0 = D \cap \{z = 0\}$ , which has equation

$$h^0 = h_1^0 h_2^0 h_3^0 h_4^0 = xy(x + y)(-\lambda x + y).$$

It is a line arrangement, and the cohomology of its complement is therefore given by the Brieskorn complex, the exterior algebra generated over  $\mathbb{C}$  by the forms  $dh_i^0/h_i^0$ , with trivial differential ([2]). This is of course a subcomplex of  $\Omega^\bullet(\log D_0)$ . Let  $V \subset \mathbb{C}^3$  be a neighbourhood of 0. Restriction from  $\mathbb{C}^3$  to  $\mathbb{C}^2 = \{z = 0\}$  gives rise to a commutative diagram

$$\begin{array}{ccccc} \wedge^p \sum_{1 \leq i \leq 4} \mathbb{C} \left\langle \frac{dh_i}{h_i} \right\rangle & \xrightarrow{a} & h^p(\Omega^\bullet(\log D)(V)) & \xrightarrow{b} & H^p(V \setminus D; \mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow \cong \\ \wedge^p \sum_{1 \leq i \leq 4} \mathbb{C} \left\langle \frac{dh_i^0}{h_i^0} \right\rangle & \xrightarrow{\cong} & h^p(\Omega^\bullet(\log D_0)(V_0)) & \xrightarrow{\cong} & H^p(V_0 \setminus D_0; \mathbb{C}). \end{array}$$

in which the left-hand horizontal morphisms are induced by the inclusion of the Brieskorn complex in the logarithmic complex, and the right-hand horizontal morphisms are de Rham maps. The lower horizontal morphisms are isomorphisms by the theorem of Brieskorn and by 1.1. The right-hand vertical morphism is an isomorphism because  $D$  is a topologically trivial deformation of  $D_0$ , so inclusion induces an isomorphism of the homology groups of the complements. The left-hand vertical morphism is evidently surjective, and thus the de Rham map  $h^p(\Omega^\bullet(\log D)(V)) \rightarrow H^p(V \setminus D; \mathbb{C})$  is surjective. As  $h^p(\Omega^\bullet(\log D)_0) = \lim_{U \ni 0} h^p(\Omega^\bullet(\log D)(V))$  and  $\lim_{U \ni 0} H^p(V \setminus D; \mathbb{C}) = H^p(\mathbb{C}^3 \setminus D; \mathbb{C})$ , then the de Rham map  $h^p(\Omega^\bullet(\log D)) \rightarrow H^p(\mathbb{C}^3 \setminus D; \mathbb{C})$  is surjective. To see that it is an isomorphism we compare dimensions. A calculation (for example, using the Brieskorn complex) gives

$$\begin{aligned}\dim_{\mathbb{C}} H^1(\mathbb{C}^2 \setminus D_0; \mathbb{C}) &= 4 \\ \dim_{\mathbb{C}} H^2(\mathbb{C}^2 \setminus D_0; \mathbb{C}) &= 3 \\ \dim_{\mathbb{C}} H^3(\mathbb{C}^2 \setminus D_0; \mathbb{C}) &= 0.\end{aligned}$$

As these are the same as the dimension of  $h^p(\Omega^\bullet(\log D)_0)$ , this completes the proof that the logarithmic comparison theorem holds for  $D$ .  $\square$

**Remark 4.1.** The calculations whose results we summarise here are not so simple as might be supposed. We have presented each image  $d_i^0(\Omega_0^k(\log D))$  as a module over  $\mathbb{C}\{z\}$  with algebraic generators, obscuring the fact that because  $D$  is not quasihomogeneous, the anti-derivatives of an algebraic exact logarithmic form are in general transcendental. For example,

$$\begin{aligned}z^k(x^2 + xy)\omega_1 \wedge \omega_2 \wedge \omega_3 &= d\left(\sum_{s=1}^{\infty} (z^{k+s}/\lambda^s(k+s))x\omega_1 \wedge \omega_2\right) \\ &= d\left(-\left(\log\left(1 - \frac{z}{\lambda}\right) + \sum_{s=1}^k (z^s/\lambda^s s)\right)\lambda^k x\omega_1 \omega_2\right)\end{aligned}$$

and

$$\begin{aligned}z^k xy\omega_1 \wedge \omega_2 \wedge \omega_3 &= d\left(\sum_{s=1}^{\infty} (z^{k+s}/(\lambda+1)^s(k+s))x(\omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3)\right) \\ &= d\left(-\left((\lambda+1)^k \log(1 - (z/(\lambda+1)))\right.\right. \\ &\quad \left.\left.+ \sum_{s=1}^k (z^s(\lambda+1)^{k-s} s)\right)x(\omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3)\right).\end{aligned}$$

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