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Autor: Dal'Bo, Françoise / Kim, Inkang
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Marked length rigidity for symmetric spaces

Françoise Dal'Bo and Inkang Kim¹

Abstract. We give conditions under which a homomorphism between two Zariski dense subgroups of connected semisimple Lie groups G and G' without compact factors and with trivial center can be extended to a continuous isomorphism between G and G' . In particular we prove the marked length rigidity and the marked translation vector rigidity. This last result was motivated by a Margulis's question.

Mathematics Subject Classification (2000). 22E40, 22F30.

Keywords. Symmetric spaces, length spectrum.

Introduction

Let G , G' be connected semisimple Lie groups without compact factors and with trivial center. The motivation of this paper is to give conditions under which a homomorphism between two Zariski dense subgroups of G and G' can be extended to a continuous isomorphism between G and G' . Much study of lattices has been done, yet the study of general co-infinite volume groups is relatively less carried out. Fix a closed Weyl chamber \mathcal{A}^+ included in the Lie algebra of G . The translation vector $v(g)$ of $g \in G$, is, by definition, the unique $a \in \mathcal{A}^+$ such that e^a is conjugate to the hyperbolic part of the Jordan decomposition of g (see section 1). The Euclidean norm of $v(g)$ is denoted $\ell(g)$ and is called the length of g . If X is a symmetric space associated to G , one has: $\ell(g) = \inf_{x \in X} d(x, g(x))$. In the particular case where $G = PSL(n, \mathbb{R})$ and \mathcal{A}^+ is the set of diagonal matrices $\text{diag}(a_1, \dots, a_n)$ with $a_1 \geq \dots \geq a_n$, one has: $v(g) = \text{diag}(\log |\lambda_1|, \dots, \log |\lambda_n|)$ where λ_i is the i^{th} complex eigenvalue of g . Let $\Gamma \subset G$, the limit cone, $\mathcal{L}(\Gamma)$, associated to Γ is, by definition, the smallest closed cone in \mathcal{A}^+ containing all $v(\gamma)$ for $\gamma \in \Gamma$. An important result due to Y. Benoist [1] says that the interior of $\mathcal{L}(\Gamma)$ is not empty, if Γ is a Zariski dense group. The originality of this paper is to explore this property to obtain strong rigidity results in a short and elementary way.

Let us give the main results.

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Theorem A. *Let $\Gamma \subset G, \Gamma' \subset G'$ be Zariski dense subgroups. If φ is a surjective homomorphism between Γ and Γ' such that $\ell(\gamma) = \ell(\varphi(\gamma))$ for any $\gamma \in \Gamma$ then φ can be extended to a continuous isomorphism between G and G' .*

Following the way of A. Parreau [15], we give applications of Theorem A to the space of representations of an abstract group into G .

Theorem A is already known for symmetric spaces of rank 1 ([4], [11]) and their products ([12]). For simple Lie groups it is shown in ([6]). Along this line, Besson, Courtois, Gallot and Hamenstädt ([2], [9]) showed that, if M is a negatively curved locally symmetric compact manifold and N is an arbitrary negatively curved manifold which has the same marked length spectrum with M , then they are isometric. Actually it is conjectured that two negatively curved compact manifolds with the same marked length spectrum are isometric. This conjecture is proved in dimension 2 ([14]).

The following theorem gives a positive answer to a Margulis's question raised during the rigidity conference at Paris in June 1998.

Theorem B. *Suppose $G = G'$ and rank $G \geq 2$. Let Γ, Γ' be Zariski dense subgroups of G . If φ is a surjective homomorphism between Γ and Γ' such that for all $\gamma \in \Gamma$ there exists $k(\gamma) \in \mathbb{R}^*$ such that $v(\varphi(\gamma)) = k(\gamma)v(\gamma)$, then φ can be extended to a continuous automorphism of G .*

We first study the simple case where G and G' are simple. Using a criterion of conjugacy proved in [6] we give a family of conditions (including conditions of Theorems A and B) under which a surjective homomorphism between Zariski dense subgroups can be extended.

1. Benoist's theorem for limit cone

An element g of a real reductive connected linear group can be uniquely written $g = ehu$ where e is elliptic (all the eigenvalues have modulus 1), u is unipotent (u -Id is nilpotent), h is hyperbolic (all the eigenvalues are real positive), and all three commute. This decomposition is called the Jordan decomposition of g . If $G = \text{KAN}$ is any Iwasawa decomposition of a connected semisimple Lie group G , then e is conjugate to an element in K , h is conjugate to an element in A and u is conjugate to an element in N ([1], [7]). Fix a closed Weyl chamber \mathcal{A}^+ in the Lie algebra of G , there exists a unique $a \in \mathcal{A}^+$, called the translation vector of g and denoted $v(g)$, such that h is conjugate to e^a . Geometrically, if X is a symmetric space associated to G , then $\|v(g)\| = \ell(g)$ where $\ell(g) = \inf_{x \in X} d(x, g(x))$ (see [15] for an interpretation of $v(g)$). Let Γ be a subgroup of G , one defines the limit cone of Γ , denoted $\mathcal{L}(\Gamma)$, as the smallest closed cone in \mathcal{A}^+ containing $v(\Gamma)$. If $G = \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ and $\mathcal{A}^+ = \{(r_1 M, r_2 M) / r_1, r_2 \in \mathbb{R}^+\}$ where $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $\mathcal{L}(\Gamma)$ is the closure

of $\{(r\ell(\gamma_1)M, r\ell(\gamma_2)M)/r \in \mathbb{R}^+, (\gamma_1, \gamma_2) \in \Gamma\}$ where $\ell(\gamma_i) = 0$ if γ_i is elliptic or parabolic and $\ell(\gamma_i) > 0$ is the displacement of γ_i if γ_i is hyperbolic. The following result, due to Y. Benoist, plays a key role in this paper.

Theorem 1.1 [1]. *If Γ is a Zariski dense subgroup of G then $\mathcal{L}(\Gamma)$ is convex and has nonempty interior.*

In the particular case where Γ is a Zariski dense subgroup of $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ associated to the diagonal action of two isomorphic Fuchsian groups $\Gamma_1 \xrightarrow{\varphi} \Gamma_2$, this theorem says that $\overline{\left\{ \frac{\ell(\gamma_1)}{\ell(\varphi(\gamma_1))}, \gamma_1 \in \Gamma_1 \right\}}$ is an interval $[a, b] \subset [0, \infty]$ with $a \neq b$. This property was already remarked in the context of rank 1 semisimple groups by M. Burger [4] (see also [5]).

2. Rigidity results for simple groups

In this section one supposes that G and G' are connected, noncompact, **simple** Lie groups with trivial center. Let $\varphi : \Gamma \rightarrow \Gamma'$ be a homomorphism between two subgroups of G and G' . One defines the graph group $\Gamma_\varphi \subset G \times G'$ by $\Gamma_\varphi = \{(\gamma, \varphi(\gamma)) / \gamma \in \Gamma\}$. The following result is proved in [6].

Criterion of conjugacy 2.1 [6]. *Let φ be a surjective homomorphism between two Zariski dense subgroups Γ, Γ' included in connected non compact simple Lie groups, G and G' , with trivial center. The following properties are equivalent:*

- 1) φ can be extended to a continuous isomorphism between G and G'
- 2) Γ_φ is not Zariski dense in $G \times G'$.

This criterion is false if G and G' are not simple. Take for example $G = \mathrm{PSL}(2, \mathbb{R})$ and $G' = G \times G$. Denote \mathcal{A}^+ the closed Weyl chamber of G defined by $\mathcal{A}^+ = \{rM / r \in \mathbb{R}^+\}$ where $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $\varphi : \Gamma_1 \rightarrow \Gamma_2$ be an isomorphism between non conjugate and non elementary Fuchsian groups. The groups Γ_1 and $\Gamma_{1\varphi}$ are Zariski dense subgroups respectively of G and G' . Consider the isomorphism $\Psi : \Gamma_1 \rightarrow \Gamma_{1\varphi}$ defined by $\Psi(\gamma) = (\gamma, \varphi(\gamma))$. The limit cone of the graph group associated to Ψ is included in $\{(rM, rM, sM) / r, s \in \mathbb{R}^+\} \subset \mathcal{A}^+ \times \mathcal{A}^+$ and hence has empty interior. According to Benoist's theorem (section 1), $\Gamma_{1\Psi}$ is not Zariski dense. On the other hand Ψ cannot be extended.

One deduces from the previous criterion the following corollary.

Corollary 2.2. *Let Ad be the adjoint representation. If there exists an algebraic relation satisfied by all $(\mathrm{Ad}(\gamma)), \mathrm{Ad}(\varphi(\gamma))$ with $\gamma \in \Gamma$, then φ can be extended to a continuous isomorphism between G and G' .*

In the case where $G = \mathrm{PSL}(n, \mathbb{R})$, $G' = \mathrm{PSL}(n', \mathbb{R})$ and φ preserves the trace, Corollary 2.2 is proved in [16].

Remark that the condition $\ell(\gamma) = \ell(\varphi(\gamma))$ for each $\gamma \in \Gamma$ is not in general an algebraic condition. But in this case, since $\|v(\gamma)\| = \|v(\varphi(\gamma))\|$ for $\gamma \in \Gamma$, the limit cone of the graph group has empty interior. Applying Benoist's theorem, one concludes that Γ_φ is not Zariski dense and hence that φ can be extended. More generally, one has the following result.

Corollary 2.3. *If the interior of $\mathcal{L}(\Gamma_\varphi)$ is empty then φ can be extended to a continuous isomorphism between G and G' .*

Let us give three different conditions under which Γ_φ is not Zariski dense and hence φ can be extended:

- 1) $\ell(\gamma) = \ell(\varphi(\gamma))$ for any $\gamma \in \Gamma$.
- 2) $v(\gamma)$ and $v(\varphi(\gamma))$ are colinear for any $\gamma \in \Gamma$.
- 3) The largest modulus of the complex eigenvalue of $\mathrm{Ad}(\gamma)$ equals the largest one of $\mathrm{Ad}(\varphi(\gamma))$ for any $\gamma \in \Gamma$.

Conditions 1) and 2) correspond to Theorems A and B when G and G' are simple. Contrary to the conditions 1) and 2), if φ satisfies condition 3) and G and G' are not simple, φ cannot be necessarily extended. For example, fix two isomorphic Schottky groups $\rho : \Gamma \rightarrow \Gamma'$ in $\mathrm{PSL}(2, \mathbb{R})$. Suppose that $\ell(\gamma) > \ell(\rho(\gamma))$ for each $\gamma \in \Gamma$ (see [5] for the construction of such groups). Consider the isomorphism $\varphi : \Gamma \rightarrow \Gamma_\rho$ defined by $\varphi(\gamma) = (\gamma, \rho(\gamma))$. The groups Γ, Γ_ρ are Zariski dense respectively in $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ and the condition 3) is satisfied but φ cannot be extended.

3. Proofs of Theorems A and B

In this section G and G' denote connected semisimple groups with trivial center and without compact factor. Such a group can be decomposed into a product of connected noncompact simple groups with trivial center.

Lemma 3.1. *Let Γ, Γ' be Zariski dense subgroups of G and G' . Suppose that φ is a surjective homomorphism between Γ and Γ' and set $\Gamma_\varphi = \{(\gamma, \varphi(\gamma)) / \gamma \in \Gamma\}$. The projections of the identity component of the Zariski closure of Γ_φ into G and G' are surjective.*

Proof. The Lie algebra \mathcal{G} of G can be decomposed into a direct sum of simple ideals $\mathcal{G} = \mathcal{F}_1 + \cdots + \mathcal{F}_n$. Moreover each ideal of \mathcal{G} is a direct sum of certain \mathcal{F}_i ([10] corollary II.6.3). Let G_i be the connected Lie subgroup in G associated to \mathcal{F}_i . Since G has trivial center, $G = G_1 \times \cdots \times G_n$. Let H be the identity component of the Zariski closure of Γ_φ . Denote p the projection of H into G and T_p its tangent map at identity. The image, \mathcal{F} , of the Lie algebra of H by T_p is a non trivial

subalgebra of \mathcal{G} normalized by Γ . Since Γ is Zariski dense, \mathcal{F} is an ideal and hence $\mathcal{F} = \mathcal{F}_{i_1} + \cdots + \mathcal{F}_{i_k}$, $k \leq n$. This implies that $p(H) = G_{i_1} \times \cdots \times G_{i_k}$. Since the index of H in the Zariski closure of Γ_φ is finite and Γ is Zariski dense, $p(H)$ is also Zariski dense. This proves that $k = n$ and thus that p is surjective. Since φ is surjective, the same argument holds for the projection of H into G' . \square

Proof of Theorem A. Denote H the identity component of the Zariski closure of Γ_φ and \mathcal{H} its Lie algebra. We want to prove that the projection p (resp. p') of H into G (resp. G') is injective. Let us first show that \mathcal{H} is semisimple. Consider its solvable radical $\mathcal{R} \subset \mathcal{H}$. The image of \mathcal{R} by the tangent map Tp of p at identity is normalized by Γ . Since Γ is Zariski dense in G , $Tp(\mathcal{R})$ is a solvable ideal. The semi simplicity of G implies that $Tp(\mathcal{R})$ is trivial. Since φ is surjective, the same argument holds for p' . This shows that \mathcal{R} is trivial. Fix a Cartan decomposition $\mathcal{H} = \mathcal{P}'' + \mathcal{T}''$ of \mathcal{H} , since $G \times G'$ is semisimple, there exists a Cartan decomposition $\mathcal{P} + \mathcal{T}$ of the Lie algebra of $G \times G'$ such that $\mathcal{P}'' \subset \mathcal{P}$ and $\mathcal{T}'' \subset \mathcal{T}$ ([10] VI exercice 8(i)). Choose a Weyl chamber $\mathcal{W} \subset \mathcal{P}''$ since $\mathcal{P}'' \subset \mathcal{P}$ one has $\mathcal{W} \subset \mathcal{A} \times \mathcal{A}'$ where \mathcal{A} and \mathcal{A}' are Cartan subalgebras of the Lie algebra $\mathcal{G}, \mathcal{G}'$ of G and G' . Let us analyze $\text{Ker } p$. This group is normalized by Γ' because φ is surjective. Since Γ' is Zariski dense and the center of G' is trivial, either $\text{Ker } p = \{\text{Id}\}$ or $\text{Ker } p$ is a normal non trivial Lie subgroup of G' . In the last case, denote \mathcal{I} the Lie algebra of the identity component of $\text{Ker } p$. One has $\mathcal{I} = \mathcal{I}'_1 + \cdots + \mathcal{I}'_p$ where \mathcal{I}'_j are noncompact simple ideals of \mathcal{G}' such that $\mathcal{G} = \mathcal{I}'_1 + \cdots + \mathcal{I}'_k$ with $k \geq p$ ([10] corollary II.6.3). It follows that \mathcal{W} contains an element $a = (0, \omega) \in \mathcal{A} \times \mathcal{A}'$ with $\|\omega\| \neq 0$. Since $\Gamma_\varphi \cap H$ is Zariski dense in H , according to Benoist's theorem, the interior of its limit cone, $\mathcal{L}^{\mathcal{W}}(\Gamma_\varphi \cap H)$, relatively to \mathcal{W} , is not empty. Moreover $\mathcal{L}^{\mathcal{W}}(\Gamma_\varphi \cap H)$ is included in $S = \{(u, u') \in \mathcal{A} \times \mathcal{A}' / \|u\| = \|u'\|\}$ because φ preserves the translation length and $\mathcal{L}^{\mathcal{W}}(\Gamma_\varphi \cap H)$ is included in the image of the limit cone of $\Gamma_\varphi \cap H$ relatively to $\mathcal{A}^+ \times \mathcal{A}'^+$ by the Weyl group. Let $b = (u, u')$ an element of the interior of $\mathcal{L}^{\mathcal{W}}(\Gamma_\varphi \cap H) \subset \mathcal{W}$. One can suppose $\|u\| = \|u'\| = 1$. Since the interior of $\mathcal{L}^{\mathcal{W}}(\Gamma_\varphi \cap H)$ in \mathcal{W} is not empty, the intersection of the plane generated by a and b with $\mathcal{L}^{\mathcal{W}}(\Gamma_\varphi \cap H)$ contains an open disc. There is a contradiction with the fact that the intersection of this plane with S is the curve $\{\alpha a + \beta b / 2\alpha\beta \langle u', \omega \rangle + \alpha^2 \|\omega\|^2 = 0\}$. In conclusion p is injective. The same argument holds for p' , because φ is surjective. Applying the lemma 3.1, one obtains that p and p' are bijective. Consider now the projections q (resp. q') of the Zariski closure $\bar{\Gamma}_\varphi^Z$ of Γ_φ into G (resp. G'). The maps q and q' are surjective. Let us prove that they are injective. Take $g \in \text{Ker } q$, for any $h \in H$ one has $q(ghg^{-1}h^{-1}) = \text{Id}$. Since H is normalized by $\bar{\Gamma}_\varphi^Z$ and p is injective, $gh = hg$. Using the fact that p' is surjective one obtains $p'(g)g' = g'p'(g)$ for any $g' \in G'$. Because the center of G' is trivial, $g = \text{Id}$. The same argument also holds for p' . Consider the map $f = p' \circ p^{-1}$, it is a continuous isomorphism between G and G' whose restriction to Γ coincides with φ . \square

Proof of Theorem B. The proof is similar to the previous one. Let us just adapt the end of the proof of Theorem A, when we suppose that $\text{Ker } p$ is nontrivial. Under this assumption one obtains that \mathcal{W} contains an element $a = (0, \omega) \in \mathcal{A} \times \mathcal{A}$ with $\omega \neq 0$. Since $v(\gamma) = k(\gamma)v(\varphi(\gamma))$ for each $\gamma \in \Gamma$, the limit cone $\mathcal{L}^{\mathcal{A}^+ \times \mathcal{A}^+}(\Gamma_\varphi \cap H)$ is included in $T = \{(u, u') \in \mathcal{A}^+ \times \mathcal{A}^+ / u \text{ and } u' \text{ are colinear}\}$ and hence $\mathcal{L}^{\mathcal{W}}(\Gamma_\varphi \cap H)$ is included in $\bigcup_{g \in \text{Weyl}} gT$ where Weyl is the Weyl group of $\mathcal{A} \times \mathcal{A}$. The interior of $\mathcal{L}^{\mathcal{W}}(\Gamma_\varphi \cap H)$ in \mathcal{W} is not empty according to Benoist's theorem. It follows that for some $g \in \text{Weyl}$, the interior I of $g(T)$ is not empty in \mathcal{W} . Let $b = (u, u') \in I$. Since $\text{rank } G \geq 2$ one can assume that u' is not colinear to u . The intersection of the plane P generated by a and b with I contains an open disc. There is a contradiction with the fact that the intersection of T with $g^{-1}(P)$ is a line. \square

4. Applications of Theorem A to the space of representations

Fix a connected semisimple Lie group G without compact factor and with trivial center, and a symmetric space X associated to G . A subgroup of G is said parabolic if it fix a point of the geometric boundary, ∂X , of X .

Proposition 4.1. *Let Γ be a nonparabolic subgroup of G and H the identity component of its identity component. If $H \neq G$ then H fix a totally geodesic submanifold $Y \subsetneq X$.*

Proof. We thank P. Eberlein for helpful arguments.

The group H is reductive or parabolic ([3] corollaire 3.3). The last case cannot happens because H is normalized by Γ which does not fix any point in ∂X . Let $H = ST$ be the Levi decomposition of H where S is a connected semisimple group and T is a torus, corresponding to the identity component of the center of H . If $T \neq \text{Id}$ there exists a flat totally geodesic submanifold $T \subset X$ such that T leaves F invariant and F/T is compact ([8]). Let C be the union of all totally geodesic submanifolds which are parallel to F . Then C is invariant under H and is isometric to $F \times N$ for some closed convex subset N of X ([7] proposition 1.6.7). The set C is a totally geodesic submanifold possible with boundary. Let Y be a complete totally geodesic submanifold of X with $\dim Y = \dim C$. Since H leaves C invariant and C contains an open subset of Y , the group H leaves Y invariant. Remark that $Y \neq X$, because Y contains an Euclidean factor. If $T = \{\text{Id}\}$ then H is semisimple, and there exists $x \in X$ such that Hx is a totally geodesic submanifold ([13] lemma 7.21). By the assumption $H \neq G$ hence $Hx \neq X$. \square

Let Γ be an abstract group and $\rho : \Gamma \rightarrow G$ be a faithful representation. One always supposes that the Zariski closure, H_ρ , of $\rho(\Gamma)$ is connected and that the representation ρ is nonparabolic (i.e. $\rho(\Gamma)$ is nonparabolic). In this case H_ρ is reductive (proof of proposition 4.1). Let $H_\rho = ST$ be the Levi decomposition

of H_ρ . The representation ρ is noncompact if S is a semisimple group without compact factor and with trivial center. Under this assumption H_ρ stabilizes a totally geodesic submanifold of X isometric to $N \times F$ where N is a symmetric space on which S acts transitively and F is a flat on which T acts by translation with compact quotient (proof of the proposition 4.1). Two faithful, nonparabolic and noncompact representations ρ and ρ' of Γ are equivalent if there exists a isometry f between $N \times F$ and $N' \times F'$ such that $f \circ \rho(\gamma) = \rho'(\gamma) \circ f$ for any $\gamma \in \Gamma$. If F and F' are empty, then ρ and ρ' are equivalent if and only if $\rho' \circ \rho^{-1}$ can be extended to a continuous isomorphism between S and S' ([7] proposition 3.9.11). Denote R_{fnpnc}/\sim the space of faithful nonparabolic, noncompact representations of Γ into G , up to the equivalence relation. The following result is an application of Theorem A to the context of representations.

Proposition 4.2. *The map $L: R_{fnpnc}/\sim \rightarrow \mathbb{R}^\Gamma$ defined by $L([\rho])(\gamma) = \ell(\rho(\gamma))$ is injective.*

Proof. Let $\rho_1, \rho_2 \in R_{fnpnc}$. Suppose $L(\rho_1) = L(\rho_2)$. For $i = 1, 2$ set $\Gamma_i = \rho_i(\Gamma)$, $H_i = H\rho_i$ and $H_i = S_i T_i$.

a) Suppose $S_1 = S_2 = \{e\}$, then T_i acts by translation on the flat $(F_i, \langle \cdot \rangle_i)$ and F_i/T_i is compact. Let us identify $\rho_i(\gamma)$ with its translation vector. Choose a basis, $\rho_1(\gamma_1), \dots, \rho_1(\gamma_n)$ of F_1 , such a basis exists because Γ_1 is Zariski dense in T_1 . For $\gamma \in \Gamma$, write $\rho_1(\gamma) = \sum_{i=1}^n a_i \rho_1(\gamma_i)$ and $\rho_2(\gamma) = \sum_{i=1}^n b_i \rho_2(\gamma_i) + \omega$ where ω is orthogonal to each $\rho_2(\gamma_i)$. Since $\|\rho_1(\gamma)\| = \|\rho_2(\gamma)\|$, one has $\langle \rho_1(\gamma), \rho_1(\gamma') \rangle_1 = \langle \rho_2(\gamma), \rho_2(\gamma') \rangle_2$ for any $\gamma, \gamma' \in \Gamma$. Put $c_{ij} = \langle \rho_1(\gamma_i), \rho_1(\gamma_j) \rangle_1 = \langle \rho_2(\gamma_i), \rho_2(\gamma_j) \rangle_2$. One has $\langle \rho_1(\gamma), \rho_1(\gamma_j) \rangle_1 = \sum_{i=1}^n a_i c_{ij}$ and $\langle \rho_2(\gamma), \rho_2(\gamma_j) \rangle_1 = \sum_{i=1}^n b_i c_{ij}$ hence $\sum_{i=1}^n (a_i - b_i) c_{ij} = 0$ for any $1 \leq j \leq n$. This proves that $a_i = b_i$. Moreover $\|\rho_1(\gamma)\| = \|\rho_2(\gamma)\|$ hence $\omega = 0$. One thus obtains $\rho_2(\gamma) = \sum_{i=1}^n a_i \rho_2(\gamma_i)$ and $\dim F_2 = n$ because Γ_2 is Zariski dense in T_2 . The linear map $f: F_1 \rightarrow F_2$ defined by $f(\rho_1(\gamma_i)) = \rho_2(\gamma_i)$ is an isometry satisfying $f \circ \rho_1(\gamma) = \rho_2(\gamma) \circ f$, hence $[\rho_1] = [\rho_2]$.

b) Suppose $S_1 \neq \{e\}$, then $S_2 \neq \{e\}$. Decompose S_i into a product of non-compact simple factors with trivial center $S_i = S_{i1} \times \dots \times S_{ik_i}$ and denote p_{is} the projection of S_i into S_{is} . Since Γ_i is Zariski dense in $S_i \times T_i$ then $p_{is}(\Gamma)$ is Zariski dense in S_{is} . Set $D = [\Gamma, \Gamma]$ and $D_i = \rho_i(D)$. The group D_i is normalized by Γ_i and is included in S_i , hence one can suppose that the Zariski closure of D_i equals $S_{i1} \times \dots \times S_{in_i}$ with $n_i \leq k_i$. Moreover $n_i = k_i$ because $p_{is}(D_i)$ is normalized by $p_{is}(\Gamma)$ which is Zariski dense in S_{is} and the center of S_{is} is trivial. In conclusion D_i is Zariski dense in S_i . By assumption $\ell(\rho_1(d)) = \ell(\rho_2(d))$ for any $d \in D$. One deduces from Theorem A that the restriction of $\rho_2 \circ \rho_1^{-1}$ to D_1 can be extended to a continuous isomorphism φ between S_1 and S_2 . Up to φ , one can suppose $S_1 = S_2$ and $\rho_1(d) = \rho_2(d)$ for any $d \in D$. Let $\gamma \in \Gamma$, since $\rho_1(\gamma d \gamma^{-1}) = \rho_2(\gamma d \gamma^{-1})$ and $\rho_1(d) = \rho_2(d)$, the projection of $\rho_2^{-1}(\gamma) \rho_1(\gamma)$ into S_1 commutes with all $\rho_1(d)$. Since D_1 is Zariski dense and the center of S_1 is trivial, the projection of $\rho_2^{-1}(\gamma) \rho_1(\gamma)$ into S_1 is trivial. Consider now the projection p_i of

Γ_i into T_i . One has $\ell(p_1 \circ \rho_1(\gamma)) = \ell(p_2 \circ \rho_2(\gamma))$, moreover $p_i(\Gamma_i)$ is Zariski dense in T_i . Using arguments developped in a), one obtains the existence of a isometry $f : F_1 \rightarrow F_2$ such that $f \circ (p_1 \circ \rho_1(\gamma)) = p_2 \circ \rho_2(\gamma) \circ f$, hence $[\rho_1] = [\rho_2]$. \square

The following part is inspired by the section 5 of A. Parreau's thesis ([15]). Let us consider the particular case where Γ is an infinite group of finite type. Fix a finite set, S , of generators. One associates to a representation $\rho : \Gamma \rightarrow G$ its minimal displacement, $\lambda(\rho) = \inf_{x \in X} (\sup_{s \in S} d(x, \rho(s)(x)))$. If $\lambda(\rho) = 0$ there exists a sequence $(x_n)_{n \geq 1}$ in X such that $\lim_n d(x_n, \rho(s)(x_n)) = 0$ for any $s \in S$. Up to a subsequence one can suppose that $(x_n)_{n \geq 1}$ converges in $X \cup \partial X$. If $\lim_n x_n = x \in X$ then $\rho(s)(x) = x$ for any $s \in S$ and hence $\rho(\Gamma)$ belongs to a compact subgroup. Otherwise $\lim_n x_n = \xi \in \partial X$ and $\rho(s)(\xi) = \xi$ for any $s \in S$. In this case ρ is parabolic. In conclusion, if $\rho \in R_{fnpnc}$ then $\lambda(\rho) > 0$. Let us consider the map $\frac{V}{\lambda} : R_{fnpnc}/\sim \rightarrow \mathbb{R}^\Gamma$ defined by $L([\rho])(\gamma) = \frac{\ell(\rho(\gamma))}{\lambda(\rho)}$. This map is continuous ([15] propositions V.2.3 and V.3.8) and its image is included in a compact set ([15] proposition V.4.1). One deduces from these properties and from the proposition 4.2 the following result.

Corollary 4.3. *The map $\frac{L}{\lambda} : R_{fnpnc}/\sim \rightarrow \mathbb{R}^\Gamma$ is injective, continuous and its image is included in a compact set.*

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Françoise Dal'Bo

Université de Rennes 1

Institut de Recherche Mathématique de Rennes

Campus de Beaulieu

35042 Rennes cedex

France

e-mail: dalbo@univ-rennes1.fr

Inkang KIM

Department of Mathematics

Seoul National University

151-741 Seoul

Korea

e-mail:inkang@math.snu.ac.kr

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