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## Marked length rigidity for symmetric spaces

Françoise Dal'Bo and In Kang Kim<sup>1</sup>

**Abstract.** We give conditions under which a homomorphism between two Zariski dense subgroups of connected semisimple Lie groups  $G$  and  $G'$  without compact factors and with trivial center can be extended to a continuous isomorphism between  $G$  and  $G'$ . In particular we prove the marked length rigidity and the marked translation vector rigidity. This last result was motivated by a Margulis's question.

**Mathematics Subject Classification (2000).** 22E40, 22F30.

**Keywords.** Symmetric spaces, length spectrum.

### Introduction

Let  $G$ ,  $G'$  be connected semisimple Lie groups without compact factors and with trivial center. The motivation of this paper is to give conditions under which a homomorphism between two Zariski dense subgroups of  $G$  and  $G'$  can be extended to a continuous isomorphism between  $G$  and  $G'$ . Much study of lattices has been done, yet the study of general co-infinite volume groups is relatively less carried out. Fix a closed Weyl chamber  $\mathcal{A}^+$  included in the Lie algebra of  $G$ . The translation vector  $v(g)$  of  $g \in G$ , is, by definition, the unique  $a \in \mathcal{A}^+$  such that  $e^a$  is conjugate to the hyperbolic part of the Jordan decomposition of  $g$  (see section 1). The Euclidean norm of  $v(g)$  is denoted  $\ell(g)$  and is called the length of  $g$ . If  $X$  is a symmetric space associated to  $G$ , one has:  $\ell(g) = \inf_{x \in X} d(x, g(x))$ . In the particular case where  $G = PSL(n, \mathbb{R})$  and  $\mathcal{A}^+$  is the set of diagonal matrices  $\text{diag}(a_1, \dots, a_n)$  with  $a_1 \geq \dots \geq a_n$ , one has:  $v(g) = \text{diag}(\text{Log} |\lambda_1|, \dots, \text{Log} |\lambda_n|)$  where  $\lambda_i$  is the  $i^{\text{th}}$  complex eigenvalue of  $g$ . Let  $\Gamma \subset G$ , the limit cone,  $\mathcal{L}(\Gamma)$ , associated to  $\Gamma$  is, by definition, the smallest closed cone in  $\mathcal{A}^+$  containing all  $v(\gamma)$  for  $\gamma \in \Gamma$ . An important result due to Y. Benoist [1] says that the interior of  $\mathcal{L}(\Gamma)$  is not empty, if  $\Gamma$  is a Zariski dense group. The originality of this paper is to explore this property to obtain strong rigidity results in a short and elementary way.

Let us give the main results.

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**Theorem A.** *Let  $\Gamma \subset G, \Gamma' \subset G'$  be Zariski dense subgroups. If  $\varphi$  is a surjective homomorphism between  $\Gamma$  and  $\Gamma'$  such that  $\ell(\gamma) = \ell(\varphi(\gamma))$  for any  $\gamma \in \Gamma$  then  $\varphi$  can be extended to a continuous isomorphism between  $G$  and  $G'$ .*

Following the way of A. Parreau [15], we give applications of Theorem A to the space of representations of an abstract group into  $G$ .

Theorem A is already known for symmetric spaces of rank 1 ([4], [11]) and their products ([12]). For simple Lie groups it is shown in ([6]). Along this line, Besson, Courtois, Gallot and Hamenstädt ([2], [9]) showed that, if  $M$  is a negatively curved locally symmetric compact manifold and  $N$  is an arbitrary negatively curved manifold which has the same marked length spectrum with  $M$ , then they are isometric. Actually it is conjectured that two negatively curved compact manifolds with the same marked length spectrum are isometric. This conjecture is proved in dimension 2 ([14]).

The following theorem gives a positive answer to a Margulis's question raised during the rigidity conference at Paris in June 1998.

**Theorem B.** *Suppose  $G = G'$  and  $\text{rank } G \geq 2$ . Let  $\Gamma, \Gamma'$  be Zariski dense subgroups of  $G$ . If  $\varphi$  is a surjective homomorphism between  $\Gamma$  and  $\Gamma'$  such that for all  $\gamma \in \Gamma$  there exists  $k(\gamma) \in \mathbb{R}^*$  such that  $v(\varphi(\gamma)) = k(\gamma)v(\gamma)$ , then  $\varphi$  can be extended to a continuous automorphism of  $G$ .*

We first study the simple case where  $G$  and  $G'$  are simple. Using a criterion of conjugacy proved in [6] we give a family of conditions (including conditions of Theorems A and B) under which a surjective homomorphism between Zariski dense subgroups can be extended.

## 1. Benoist's theorem for limit cone

An element  $g$  of a real reductive connected linear group can be uniquely written  $g = ehu$  where  $e$  is elliptic (all the eigenvalues have modulus 1),  $u$  is unipotent ( $u - \text{Id}$  is nilpotent),  $h$  is hyperbolic (all the eigenvalues are real positive), and all three commute. This decomposition is called the Jordan decomposition of  $g$ . If  $G = \text{KAN}$  is any Iwasawa decomposition of a connected semisimple Lie group  $G$ , then  $e$  is conjugate to an element in  $K$ ,  $h$  is conjugate to an element in  $A$  and  $u$  is conjugate to an element in  $N$  ([1], [7]). Fix a closed Weyl chamber  $\mathcal{A}^+$  in the Lie algebra of  $G$ , there exists a unique  $a \in \mathcal{A}^+$ , called the translation vector of  $g$  and denoted  $v(g)$ , such that  $h$  is conjugate to  $e^a$ . Geometrically, if  $X$  is a symmetric space associated to  $G$ , then  $\|v(g)\| = \ell(g)$  where  $\ell(g) = \inf_{x \in X} d(x, g(x))$  (see [15] for an interpretation of  $v(g)$ ). Let  $\Gamma$  be a subgroup of  $G$ , one defines the limit cone of  $\Gamma$ , denoted  $\mathcal{L}(\Gamma)$ , as the smallest closed cone in  $\mathcal{A}^+$  containing  $v(\Gamma)$ . If  $G = \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$  and  $\mathcal{A}^+ = \{(r_1 M, r_2 M) / r_1, r_2 \in \mathbb{R}^+\}$  where  $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , then  $\mathcal{L}(\Gamma)$  is the closure

of  $\{(r\ell(\gamma_1)M, r\ell(\gamma_2)M)/r \in \mathbb{R}^+, (\gamma_1, \gamma_2) \in \Gamma\}$  where  $\ell(\gamma_i) = 0$  if  $\gamma_i$  is elliptic or parabolic and  $\ell(\gamma_i) > 0$  is the displacement of  $\gamma_i$  if  $\gamma_i$  is hyperbolic. The following result, due to Y. Benoist, plays a key role in this paper.

**Theorem 1.1** [1]. *If  $\Gamma$  is a Zariski dense subgroup of  $G$  then  $\mathcal{L}(\Gamma)$  is convex and has nonempty interior.*

In the particular case where  $\Gamma$  is a Zariski dense subgroup of  $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$  associated to the diagonal action of two isomorphic Fuchsian groups  $\Gamma_1 \xrightarrow{\varphi} \Gamma_2$ , this theorem says that  $\left\{ \frac{\ell(\gamma_1)}{\ell(\varphi(\gamma_1))}, \gamma_1 \in \Gamma_1 \right\}$  is an interval  $[a, b] \subset [0, \infty]$  with  $a \neq b$ . This property was already remarked in the context of rank 1 semisimple groups by M. Burger [4] (see also [5]).

## 2. Rigidity results for simple groups

In this section one supposes that  $G$  and  $G'$  are connected, noncompact, **simple** Lie groups with trivial center. Let  $\varphi : \Gamma \rightarrow \Gamma'$  be a homomorphism between two subgroups of  $G$  and  $G'$ . One defines the graph group  $\Gamma_\varphi \subset G \times G'$  by  $\Gamma_\varphi = \{(\gamma, \varphi(\gamma))/\gamma \in \Gamma\}$ . The following result is proved in [6].

**Criterion of conjugacy 2.1** [6]. *Let  $\varphi$  be a surjective homomorphism between two Zariski dense subgroups  $\Gamma, \Gamma'$  included in connected non compact simple Lie groups,  $G$  and  $G'$ , with trivial center. The following properties are equivalent:*

- 1)  $\varphi$  can be extended to a continuous isomorphism between  $G$  and  $G'$
- 2)  $\Gamma_\varphi$  is not Zariski dense in  $G \times G'$ .

This criterion is false if  $G$  and  $G'$  are not simple. Take for example  $G = \mathrm{PSL}(2, \mathbb{R})$  and  $G' = G \times G$ . Denote  $\mathcal{A}^+$  the closed Weyl chamber of  $G$  defined by  $\mathcal{A}^+ = \{rM/r \in \mathbb{R}^+\}$  where  $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . Let  $\varphi : \Gamma_1 \rightarrow \Gamma_2$  be an isomorphism between non conjugate and non elementary Fuchsian groups. The groups  $\Gamma_1$  and  $\Gamma_{1\varphi}$  are Zariski dense subgroups respectively of  $G$  and  $G'$ . Consider the isomorphism  $\Psi : \Gamma_1 \rightarrow \Gamma_{1\varphi}$  defined by  $\Psi(\gamma) = (\gamma, \varphi(\gamma))$ . The limit cone of the graph group associated to  $\Psi$  is included in  $\{(rM, rM, sM)/r, s \in \mathbb{R}^+\} \subset \mathcal{A}^+ \times \mathcal{A}^+$  and hence has empty interior. According to Benoist's theorem (section 1),  $\Gamma_{1\Psi}$  is not Zariski dense. On the other hand  $\Psi$  cannot be extended.

One deduces from the previous criterion the following corollary.

**Corollary 2.2.** *Let  $\mathrm{Ad}$  be the adjoint representation. If there exists an algebraic relation satisfied by all  $(\mathrm{Ad}(\gamma)), \mathrm{Ad}(\varphi(\gamma))$  with  $\gamma \in \Gamma$ , then  $\varphi$  can be extended to a continuous isomorphism between  $G$  and  $G'$ .*

In the case where  $G = \mathrm{PSL}(n, \mathbb{R})$ ,  $G' = \mathrm{PSL}(n', \mathbb{R})$  and  $\varphi$  preserves the trace, Corollary 2.2 is proved in [16].

Remark that the condition  $\ell(\gamma) = \ell(\varphi(\gamma))$  for each  $\gamma \in \Gamma$  is not in general an algebraic condition. But in this case, since  $\|v(\gamma)\| = \|v(\varphi(\gamma))\|$  for  $\gamma \in \Gamma$ , the limit cone of the graph group has empty interior. Applying Benoist's theorem, one concludes that  $\Gamma_\varphi$  is not Zariski dense and hence that  $\varphi$  can be extended. More generally, one has the following result.

**Corollary 2.3.** *If the interior of  $\mathcal{L}(\Gamma_\varphi)$  is empty then  $\varphi$  can be extended to a continuous isomorphism between  $G$  and  $G'$ .*

Let us give three different conditions under which  $\Gamma_\varphi$  is not Zariski dense and hence  $\varphi$  can be extended:

- 1)  $\ell(\gamma) = \ell(\varphi(\gamma))$  for any  $\gamma \in \Gamma$ .
- 2)  $v(\gamma)$  and  $v(\varphi(\gamma))$  are colinear for any  $\gamma \in \Gamma$ .
- 3) The largest modulus of the complex eigenvalue of  $\mathrm{Ad}(\gamma)$  equals the largest one of  $\mathrm{Ad}(\varphi(\gamma))$  for any  $\gamma \in \Gamma$ .

Conditions 1) and 2) correspond to Theorems A and B when  $G$  and  $G'$  are simple. Contrary to the conditions 1) and 2), if  $\varphi$  satisfies condition 3) and  $G$  and  $G'$  are not simple,  $\varphi$  cannot be necessarily extended. For example, fix two isomorphic Schottky groups  $\rho : \Gamma \rightarrow \Gamma'$  in  $\mathrm{PSL}(2, \mathbb{R})$ . Suppose that  $\ell(\gamma) > \ell(\rho(\gamma))$  for each  $\gamma \in \Gamma$  (see [5] for the construction of such groups). Consider the isomorphism  $\varphi : \Gamma \rightarrow \Gamma_\rho$  defined by  $\varphi(\gamma) = (\gamma, \rho(\gamma))$ . The groups  $\Gamma, \Gamma_\rho$  are Zariski dense respectively in  $\mathrm{PSL}(2, \mathbb{R})$  and  $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$  and the condition 3) is satisfied but  $\varphi$  cannot be extended.

### 3. Proofs of Theorems A and B

In this section  $G$  and  $G'$  denote connected semisimple groups with trivial center and without compact factor. Such a group can be decomposed into a product of connected noncompact simple groups with trivial center.

**Lemma 3.1.** *Let  $\Gamma, \Gamma'$  be Zariski dense subgroups of  $G$  and  $G'$ . Suppose that  $\varphi$  is a surjective homomorphism between  $\Gamma$  and  $\Gamma'$  and set  $\Gamma_\varphi = \{(\gamma, \varphi(\gamma)) / \gamma \in \Gamma\}$ . The projections of the identity component of the Zariski closure of  $\Gamma_\varphi$  into  $G$  and  $G'$  are surjective.*

*Proof.* The Lie algebra  $\mathcal{G}$  of  $G$  can be decomposed into a direct sum of simple ideals  $\mathcal{G} = \mathcal{F}_1 + \cdots + \mathcal{F}_n$ . Moreover each ideal of  $\mathcal{G}$  is a direct sum of certain  $\mathcal{F}_i$  ([10] corollary II.6.3). Let  $G_i$  be the connected Lie subgroup in  $G$  associated to  $\mathcal{F}_i$ . Since  $G$  has trivial center,  $G = G_1 \times \cdots \times G_n$ . Let  $H$  be the identity component of the Zariski closure of  $\Gamma_\varphi$ . Denote  $p$  the projection of  $H$  into  $G$  and  $T_p$  its tangent map at identity. The image,  $\mathcal{F}$ , of the Lie algebra of  $H$  by  $T_p$  is a non trivial

subalgebra of  $\mathcal{G}$  normalized by  $\Gamma$ . Since  $\Gamma$  is Zariski dense,  $\mathcal{F}$  is an ideal and hence  $\mathcal{F} = \mathcal{F}_{i_1} + \cdots + \mathcal{F}_{i_k}$ ,  $k \leq n$ . This implies that  $p(H) = G_{i_1} \times \cdots \times G_{i_k}$ . Since the index of  $H$  in the Zariski closure of  $\Gamma_\varphi$  is finite and  $\Gamma$  is Zariski dense,  $p(H)$  is also Zariski dense. This proves that  $k = n$  and thus that  $p$  is surjective. Since  $\varphi$  is surjective, the same argument holds for the projection of  $H$  into  $G'$ .  $\square$

*Proof of Theorem A.* Denote  $H$  the identity component of the Zariski closure of  $\Gamma_\varphi$  and  $\mathcal{H}$  its Lie algebra. We want to prove that the projection  $p$  (resp.  $p'$ ) of  $H$  into  $G$  (resp.  $G'$ ) is injective. Let us first show that  $\mathcal{H}$  is semisimple. Consider its solvable radical  $\mathcal{R} \subset \mathcal{H}$ . The image of  $\mathcal{R}$  by the tangent map  $Tp$  of  $p$  at identity is normalized by  $\Gamma$ . Since  $\Gamma$  is Zariski dense in  $G$ ,  $Tp(\mathcal{R})$  is a solvable ideal. The semi simplicity of  $G$  implies that  $Tp(\mathcal{R})$  is trivial. Since  $\varphi$  is surjective, the same argument holds for  $p'$ . This shows that  $\mathcal{R}$  is trivial. Fix a Cartan decomposition  $\mathcal{H} = \mathcal{P}'' + \mathcal{T}''$  of  $\mathcal{H}$ , since  $G \times G'$  is semisimple, there exists a Cartan decomposition  $\mathcal{P} + \mathcal{T}$  of the Lie algebra of  $G \times G'$  such that  $\mathcal{P}'' \subset \mathcal{P}$  and  $\mathcal{T}'' \subset \mathcal{T}$  ([10] VI exercise 8(i)). Choose a Weyl chamber  $\mathcal{W} \subset \mathcal{P}''$  since  $\mathcal{P}'' \subset \mathcal{P}$  one has  $\mathcal{W} \subset \mathcal{A} \times \mathcal{A}'$  where  $\mathcal{A}$  and  $\mathcal{A}'$  are Cartan subalgebras of the Lie algebra  $\mathcal{G}, \mathcal{G}'$  of  $G$  and  $G'$ . Let us analyze  $\text{Ker } p$ . This group is normalized by  $\Gamma'$  because  $\varphi$  is surjective. Since  $\Gamma'$  is Zariski dense and the center of  $G'$  is trivial, either  $\text{Ker } p = \{\text{Id}\}$  or  $\text{Ker } p$  is a normal non trivial Lie subgroup of  $G'$ . In the last case, denote  $\mathcal{I}$  the Lie algebra of the identity component of  $\text{Ker } p$ . One has  $\mathcal{I} = \mathcal{I}'_1 + \cdots + \mathcal{I}'_p$  where  $\mathcal{I}'_j$  are noncompact simple ideals of  $\mathcal{G}'$  such that  $\mathcal{G} = \mathcal{I}'_1 + \cdots + \mathcal{I}'_k$  with  $k \geq p$  ([10] corollary II.6.3). It follows that  $\mathcal{W}$  contains an element  $a = (0, \omega) \in \mathcal{A} \times \mathcal{A}'$  with  $\|\omega\| \neq 0$ . Since  $\Gamma_\varphi \cap H$  is Zariski dense in  $H$ , according to Benoist's theorem, the interior of its limit cone,  $\mathcal{L}^{\mathcal{W}}(\Gamma_\varphi \cap H)$ , relatively to  $\mathcal{W}$ , is not empty. Moreover  $\mathcal{L}^{\mathcal{W}}(\Gamma_\varphi \cap H)$  is included in  $S = \{(u, u') \in \mathcal{A} \times \mathcal{A}' / \|u\| = \|u'\|\}$  because  $\varphi$  preserves the translation length and  $\mathcal{L}^{\mathcal{W}}(\Gamma_\varphi \cap H)$  is included in the image of the limit cone of  $\Gamma_\varphi \cap H$  relatively to  $\mathcal{A}^+ \times \mathcal{A}'^+$  by the Weyl group. Let  $b = (u, u')$  an element of the interior of  $\mathcal{L}^{\mathcal{W}}(\Gamma_\varphi \cap H) \subset \mathcal{W}$ . One can suppose  $\|u\| = \|u'\| = 1$ . Since the interior of  $\mathcal{L}^{\mathcal{W}}(\Gamma_\varphi \cap H)$  in  $\mathcal{W}$  is not empty, the intersection of the plane generated by  $a$  and  $b$  with  $\mathcal{L}^{\mathcal{W}}(\Gamma_\varphi \cap H)$  contains an open disc. There is a contradiction with the fact that the intersection of this plane with  $S$  is the curve  $\{\alpha a + \beta b / 2\alpha\beta \langle u', \omega \rangle + \alpha^2 \|\omega\|^2 = 0\}$ . In conclusion  $p$  is injective. The same argument holds for  $p'$ , because  $\varphi$  is surjective. Applying the lemma 3.1, one obtains that  $p$  and  $p'$  are bijective. Consider now the projections  $q$  (resp.  $q'$ ) of the Zariski closure  $\overline{\Gamma}_\varphi^{\mathbb{Z}}$  of  $\Gamma_\varphi$  into  $G$  (resp.  $G'$ ). The maps  $q$  and  $q'$  are surjective. Let us prove that they are injective. Take  $g \in \text{Ker } q$ , for any  $h \in H$  one has  $q(ghg^{-1}h^{-1}) = \text{Id}$ . Since  $H$  is normalized by  $\overline{\Gamma}_\varphi^{\mathbb{Z}}$  and  $p$  is injective,  $gh = hg$ . Using the fact that  $p'$  is surjective one obtains  $p'(g)g' = g'p'(g)$  for any  $g' \in G'$ . Because the center of  $G'$  is trivial,  $g = \text{Id}$ . The same argument also holds for  $p'$ . Consider the map  $f = p' \circ p^{-1}$ , it is a continuous isomorphism between  $G$  and  $G'$  whose restriction to  $\Gamma$  coincides with  $\varphi$ .  $\square$

*Proof of Theorem B.* The proof is similar to the previous one. Let us just adapt the end of the proof of Theorem A, when we suppose that  $\text{Ker } p$  is nontrivial. Under this assumption one obtains that  $\mathcal{W}$  contains an element  $a = (0, \omega) \in \mathcal{A} \times \mathcal{A}$  with  $\omega \neq 0$ . Since  $v(\gamma) = k(\gamma)v(\varphi(\gamma))$  for each  $\gamma \in \Gamma$ , the limit cone  $\mathcal{L}^{\mathcal{A}^+ \times \mathcal{A}^+}(\Gamma_\varphi \cap H)$  is included in  $T = \{(u, u') \in \mathcal{A}^+ \times \mathcal{A}^+ / u \text{ and } u' \text{ are colinear}\}$  and hence  $\mathcal{L}^{\mathcal{W}}(\Gamma_\varphi \cap H)$  is included in  $\bigcup_{g \in \text{Weyl}} gT$  where Weyl is the Weyl group of  $\mathcal{A} \times \mathcal{A}$ . The interior of  $\mathcal{L}^{\mathcal{W}}(\Gamma_\varphi \cap H)$  in  $\mathcal{W}$  is not empty according to Benoist's theorem. It follows that for some  $g \in \text{Weyl}$ , the interior  $I$  of  $g(T)$  is not empty in  $\mathcal{W}$ . Let  $b = (u, u') \in I$ . Since rank  $G \geq 2$  one can assume that  $u'$  is not colinear to  $w$ . The intersection of the plane  $P$  generated by  $a$  and  $b$  with  $I$  contains an open disc. There is a contradiction with the fact that the intersection of  $T$  with  $g^{-1}(P)$  is a line.  $\square$

#### 4. Applications of Theorem A to the space of representations

Fix a connected semisimple Lie group  $G$  without compact factor and with trivial center, and a symmetric space  $X$  associated to  $G$ . A subgroup of  $G$  is said parabolic if it fix a point of the geometric boundary,  $\partial X$ , of  $X$ .

**Proposition 4.1.** *Let  $\Gamma$  be a nonparabolic subgroup of  $G$  and  $H$  the identity component of its identity component. If  $H \neq G$  then  $H$  fix a totally geodesic submanifold  $Y \subsetneq X$ .*

*Proof.* We thank P. Eberlein for helpful arguments.

The group  $H$  is reductive or parabolic ([3] corollaire 3.3). The last case cannot happens because  $H$  is normalized by  $\Gamma$  which does not fix any point in  $\partial X$ . Let  $H = ST$  be the Levi decomposition of  $H$  where  $S$  is a connected semisimple group and  $T$  is a torus, corresponding to the identity component of the center of  $H$ . If  $T \neq \text{Id}$  there exists a flat totally geodesic submanifold  $T \subset X$  such that  $T$  leaves  $F$  invariant and  $F/T$  is compact ([8]). Let  $C$  be the union of all totally geodesic submanifolds which are parallel to  $F$ . Then  $C$  is invariant under  $H$  and is isometric to  $F \times N$  for some closed convex subset  $N$  of  $X$  ([7] proposition 1.6.7). The set  $C$  is a totally geodesic submanifold possible with boundary. Let  $Y$  be a complete totally geodesic submanifold of  $X$  with  $\dim Y = \dim C$ . Since  $H$  leaves  $C$  invariant and  $C$  contains an open subset of  $Y$ , the group  $H$  leaves  $Y$  invariant. Remark that  $Y \neq X$ , because  $Y$  contains an Euclidean factor. If  $T = \{\text{Id}\}$  then  $H$  is semisimple, and there exists  $x \in X$  such that  $Hx$  is a totally geodesic submanifold ([13] lemma 7.21). By the assumption  $H \neq G$  hence  $Hx \neq X$ .  $\square$

Let  $\Gamma$  be an abstract group and  $\rho : \Gamma \rightarrow G$  be a faithful representation. One always supposes that the Zariski closure,  $H_\rho$ , of  $\rho(\Gamma)$  is connected and that the representation  $\rho$  is nonparabolic (i.e.  $\rho(\Gamma)$  is nonparabolic). In this case  $H_\rho$  is reductive (proof of proposition 4.1). Let  $H_\rho = ST$  be the Levi decomposition

of  $H_\rho$ . The representation  $\rho$  is noncompact if  $S$  is a semisimple group without compact factor and with trivial center. Under this assumption  $H_\rho$  stabilizes a totally geodesic submanifold of  $X$  isometric to  $N \times F$  where  $N$  is a symmetric space on which  $S$  acts transitively and  $F$  is a flat on which  $T$  acts by translation with compact quotient (proof of the proposition 4.1). Two faithful, nonparabolic and noncompact representations  $\rho$  and  $\rho'$  of  $\Gamma$  are equivalent if there exists an isometry  $f$  between  $N \times F$  and  $N' \times F'$  such that  $f \circ \rho(\gamma) = \rho'(\gamma) \circ f$  for any  $\gamma \in \Gamma$ . If  $F$  and  $F'$  are empty, then  $\rho$  and  $\rho'$  are equivalent if and only if  $\rho' \circ \rho^{-1}$  can be extended to a continuous isomorphism between  $S$  and  $S'$  ([7] proposition 3.9.11). Denote  $R_{fnpnc}/\sim$  the space of faithful nonparabolic, noncompact representations of  $\Gamma$  into  $G$ , up to the equivalence relation. The following result is an application of Theorem A to the context of representations.

**Proposition 4.2.** *The map  $L: R_{fnpnc}/\sim \rightarrow \mathbb{R}^\Gamma$  defined by  $L([\rho])(\gamma) = \ell(\rho(\gamma))$  is injective.*

*Proof.* Let  $\rho_1, \rho_2 \in R_{fnpnc}$ . Suppose  $L(\rho_1) = L(\rho_2)$ . For  $i = 1, 2$  set  $\Gamma_i = \rho_i(\Gamma)$ ,  $H_i = H_{\rho_i}$  and  $T_i = S_i T_i$ .

a) Suppose  $S_1 = S_2 = \{e\}$ , then  $T_i$  acts by translation on the flat  $(F_i, \langle \cdot \rangle_i)$  and  $F_i/T_i$  is compact. Let us identify  $\rho_i(\gamma)$  with its translation vector. Choose a basis,  $\rho_1(\gamma_1), \dots, \rho_1(\gamma_n)$  of  $F_1$ , such a basis exists because  $\Gamma_1$  is Zariski dense in  $T_1$ . For  $\gamma \in \Gamma$ , write  $\rho_1(\gamma) = \sum_{i=1}^n a_i \rho_1(\gamma_i)$  and  $\rho_2(\gamma) = \sum_{i=1}^n b_i \rho_2(\gamma_i) + \omega$  where  $\omega$  is orthogonal to each  $\rho_2(\gamma_i)$ . Since  $\|\rho_1(\gamma)\| = \|\rho_2(\gamma)\|$ , one has  $\langle \rho_1(\gamma), \rho_1(\gamma') \rangle_1 = \langle \rho_2(\gamma), \rho_2(\gamma') \rangle_2$  for any  $\gamma, \gamma' \in \Gamma$ . Put  $c_{ij} = \langle \rho_1(\gamma_i), \rho_1(\gamma_j) \rangle_1 = \langle \rho_2(\gamma_i), \rho_2(\gamma_j) \rangle_2$ . One has  $\langle \rho_1(\gamma), \rho_1(\gamma_j) \rangle_1 = \sum_{i=1}^n a_i c_{ij}$  and  $\langle \rho_2(\gamma), \rho_2(\gamma_j) \rangle_2 = \sum_{i=1}^n b_i c_{ij}$  hence  $\sum_{i=1}^n (a_i - b_i) c_{ij} = 0$  for any  $1 \leq j \leq n$ . This proves that  $a_i = b_i$ . Moreover  $\|\rho_1(\gamma)\| = \|\rho_2(\gamma)\|$  hence  $\omega = 0$ . One thus obtains  $\rho_2(\gamma) = \sum_{i=1}^n a_i \rho_2(\gamma_i)$  and  $\dim F_2 = n$  because  $\Gamma_2$  is Zariski dense in  $T_2$ . The linear map  $f: F_1 \rightarrow F_2$  defined by  $f(\rho_1(\gamma_i)) = \rho_2(\gamma_i)$  is an isometry satisfying  $f \circ \rho_1(\gamma) = \rho_2(\gamma) \circ f$ , hence  $[\rho_1] = [\rho_2]$ .

b) Suppose  $S_1 \neq \{e\}$ , then  $S_2 \neq \{e\}$ . Decompose  $S_i$  into a product of noncompact simple factors with trivial center  $S_i = S_{i1} \times \dots \times S_{ik_i}$  and denote  $p_{is}$  the projection of  $S_i$  into  $S_{is}$ . Since  $\Gamma_i$  is Zariski dense in  $S_i \times T_i$  then  $p_{is}(\Gamma)$  is Zariski dense in  $S_{is}$ . Set  $D = [\Gamma, \Gamma]$  and  $D_i = \rho_i(D)$ . The group  $D_i$  is normalized by  $\Gamma_i$  and is included in  $S_i$ , hence one can suppose that the Zariski closure of  $D_i$  equals  $S_{i1} \times \dots \times S_{in_i}$  with  $n_i \leq k_i$ . Moreover  $n_i = k_i$  because  $p_{is}(D_i)$  is normalized by  $p_{is}(\Gamma)$  which is Zariski dense in  $S_{is}$  and the center of  $S_{is}$  is trivial. In conclusion  $D_i$  is Zariski dense in  $S_i$ . By assumption  $\ell(\rho_1(d)) = \ell(\rho_2(d))$  for any  $d \in D$ . One deduces from Theorem A that the restriction of  $\rho_2 \circ \rho_1^{-1}$  to  $D_1$  can be extended to a continuous isomorphism  $\varphi$  between  $S_1$  and  $S_2$ . Up to  $\varphi$ , one can suppose  $S_1 = S_2$  and  $\rho_1(d) = \rho_2(d)$  for any  $d \in D$ . Let  $\gamma \in \Gamma$ , since  $\rho_1(\gamma d \gamma^{-1}) = \rho_2(\gamma d \gamma^{-1})$  and  $\rho_1(d) = \rho_2(d)$ , the projection of  $\rho_2^{-1}(\gamma) \rho_1(\gamma)$  into  $S_1$  commutes with all  $\rho_1(d)$ . Since  $D_1$  is Zariski dense and the center of  $S_1$  is trivial, the projection of  $\rho_2^{-1}(\gamma) \rho_1(\gamma)$  into  $S_1$  is trivial. Consider now the projection  $p_i$  of



$\Gamma_i$  into  $T_i$ . One has  $\ell(p_1 \circ \rho_1(\gamma)) = \ell(p_2 \circ \rho_2(\gamma))$ , moreover  $p_i(\Gamma_i)$  is Zariski dense in  $T_i$ . Using arguments developed in a), one obtains the existence of a isometry  $f : F_1 \rightarrow F_2$  such that  $f \circ (p_1 \circ \rho_1(\gamma)) = p_2 \circ \rho_2(\gamma) \circ f$ , hence  $[\rho_1] = [\rho_2]$ .  $\square$

The following part is inspired by the section 5 of A. Parreau's thesis ([15]). Let us consider the particular case where  $\Gamma$  is an infinite group of finite type. Fix a finite set,  $S$ , of generators. One associates to a representation  $\rho : \Gamma \rightarrow G$  its minimal displacement,  $\lambda(\rho) = \inf_{x \in X} (\sup_{s \in S} d(x, \rho(s)(x)))$ . If  $\lambda(\rho) = 0$  there exists a sequence  $(x_n)_{n \geq 1}$  in  $X$  such that  $\lim_n d(x_n, \rho(s)(x_n)) = 0$  for any  $s \in S$ . Up to a subsequence one can suppose that  $(x_n)_{n \geq 1}$  converges in  $X \cup \partial X$ . If  $\lim_n x_n = x \in X$  then  $\rho(s)(x) = x$  for any  $s \in S$  and hence  $\rho(\Gamma)$  belongs to a compact subgroup. Otherwise  $\lim_n x_n = \xi \in \partial X$  and  $\rho(s)(\xi) = \xi$  for any  $s \in S$ . In this case  $\rho$  is parabolic. In conclusion, if  $\rho \in R_{fnpc}$  then  $\lambda(\rho) > 0$ . Let us consider the map  $\frac{V}{\lambda} : R_{fnpc}/ \sim \rightarrow \mathbb{R}^\Gamma$  defined by  $L([\rho])(\gamma) = \frac{\ell(\rho(\gamma))}{\lambda(\rho)}$ . This map is continuous ([15] propositions V.2.3 and V.3.8) and its image is included in a compact set ([15] proposition V.4.1). One deduces from these properties and from the proposition 4.2 the following result.

**Corollary 4.3.** *The map  $\frac{L}{\lambda} : R_{fnpc}/ \sim \rightarrow \mathbb{R}^\Gamma$  is injective, continuous and its image is included in a compact set.*

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## References

- [1] Y. Benoist, Propriétés asymptotiques des groupes linéaires, *GAF* **7** (1997), 1–47.
- [2] G. Besson, G. Courtois and S. Gallot, Entropies et rigidités des espaces localement symétriques de courbure strictement négative, *GAF* **5** (1997), 731–799.
- [3] A. Borel and J. Tits, Eléments unipotents et sous-groupes paraboliques de groupes réductifs. *Inv. math.* **12** (1971), 95–104.
- [4] M. Burger, Intersection, the Manhattan curve and Patterson–Sullivan theory in Rank 2, *International Mathematics Research Notices* **7** (1993), 217–225.
- [5] F. Dal'Bo, Famille de groupes agissant sur le produit de deux variétés de Hadamard, *Séminaire de théorie spectrale et géométrie de Grenoble* (1997).
- [6] F. Dal'Bo and I. Kim, A criterion of conjugacy for Zariski dense subgroups, *CRAS* t. 330, série I (2000), 647–650.
- [7] P. Eberlein, Geometry of nonpositively curved manifolds, *University of Chicago Press*, 1996.
- [8] D. Gromoll and J. Wolf, Some relations between the metric structure and the algebraic structure of the fundamental group in manifolds of nonpositive curvature, *Bull. Amer. Math. Soc.* **77** (1971), 545–552.

- [9] U. Hamenstadt, Cocycles, symplectic structures and intersection, *GAF* **9** (1) (1999), 90–140.
- [10] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, 1978.
- [11] I. Kim, Marked length rigidity of rank one symmetric spaces, *Topology*, **40** n°6 (2001), 1295–1323.
- [12] I. Kim, Ergodic theory and rigidity on the symmetric space of noncompact type, *Ergodic theory and dynamical systems*, **21** n°1 (2001), 93–114.
- [13] G. Margulis, Discrete subgroups of semisimple Lie group, Springer-Verlag, 1991.
- [14] J.-P. Otal, Le spectre marqué des surfaces à courbure négative, *Ann. of Math.* (2) **131** (1990), 151–162.
- [15] A. Parreau, Dégénérescences de sous-groupes discrets de groupes de Lie semisimples et actions de groupes sur des espaces affines, Thèse, Orsay, 2000.
- [16] J. H. Sampson, Sous-groupes conjugués d'un groupe linéaire, *Ann. Inst. Fourier* **26** (2) (1976), 1–6.

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