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## Some properties of locally conformal symplectic structures

Augustin Banyaga

**Abstract.** We show that the  $d_\omega$ -cohomology is isomorphic to a conformally invariant usual de Rham cohomology of an appropriate cover. We also prove a Moser theorem for locally conformal symplectic (lcs) forms. We point out a connection between lcs geometry and contact geometry. Finally, we show the connections between first kind, second kind, essential, inessential, local, and global conformal symplectic structures through several invariants.

**Mathematics Subject Classification (2000).** 53C12; 53C15.

**Keywords.** Locally conformal symplectic structures, Lee form, extended Lee homomorphism, de Rham invariant, Gelfand–Fuchs invariant, Lee invariant, conformal invariants, essential/inessential conformal structures, the  $d_\omega$  cohomology, the  $c\mathcal{A}$ -cohomology.

### 1. Preliminaries

A *locally conformal symplectic* (lcs) form on a smooth manifold  $M$  is a non-degenerate 2-form  $\Omega$  such that there exists an open cover  $\mathcal{U} = (U_i)$  and smooth positive functions  $\lambda_i$  on  $U_i$  such that

$$\Omega_i = \lambda_i(\Omega|_{U_i})$$

is a symplectic form on  $U_i$ . If for all  $i$ ,  $\lambda_i = 1$ , the form  $\Omega$  is a symplectic form. Lee [15] observed that the 1-forms  $\{d(\ln \lambda_i)\}$  fit together into a closed 1-form  $\omega$  such that

$$d\Omega = -\omega \wedge \Omega. \tag{1}$$

Such 1-form is uniquely determined by  $\Omega$  and is called the Lee form of  $\Omega$ .

Conversely, if a non-degenerate 2-form  $\Omega$  satisfies (1), and  $\mathcal{U} = (U)_i$  is an open cover with contractible open sets, then  $\omega|_{U_i} = d\ln \lambda_i$ , for some positive function  $\lambda_i$  on  $U_i$  and  $\lambda_i \Omega|_{U_i}$  is symplectic.

Two lcs forms  $\Omega$ ,  $\Omega'$  on a smooth manifold  $M$  are said to be (conformally) equivalent if  $\Omega' = f\Omega$ , for some positive function  $f$  on  $M$ .

A locally conformal symplectic (lcs) structure  $\mathcal{S}$  on a smooth manifold  $M$  is an equivalence class of lcs forms.

The couple  $(M, \mathcal{S})$  is called a *lcs manifold*. If  $\Omega$  is a representative of  $\mathcal{S}$ , we write  $\Omega \in \mathcal{S}$ . If  $\omega = 0$  in the definition above, then  $\Omega$  is a symplectic form. In that case the lcs structure  $\mathcal{S}$  is said to be a *global conformal symplectic* (gcs) structure and we write  $\mathcal{S} = \mathcal{O}$ .

Let  $(M, \mathcal{S})$  be a lcs manifold, and let  $\Omega \in \mathcal{S}$  and  $\omega$  its Lee form. If  $\Omega' = \lambda\Omega$  for some positive function  $\lambda$ , then an immediate calculation shows that the Lee form of  $\Omega'$  is  $\omega' = \omega - d\ln(\lambda)$ .

Hence the cohomology class  $[\omega] \in H^1(M, \mathbb{R})$  is an invariant  $\mathcal{L}_{\mathcal{S}}$  of  $\mathcal{S}$ , we call the Lee class of  $\mathcal{S}$ . Clearly,  $\mathcal{S} = \mathcal{O}$  iff  $\mathcal{L}_{\mathcal{S}} = 0$ .

Locally conformal symplectic forms were introduced by Lee [15], and have been extensively studied by Vaisman [18], [19]. The first properties of their automorphism groups were established by Lefebvre [16].

We will assume that all manifolds considered are connected, but not necessarily compact, and have dimension at least 4. (In dimension 2, a lcs form is simply a volume-form, and the corresponding structure is an orientation.)

For any closed 1-form  $\omega$  on a smooth manifold  $M$ , the operator  $d_{\omega}$  which assigns to a  $p$ -form  $\gamma$  the  $(p+1)$ -form

$$d_{\omega}\gamma = d\gamma + \omega \wedge \gamma$$

is a coboundary operator, i.e.  $d_{\omega} \circ d_{\omega} = 0$ .

The cohomology of differential forms with this coboundary operator will be denoted by  $H_{\omega}^*(M)$  and will be called the  $d_{\omega}$ -cohomology. For more information on this cohomology, see [11] or [19].

A lcs form  $\Omega$  is precisely a non-degenerate  $d_{\omega}$  closed 2-form (where  $\omega$  is the Lee form).

This cohomology is “almost” an invariant of the lcs structure  $\mathcal{S} = [\Omega]$ : given  $\Omega' \in \mathcal{S}$ , there is an isomorphism between  $H_{\omega}(M)$  and  $H_{\omega'}(M)$ , ( $\omega'$  the Lee form of  $\Omega'$ ), depending on the choice of  $\lambda$  such that  $\omega' = \omega - d\ln\lambda$ . More precisely the isomorphism is given by  $\alpha \mapsto \lambda\alpha$ .

In section 3, we show that the  $c\mathcal{A}$  cohomology constructed in [5], [6], is isomorphic to  $H_{\omega}(M)$ . This shows that the  $d_{\omega}$  cohomology (which is a sort of twisted de Rham cohomology of  $M$ ) is a conformally invariant usual de Rham cohomology of an appropriate cover of  $M$ .

Let  $\text{Diff}_{\mathcal{S}}(M)$  be the group of all automorphisms of a lcs structure  $\mathcal{S}$  on a smooth manifold  $M$ . It is clear that for any representative  $\Omega \in \mathcal{S}$ , then  $\text{Diff}_{\mathcal{S}}(M)$  is the set of all diffeomorphisms  $\phi$  of  $M$  such that  $\phi^*\Omega = f_{\phi}\Omega$ , where  $f_{\phi}$  is a nowhere zero (positive) smooth function on  $M$ .

We also may choose (or fix) an underlying  $\Omega \in \mathcal{S}$ , and consider the group  $G_{\Omega}(M)$  of diffeomorphisms of  $M$  which preserve the form  $\Omega$ . This is a non-invariant subgroup of  $\text{Diff}_{\mathcal{S}}(M)$ .

The Lie algebra  $\mathcal{X}_{\mathcal{S}}(M)$  of infinitesimal automorphisms of  $\mathcal{S}$ , consists of vector fields  $X$  on  $M$  such that  $L_X\Omega = (u_{\Omega}(X))\Omega$ , where  $u_{\Omega}(X)$  is a smooth function on  $M$ . Here  $L_X$  stands for the Lie derivative in the direction  $X$ . We denote  $\mathcal{X}_{\mathcal{S}}(M)_c$

the subalgebra of compact supported automorphisms. We will also consider the subalgebra  $\mathcal{X}_\Omega(M)$  of  $\mathcal{X}_S(M)$  consisting of vector fields  $X$  such that  $L_X\Omega = 0$ .

**Definition.** A lcs form  $\Omega$  on  $M$  is said to be of the first kind if there exists  $X \in \mathcal{X}_\Omega(M)$ , with  $\omega(X) \neq 0$ , where  $\omega$  is the corresponding Lee form. Otherwise it is said to be of the second kind [18].

A lcs structure  $S$  on  $M$  is said to be of the first kind if there is a representative  $\Omega \in S$  of the first kind. The lcs structure  $S$  is said to be of the second kind otherwise.

**Warning.** Vaisman [18] observed that a first kind lcs structure admits representatives which are second kind lcs forms.

For  $X \in \mathcal{X}_\Omega(M)$ , and  $M$  connected,  $\omega(X)$ , is a constant number since:

$$0 = dL_X\Omega = L_Xd\Omega = L_X(-\omega \wedge \Omega) = -((L_X\omega) \wedge \Omega + \omega \wedge L_X\Omega) = -(di(X)\omega) \wedge \Omega$$

and  $\Omega$  is non-degenerate.

Hence if  $\Omega$  is a first kind lcs form with Lee form  $\omega$ , the condition:

$$\text{There is } X \in \mathcal{X}_\Omega(M), \text{ with } \omega(X) \neq 0$$

is equivalent to saying that there a 1-form  $\theta$  such that

$$\Omega = d\theta + \omega \wedge \theta$$

Indeed just normalize  $X$  as above so that  $\omega(X) = 1$  and set  $\theta = i(X)\Omega$ . First kind lcs forms are  $d_\omega$  exact.

## 2. Examples

We describe here a few examples of lcs forms. The reader can consult the book [9] for more examples.

### 2.1. Examples connected with Contact Geometry

A contact form  $\alpha$  on a  $(2n+1)$  dimensional manifold  $N$  is a 1-form  $\alpha$  such that  $\alpha \wedge (d\alpha)^n$  is everywhere non-zero. Two contact forms  $\alpha$  and  $\alpha'$  are equivalent if there is a smooth positive function  $f$  on  $N$  such that  $\alpha' = f\alpha$ . The contact structure  $\mathcal{C}(\alpha)$ , determined by  $\alpha$  is the equivalence class of  $\alpha$ .

Consider the cartesian product  $M = N \times S^1$ , and the projections  $p_1 : M \rightarrow N$ ,  $p_2 : M \rightarrow S^1$ . Let  $\beta$  be the canonical 1-form on  $S^1$  with integral 1. If we set  $\theta = p_1^*\alpha$  and  $\omega = p_2^*\beta$ , then

$$\Omega = d\theta + \omega \wedge \theta$$

is non-degenerate and  $d\Omega = -\omega \wedge d\theta = -\omega \wedge (\Omega - \omega \wedge \theta) = -\omega \wedge \Omega + \omega \wedge \omega \wedge \theta = -\omega \wedge \Omega$ . Hence the conformal class of  $\Omega$  is a lcs structure on  $M$ , we denote  $\mathcal{S}(\alpha)$ . This structure is of the first kind.

The following result will be proved in section 4.

**Theorem 1.** *The lcs structure  $\mathcal{S}(\alpha)$  depends only on the contact structure  $\mathcal{C}(\alpha)$ . In fact there is a well defined mapping from the group  $\text{Diff}_{\mathcal{C}(\alpha)}(M)$  of automorphisms of the contact structure  $\mathcal{C}(\alpha)$  (the group of contact diffeomorphisms of  $(M, \alpha)$ ) to the group  $\text{Diff}_{\mathcal{S}(\alpha)}(M \times S^1)$ .*

## 2.2. Deformations of lcs structures

If we add a 2-form  $\eta_\epsilon$   $C^0$  close to 0 to a lcs form  $\Omega$ , the resulting form  $\Omega_\epsilon = \Omega + \eta_\epsilon$  is again non-degenerate. An immediate calculation gives:

$$d\Omega_\epsilon = -\omega \wedge \Omega_\epsilon + (d\eta_\epsilon + \omega \wedge \eta_\epsilon) = -\omega \wedge \Omega_\epsilon + d_\omega \eta_\epsilon.$$

Hence if  $\eta_\epsilon$  is  $d_\omega$  closed, then  $\Omega_\epsilon$  is a lcs form with  $\omega$  as Lee form. For instance take  $\eta_\epsilon = d_\omega \gamma_\epsilon$  where  $\gamma_\epsilon$  is  $C^1$  close to zero.

To construct general deformations of a lcs form  $\Omega$ , with Lee form  $\omega$ , we may look for 2-forms  $\eta_\epsilon$   $C^0$  closed to zero, and closed 1-forms  $\rho$  (not necessarily small) such that  $d\Omega_\epsilon = -(\omega + \rho) \wedge \Omega_\epsilon$ . In that connection, we note that if  $\mathcal{L}_{cs}(M)$  is the set of all lcs forms on a smooth manifold  $M$ , and  $\mathcal{F}^*(M)$  the space of differential forms, both with the  $C^\infty$  topology,  $\mathcal{L}_{cs}(M)$  is not an open subset of  $\mathcal{F}^*(M)$ .

Note that if the lcs form  $\Omega$  is of first kind and we add to it a non- $d_\omega$ -exact form, the resulting lcs form is not  $d_\omega$ -exact, hence of the second kind.

We have the following fact:

**Theorem 2.** *Let  $(M, \mathcal{S})$  be a compact lcs manifold, and let  $\Omega \in \mathcal{S}$  be a representative, with Lee form  $\omega$ . Then for any  $d_\omega$  exact 2-form  $\eta_\epsilon$ ,  $C^0$  close to zero, the lcs form  $\Omega_\epsilon = \Omega + \eta_\epsilon$  represents a lcs structure equivalent to  $\mathcal{S}$ .*

Hence the non-trivial deformations of lcs structures are parametrized by elements of the second cohomology group  $H_\omega^2(M)$ .

## 2.3. Lcs on cotangent bundles [12]

Let  $M = T^*(N)$  be the total space of the cotangent bundle  $\pi : T^*(N) \rightarrow N$  over a smooth manifold  $N$ . Let  $\Lambda_N$  be the Liouville 1-form on  $M$  and  $\alpha$  a closed 1-form on  $N$ , then

$$\Omega_\alpha = d_\omega \Lambda_N$$

where  $\omega = \pi^* \alpha$ , is a lcs form on  $M$ . The conformal structure defined by this lcs form depends only on the cohomology class of  $\alpha$ .

### 3. The $c\mathcal{A}$ -cohomology and the $d_\omega$ -cohomology

For any closed 1-form  $\omega$  on a smooth manifold  $M$ , the operator  $d_\omega$  which assigns to a  $p$ -form  $\gamma$  the  $(p+1)$ -form

$$d_\omega \gamma = d\gamma + \omega \wedge \gamma$$

is a coboundary operator, i.e.  $d_\omega \circ d_\omega = 0$ .

The cohomology of differential forms with this coboundary operator will be denoted by  $H_\omega^*(M)$  and will be called the  $d_\omega$ -cohomology. For more information on this cohomology, see [11] or [19]. For instance, it was proved in [19] that the groups  $H_\omega^p(M)$  are isomorphic to the cohomology groups of  $M$  with coefficients in the sheaf  $\mathcal{F}_\omega(M)$  of germs of smooth functions  $f$  on  $M$  such that  $d_\omega f = 0$ .

In this section, we give another interpretation of the  $d_\omega$  cohomology.

One associates with a closed 1-form  $\omega$  on a smooth manifold  $M$  the minimum regular cover  $\pi : \tilde{M} \rightarrow M$  over which the 1-form  $\omega$  pulls back to an exact 1-form. The manifold  $\tilde{M}$  is a connected component of the sheaf of germs of smooth functions  $f$  on  $M$  such that  $\omega = df$  [10].

Let  $\lambda : \tilde{M} \rightarrow \mathbb{R}$  be a positive function on  $\tilde{M}$  such that

$$\pi^* \omega = d(\ln \lambda).$$

It is well known that the group  $\mathcal{A}$  of automorphisms of the covering  $\tilde{M}$ , is isomorphic to the group of periods of  $\omega$  [10]. We will need the following:

**Lemma 1** [6]. *For any  $\tau \in \mathcal{A}$ , the function*

$$(\lambda \circ \tau) / \lambda$$

*is a constant, we denote  $c_\tau$ , independent of the choice of  $\lambda$  and*

$$\tau \mapsto c_\tau$$

*is a group homomorphism  $c$  from  $\mathcal{A}$  to the multiplicative group  $\mathbb{R}^+$  of positive real numbers.*

For the convenience of the reader, we give here the proof [6].

*Proof.* Clearly if  $\lambda' = a\lambda$  for some constant  $a$ ,  $\lambda' \circ \tau / \lambda' = \lambda \circ \tau / \lambda$ .

For any  $\tau \in \mathcal{A}$ , we have:

$$d(\ln(\lambda \circ \tau) - \ln \lambda) = \tau^* \pi^* \omega - \pi^* \omega = (\pi \tau)^* \omega - \pi^* \omega = \pi^* \omega - \pi^* \omega = 0.$$

Hence  $\ln(\lambda \circ \tau / \lambda) = K$ , a constant and  $\lambda \circ \tau / \lambda = e^K = c_\tau$ .

If  $\tau, \tau' \in \mathcal{A}$ :

$$\begin{aligned} c_{\tau\tau'} &= (\lambda \circ \tau\tau')/\lambda = ((\lambda \circ (\tau\tau'))/(\lambda \circ \tau')).(\lambda \circ \tau')/\lambda \\ &= ((\lambda \circ \tau)/\lambda) \circ \tau'.((\lambda \circ \tau')/\lambda) = ((\lambda \circ \tau)/\lambda).((\lambda \circ \tau')/\lambda) = c_\tau \cdot c_{\tau'}. \quad \square \end{aligned}$$

The set  $\mathcal{F}_{c\mathcal{A}}^*(M)$  of all differential forms  $\alpha$  on  $\tilde{M}$  such that  $\tau^* \alpha = c_\tau \alpha$  for all  $\tau \in \mathcal{A}$ , is a subcomplex of the de Rham complex of  $\tilde{M}$ . We denote its cohomology by  $H_{c\mathcal{A}}^*(M)$  and call it the conformally  $\mathcal{A}$ -invariant cohomology of  $M$ . Clearly, if the cohomology class of  $\omega$  is trivial, then  $H_{c\mathcal{A}}^*(M)$  coincides with the de Rham cohomology of  $M$ .

**Remark 1.** For any differential form  $\alpha$  on  $M$ , then  $U_\alpha = \lambda \pi^* \alpha \in \mathcal{F}_{c\mathcal{A}}^*(M)$

Indeed, for any  $\tau \in \mathcal{A}$ ,

$$\tau^* U_\alpha = \lambda \circ \tau \cdot \tau^* \pi^* \alpha = \frac{\lambda \circ \tau}{\lambda} \cdot \lambda \cdot (\pi \circ \tau)^* \alpha = c_\tau (\lambda \pi^* \alpha) = c_\tau U_\alpha. \quad \square$$

**Lemma 2.** For any differential form,  $\alpha$ ,  $d_\omega \alpha = 0$  if and only if  $d(\lambda \pi^* \alpha) = 0$ .

*Proof.* Suppose  $d_\omega \alpha = 0$ . Then:  $d(\lambda \pi^* \alpha) = d\lambda \wedge \pi^* \alpha + \lambda \pi^* (-\omega \wedge \alpha) = d\lambda \wedge \pi^* \alpha - \lambda d(\ln \lambda) \wedge \pi^* \alpha = 0$ .

Suppose now  $d(\lambda \pi^* \alpha) = 0$ , and compute:

$$\lambda \pi^* (d_\omega \alpha) = \lambda \pi^* d\alpha + \lambda \pi^* \omega \wedge \pi^* \alpha = \lambda \pi^* d\alpha + \lambda d(\ln \lambda) \wedge \pi^* \alpha = d(\lambda \pi^* \alpha) = 0.$$

Since  $\lambda$  is a positive function and  $\pi$  is a local diffeomorphism,  $d_\omega \alpha = 0$ .  $\square$

**Theorem 3.**  $H_{c\mathcal{A}}^*(M)$  is (non-canonically) isomorphic with  $H_\omega^*(M)$

*Proof.* The natural homomorphism

$$H_\omega^*(M) \rightarrow H_{c\mathcal{A}}^*(M) \quad [\alpha] \mapsto [\lambda \pi^* \alpha]$$

is onto: indeed, let  $\beta$  be a form such that  $d\beta = 0$  and  $\tau^* \beta = c_\tau \beta$  for all  $\tau \in \mathcal{A}$ . Then:

$$\tau^* (\beta/\lambda) = \tau^* \beta/\lambda \circ \tau = (c_\tau \cdot \beta/\lambda) \cdot (\lambda/\lambda \circ \tau) = \beta/\lambda$$

for all  $\tau \in \mathcal{A}$ . Hence  $\beta/\lambda$  is basic, i.e. there is a form  $\alpha$  on  $M$  such that  $\beta/\lambda = \pi^* \alpha$ . Since  $\beta = \lambda \pi^* \alpha$  is closed,  $\alpha$  is  $d_\omega$  closed, by Lemma 2.

It is also one-to-one: suppose  $d_\omega \alpha = 0$  and  $\lambda \pi^* \alpha = d\rho$  with  $\tau^* \rho = c_\tau \rho$  for all  $\tau \in \mathcal{A}$ . Then: rewriting the equations above with  $\beta$  replaced by  $\rho$ , we see that  $\rho/\lambda$  is basic, i.e. there is a form  $\gamma$  on  $M$  such that  $\rho/\lambda = \pi^* \gamma$ .

Let us now compute:  $\pi^* (d_\omega \gamma) = \pi^* (d\gamma + \omega \wedge \gamma) = d(\rho/\lambda) + d\ln \lambda \wedge \rho/\lambda = d\rho/\lambda - d\lambda/(\lambda)^2 \wedge \rho + (d\lambda/\lambda) \wedge \rho/\lambda = d\rho/\lambda = \pi^* \alpha$ .

Since  $\pi$  is a covering map,  $\alpha = d_\omega \gamma$ .  $\square$

In [5], [6], we had already observed that  $H_{c\mathcal{A}}(M)$  is a quotient of  $H_\omega(M)$ . We deduce the following well known fact ([11])

**Corollary.** *If  $\omega$  is a non-exact 1-form on a smooth manifold  $M$ ,  $H_\omega^0(M) = 0$ .*

*Proof.* An element of  $H_\omega^0(M) \approx H_{c\mathcal{A}}^0(M)$  is represented by a constant  $K$  such that  $K \circ \tau = K = c_\tau K$  for all  $\tau \in \mathcal{A}$ . Since  $\omega$  is not exact, there is a  $\tau \in \mathcal{A}$  with  $c_\tau \neq 1$ . Hence  $K = 0$ .  $\square$

Let  $(M, \mathcal{S})$  be a lcs manifold,  $\Omega \in \mathcal{S}$  a representative, with Lee form  $\omega$ . Let  $\pi : \tilde{M} \rightarrow M$  be the minimum regular covering of  $M$  associated with the 1-form  $\omega$  and let  $\lambda : \tilde{M} \rightarrow \mathbb{R}$  be a positive function on  $\tilde{M}$  such that

$$\pi^* \omega = d(\ln \lambda).$$

Then  $\tilde{\Omega} = \lambda(\pi^* \Omega)$  is a symplectic form on  $\tilde{M}$  and its conformal class  $\tilde{\mathcal{S}}$  is independent of the choice of  $\Omega \in \mathcal{S}$  and of  $\lambda$ .

Note that given a lcs  $\Omega \in \mathcal{S}$ , with Lee form  $\omega$ , the cohomology classes  $[\Omega] \in H_\omega^2(M)$  and  $[\lambda \pi^* \Omega] \in H_{c\mathcal{A}}^2(M)$  are not invariants of the lcs structure  $\mathcal{S}$ .

The cohomology groups  $H_{c\mathcal{A}}^*(M)$  and the  $d_\omega$  cohomology are “almost” invariants of the lcs structure: since if  $\omega$  and  $\omega' = \omega - d \ln \lambda$  are two Lee forms, then  $H_\omega(M)$  is isomorphic to  $H_{\omega'}(M)$ , by the isomorphism  $\alpha \rightarrow \lambda \alpha$ , which unfortunately depends on the choice of  $\lambda$ . Two such  $\lambda$ ’s differ by a constant.

#### 4. Equivalence of lcs structures

We have the following Moser type result:

**Theorem 4.** *Let  $\Omega_t$  be a smooth family of lcs forms on a compact manifold  $M$ . Suppose that for all  $t$ , the Lee form of  $\Omega_t$  is the same 1-form  $\omega$  and that  $\Lambda_t = \Omega_t - \Omega_0$  is  $d_\omega$ -exact, then there exist a smooth family of diffeomorphisms  $\phi_t$  with  $\phi_0 = id$  and a smooth family of functions  $f_t$  such that  $\phi_t^* \Omega_t = f_t \Omega_0$ .*

**Remark 2.** If the smooth family of lcs forms  $\Omega_t$  has a smooth family  $\omega_t$  of corresponding Lee forms, and we write  $\omega_t = \omega_0 + d \ln u_t$  for some positive functions  $u_t$  (see the beginning of the proof of Theorem 5), then  $\Omega'_t = u_t \Omega_t$  has  $\omega_0$  as Lee form for all  $t$ . Hence assuming  $\Lambda'_t = \Omega'_t - \Omega'_0$  to be  $d_{\omega_0}$ -exact, yields that  $\Omega_t$  represent equivalent lcs structures for all  $t$ .

*Proof.* By assumption,  $\partial/\partial t(\Omega_t)$  is  $d_\omega$  exact for all  $t$ . A result of [12], (Lemma 1.9) asserts that there exists a smooth family of 1-forms  $\eta_t$  such that

$$\partial/\partial t(\Omega_t) = d_\omega \eta_t.$$

The argument used to find a smooth lifting of  $d_\omega$ -coboundaries is the same as in [1], (Lemma II.2.2), which is an application of Grothendieck's theory of nuclear topological vector spaces. This replaces the Hodge-de Rham theorem in Moser's theorem for symplectic forms [17].

Let  $\tilde{\Omega}_t = \lambda\pi^*\Omega_t$ , where  $\pi : M \rightarrow M$  is the minimum regular cover and  $\lambda$  is such that  $\pi^*\omega = d\ln\lambda$ . We define a smooth family of vector fields  $X_t$  on  $\tilde{M}$  by:

$$i(X_t)\tilde{\Omega}_t = -\lambda\pi^*\eta_t$$

Since  $d(\lambda\pi^*\eta_t) = \lambda\pi^*d_\omega\eta_t$ , we have:

$$L_{X_t}\tilde{\Omega}_t + \partial/\partial t(\tilde{\Omega}_t) = 0.$$

We claim that  $X_t$  is complete. Hence it defines a smooth family of diffeomorphisms  $\psi_t$  of  $\tilde{M}$  such that  $\psi_t^*\tilde{\Omega}_t = \tilde{\Omega}_0$ .

This argument is Moser's standard path method [17].

To prove that  $X_t$  is complete, it is enough to show that it is basic, i.e., there is a family of vector fields  $Y_t$  on  $M$  such that  $\pi_*X_t = Y_t$ . Since  $M$  is compact,  $Y_t$  is integrable, and so will be  $X_t$ .

For any  $\tau \in \mathcal{A}$ , we easily see that:

$$\tau^*\tilde{\Omega}_t = c_\tau\tilde{\Omega}_t,$$

and

$$\tau^*(\lambda\pi^*\eta_t) = c_\tau(\lambda\pi^*\eta_t).$$

We therefore have:

$$\begin{aligned} -c_\tau i(X_t)\tilde{\Omega}_t &= \tau^*(\lambda\pi^*\eta_t) = -\tau^*(i(X_t)\tilde{\Omega}_t) = -i((\tau)^{-1})_*X_t(\tau^*\tilde{\Omega}_t) \\ &= -i((\tau)^{-1})_*X_t(c_\tau\tilde{\Omega}_t) = -c_\tau i((\tau)^{-1})_*X_t(\tilde{\Omega}_t). \end{aligned}$$

Hence

$$c_\tau i((\tau)^{-1})_*X_t(\tilde{\Omega}_t) = c_\tau i(X_t)\tilde{\Omega}_t.$$

Since  $c_\tau \neq 0$ , we have:  $i((\tau)^{-1})_*X_t(\tilde{\Omega}_t) = i(X_t)\tilde{\Omega}_t$ . Therefore  $((\tau)^{-1})_*X_t = X_t$ .

Let now  $\phi_t$  be the family of diffeomorphisms of  $M$  covered by  $\psi_t$ , i.e.  $\pi \circ \psi_t = \phi_t \circ \pi$ , then  $\psi_t^*\tilde{\Omega}_t = (\lambda_t \circ \psi_t).\pi^*(\phi_t^*\Omega_t) = \lambda_0\pi^*\Omega_0$ . Hence  $\pi^*(\phi_t^*\Omega_t) = (\lambda_0/(\lambda_t \circ \phi_t))\pi^*\Omega_0$ . For all  $\tau \in \mathcal{A}$ , we have:

$$(\lambda_0/(\lambda_t \circ \phi_t))\pi^*\Omega_0 = \pi^*(\phi_t^*\Omega_t) = \tau^*\pi^*(\phi_t^*\Omega_t) = ((\lambda_0/(\lambda_t \circ \phi_t))\circ\tau)\pi^*\Omega_0.$$

Therefore,  $(\lambda_0/\lambda_t \circ \phi_t)$  is invariant by all  $\tau \in \mathcal{A}$ , hence  $(\lambda_0/\lambda_t \circ \phi_t) = f_t \circ \pi$  for some function  $f_t$  on  $M$ . We thus get that  $\pi^*(\phi_t^*\Omega_t) = \pi^*(f_t\Omega_0)$ , and hence  $\phi_t^*\Omega_t = f_t\Omega_0$ .

This finishes the proof of Theorem 4.  $\square$

Exactly like in Moser's theorem in Symplectic Geometry [17], there are examples in which we get smooth liftings of the coboundaries  $\Lambda_t$  without using the deep lemma (which is an application of Grothendieck's theory of topological vector spaces). The most trivial example is provided by Theorem 2: if  $\eta_\epsilon = d_\omega \gamma_\epsilon$ , then  $\Lambda_t = d_\omega(t\gamma_\epsilon)$

In the following situation, we also have an immediate smooth lifting of the coboundaries  $\Lambda_t$ .

**Theorem 5.** *Let  $\Omega_t$  be a smooth family of lcs forms on a compact manifold  $M$ , with a smooth family  $\omega_t$  of Lee forms having a fixed de Rham cohomology, i.e.  $[\omega_0] = [\omega_t], \forall t$ , and such that there exists a smooth family  $\theta_t$ , with  $\Omega_t = d\theta_t + \omega_t \wedge \theta_t$ , then the lcs forms  $\Omega_t$  define equivalent lcs structures.*

*Proof.* There is a smooth family of positive functions  $u_t$  on  $M$  with  $\omega_t = \omega_0 + d\ln(u_t)$  and  $u_0 = 1$ . Indeed, since  $(\partial/\partial t)(\omega_t)$  is exact, there is a smooth family of positive functions  $v_t$  such that  $(\partial/\partial t)(\omega_t) = d\ln(v_t)$ . Use for instance the Hodge-de Rham decomposition theorem. Now integrate both side and set  $u_t = \int_0^t (v_s) ds$ .

Let  $\pi : \tilde{M} \rightarrow M$  be the minimum cover associated with  $\omega_0$ , and let  $\lambda_0 : \tilde{M} \rightarrow \mathbb{R}$  be a positive function such that  $\pi^* \omega_0 = d\ln \lambda_0$ . Then  $\pi^* \omega_t = d\ln \lambda_0 + d\ln(u_t \circ \pi) = d\ln \lambda_t$  with  $\lambda_t = \lambda_0 \circ (u_t \circ \pi)$ . We have:

$$\tilde{\Omega}_t = \lambda_t \pi^* \Omega_t = \lambda_t \pi^*(d\theta_t) + \lambda_t d\ln \lambda_t \wedge \pi^* \theta_t = d(\lambda_t \pi^* \theta_t).$$

Setting  $\partial/\partial t(\lambda_t \pi^* \theta_t) = \rho_t$ , we define a smooth family of vector fields  $X_t$  on  $\tilde{M}$  by:

$$i(X_t) \tilde{\Omega}_t = -\rho_t.$$

We have:

$$L_{X_t} \tilde{\Omega}_t + \partial/\partial t(\tilde{\Omega}_t) = 0.$$

We claim that  $X_t$  is complete. Hence it defines a smooth family of diffeomorphisms  $\psi_t$  of  $\tilde{M}$  such that  $\psi_t^* \tilde{\Omega}_t = \tilde{\Omega}_0$ .

From here proceed like in the proof of Theorem 3.  $\square$

**Remark 3.** Let  $u_t$  be a smooth family of positive functions such that  $\omega_t = \omega_0 + d\ln u_t$ . Then  $\Omega'_t = u_t \Omega_t$  has  $\omega_0$  as Lee form for all  $t$ . Moreover setting  $\theta'_t = u_t \theta_t$ , we have:

$$d_{\omega_0}(\theta'_t) = u_t d\theta_t + \frac{du_t}{u_t} \wedge (u_t \theta_t) + \omega_0 \wedge u_t \theta_t = u_t (d\theta_t + (d\ln u_t + \omega_0) \wedge \theta_t) = u_t \Omega_t = \Omega'_t.$$

Hence  $\Omega'_t = d_{\omega_0}(\theta'_t)$ . The coboundary  $\Lambda'_t = \Omega'_t - \Omega'_0$  has the smooth lifting  $d_{\omega_0}(\theta'_t - \theta'_0)$ .

*Proof of Theorem 1.* Theorem 1 is a consequence of Theorem 5 since two contact forms  $\alpha, \alpha'$  define the same contact structure if  $\alpha' = w\alpha$ , with  $w$  a smooth positive

function. Now set  $\alpha_t = \exp(t \ln(w))\alpha$ . The family of lcs forms is  $\Omega_t = d\theta_t + \omega \wedge \theta_t$  with  $\theta_t = p_1^* \alpha_t$ .

The mapping  $\rho : \text{Diff}_{C(\alpha)}(M) \rightarrow \text{Diff}_{S(\alpha)}(M \times S^1)$  comes from the proof. For  $h \in \text{Diff}_{C(\alpha)}(M)$ ,  $h^* \alpha = w \cdot \alpha$ , then the diffeomorphism  $\phi_1$  above obtained using  $\Omega_t = d\theta_t + \omega \wedge \theta_t$ , with  $\theta_t = p_1^* \alpha_t$  and  $\alpha_t = \exp(t \ln(w))\alpha$ , takes  $\Omega_1$  to  $a\Omega_0$ . Taking a path from  $h\alpha$  to  $\alpha$ , which does not reverse the first one, for instance  $\alpha'_t = (t + (1-t)h)\alpha$ ,  $\theta'_t = p_1^* \alpha'_t$  and  $\Omega'_t = d\theta'_t + \omega \wedge \theta'_t$ , get a diffeomorphism  $\phi_1$  taking  $\Omega_0$  back to a multiple of  $\Omega_1$ . Now set  $\rho(h) = \phi_1 \circ \psi_1$ .  $\square$

## 5. Invariants of lcs structures

Given a lcs manifold  $(M, \mathcal{S})$ , we have considered the following objects attached to  $\mathcal{S}$ :

1. The cohomology class of the Lee form  $\omega$  of any representative lcs form  $\Omega \in \mathcal{S}$ . We saw that this is an invariant  $\mathcal{L}_{\mathcal{S}}$ , we called the Lee class of  $\mathcal{S}$ . The group  $\mathcal{A}$  of periods of  $\omega$  is an object depending only on the conformal class  $\mathcal{S}$ .
2. We considered the minimum cover of  $M$  which has a group of deck transformations isomorphic with the group  $\mathcal{A}$  of periods of  $\omega$  as group of automorphisms, and the  $c\mathcal{A}$  cohomology.

In Proposition 1, we gather other invariants built using the automorphisms of the lcs structure.

If  $\mathcal{G}$  is a Lie algebra and  $K$  is a  $\mathcal{G}$ -module, we denote by  $H^*(\mathcal{G}, K)$ , the cohomology of  $\mathcal{G}$  with coefficients in  $K$  [14]. This is the cohomology of the complex  $(C^*(\mathcal{G}, K), \delta)$  where  $p$ -cochains are  $p$ -linear alternating mappings on  $\mathcal{G}$  with values in  $K$  and the coboundary operator is given by:

$$\begin{aligned} \partial f(X_1, \dots, X_{p+1}) &= \sum_i (-1)^{i+1} X_i \cdot f(X_1, \dots, \hat{X}_i, \dots, X_{p+1}) \\ &\quad + \sum_{i \leq j} (-1)^{i+j} f([X_i, X_j], \dots, \hat{X}_i, \dots, \hat{X}_j, \dots). \end{aligned}$$

We also consider the cohomology  $H^*(G, K)$  of an (abstract) group  $G$  into a  $G$ -module  $K$  [13]. The  $p$ -cochains now are mappings from  $G^p$  to  $K$  and the coboundary operator  $\delta$  is given by

$$\begin{aligned} \delta g(a_0, \dots, a_p) &= a_0 \cdot c(a_1, \dots, a_p) - \left( \sum_i (-1)^i c(a_0, \dots, a_i a_{i+1}, \dots, a_p) \right) \\ &\quad + (-1)^{p+1} c(a_0, \dots, a_{p-1}). \end{aligned}$$

$H^1(G, K)$  is the quotient of derivations (1-cocycles) by inner derivations (co-boundaries). Recall that derivations are maps  $d : G \rightarrow K$  such that  $d(gh) = g.d(h) + dg$  and an inner derivation is a map  $v : G \rightarrow K$  such that there exists  $k \in K$  such that  $v(g) = g \cdot k - k$ .

$H^1(\mathcal{G}, K)$  is the quotient of the space of linear maps  $v : \mathcal{G} \rightarrow K$  such that  $v([X, Y]) = X.v(Y) - Y.v(X)$  (1-cocycles), modulo (the coboundaries) consisting of linear maps  $v$  such that there exists  $k \in K$  with  $v(X) = X.k$ , for all  $X, Y \in \mathcal{G}$ .

**Proposition 1.** *Let  $\mathcal{S}$  be a lcs structure on  $M$ , and  $\Omega \in \mathcal{S}$  with Lee form  $\omega$ .*

1. *The map  $D_\Omega : \text{Diff}_\mathcal{S}(M) \rightarrow C^\infty(M)$ ,  $\phi \mapsto \ln(f_{\phi-1})$ , if  $\phi^*\Omega = f_\phi\Omega$  is a 1-cocycle on  $\text{Diff}_\mathcal{S}(M)$  whose cohomology class  $a_\mathcal{S} \in H^1(\text{Diff}_\mathcal{S}(M), C^\infty(M))$  is independent of the choice of  $\Omega \in \mathcal{S}$ , i.e. an invariant of  $\mathcal{S}$ .*

2. *The map  $d_\Omega : \mathcal{X}_\mathcal{S}(M) \rightarrow C^\infty(M)$ ,  $X \mapsto u_\Omega(X)$ , where  $L_X\Omega = (u_\Omega(X))\Omega$ , is a 1-cocycle, whose cohomology class  $b_\mathcal{S} \in H^1(\mathcal{X}_\mathcal{S}(M), C^\infty(M))$  is independent of the choice of  $\Omega \in \mathcal{S}$ , i.e., an invariant of  $\mathcal{S}$ .*

3. *The map  $\hat{\omega} : \mathcal{X}_\mathcal{S}(M) \rightarrow C^\infty(M)$ ,  $X \mapsto \omega(X)$  is a 1-cocycle, whose cohomology class  $c_\mathcal{S} \in H^1(\mathcal{X}_\mathcal{S}(M), C^\infty(M))$  is independent of the choice of  $\Omega \in \mathcal{S}$ , i.e. an invariant of  $\mathcal{S}$ .*

4. *The sum  $d_\Omega + \hat{\omega}$  is a 1-cocycle on  $\mathcal{X}_\mathcal{S}(M)$  with values in  $\mathbb{R}$ , hence a homomorphism  $l$ , called the extended Lee homomorphism, an invariant of  $\mathcal{S}$ .*

5. *Suppose  $M$  is compact and fix a riemannian metric. For each  $h \in \text{Diff}_\mathcal{S}(M)$  (not even homotopic to the identity)  $h^*\omega - \omega$  is an exact 1-form. Let  $u_h$  be the unique function provided by the Hodge decomposition of  $h^*\omega - \omega$  such that  $h^*\omega - \omega = du_h$ .*

*For  $h, h' \in \text{Diff}_\mathcal{S}(M)$ :*

$$(h, h') \mapsto u_h \circ h' + u_{h'} - u_{hh'}$$

*is a 2-cocycle  $K_\omega$  with values in  $\mathbb{R}$ . Its cohomology class in  $H^2(\text{Diff}_\mathcal{S}(M), \mathbb{R})$  is an invariant  $\mathcal{K}_\mathcal{S}$  of  $\mathcal{S}$ .*

Statements 1, and 2 have been observed in [2]. The statement 3 is obvious, since the coboundary operator in the Gelfand–Fucks cohomology (cohomology on Lie algebras of vector fields) is the same as in the de Rham cohomology.

The class  $c_\mathcal{S}$  may be called the Gelfand–Fucks class of  $\mathcal{S}$ .

Statement 4 was proved by Vaisman [18]. See also [6].

Statement 5 was proved in [8]. The Hodge–de Rham theory gives a smooth lifting of de Rham coboundaries: i.e. any exact  $p$ -form  $\theta$  determines uniquely a  $(p-1)$ -form  $\alpha$  such that  $\theta = d\alpha$  as follows: let  $\delta$  be the codifferential, and  $G$  the Green operator defined by a riemannian metric, then  $\alpha = \delta G(\theta)$ . Here the function  $u_h$  is  $u_h = \delta(G(h^*\omega - \omega))$ . See for instance [3].

**Remark 4.** We can define similar invariants using objects with compact support, and denote them by  $a_\mathcal{S}^c$ ,  $b_\mathcal{S}^c$ ,  $c_\mathcal{S}^c$ .

**Definition.** *The structure  $\mathcal{S}$  is called **inessential** if there exists  $\Omega_* \in \mathcal{S}$  such that  $G_{\Omega_*}(M) = \text{Diff}_\mathcal{S}(M)$ . The structure  $\mathcal{S}$  is called **essential** otherwise.*

The following fact was observed in [4]:

**Proposition 2.** *Let  $(M, \mathcal{S})$  be a lcs manifold. Then  $\mathcal{S}$  is inessential iff  $a_{\mathcal{S}} = 0$ .*

The connection between these invariants, and the problem of essentiality, and globality of locally conformal structure is given by the following:

**Theorem 6.** *Let  $(M, \mathcal{S})$  be a lcs manifold.*

1. *If  $a_{\mathcal{S}} = 0$ , then  $\mathcal{S} = \mathcal{O}$ . Furthermore, the Lee homomorphism is trivial, and the structure  $\mathcal{S}$  is of the second kind. Thus inessential structures are of the second kind. This also says that if  $\mathcal{S}$  is of the first kind, then  $a_{\mathcal{S}} \neq 0$ .*
2. *If  $M$  is compact, then  $\mathcal{S} = \mathcal{O}$  implies that  $a_{\mathcal{S}} = 0$ .*
3. *The Gelfand–Fuchs class  $c_{\mathcal{S}}$  vanishes iff the Lee class  $\mathcal{L}_{\mathcal{S}}$  does.*
4. *If  $M$  is compact, the vanishing of one of the four classes  $a_{\mathcal{S}}$ ,  $b_{\mathcal{S}}$ ,  $c_{\mathcal{S}}$ ,  $\mathcal{L}_{\mathcal{S}}$ , implies the vanishing of the remaining three classes.*

We will need the following “local transitivity” result. Lefebvre’s [16] proved it away from the zeros of the Lee form. Since for any point, the lcs structure can be represented by a lcs form with Lee form not vanishing at that point, Lefebvre’s argument applies. For the convenience of the reader, we rewrote it in our style.

**Theorem 7.** *Let  $(M, \mathcal{S})$  be a lcs manifold of dimension  $2n$ . For each  $x \in M$ , there exist  $2n$  vector fields  $V_j^x \in \mathcal{X}_{\mathcal{S}}(M)$  with arbitrarily small compact support in an open neighborhood of  $x$  and such that  $\{V_j^x(x)\}_{j=1, \dots, 2n}$  form a basis of the tangent space  $T_x M$ .*

*Proof.* 1. For each point  $x \in M$ , there is  $\Omega \in \mathcal{S}$ , with Lee form  $\omega$  such that  $\omega(x) \neq 0$ . Indeed, if the Lee form  $\omega$  of  $\Omega \in \mathcal{S}$  vanishes at  $x$ , consider a contractible neighborhood  $U$  of  $x$  at which  $\omega|_U = d\ln(\lambda)$ , and choose a smooth positive function  $\rho$ , constant outside of  $U$  with  $d\rho(x) \neq 0$  and  $d\ln \lambda \neq d\ln \rho$  on a neighborhood of  $x$ . The form  $\rho\Omega \in \mathcal{S}$  and has Lee form  $\omega' = \omega - d\ln(\rho)$ . The new Lee form does not vanish at  $x$  (and in a neighborhood).

2. Any function  $u$  on an open set  $U$  where  $f\Omega|_U$  is symplectic defines a vector field  $X_u$  on  $U$  by the equation:

$$i(X_u)f\Omega|_U = d(fu).$$

A direct calculation shows that  $L_{X_u}\Omega|_U = (-X_u \cdot \ln f)\Omega$  [18].

3. The form  $\Omega \in \mathcal{S}$  above has a Lee form  $\omega$  not vanishing on an open neighborhood  $V \subset U$  of  $x$ . Hence, there are local coordinates  $(x_1, \dots, x_n, y_1, \dots, y_n)$  defined on a smaller neighborhood  $V_1$  of  $x$  such that  $y_1 \neq 0$ , and

$$\Omega|_{V_1} = y_1 \left( \sum_{k=1}^n dx_k \wedge dy_k \right).$$

Let  $\mu$  be a smooth function, supported in  $V_2$  and which is equal to 1 on a closed neighborhood  $F$  of  $x$ , where  $F \subset V_2 \subset V_1$ .

We define 2n vector fields by:

$$i(Y_1)\left(\frac{1}{y_1}\Omega|_{V_1}\right) = d\left(\mu\frac{y_1^2}{y_1}\right) = d(\mu y_1)$$

and for  $j = 2, \dots, n$ ,

$$i(Y_j)\left(\frac{1}{y_1}\Omega|_{V_1}\right) = d\left(\mu\frac{y_j}{y_1}\right).$$

For  $j = 1, \dots, n$  define  $X_j$  by:

$$i(X_j)\left(\frac{1}{y_1}\Omega|_{V_1}\right) = d\left(\mu\frac{x_j}{y_1}\right).$$

Then  $X_i, Y_i$  are smooth vector fields on  $M$  with compact support in  $V_1$ , which all belong to  $\mathcal{X}_S(M)_c$ .

Let us note  $e_j = \partial/\partial x_j$  and  $e'_j = \partial/\partial y_j$ , then on  $F$ , we have

$$\begin{aligned} Y_1 &= e_1, & Y_j &= \frac{1}{y_1}e_j - \frac{y_j}{y_1^2}e_1, & j &= 2, \dots, n \\ X_j &= -\frac{1}{y_1}e'_j - \frac{x_j}{y_1^2}e_1, & j &= 1, \dots, n. \end{aligned}$$

Writing that  $\sum_{i=1}^n (a_i X_i + b_i Y_i) = 0$ , gives immediately that  $b_i = 0$  and  $a_i = 0$ , i.e. these vector fields are linearly independent near  $x$ .  $\square$

*Proof of Theorem 6.1.* Suppose that  $a_S = 0$ , that is  $S$  is inessential (Proposition 2). Let  $\Omega_* \in S$  with  $\text{Diff}_S(M) = G_{\Omega_*}(M)$ , and let  $\omega_*$  be the corresponding Lee form. It follows that

$$\mathcal{X}_S(M)_c = \mathcal{X}_{\Omega_*}(M)_c.$$

Let us now show that  $\omega_* = 0$ .

For each  $x \in M$ , and any tangent vector  $\xi \in T_x M$ , we want to show that  $\omega_*(x)(\xi) = 0$ . By Theorem 7,  $\xi = \sum_{j=1}^{2n} c_j(x) V_j^x(x)$ . Extend now the coefficients  $c_j(x)$  into smooth functions  $c_j$  with compact support near  $x$ . We get a smooth vector field with compact support  $V = \sum_{j=1}^{2n} c_j V_j^x$ , which coincides with  $\xi$  at  $x \in M$ . Therefore,

$$\omega_*(x)(\xi) = \omega_*(x)(V(x)) = (\omega_*(V))(x) = \sum_{j=1}^{2n} (c_j \omega_*(V_j^x))(x).$$

Since  $V_j^x \in \mathcal{X}_S(M)_c = \mathcal{X}_{\Omega_*}(M)_c$ ,  $\omega_*(V_j^x)$  is a constant function (see Remark 5.3) with compact support, and hence identically zero. This proves that  $\omega_*(x) = 0$ .

This implies that  $S = \mathcal{O}$ .

Since the Lee homomorphism can be computed using  $\Omega_*$  and  $\omega_*$ , we see that

$$l = \hat{\omega}_* = 0.$$

This implies that the structure is of the second kind. Indeed, if  $\Omega$  is any representative of  $\mathcal{S}$  with Lee form  $\omega$  and  $X \in \mathcal{X}_\Omega(M)$ , then  $l(X) = \omega(X) = 0$ .

2. If  $\mathcal{S} = \mathcal{O}$ , there is a symplectic form  $\Omega \in \mathcal{S}$ . If  $\phi \in \text{Diff}_\mathcal{S}(M)$ , then  $\phi^*\Omega = f\Omega$ . By the classical theorem of Libermann (see [6]),  $f$  is a constant, provided that the dimension of  $M$  is at least 4, (which is assumed here) and if  $M$  is compact, this constant must be 1. This follows from the fact that  $\int_M \phi^*\Omega^n = f^n \int_M \Omega^n$  and by the formula of change of variable, we have equality with  $\int_M \Omega^n$ . Hence  $f = 1$  and therefore  $a_{\mathcal{S}} = 0$ .

3. It is clear that  $[\omega] = 0$  implies that  $[\hat{\omega}] = 0$ . Conversely, suppose there exists a smooth function  $u$  such that  $\omega(X) = X.u = du(X)$  for all  $X \in \mathcal{X}_\mathcal{S}(M)$ . We show that indeed  $\omega(\xi) = du(\xi)$  for all vector fields  $\xi$ , i.e. that  $\omega = du$ . For each point  $x \in M$ , we need to show that  $\omega(\xi)(x) = (du(\xi)(x))$ .

As above, we consider the vector field  $V = \sum_{j=1}^{2n} c_j V_j^x$ , which is equal to  $\xi$  at  $x$ . Then, like above:  $\omega(\xi)(x) = \sum_{j=1}^{2n} (c_j \omega(V_j^x))(x) = \sum_{j=1}^{2n} (c_j du(x)(V_j^x)) = du(x)(\sum_{j=1}^{2n} c_j V_j^x) = du(x)(V) = du(x)(\xi)$ . Therefore the de Rham class of  $\omega$  is trivial.

4. In the compact case  $(a_{\mathcal{S}} = 0) \Leftrightarrow (\mathcal{S} = \mathcal{O})$  and  $(a_{\mathcal{S}} = 0) \Leftrightarrow (b_{\mathcal{S}} = 0)$ .

We also have that in general,  $(\mathcal{S} = \mathcal{O} \Leftrightarrow (\mathcal{L}_\mathcal{S} = 0))$  and  $(c_{\mathcal{S}} = 0) \Leftrightarrow (\mathcal{L}_\mathcal{S} = 0)$

Putting these facts together, yields the last assertion of Theorem 5.  $\square$

**Remarks.** 1. If  $M$  is not compact,  $\mathcal{S} = 0$  does not imply that  $a_{\mathcal{S}} = 0$ . Take for instance the global conformal symplectic structure defined by the standard symplectic form on  $\mathbb{R}^{2n}$ , and more generally non-compact manifolds with complete Liouville vector fields, like Stein manifolds [4].

2. The vanishing of the compactly supported invariant  $a_{\mathcal{S}}^c$  also implies that  $\mathcal{S} = 0$ . This was proved in [12].

## 6. Concluding remarks and questions

1. The mapping  $L : \mathcal{L}_{cs}(M) \rightarrow \mathcal{F}^1(M)$  assigning to a lcs form its Lee form is not continuous in the  $C^0$  topology. Indeed if  $u$  is a smooth function which is  $C^0$  close to 1 and  $C^1$  far from 0, then the Lee forms of  $u\Omega$  and  $\Omega$ , are far apart. How about the continuity for the  $C^\infty$  topology?

If  $M$  has a complex structure  $J$  and a hermitian metric  $g$  such that the lcs form  $\Omega$  is given by  $\Omega(X, Y) = g(X, JY)$  ( $M$  is said to be a locally conformal Kähler manifold), then  $L$  is continuous for the  $C^\infty$  topology. Indeed in that case we have an explicit formula for  $L(\Omega)$  [9]:

$$L(\Omega) = \frac{1}{n-1}(\delta\Omega \circ J).$$

Here  $\delta$  is the codifferential with respect to the metric  $g$ , and  $2n$  is the dimension of  $M$ .

2. The Lee homomorphism  $l : \mathcal{X}_S(M) \rightarrow \mathbb{R}$  can be integrated into a homomorphism  $\mathcal{L} : \text{Diff}_S(M)_+ \rightarrow \mathbb{R}/\Delta$  (where  $\Delta$  is some countable subgroup of  $\mathbb{R}$ ), and  $\text{Diff}_S(M)_+$  is the group of automorphisms of  $S$  which admit a lift to the minimal regular cover  $\tilde{M}$  [6].

If  $\alpha$  is a contact form on a compact manifold  $M$ , we constructed in Theorem 1 a map  $\rho : \text{Diff}_{C(\alpha)}(M) \rightarrow \text{Diff}_{S(\alpha)}(M \times S^1)_+$ . Composing  $\rho$  with the extended global Lee homomorphism, we get a map:

$$\mu = \mathcal{L} \circ \rho : \text{Diff}_{C(\alpha)}(M) \rightarrow \mathbb{R}/\Delta.$$

This map is not a group homomorphism. This allows us to define a 2-cocycle  $\eta$  on the group  $\text{Diff}_{C(\alpha)}(M)$ :

$$\eta(\phi, \psi) = \rho(\phi) \cdot \rho(\psi) \cdot (\rho(\phi\psi))^{-1}$$

for all  $\phi, \psi \in \text{Diff}_{C(\alpha)}(M)$ .

What is the meaning of that cocycle?

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