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On the triple points of singular maps

Tobias Ekholm and András Szűcs

Abstract. The number of triple points (mod 2) of a self-transverse immersion of a closed $2n$ -manifold M into $3n$ -space are known to equal one of the Stiefel–Whitney numbers of M . This result is generalized to the case of generic (i.e. stable) maps with singularities. Besides triple points and Stiefel–Whitney numbers, a certain linking number of the manifold of singular values with the rest of the image is involved in the generalized equation which corrects an erroneous formula in [9].

If n is even and the closed manifold is oriented then the equations mentioned above make sense over the integers. Together, the integer- and mod 2 generalized equations imply that a certain Stiefel–Whitney number of closed oriented $4k$ -manifolds vanishes. This Stiefel–Whitney number is in fact the first in a family which vanish on such manifolds.

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1. Introduction

In his classical paper [10] of 1946, Whitney showed that the number of double points of a self-transverse immersion of an n -manifold into $2n$ -space is related to the Euler number of its normal bundle. Since then many results of a similar nature have been found. This paper deals with a generalization of one of these results, the Herbert–Ronga formula [5] which expresses the number of triple points of a self-transverse immersion of a closed $2n$ -manifold into $3n$ -space in terms of one of its characteristic numbers. More precisely, the Herbert–Ronga formula is extended to singular generic (i.e. stable) maps of $2n$ -manifolds into $3n$ -space. (In this paper all manifolds and maps are assumed to be C^∞ -smooth, unless otherwise explicitly stated.) To state the formula, some notation is needed:

Let M be a closed $2n$ -manifold and let $f: M \rightarrow \mathbb{R}^{3n}$ be a generic map. If $\Delta(f) \subset \mathbb{R}^{3n}$ denotes the set of double points of f then $\Delta(f)$ is an immersed n -dimensional submanifold with boundary. The self-intersection points of $\Delta(f)$ are the triple points of f . The boundary of $\Delta(f)$ is $\Sigma(f)$, the set of singular values of f .

Define $t_2(f) \in \mathbb{Z}_2$ as the mod 2-number of triple points of f . Let $\Sigma'(f)$ denote the $(n-1)$ -dimensional submanifold of \mathbb{R}^{3n} which is obtained by shifting $\Sigma(f)$ slightly along its outward normal vector field in $\Delta(f)$. Then $\Sigma'(f) \cap f(M) = \emptyset$. Define $l_2(f) \in \mathbb{Z}_2$ as the mod 2-linking number of the cycles $f(M)$ and $\Sigma'(f)$ in \mathbb{R}^{3n} . If $i_1 + \dots + i_m = 2n$ then let $\bar{w}_{i_1} \dots \bar{w}_{i_m}[M] \in \mathbb{Z}_2$ denote the product of the normal Stiefel–Whitney classes of M in dimensions i_1, \dots, i_m evaluated on the fundamental homology class of M .

Theorem 1. *Let M be a closed manifold of dimension $2n$ and let $f: M \rightarrow \mathbb{R}^{3n}$ be a generic map. Then*

$$t_2(f) + l_2(f) = \bar{w}_n^2[M] + \bar{w}_{n+1}\bar{w}_{n-1}[M] \quad (1)$$

Theorem 1 is proved in Section 2. It corrects the erroneous theorem on the second page of [9], in which the second term in the right hand side of Equation (1) is missing.

For closed oriented $4k$ -manifolds Equation (1) can be lifted to an integer equation: If $n = 2k$ is even and M is oriented then there is an induced orientation on $\Delta(f)$ as well as on the triple points of f . Define $t(f) \in \mathbb{Z}$ as the algebraic number of triple points of f . The orientation of $\Delta(f)$ induces an orientation of its boundary $\Sigma(f)$ which in turn induces an orientation of $\Sigma'(f)$. Define $l(f) \in \mathbb{Z}$ as the linking number of the oriented cycles $f(M)$ and $\Sigma'(f)$ in \mathbb{R}^{6k} . Let $\bar{p}_k[M^{4k}]$ denote the k^{th} normal Pontryagin number of M . The following theorem is Lemma 4 in [1].

Theorem 2. *Let M be a closed oriented manifold of dimension $4k$ and let $f: M \rightarrow \mathbb{R}^{6k}$ be a generic map. Then*

$$3t(f) - 3l(f) = \bar{p}_k[M]. \quad (2)$$

Equation (2) turned out to be very useful: It is used in the derivation of a geometric formula for Smale invariants of immersions of spheres, see [1] and [2], and in the study of geometric features of the regular homotopy classification of immersions of 3-manifolds in 5-space, see [7].

If M is a closed oriented $4k$ -manifold then the mod 2-reduction of $\bar{p}_k[M]$ equals $\bar{w}_{2k}^2[M]$. Hence Theorems 1 and 2 together imply that

$$\bar{w}_{2k+1}\bar{w}_{2k-1}[M] = 0 \quad (3)$$

for any closed oriented $4k$ -manifold M . In fact, $\bar{w}_{2k+1}\bar{w}_{2k-1}[M]$ is the first in a sequence of Stiefel–Whitney numbers which vanish on closed oriented $4k$ -manifolds. More precisely,

Theorem 3. (Stong). *If M is an oriented $4k$ -manifold and $(2k_1 + 1) + \dots + (2k_r + 1) = 4k$ then*

$$\bar{w}_{2k_1+1} \dots \bar{w}_{2k_r+1}[M] = 0.$$

This theorem was communicated by R. Stong to the second author together with a proof of the first case (3). A proof of Theorem 3 is presented in Section 3.

2. Proof of Theorem 1

Fix a generic map $f: M \rightarrow \mathbb{R}^{3n}$ of a closed $2n$ -manifold. Let $\tilde{\Sigma} \subset M$ denote the $(n-1)$ -dimensional submanifold of singular points of f and let $\Sigma = f(\tilde{\Sigma})$. Then f maps $\tilde{\Sigma}$ diffeomorphically to Σ .

Let $\tilde{\Delta} \subset M$ denote the closure of the preimages of multiple points of f . Then $\tilde{\Delta}$ is an immersed closed n -dimensional manifold with transverse double points at the preimages of triple points of f . Let $\tilde{\Delta}_{\text{res}}$ denote the resolution of $\tilde{\Delta}$ and let $\tilde{\iota}: \tilde{\Delta}_{\text{res}} \rightarrow M$ denote the natural immersion with image $\tilde{\Delta} \subset M$.

There is a natural involution $T: \tilde{\Delta}_{\text{res}} \rightarrow \tilde{\Delta}_{\text{res}}$ such that $f \circ \tilde{\iota} \circ T = f \circ \tilde{\iota}$. Since no triple point of f is singular we have a natural embedding $\tilde{\Sigma} \subset \tilde{\Delta}_{\text{res}}$ and $\tilde{\Sigma}$ is the fix point set of T .

Let $\nu(\tilde{\iota})$ denote the normal bundle of the immersion $\tilde{\iota}$ and let ν denote its restriction to $\tilde{\Sigma}$. Since ν is an n -dimensional vector bundle over an $(n-1)$ -manifold there exists a non-zero section. Let \tilde{s} be such a section.

A standard transversality argument allows us to extend \tilde{s} to a section \tilde{S} of $\nu(\tilde{\iota})$ which is transverse to the 0-section and which satisfies the following two conditions:

- If x is a double point of $\tilde{\iota}$ then $\tilde{S}(x) \neq 0$.
- If $\tilde{S}(x) = 0$ then $\tilde{S}(T(x)) \neq 0$.

Let $\Delta \subset \mathbb{R}^{3n}$ denote the closure of the double points of f . Then Δ is an immersed submanifold with boundary Σ and Δ has triple points at the triple points of f . Let Δ_{res} denote the resolution of Δ and let $\iota: \Delta_{\text{res}} \rightarrow \mathbb{R}^{3n}$ denote the natural immersion with image Δ . Let $\nu(\iota)$ denote the normal bundle of the immersion ι . Note that there is a natural map $\Pi: \tilde{\Delta}_{\text{res}} \rightarrow \Delta_{\text{res}}$ which is a double cover of $\Delta_{\text{res}} - \Sigma$ when restricted to $\tilde{\Delta}_{\text{res}} - \tilde{\Sigma}$, and which maps $\tilde{\Sigma}$ diffeomorphically onto Σ .

Define the section S of $\nu(\iota)$ as follows:

$$S(y) = \begin{cases} df(\tilde{S}(y_1)) + df(\tilde{S}(y_2)) & \text{if } y \in \Delta_{\text{res}} - \Sigma, \text{ where } y_1 \neq y_2, \Pi(y_1) = \Pi(y_2) = y, \\ 2df(\tilde{S}(y_1)) & \text{if } y \in \Sigma, \text{ where } \Pi(y_1) = y. \end{cases}$$

Let $C(\Sigma) \subset \Delta_{\text{res}}$ be a small open collar on the boundary Σ of Δ_{res} . Let Δ'' denote the image of the immersion $y \mapsto \iota(y) + \epsilon S(y)$, $y \in \Delta_{\text{res}} - C(\Sigma)$ for some small $\epsilon > 0$. Then, if ϵ and the collar $C(\Sigma)$ are small enough, Δ'' is a chain with boundary $\partial\Delta'' = \Sigma''$ satisfying $\Sigma'' \cap f(M) = \emptyset$. If lk_2 denotes the mod 2-linking number, \bullet denotes the mod 2-intersection number, and $\sharp(F)$ denotes the mod 2-number of elements in the finite set F , then

$$\text{lk}_2(\Sigma'', f(M)) = \Delta'' \bullet f(M) = \sharp(\tilde{S}^{-1}(0)) + t_2(f), \quad (4)$$

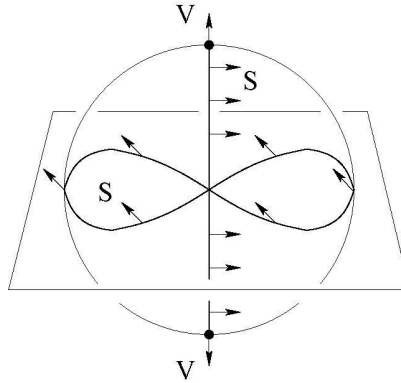


Figure 1. A piece of $f(M)$ (represented by a 2-sphere and a piece of a plane) with the double point set Δ (fat lines), its normal field S , and singularity set Σ (dots) with its outward normal field V in Δ .

since near each zero z of \tilde{S} there is a unique intersection point of Δ'' and $f(M)$ near $f(z)$, and near each triple point of f there are exactly three such intersection points.

The homology class of the cycle $\tilde{\Delta}$ in M is Poincaré dual to n^{th} normal Stiefel–Whitney class \bar{w}_n of M , see [6]. Thus

$$\bar{w}_n^2[M] = \tilde{\Delta} \bullet \tilde{\Delta} = \sharp(\tilde{S}^{-1}(0)), \quad (5)$$

since the image of a slight shift of the immersion $\tilde{\iota}$ along \tilde{S} intersects $\tilde{\Delta}$ near each zero of \tilde{S} and in *two* points near each double point of $\tilde{\iota}$.

Equations (4) and (5) imply

$$\text{lk}_2(\Sigma'', f(M)) = \bar{w}_n^2[M] + t_2(f). \quad (6)$$

Recall that $\Sigma' \subset \mathbb{R}^{3n}$ is the submanifold which results when Σ is shifted slightly along its unit outward normal vector field V in Δ , and that $\Sigma' \cap f(M) = \emptyset$. We compare the linking numbers $\text{lk}_2(\Sigma'', f(M))$ and $\text{lk}_2(\Sigma', f(M))$:

Let $\tilde{\Sigma}_0 \subset M$ be the submanifold which results when $\tilde{\Sigma}$ is shifted a small distance along \tilde{S} . Let $\Sigma_0 = f(\tilde{\Sigma}_0)$ and for $p \in \Sigma$, let $p_0 = f(\tilde{p}_0)$ where \tilde{p}_0 is the point in $\tilde{\Sigma}_0$ corresponding to $\tilde{p} \in \tilde{\Sigma}$ with $f(\tilde{p}) = p$.

For small $\epsilon > 0$ and $p \in \Sigma$ let $l_p(\epsilon)$ be the segment of the straight line through $p + \epsilon V(p)$ and p_0 of length 2ϵ and centered at p_0 . For $\epsilon > 0$ and the shifting of $\tilde{\Sigma}$ in M small enough,

$$\Gamma = \bigcup_{p \in \Sigma} l_p(\epsilon)$$

is a submanifold of \mathbb{R}^{3n} . If the collar $C(\Sigma)$ is chosen small enough and if the

shifting distance along S is small enough then the boundary $\partial\Gamma$ of Γ is isotopic to $\Sigma' \cup \Sigma''$ in $\mathbb{R}^{3n} - f(M)$. Thus

$$\mathrm{lk}_2(\Sigma', f(M)) = \mathrm{lk}_2(\Sigma_0, f(M)) + \Gamma \bullet f(M) = \mathrm{lk}_2(\Sigma'', f(M)) + \Gamma \bullet f(M). \quad (7)$$

We compute $\Gamma \bullet f(M)$: The intersection $\Gamma \cap f(M)$ is a clean intersection. That is, $\Gamma \cap f(M) = \Sigma_0$ is a manifold and the tangent bundle

$$T\Sigma_0 = T f(M) \cap T\Gamma \subset T\mathbb{R}^{3n}, \quad (8)$$

where all bundles in the left hand side are restricted to Σ_0 .

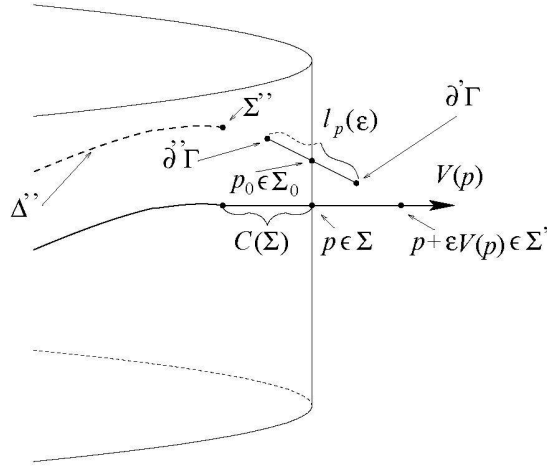


Figure 2. The normal space of Σ in \mathbb{R}^{3n} at $p \in \Sigma$. In the figure the boundary of Γ is the union of $\partial'\Gamma$, isotopic to Σ' in $\mathbb{R}^{3n} - f(M)$, and $\partial''\Gamma$ isotopic to Σ'' in $\mathbb{R}^{3n} - f(M)$.

As in [4], we find

$$\Gamma \bullet f(M) = w_{n-1}(\xi),$$

where ξ is the so called excess bundle over Σ_0 :

$$\xi = T\mathbb{R}^{3n} / (T\Gamma + T f(M)),$$

where all bundles are restricted to Σ_0 .

To finish the proof it remains to calculate $w_{n-1}(\xi)$. Note that

$$T\Gamma|_{\Sigma_0} = T\Sigma_0 \oplus \epsilon^1,$$

where ϵ^1 is the trivial line bundle directed along the intervals $l_p(\epsilon)$. Thus, by (8),

$$\xi \oplus T f(M)|_{\Sigma_0} \oplus \epsilon^1 = T\mathbb{R}^{3n}|_{\Sigma_0}. \quad (9)$$

The bundle $T f(M)|_{\Sigma_0}$ is identified with $TM|_{\tilde{\Sigma}_0}$ by the differential of f . Hence if $i_0: \tilde{\Sigma}_0 \rightarrow M$ denotes the inclusion then $w(\xi) = i_0^* w(M)$. Therefore, if F_V denotes

the fundamental homology class of the manifold V and PD denotes the Poincaré duality operator,

$$\begin{aligned}\langle w_{n-1}(\xi), F_{\Sigma_0} \rangle &= \langle i_0^* \bar{w}(M), F_{\Sigma_0} \rangle = \langle \bar{w}(M), i_{0*}(F_{\Sigma_0}) \rangle = \langle \bar{w}(M), \text{PD } \bar{w}_{n+1}(M) \rangle \\ &= \langle \bar{w}(M) \cup \bar{w}_{n+1}(M), F_M \rangle = \bar{w}_{n-1} \bar{w}_{n+1}[M].\end{aligned}\quad (10)$$

Here, the third equality follows from the well-known formula $\text{PD } \bar{w}_{n+1}(M) = i_* F_{\tilde{\Sigma}}$, where $i: \tilde{\Sigma} \rightarrow M$ denotes the inclusion, together with $i_* F_{\tilde{\Sigma}} = i_{0*} F_{\Sigma_0}$. Equations (6), (7), and (10) prove the theorem. \square

3. Proof of Theorem 3

Let \mathfrak{N}_* , Ω_* , and Ω_*^U denote the cobordism ring, the oriented cobordism ring, and the complex cobordism ring, respectively. Note that there are natural forgetting homomorphisms

$$\Omega_*^U \longrightarrow \Omega_* \longrightarrow \mathfrak{N}_*.$$

For a manifold M , let $[M]$ denote its cobordism class.

Using some facts from cobordism theory which can all be found in Chapter 4 of Stong's book [8], we show that it is enough to prove the theorem for oriented $4k$ -manifolds M such that either

- (a) $[M] \in \Omega_{4k}$ maps to a square $[N \times N] \in \mathfrak{N}_{4k}$, or
- (b) $[M]$ is a torsion element of Ω_{4k} (in fact, $[M]$ torsion implies $2 \cdot [M] = 0$):

Let $\text{Tors}(\Omega_*)$ denote the torsion subgroup of Ω_* . The homomorphism $\Omega_*^U \rightarrow \Omega_*$ induces an epimorphism

$$\Omega_*^U \longrightarrow \Omega_* / \text{Tors}(\Omega_*).$$

and the image $\Omega_*^U \rightarrow \mathfrak{N}_*$ consists of squares of elements in \mathfrak{N}_* .

Hence, if M is any oriented $4k$ -manifold then there exists some oriented $4k$ -manifold V such that $[V]$ is torsion in Ω_{4k} and $[M] + [V] = [N \times N]$ in \mathfrak{N}_{4k} . This implies that the theorem follows once it is proved for manifolds satisfying (a) or (b) above.

First consider (a): let $M = N \times N$. Then $\bar{w}(M) = \bar{w}(N) \times \bar{w}(N)$ and hence

$$\bar{w}_{2k+1}(M) = \sum_{i+j=2k+1} \bar{w}_i(N) \times \bar{w}_j(N).$$

Thus

$$\begin{aligned}\langle \bar{w}_{2k_1+1}(M) \dots \bar{w}_{2k_r+1}(M), F_M \rangle &= \\ &= \sum \langle \bar{w}_{i_1}(N) \dots \bar{w}_{i_r}(N), F_N \rangle \cdot \langle \bar{w}_{j_1}(N) \dots \bar{w}_{j_r}(N), F_N \rangle.\end{aligned}\quad (11)$$

Since $i_s + j_s$ is odd for all i_s, j_s there is a fixed point free involution T acting on

the set of the terms in the sum in (11) such that $i_1 + \dots + i_r = 2k = j_1 + \dots + j_r$:

$$\begin{aligned} T: & \langle \bar{w}_{i_1}(N) \dots \bar{w}_{i_r}(N), F_N \rangle \cdot \langle \bar{w}_{j_1}(N) \dots \bar{w}_{j_r}(N), F_N \rangle \\ & \mapsto \langle \bar{w}_{j_1}(N) \dots \bar{w}_{j_r}(N), F_N \rangle \cdot \langle \bar{w}_{i_1}(N) \dots \bar{w}_{i_r}(N), F_N \rangle. \end{aligned}$$

Thus the terms in the left hand side of (11) which does not vanish for dimensional reasons cancel in pairs and hence $\bar{w}_{2k_1+1} \dots \bar{w}_{2k_r+1}[M] = 0$.

Next consider (b): let $u: \mathfrak{N}_{4k} \rightarrow \mathbb{Z}_2$ denote the homomorphism induced by the product of odd-dimensional normal Stiefel–Whitney classes $\bar{w}_{2k_1+1} \dots \bar{w}_{2k_r+1}$, $\sum 2k_j + 1 = 4k$. Odd-dimensional Stiefel–Whitney classes are mod 2-reductions of twisted integer classes, see [3], p. 140. Hence, a product of an even number of such classes is an integer class so the map

$$\Omega_{4k} \xrightarrow{\pi} \mathfrak{N}_{4k} \xrightarrow{u} \mathbb{Z}_2$$

lifts to a homomorphism

$$\Omega_{4k} \xrightarrow{U} \mathbb{Z}.$$

Thus U and therefore $u \circ \pi$ is zero on any torsion element of Ω_{4k} . \square

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