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## A gap theorem for hypersurfaces of the sphere with constant scalar curvature one

Hilário Alencar\*, Manfredo do Carmo\* and Walcy Santos\*

**Abstract.** We consider closed hypersurfaces of the sphere with scalar curvature one, prove a gap theorem for a modified second fundamental form and determine the hypersurfaces that are at the end points of the gap. As an application we characterize the closed, two-sided index one hypersurfaces with scalar curvature one in the real projective space.

**Mathematics Subject Classification (2000).** 53C42.

**Keywords.** Scalar curvature, sphere, Clifford torus, index one, projective space.

### 1. Introduction

To state our main result we need some notation.

$x : M^n \rightarrow S^{n+1}(1)$  will be a closed (compact without boundary) hypersurface of the unit sphere  $S^{n+1}(1)$ . We denote by  $A$  the linear map associated to the second fundamental form and by  $k_1, \dots, k_n$  its eigenvalues (principal curvatures of  $M$ ). We will use the first two elementary symmetric function of the principal curvatures:

$$S_1 = \sum_{i=1}^n k_i, \quad S_2 = \sum_{i < j=1}^n k_i k_j.$$

We will also use the normalized means: *the mean curvature*  $H = \frac{1}{n}S_1$  and the *scalar curvature*  $R$ , given by  $n(n-1)(R-1) = S_2$ . Finally, we introduce the first two *Newton tensors* by

$$P_0 = Id, \quad P_1 = S_1 Id - A.$$

Clearly  $P_1$  commutes with  $A$  and it is also a self-adjoint operator. We will show later (see Remark 2.1) that if  $R = 1$  and  $S_1 \geq 0$ , then all eigenvalues of  $P_1$  are nonnegative, hence we can consider  $\sqrt{P_1}$ .

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We can now state our gap theorem.

**Theorem 1.** *Let  $x : M^n \rightarrow S^{n+1}(1)$  be a closed orientable hypersurface with scalar curvature  $R = 1$  (equivalently,  $S_2 = 0$ ). Assume that  $S_1$  does not change sign and choose the orientation such that  $S_1 \geq 0$ . Assume further that*

$$\|\sqrt{P_1}A\|^2 \leq \text{trace}P_1.$$

*Then:*

- (i)  $\|\sqrt{P_1}A\|^2 = \text{trace}P_1$ .
- (ii)  $M^n$  is either a totally geodesic submanifold or  $M^n = S^{n_1}(r_1) \times S^{n_2}(r_2) \subset S^{n+1}(1)$ , where  $n_1 + n_2 = n$ ,  $r_1^2 + r_2^2 = 1$  and  $\left(\frac{r_2}{r_1}\right)^2 = \beta$  satisfies the quadratic equation:

$$n_1(n_1 - 1)\beta^2 - 2n_1n_2\beta + n_2(n_2 - 1) = 0.$$

Our theorem was inspired by a similar theorem on minimal submanifolds of the sphere first proved by J. Simons [S] (part (i)) and latter completed (part (ii)) by S. S. Chern, M. do Carmo and Kobayashi [CdCK] and, independently, by H. B. Lawson [L].

**Remark.** The condition on the modified second fundamental form in above theorem can not be dropped, as can be seen by the following example: Let  $M^6 \rightarrow S^7(1)$  be an isoparametric hypersurface with principal curvatures given by

$$\lambda_1 = \lambda_2 = \theta, \quad \lambda_3 = \frac{\theta + 1}{1 - \theta}, \quad \lambda_4 = \lambda_5 = -\frac{1}{\theta} \quad \text{and} \quad \lambda_6 = -\frac{1 - \theta}{1 + \theta},$$

where  $\theta$  is given by  $\theta = \sqrt{\frac{13 + \sqrt{165}}{2}}$  (see [M]). It is easy to see that  $M^6$  has  $R = 1$  and  $S_1 > 0$ . We would like to thank Luiz Amancio de Sousa Junior for showing us this example.

As an application of Theorem 1, we will present a characterization of index one closed hypersurfaces with constant scalar curvature one of the real projective space  $\mathbb{P}(\mathbb{R})^{n+1}$ . For minimal submanifolds this result was obtained recently by M. do Carmo, M. Ritoré and A. Ros [dCRR].

Before giving a formal statement we need some considerations. Hypersurfaces of a curvature one space form with constant scalar curvature one are solutions to a variational problem (see [Re], [Ro], [BC]) whose *Jacobi equation* is

$$T_1 f = L_1 f + \{\|\sqrt{P_1}A\|^2 + \text{trace}P_1\}f = 0.$$

Here  $f \in C^\infty(M)$  and  $L_1$  is a second order differential operator given by

$$L_1 f = \text{div}(P_1 \nabla f),$$

where  $\nabla f$  is the gradient of  $f$ . Notice that  $L_1$  generalizes the Laplacian. However, differently from the Laplacian,  $L_1$  is not always elliptic. J. Hounie and M. L. Leite [HL] have proved that if  $S_3 \neq 0$  everywhere, then  $L_1$  is elliptic. Of course, from the definition of  $L_1$ , it follows that  $L_1$  is elliptic if and only if  $P_1$  is positive definite (or negative definite). For the next theorem we will assume that  $L_1$  is elliptic and  $P_1$  is positive definite. Denote by  $Ind(M)$  the *Morse index* of  $M$ , i.e., the number of negative eigenvalues of  $T_1$ .

**Theorem 2.** *Let  $x : M^n \rightarrow \mathbb{P}(\mathbb{R})^{n+1}(1)$  be a closed two-sided hypersurface with scalar curvature one. Then  $Ind(M) \geq 1$  and if  $Ind(M) = 1$ ,  $M$  is the Clifford hypersurfaces obtained by the projection of the Clifford torus of Theorem 1.*

## 2. Preliminaries

In this section we will present some properties of the  $r^{th}$  Newton tensors in  $M$  and describe the Clifford hypersurfaces of  $\mathbb{P}(\mathbb{R})^{n+1}$ .

### 2.1. The $r^{th}$ Newton tensors

We introduce the  $r^{th}$  Newton tensors,  $P_r : T_p M \rightarrow T_p M$ , which are defined inductively by

$$\begin{aligned} P_0 &= I, \\ P_r &= S_r I - AP_{r-1}, \quad r > 1, \end{aligned}$$

where  $S_r = \sum_{i_1 < \dots < i_r} k_{i_1} \dots k_{i_r}$  is the  $r^{th}$  symmetric function of the principal curvatures  $k_1, \dots, k_n$ .

It is easy to see that each  $P_r$  commutes with  $A$  and if  $e_i$  an eigenvector of  $A$  associated to principal curvature  $k_i$ , then

$$P_1(e_i) = \mu_i e_i = (S_1 - k_i)e_i.$$

In [Re], Reilly showed that the  $P_r$ 's satisfy the following

**Proposition 2.1** ([Re], see also [BC] – Lemma 2.1). *Let  $x : M^n \rightarrow N^{n+1}$  be an isometric immersion between two Riemannian manifolds and let  $A$  be its second fundamental form. The  $r^{th}$  Newton tensor  $P_r$  associated to  $A$  satisfies:*

1.  $\text{trace}(P_r) = (n - r)S_r$ ,
2.  $\text{trace}(AP_r) = (r + 1)S_{r+1}$ ,
3.  $\text{trace}(A^2 P_r) = S_1 S_{r+1} - (r + 2)S_{r+2}$ .

It follows from (3) that if  $S_2 = 0$ ,  $\text{trace}(A^2 P_1) = -3S_3$ .

**Remark 2.1.** Observe that if  $S_2 = 0$ , we have that

$$S_1^2 = |A|^2 + 2S_2 \geq k_i^2, \text{ for all } i.$$

Thus,  $0 \leq (S_1^2 - k_i^2) = (S_1 - k_i)(S_1 + k_i)$ , what implies that all eigenvalues of  $P_1$  are nonnegative if  $S_1 \geq 0$ , that is,  $P_1$  is a nonnegative operator. We also remark that if  $S_2 = 0$  and  $P_1$  has one eigenvalue equal to zero, then

$$P_1 A \equiv 0. \quad (1)$$

In fact, if  $\mu_{i_0} = 0$ , then  $k_{i_0} = S_1$ . As  $S_1^2 = |A|^2$ , we get

$$\sum_{i \neq i_0} k_{i_0}^2 = 0.$$

So  $k_i = 0$ , for all  $i \neq i_0$ , hence  $P_1 A \equiv 0$ .

Associated to each Newton tensor  $P_r$ , we define a second order differential operator

$$L_r(f) = \text{trace}(P_r \text{Hess } f).$$

If  $N^{n+1}$  has constant sectional curvature, it follows from Codazzi equation (see Rosenberg [Ro], p. 225) that  $L_r$  is

$$L_r(f) = \text{div}_M(P_r \nabla f).$$

Hence  $L_r$  is a self-adjoint operator and for any differentiable functions  $f$  and  $g$  on  $M^n$ ,

$$\int_M f L_r g dM = \int_M g L_r f dM \quad (2)$$

We observe that for  $r = 0$ ,  $L_0$  is the Laplacian which is always an elliptic operator. For  $r > 0$  we have to add some extra condition in order to ensure that  $L_r$  is elliptic. For hypersurfaces of  $\mathbb{R}^{n+1}$  with  $S_r = 0$ , Hounie and Leite, [HL], were able to give a geometric condition that is equivalent to  $L_r$  being elliptic. In fact their proof can be generalized to hypersurfaces of the sphere and we have that

**Theorem 2.1** ([HL] – Proposition 1.5). *Let  $M$  be a hypersurface in  $\mathbb{R}^{n+1}$  or  $S^{n+1}$  with  $S_r = 0$ ,  $2 \leq r < n$ . Then the operator  $L_{r-1}(f) = \text{div}(P_{r-1} \nabla f)$  is elliptic at  $p \in M$  if and only if  $S_{r+1}(p) \neq 0$ .*

Thus, for hypersurfaces with  $S_2 = 0$ ,  $L_1$  is an elliptic operator if and only if  $S_3 \neq 0$ . Since  $L_1(f) = \text{div}_M(P_1 \nabla f)$ , it follows that the ellipticity of  $L_1$  implies that  $P_1$  is definite, hence then  $S_1 \neq 0$ .

Let  $a \in \mathbb{R}^{n+2}$  be a fixed vector. Let  $x : M \rightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2}$  be an isometric immersion with  $S_2 = 0$  and let  $N$  be its unit normal vector. The functions  $f = \langle N, a \rangle$  and  $g = \langle x, a \rangle$  satisfy (see [BC], lemma 5.2)

$$L_1(g) = -(n-1)S_1g \quad (3)$$

and

$$L_1(f) = 3S_3 f. \quad (4)$$

## 2.2. Clifford hypersurfaces of $\mathbb{P}(\mathbb{R})^{n+1}$

We are now going to describe some properties of the *Clifford hypersurface* in  $\mathbb{P}(\mathbb{R})^{n+1}$ . A Clifford torus in  $S^{n+1}(1)$  is given by the product immersion of  $M = S^{n_1}(r_1) \times S^{n_2}(r_2)$ , with  $n_1 + n_2 = n$  and  $r_1^2 + r_2^2 = 1$ , which is a closed hypersurface of  $S^{n+1}(1)$ . It is easy to see that this immersion is invariant under the antipodal map, hence it induces an immersion of  $M$  into  $\mathbb{P}(\mathbb{R})^{n+1}$ . This hypersurface will be called *Clifford hypersurface*. If  $x : S^{n_1}(r_1) \times S^{n_2}(r_2) \rightarrow S^{n+1}(1)$  is a Clifford torus, then the unit normal vector at a point  $p = (p_1, p_2) \in S^{n_1}(r_1) \times S^{n_2}(r_2)$  is given by

$$N = \left( -\frac{r_2}{r_1} p_1, \frac{r_1}{r_2} p_2 \right).$$

Thus, the principal curvatures of  $M$  are  $\frac{r_2}{r_1}$  with multiplicity  $n_1$  and  $-\frac{r_1}{r_2}$  with multiplicity  $n_2$ . It is easily checked that the scalar curvature of  $M$  is equal to one ( $S_2 = 0$ ) if and only if  $\left(\frac{r_2}{r_1}\right)^2 = \beta$  satisfies the quadratic equation:

$$n_1(n_1 - 1)\beta^2 - 2n_1n_2\beta + n_2(n_2 - 1) = 0. \quad (5)$$

We will show in a while that only one of the torus given by (5) yields  $S_1 > 0$ . Notice that  $L_1$  is an elliptic operator and in order to calculate the index of  $M$ , we first observe that in a principal basis,  $P_1$  is a diagonal matrix whose elements are

$$\left\{ (n_1 - 1)\frac{r_2}{r_1} - n_2\frac{r_1}{r_2} \right\} \text{ with multiplicity } n_1$$

and

$$\left\{ n_1\frac{r_2}{r_1} - (n_2 - 1)\frac{r_1}{r_2} \right\} \text{ with multiplicity } n_2.$$

Thus,

$$\text{trace}P_1 = (n - 1)S_1 = (n - 1) \left( n_1\frac{r_2}{r_1} - n_2\frac{r_1}{r_2} \right).$$

We will need the following relation:

$$\|\sqrt{P_1}A\|^2 = -3S_3 = (n - 1)S_1.$$

The first equality is a general fact that follows from Proposition 2.1, part 3, by setting  $r = 1$  and  $S_2 = 0$ . The second equality is specific for Clifford tori with

$S_2 = 0$  and can be proved as follows. Write:

$$\begin{aligned} S_1 &= n_1 \frac{r_2}{r_1} - n_2 \frac{r_1}{r_2}, \\ S_2 &= \frac{n_1(n_1-1)}{2} \left( \frac{r_2}{r_1} \right)^2 + \frac{n_2(n_2-1)}{2} \left( \frac{r_1}{r_2} \right)^2 - n_1 n_2, \\ S_3 &= \frac{n_1(n_1-1)(n_1-2)}{6} \left( \frac{r_2}{r_1} \right)^3 - \frac{n_2(n_2-1)(n_2-2)}{6} \left( \frac{r_1}{r_2} \right)^3 \\ &\quad + \frac{n_1 n_2 (n_2-1)}{2} \left( \frac{r_1}{r_2} \right)^2 \frac{r_2}{r_1} - \frac{n_1 n_2 (n_1-1)}{2} \left( \frac{r_2}{r_1} \right)^2 \frac{r_1}{r_2}. \end{aligned}$$

By introducing the condition  $S_2 = 0$  into  $S_3$ , we obtain, after a long but straightforward computation, that

$$3S_3 = \frac{1}{2} \left[ -2(n-1)n_1 \frac{r_2}{r_1} + 2(n-1)n_2 \frac{r_1}{r_2} \right] = -(n-1)S_1,$$

and this proves our claim. Thus the *Jacobi operator* reduces to

$$T_1(f) = L_1(f) + \{\|\sqrt{P_1}A\|^2 + \text{trace}P_1\}f = L_1(f) + 2(n-1)S_1f.$$

If  $\varphi = \text{const.}$ ,  $L_1(\varphi) = 0$  and

$$T_1(\varphi) + 2(n-1)S_1\varphi = 0.$$

Thus the first eigenvalue of  $T_1$  is negative, hence  $\text{Ind}(M)$  is at least 1. Now let us look at the second eigenvalue of  $T_1$ . By using the expression of the eigenvalues of  $P_1$  given above, we have that

$$\begin{aligned} L_1(f) &= \text{div}(P_1 \nabla f) \\ &= \left\{ (n_1-1) \frac{r_2}{r_1} - n_2 \frac{r_1}{r_2} \right\} \Delta^{n_1}(f) + \left\{ n_1 \frac{r_2}{r_1} - (n_2-1) \frac{r_1}{r_2} \right\} \Delta^{n_2}(f), \end{aligned}$$

where  $\Delta^{n_i}$  is the Laplacian in  $S^{n_i}(r_i)$ ,  $i = 1, 2$ . Thus the second eigenvalue of  $L_1$  is given by

$$\lambda_2 = - \left\{ (n_1-1) \frac{r_2}{r_1} - n_2 \frac{r_1}{r_2} \right\} \nu_2^{\Delta^{n_1}} + \left\{ n_1 \frac{r_2}{r_1} - (n_2-1) \frac{r_1}{r_2} \right\} \nu_2^{\Delta^{n_2}},$$

where  $\nu_2^{\Delta^{n_i}}$  is the first nonzero eigenvalue of  $\Delta^{n_i}$  that corresponds to an eigenfunction which is invariant by the antipodal map (see [BGM] chap III, CII). Thus

$$\begin{aligned} \lambda_2 &= - \left[ \left\{ (n_1-1) \frac{r_2}{r_1} - n_2 \frac{r_1}{r_2} \right\} \frac{n_1}{r_1^2} + \left\{ n_1 \frac{r_2}{r_1} - (n_2-1) \frac{r_1}{r_2} \right\} \frac{n_2}{r_2^2} \right] \\ &= \frac{-1}{r_1^3 r_2^3} \{ [n_1(n_1-1) - n_1(n-1)r_1^2]r_2^2 + [n_2(n-1)r_2^2 - n_2(n_2-1)]r_1^2 \}. \end{aligned} \tag{6}$$

Observe that

$$S_1 = n_1 \frac{r_2}{r_1} - n_2 \frac{r_1}{r_2} = \frac{n_1 r_2^2 - n_2 r_1^2}{r_1 r_2}. \quad (7)$$

The fact that  $S_2 = 0$  is equivalent to

$$n(n-1)r_1^4 - 2n_1(n-1)r_1^2 + n_1(n_1-1) = n(n-1)r_2^4 - 2n_2(n-1)r_2^2 + n_2(n_2-1) = 0. \quad (8)$$

By using (7) and (8), we have that

$$[n_1(n_1-1) - n_1(n-1)r_1^2]r_2^2 = (n-1)S_1 r_1^3 r_2^3$$

and

$$[n_2(n-1)r_2^2 - n_2(n_2-1)]r_1^2 = (n-1)S_1 r_1^3 r_2^3.$$

Thus,

$$\lambda_2 = -2(n-1)S_1.$$

Since the second eigenvalue of  $T_1$  is given by  $\lambda_2 + 2(n-1)S_1$ , it is equal to zero. This shows then that the Clifford hypersurfaces of  $\mathbb{P}(\mathbb{R})^{n+1}$  have index one.

**Remark.** Observe that, by equation (7), the condition  $S_1 \geq 0$  means that

$$n_1 r_2^2 - n_2 r_1^2 \geq 0.$$

On the other hand, since  $\beta = \left(\frac{r_2}{r_1}\right)^2$ , the above inequality implies that

$$n_1 \beta \geq n_2. \quad (9)$$

The condition  $S_2 = 0$  is equivalent to

$$n_1(n_1-1)\beta^2 - 2n_1n_2\beta + n_2(n_2-1) = 0, \quad (10)$$

and one can easily see that only one solution of (10) is compatible with (9).

### 3. A gap theorem for hypersurfaces of the sphere with $R = 1$

In this section we prove a gap theorem for hypersurfaces of the sphere with  $R = 1$ .

**Theorem 3.1** (Theorem 1 of the Introduction). *Let  $x : M^n \rightarrow S^{n+1}(1)$  be a closed orientable hypersurface with scalar curvature  $R = 1$  (equivalently,  $S_2 = 0$ ). Assume that  $S_1$  does not change sign and choose the orientation such that  $S_1 \geq 0$ . Assume further that*

$$\|\sqrt{P_1}A\|^2 \leq \text{trace}P_1.$$

*Then:*

$$(i) \|\sqrt{P_1}A\|^2 = \text{trace}P_1.$$

(ii)  $M^n$  is either a totally geodesic submanifold or  $M^n = S^{n_1}(r_1) \times S^{n_2}(r_2) \subset S^{n+1}(1)$ , where  $n_1 + n_2 = n$ ,  $r_1^2 + r_2^2 = 1$  and  $\left(\frac{r_2}{r_1}\right)^2 = \beta$  satisfies the quadratic equation:

$$n_1(n_1 - 1)\beta^2 - 2n_1n_2\beta + n_2(n_2 - 1) = 0.$$

*Proof.* Let us calculate  $\frac{1}{2}L_1\|A\|^2$ . Since  $R = 1$ ,  $S_2 = n(n-1)(R-1) = 0$ , by the Gauss' formula. Thus  $\|A\|^2 = (nH)^2 = S_1^2$ . Hence,

$$\frac{1}{2}L_1\|A\|^2 = \frac{1}{2}L_1S_1^2 = S_1L_1S_1 + \langle P_1\nabla S_1, \nabla S_1 \rangle.$$

From [AdCC](Lemma 3.7), by using that  $2S_2 = n(n-1)(R-1) = 0$ , we have

$$L_1S_1 = |\nabla A|^2 - |\nabla S_1|^2 + n\|A\|^2 - S_1^2 + 3S_1S_3.$$

Therefore,

$$L_1S_1 = |\nabla A|^2 - |\nabla S_1|^2 + (n-1)S_1^2 + 3S_1S_3. \quad (11)$$

Now, by using Proposition 2.1 (3), we obtain that

$$\|\sqrt{P_1}A\|^2 = \text{trace}P_1A^2 = -3S_3.$$

Then, equation (11) becomes

$$L_1S_1 = |\nabla A|^2 - |\nabla S_1|^2 + (n-1)S_1^2 - S_1\|\sqrt{P_1}A\|^2.$$

Thus,

$$\begin{aligned} \frac{1}{2}L_1\|A\|^2 &= S_1L_1S_1 + \langle P_1\nabla S_1, \nabla S_1 \rangle \\ &= S_1(|\nabla A|^2 - |\nabla S_1|^2 + (n-1)S_1^2 - 3S_1\|\sqrt{P_1}A\|^2) + \langle P_1\nabla S_1, \nabla S_1 \rangle \\ &= S_1(|\nabla A|^2 - |\nabla S_1|^2) + S_1^2((n-1)S_1 - \|\sqrt{P_1}A\|^2) + \langle P_1\nabla S_1, \nabla S_1 \rangle. \end{aligned}$$

Since  $M$  is compact, we obtain

$$\begin{aligned} 0 &= \frac{1}{2} \int_M L_1\|A\|^2 dM \\ &= \int_M \{S_1(|\nabla A|^2 - |\nabla S_1|^2) + S_1^2((n-1)S_1 - \|\sqrt{P_1}A\|^2) + \langle P_1\nabla S_1, \nabla S_1 \rangle\} dM. \end{aligned} \quad (12)$$

We recall the following result (see [AdCC] – Lemma 4.1):

**Lemma 3.1** ([AdCC]). *Let  $M$  be an  $n$ -dimensional compact hypersurface in an  $(n+1)$ -dimensional unit sphere  $S^{n+1}$ . If the normalized scalar curvature  $R$  is constant and  $R-1 \geq 0$ , then*

$$|\nabla A|^2 - |\nabla S_1|^2 \geq 0. \quad (13)$$

Since  $S_1 \geq 0$  and  $P_1$  is positive, we have that

$$\langle P_1 \nabla S_1, \nabla S_1 \rangle = \|\sqrt{P_1} \nabla S_1\|^2 \geq 0. \quad (14)$$

Our hypothesis and inequalities (13) and (14) implies that the right-hand side of (12) is non-negative. Thus we conclude that

$$S_1(|\nabla A|^2 - |\nabla S_1|^2) + S_1^2((n-1)S_1 - \|\sqrt{P_1} A\|^2) + \langle P_1 \nabla S_1, \nabla S_1 \rangle = 0. \quad (15)$$

Since each term in above equation is non-negative, we have

$$S_1((n-1)S_1 - \|\sqrt{P_1} A\|^2) = 0.$$

Observe that when  $S_1 = 0$ ,  $\|A\|^2 = 0$  and  $\|\sqrt{P_1} A\|^2 = 0$ . Since by Lemma 2.1,  $\text{trace} P_1 = (n-1)S_1$ , the first part of the theorem is proved.

Now, let us assume that  $\|\sqrt{P_1} A(p)\|^2 = (n-1)S_1(p)$ , for all  $p \in M$ . If  $S_1(p) = 0$  for all  $p \in M$ , since  $S_2 = 0$ ,  $\|A\|^2 = 0$  and  $M$  is totally geodesic. Let us suppose that there exists a point  $p_0$  in  $M$  such that  $S_1(p_0) > 0$ . So the set  $\mathcal{A} \subset M$  where  $S_1(p) > 0$  is an open and non-void set of  $M$ . We claim that  $P_1$  is positive definite in  $\mathcal{A}$ . In fact, if  $P_1$  has one eigenvalue equal to zero, then by Remark 2.1,  $P_1 A \equiv 0$  and since  $\|\sqrt{P_1} A(p)\|^2 = (n-1)S_1(p)$ , we conclude that  $S_1 = 0$ , which is a contradiction. On each connected component of  $\mathcal{A}$ , we have that

$$\langle P_1 \nabla S_1, \nabla S_1 \rangle = 0$$

and

$$|\nabla A|^2 - |\nabla S_1|^2 = 0.$$

Since  $P_1$  is positive definite, the first equation implies that  $\nabla S_1 = 0$ . This implies that  $|\nabla A|^2 = 0$ , by the second equation, i.e., the second fundamental form of  $M$  is covariant constant. It follows that the component  $\mathcal{A}$  is a piece of a Clifford torus, by using the following theorem of H. B. Lawson ([L] – Theorem 4, see also [CdCK] Lemma 3).

**Theorem 3.2** [L]. *Let  $M^n$  be an isometrically immersed hypersurface of  $S^{n+1}$ , over which the second fundamental form is covariant constant. Then, up to isometries of  $S^{n+1}$ ,  $M^n$  is an open set of  $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$ .*

Finally, since along the boundary of  $\mathcal{A}$ ,  $\|A\|^2 = S_1^2 = 0$ , we conclude that  $\partial \mathcal{A} = \emptyset$  and  $M$  is a Clifford torus.  $\square$

#### 4. Characterization of index one closed hypersurfaces with $R = 1$ in the real projective space

In this section we will assume that the operator  $L_1$  is elliptic and will describe the index of closed hypersurfaces in the real projective space  $\mathbb{P}(\mathbb{R})^{n+1}$ . In order to do that we are going to use the covering map of  $S^{n+1}$  onto  $\mathbb{P}(\mathbb{R})^{n+1}$ . The following result will be needed.

**Lemma 4.1.** *Let  $M^n \rightarrow S^{n+1}$  is a closed orientable hypersurface with  $R = 1$ . Then the index of the quadratic form*

$$\begin{aligned} I(f, f) &= - \int_M f T_1 f dM \\ &= - \int_M f L_1 f + ((n-1)S_1 - 3S_3)f^2 dM \end{aligned}$$

*is greater than one.*

*Proof.* First of all observe that for constant functions  $f = \text{const.}$ , we have that

$$\begin{aligned} I(f, f) &= - \int_M f L_1 f + ((n-1)S_1 - 3S_3)f^2 dM \\ &= - \int_M ((n-1)S_1 - 3S_3)f^2 dM < 0. \end{aligned}$$

Thus  $\text{ind}(M) \geq 1$ .

Suppose that this index is equal to one. Let  $\{e_1, \dots, e_{n+2}\}$  be an orthonormal basis of  $\mathbb{R}^{n+2}$ . If we write the normal vector field of the immersion as  $N = \sum_{i=1}^{n+2} n_i e_i$ , we obtain that

$$L_1(n_i) = 3S_3 n_i, \quad \text{for all } i = 1, \dots, n+2.$$

Thus

$$I(n_i, n_i) = - \int_M ((n-1)S_1)n_i^2 dM \leq 0.$$

Since the functions  $n_i$  are linearly independent, the index one hypothesis implies that  $(n-1)$  of the  $n_i$ 's have to be null and since  $|N| = 1$ , after reordering if necessary, we have  $n_1 = 1$  and  $n_i = 0$  for  $i = 2, \dots, n+2$ . Thus the normal vector field  $N = e_1$ . This implies that  $M^n$  is totally geodesic. On the other hand, since  $L_1$  is elliptic, we have that  $S_1 > 0$ , and this contradicts the fact that  $M^n$  is totally geodesic. We conclude then that  $\text{ind}(M) > 1$ .

The main result of this section is the following characterization of index one closed hypersurfaces of  $\mathbb{P}(\mathbb{R})^{n+1}$ .

**Theorem 4.1** (Theorem 2 of the introduction). *Let  $x : M^n \rightarrow \mathbb{P}(\mathbb{R})^{n+1}(1)$  be a closed two-sided hypersurface with scalar curvature one. Then  $\text{Ind}(M) \geq 1$  and if  $\text{Ind}(M) = 1$ ,  $M$  is the Clifford hypersurfaces obtained by the projection of the Clifford torus of Theorem 3.1.*

*Proof.* The proof is inspired by the proof of the minimal case in [dCRR]. Observe that the index one hypothesis implies that  $M$  must be connected. Since, by lemma 4.1,  $S^{n+1}$  does not have an index one hypersurface with  $R = 1$ ,  $x$  cannot lift to an

immersion of  $M$  into  $S^{n+1}$ . Thus we obtain that there exists a connected twofold covering  $\tilde{M} \rightarrow M$  and an isometric immersion  $\tilde{x} : \tilde{M} \rightarrow S^{n+1}$  which is locally congruent to the immersion of  $M$  in  $\mathbb{P}(\mathbb{R})^{n+1}$ . An object in  $\tilde{M}$  that corresponds to an object in  $M$  will be denoted by the same notation as in  $M$ . If we denote by  $\pi : \tilde{M} \rightarrow M$  the isometric involution induced by the covering, then  $\tilde{x}$  must satisfy

$$\tilde{x} \circ \pi = -\tilde{x}$$

and, since  $\tilde{x}(M)$  is two-sided,  $\tilde{M}$  is orientable, and

$$N \circ \pi = -N,$$

where  $N$  is the unit normal vector field of the immersion. We have that the immersion  $\tilde{x}$  is such that  $R = 1$  and  $S_3 \neq 0$ . By ellipticity we can choose the orientation of  $\tilde{M}$  in such way that  $S_1 > 0$ .

Let  $\lambda_1$  be the first eigenvalue of the operator

$$T_1(\varphi) = L_1(\varphi) + ((n-1)S_1 + 3S_3)\varphi.$$

We know that its first eigenspace is one-dimensional and generated by a function  $\varphi$  that does not change sign on  $\tilde{M}$ . Now, let  $\varphi_1 = \varphi \circ \pi$ . Since  $\pi$  is an isometry, we obtain that  $T_1(\varphi_1) = \lambda_1\varphi_1$ . This implies that  $\varphi = \pm\varphi \circ \pi$ . Observe that if  $\varphi = -\varphi \circ \pi$ ,  $\varphi$  has to change sign on  $\tilde{M}$ . Thus  $\varphi = \varphi \circ \pi$ .

From the fact that  $Ind(M) = 1$ , we obtain that any function  $u : \tilde{M} \rightarrow \mathbb{R}$  such that  $u \circ \pi = u$  and  $\int_{\tilde{M}} u\varphi d\tilde{M} = 0$  satisfies

$$I(u, u) = -\int_{\tilde{M}} \{uL_1u + ((n-1)S_1 + 3S_3)u^2\} d\tilde{M} \geq 0.$$

Moreover, if such a function  $u$  satisfies  $I(u, u) = 0$ , then  $u$  is a Jacobi function, that is,

$$L_1u + ((n-1)S_1 + 3S_3)u = 0.$$

Given  $a, b \in \mathbb{R}^{n+2}$ , let  $\phi_{a,b} : \tilde{M} \rightarrow \mathbb{R}^{n+2}$  be defined by

$$\phi_{a,b} = \langle \tilde{x}, a \rangle \tilde{x} + \langle N, a \rangle N + \langle \tilde{x}, b \rangle N.$$

By doing the calculation coordinatewise and using equations (3) and (4) we have that

$$L_1(\tilde{x}) = -(n-1)S_1\tilde{x}$$

and

$$L_1(N) = 3S_3N.$$

Thus,

$$L_1(\langle \tilde{x}, a \rangle \tilde{x}) = -2(n-1)S_1\langle \tilde{x}, a \rangle \tilde{x} - P_1A(a^t),$$

$$L_1(\langle N, a \rangle N) = 6S_3\langle N, a \rangle N - P_1A^2(a^t)$$

and

$$L_1(\langle \tilde{x}, b \rangle N) = [-(n-1)S_1 + 3S_3]\langle \tilde{x}, b \rangle N - P_1A(b^t),$$

where  $a^t, b^t$  are the tangent projection of  $a$  and  $b$ . This implies that

$$T_1(\phi_{a,b}) = -[(n-1)S_1 + 3S_3][\langle \tilde{x}, a \rangle \tilde{x} - \langle N, a \rangle N] + X_{a,b}, \quad (16)$$

where  $X_{a,b}$  is a tangent vector field. Then,

$$\begin{aligned} & - \int_{\tilde{M}} \langle T_1(\phi_{a,b}), \phi_{a,b} \rangle d\tilde{M} \\ &= \int_{\tilde{M}} [(n-1)S_1 + 3S_3][\langle \tilde{x}, a \rangle^2 - \langle N, a \rangle^2 - \langle \tilde{x}, b \rangle \langle N, a \rangle] d\tilde{M}. \end{aligned}$$

Now, by (2), we have

$$\begin{aligned} & \int_{\tilde{M}} [(n-1)S_1 + 3S_3] \langle \tilde{x}, b \rangle \langle N, a \rangle d\tilde{M} \\ &= - \int_{\tilde{M}} \{ \langle N, a \rangle L_1(\langle \tilde{x}, b \rangle) - \langle \tilde{x}, b \rangle L_1(\langle N, a \rangle) \} d\tilde{M} = 0. \end{aligned}$$

Thus,

$$- \int_{\tilde{M}} \langle T_1(\phi_{a,b}), \phi_{a,b} \rangle d\tilde{M} = \int_{\tilde{M}} [(n-1)S_1 + 3S_3][\langle \tilde{x}, a \rangle^2 - \langle N, a \rangle^2] d\tilde{M}. \quad (17)$$

Observe that the above expression does not depend on  $b$ . We are going to show that for any  $a \in \mathbb{R}^{n+2}$ , it is possible to choose  $b \in \mathbb{R}^{n+2}$  such that  $\int_{\tilde{M}} \varphi \phi_{a,b} d\tilde{M} = 0$ . To do this, consider a linear map  $F : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+2}$  given by

$$F(b) = \int_{\tilde{M}} \varphi \langle \tilde{x}, b \rangle N d\tilde{M}.$$

We claim that  $F$  is injective (thus a linear isomorphism). In fact, if  $b \neq 0$  is such that  $F(b) = 0$ , one has that (17), with  $\phi = \phi_{0,b} = \langle \tilde{x}, b \rangle N$ , implies that

$$I(\phi, \phi) = 0.$$

Then,  $T_1(\phi) = 0$ . On the other hand, for  $a = 0$ ,

$$T_1(\phi) = X_{0,b} = -P_1 A(b^t) = 0, \quad (18)$$

where  $b^t$  is the tangent projection of  $b$  along  $\tilde{M}$ . Since  $P_1$  is positive definite, (18) says that  $A(b^t) = 0$  on  $\tilde{M}$ , which is the same that  $\langle N, b \rangle$  is constant along  $\tilde{M}$ . As we have that  $N \circ \pi = -N$ , we get that  $\langle N, b \rangle = 0$ . This implies that the function  $u = \langle \tilde{x}, b \rangle$  satisfies that  $\text{Hess}_u(X, Y) = \langle X, Y \rangle u$ . We need the following result of M. Obata.

**Theorem 4.2** ([O] – Theorem A). *In order that a complete Riemannian manifold of dimension  $n \geq 2$  admit a non-constant function  $\phi$  with  $\text{Hess}\phi(X, Y) = c^2 \phi \langle X, Y \rangle$ , it is necessary and sufficient that the manifold be isometric to a sphere  $S^n(c)$  of radius  $\frac{1}{c}$  in the  $(n+1)$  Euclidean space.*

Thus, if  $u$  is non-constant, then  $\widetilde{M}$  is isometric to a unit sphere and since  $\widetilde{M}$  is isometrically immersed in  $S^{n+1}(1)$ , this implies that  $\widetilde{M}$  is totally geodesic. On the other hand, if  $u$  is constant,  $M$  is totally umbilic. Since  $S_2 = 0$ ,  $\widetilde{M}$  is again totally geodesic. In both cases,  $S_1^2 = |A|^2 = 0$ , which is a contradiction to the fact that  $S_1 > 0$ . Thus the claim is proved.

Take an orthonormal basis  $\{a_1, \dots, a_{n+2}\}$  of  $\mathbb{R}^{n+2}$ . By using the isomorphism  $F$ , for any  $i = 1, \dots, n+2$ , it is possible to find  $b_i \in \mathbb{R}^{n+2}$  such that  $\int_{\widetilde{M}} \varphi \phi_{a_i, b_i} d\widetilde{M} = 0$ . Thus each coordinate  $\phi_{ij}$  of  $\phi_{a_i, b_i}$  is such that  $\int_{\widetilde{M}} \varphi \phi_{ij} d\widetilde{M} = 0$ . Then,  $I(\phi_{ij}, \phi_{ij}) \geq 0$ . From equation (17), we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n+2} \int_{\widetilde{M}} [(n-1)S_1 + 3S_3][\langle \widetilde{x}, a_i \rangle^2 - \langle N, a_i \rangle^2] d\widetilde{M} \\ &= \sum_{i=1}^{n+2} \int_{\widetilde{M}} [(n-1)S_1 + 3S_3](|\widetilde{x}|^2 - |N|^2) d\widetilde{M} = 0. \end{aligned}$$

This implies that  $T_1(\phi_{a_i, b_i}) = 0$ ,  $i = 1, \dots, n+2$ . Hence,  $\langle T_1(\phi_{a_i, b_i}), \widetilde{x} \rangle = 0$  and, by equation (16), we obtain that

$$[(n-1)S_1 + 3S_3]\langle \widetilde{x}, a_i \rangle = 0, \quad i = 1, \dots, n+2.$$

But this is only possible if  $(n-1)S_1 + 3S_3 = 0$ . Since  $\|\sqrt{P_1}A\|^2 = -3S_3 = (n-1)S_1$ , theorem (3) implies that  $\widetilde{M}$  is a Clifford torus.  $\square$

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