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## Closed incompressible surfaces in the complements of positive knots

Makoto Ozawa

*This paper is dedicated to Professor Shin'ichi Suzuki for his 60th birthday*

**Abstract.** We show that any closed incompressible surface in the complement of a positive knot is algebraically non-split from the knot, positive knots cannot bound non-free incompressible Seifert surfaces and that the splittability and the primeness of positive knots and links can be seen from their positive diagrams.

**Mathematics Subject Classification (2000).** 57M25.

**Keywords.** Positive knot, closed incompressible surface, order, free Seifert surface, splittability, primeness.

### 1. Introduction

A knot  $K$  in the 3-sphere  $S^3$  is called *positive* if it has an oriented diagram all crossings of which are positive crossings. For a closed surface  $F$  in  $S^3 - K$ , we define the *order*  $o(F; K)$  of  $F$  for  $K$  as follows ([5]). Let  $i : F \rightarrow S^3 - K$  be the inclusion map and let  $i_* : H_1(F) \rightarrow H_1(S^3 - K)$  be the induced homomorphism. Since  $\text{Im}(i_*)$  is a subgroup of  $H_1(S^3 - K) = \mathbb{Z}\langle \text{meridian} \rangle$ , there is an integer  $m$  such that  $\text{Im}(i_*) = m\mathbb{Z}$ . Then we define  $o(F; K) = m$ .

The positive knot complements have the following special properties.

**Theorem 1.1.** *Any closed incompressible surface in a positive knot complement has non-zero order.*

A Seifert surface  $F$  for a knot is said to be *free* if  $\pi_1(S^3 - F)$  is a free group. In [5, Theorem 1.1], it is shown that a knot bounds a non-free incompressible Seifert surface if and only if there exists a closed incompressible surface in the

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knot complement whose order is equal to zero. Therefore, Theorem 1.1 gives us the next corollary.

**Corollary 1.2.** *Positive knots cannot bound non-free incompressible Seifert surfaces.*

Although positive links which have connected positive diagrams are non-split because they have positive linking numbers, we can give another geometrical proof of this fact.

**Theorem 1.3.** *Positive links are non-split if their positive diagrams are connected.*

Positive diagrams of positive knots or links also tell us their primeness. We say that a knot or link diagram  $\tilde{K}$  on the 2-sphere  $S$  is *prime* if for any loop  $l$  in  $S$  intersecting  $\tilde{K}$  in 2 points,  $l$  bounds a disk intersecting  $\tilde{K}$  in an arc.

**Theorem 1.4.** *Non-trivial positive knots or links are prime if their positive diagrams are connected and prime.*

**Remark 1.5.** The referee suggested that one can show that: *A non-trivial positive link is prime iff its positive diagram is connected and prime*, with the addition of the assumption that the positive link projections contain no nugatory crossings. In fact, the converse of Theorem 1.3 and 1.4 is true, but it needs [2, Theorem 3].

There are other results about determining when a link projection represents a non-split or prime link.

For the splittability,

- alternating links ([1, Theorem 10.2], [4, Theorem 1 (a)]);
- almost alternating links ([6]);
- homogeneous links ([2, Corollary 3.1]).

For the primeness,

- alternating links ([4, Theorem 1 (b)]);
- positive braids ([3, 1.2 Theorem]).

## 2. Proof of Theorem 1.1 and 1.3

Theorem 1.1 and 1.3 follow from the next Theorem.

**Theorem 2.1.** *Let  $K$  be a positive knot or link in the 3-sphere  $S^3$  and  $F$  a closed incompressible surface in the complement of  $K$ . Then one of the following conclusions hold.*

- (1) *There exists a loop  $l$  in  $F$  such that  $lk(l, K) \neq 0$ .*  
 (2)  *$F$  is a splitting sphere for  $K$ , and any positive diagram of  $K$  is disconnected.*

Henceforth, we shall prove Theorem 2.1.

Let  $S$  be a 2-sphere in  $S^3$  and  $p : S^3 - \{2 \text{ points}\} \cong S \times R \rightarrow S$  a projection. Put  $K$  so that  $p(K)$  is a positive diagram. As usual way, we express  $K$  in a bridge presentation. Thus we have the following data (see Figure 1).

- $S^3 = B^+ \cup_S B^-$  ( $S$  decomposes  $S^3$  into two 3-balls).
- $K = K^+ \cup_S K^-$ , where  $K^\pm \subset B^\pm$  ( $S$  cuts  $K$  into over bridges and under bridges).
- $K^\pm = K_1^\pm \cup K_2^\pm \cup \dots \cup K_n^\pm$  ( $K$  is presented as  $n$  over bridges and  $n$  under bridges).
- $D^\pm = D_1^\pm \cup D_2^\pm \cup \dots \cup D_n^\pm$  (each  $K_i^\pm \cup p(K_i^\pm)$  bounds a disk  $D_i^\pm$  such that  $p(D_i^\pm) = p(K_i^\pm)$ ).

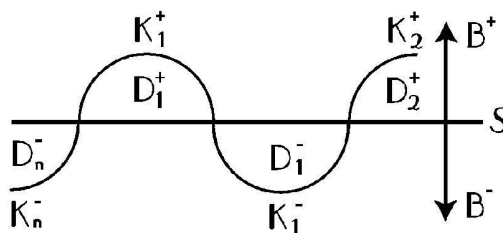


Figure 1. View from level surface

We take  $n$  minimal over all bridge presentations of  $p(K)$ .

**Lemma 2.2.** *We may assume that:*

- (a)  $F \cap D^- = \emptyset$ ,  
 (b)  $F \cap B^-$  consists of disks,  
 (c)  $F \cap D^+$  consists of arcs, and  
 (d) any component of  $F \cap B^+ - D^+$  is a disk.

*Proof.* (a): Simply push out  $F$  near  $D^-$  into  $B^+$ .

(b): If there exists a component of  $F \cap B^-$  which is not a disk, then  $F \cap B^-$  has a compressing disk  $E$  in  $B - N(D^-)$  since  $B - N(D^-)$  is a 3-ball. By the incompressibility of  $F$  in  $S^3 - K$ ,  $\partial E$  bounds a disk in  $F$ . Then by cutting and pasting  $F$  along  $E$ , we have a new incompressible surface  $F'$  and a sphere  $F''$ . Replace  $F$  with  $F'$  and continue this operation.

(c): Suppose there exists a loop component of  $F \cap D^+$  and let  $E$  be an innermost disk in  $D^+$ . Then the similar argument to (b) passes by using  $E$ .

(d): If there exists a component of  $F \cap B^+ - D^+$  which is not a disk, then  $F \cap B^+ - D^+$  has a compressing disk  $E$  in  $B^+ - D^+$ . By using  $E$ , we can show (d) similarly.  $\square$

We take a 2-tuple lexicographically ordered complexity measure  $(|F \cap B^-|, |F \cap D^+|)$  minimal. Note that the complexity measure is not  $(0, *)$ . For  $(0, *)$ ,  $F$  fails to be incompressible in  $S^3 - K$  since  $(B^+, K^+)$  is a trivial tangle. If the complexity measure is  $(1, 0)$ , then we have the conclusion (2).

Hereafter, we suppose that the complexity  $(|F \cap B^-|, |F \cap D^+|) \geq (1, 1)$ .

Then we obtain a connected graph  $G$  in  $F$  by regarding  $F \cap B^-$  and  $F \cap D^+$  as vertices and edges respectively. Note that every vertex has a positive even valency by the construction.

An arc  $\alpha_j$  of  $F \cap D_i^+$  divides  $D_i^+$  into two disks  $\delta_j$  and  $\delta'_j$ , where  $\delta'_j$  contains  $K_i^+$ . Put  $\beta_j = \delta_j \cap S$ . We may assume that  $p(\alpha_j) = p(\delta_j) = \beta_j$  for all  $\alpha_j$ . We assign an orientation endowed from  $K_i$  to  $\alpha_j$  and  $\beta_j$  naturally (see Figure 2).

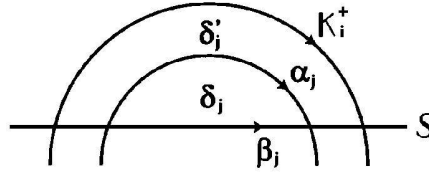


Figure 2.  $\alpha_j$  and  $\beta_j$  have the orientation

**Lemma 2.3.** *For any arc  $\alpha_j$  of  $F \cap D_i^+$ ,  $\beta_j \cap p(K^-) \neq \emptyset$ .*

*Proof.* Suppose that there exists an arc  $\alpha_j$  of  $F \cap D_i^+$  such that  $\beta_j \cap p(K^-) = \emptyset$ . By exchanging  $\alpha_j$  if necessary, we may assume that  $\alpha_j$  is outermost in  $D_i^+$ , that is,  $\text{int } \delta_j \cap F = \emptyset$ . If  $\alpha_j$  connects different vertices, then a  $\partial$ -compression of  $F$  along  $\delta_j$  reduces the complexity. Otherwise,  $\alpha_j$  incidents a single vertex, say  $D_k^-$ . We perform a  $\partial$ -compression of  $F$  along  $\delta_j$ , and obtain an annulus  $A$  consisting of the disk  $D_k^-$  and the resultant band  $b$ . Since we chose an outermost arc  $\alpha_j$  and  $\beta_j \cap p(K^-) = \emptyset$ , there exists a compressing disk for  $A$  in  $B^- - K^-$ . By retaking  $F$  along the compressing disk, we can reduce the complexity. In both cases, there is a contradiction in the assumption the complexity is minimal.  $\square$

Now we pay attention to a face  $f$  of  $G$  in  $F$ . A *corner* is a subarc of  $\partial(F \cap B^-) - (F \cap D^+)$ . The *cycle*  $\partial f$  for  $f$  is a loop consisting of edges and corners such that it bounds  $f$ . The edges have orientations as previously mentioned.

**Lemma 2.4** (The cycle lemma). *For any face  $f$ , the cycle  $\partial f$  can not be oriented.*

*Proof.* Suppose that there is a face  $f$  such that  $\partial f$  can be oriented. Then, since no corner of  $\partial f$  intersects  $p(K)$ , and by Lemma 2.3,  $p(\partial f)$  has non-zero intersection number with  $p(K^-)$  on  $S$ . Figure 3 illustrates the projection of  $f$  and  $K^-$  on  $S$ . This is a contradiction.  $\square$

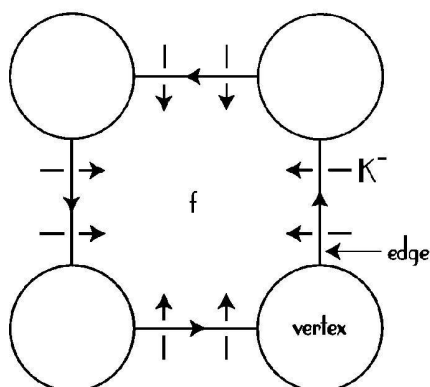


Figure 3.  $p(\partial f)$  has non-zero intersection number

For each face  $f$  of  $G$  and any point in the interior of any edge of  $\partial f$ , we can find an arc  $\gamma$  on  $f$  satisfying the following property.

- (\*)  $\gamma$  connects two edges of  $\partial f$  whose orientations are different in  $\partial f$ .

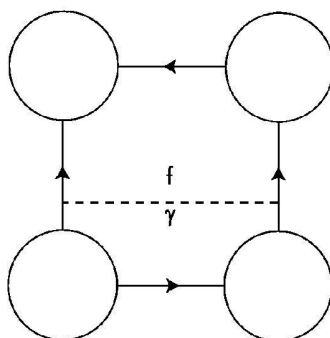


Figure 4.  $\gamma$  with the property (\*)

Lemma 2.4 assures the existence of such an arc  $\gamma$ .

To find a loop  $l$  on  $F$  with  $lk(l, K) \neq 0$ , we depart a point in the interior of any edge of  $G$ , trace arcs with the property (\*), and will arrive at the face on which we have walked. Connecting these arcs, we will obtain an oriented loop  $l$  in  $F \cap B^+$  with a suitable orientation such that  $l$  has a positive intersection number with edges of  $G$  on  $F$ . Thus we got an oriented loop  $l$  in  $F$  which has non-zero linking number with  $K$ . Since any loop in a splitting sphere is contractible in  $S^3 - K$ , we have the conclusion (1).

This completes the proof of Theorem 2.1.

### 3. Proof of Theorem 1.4

Let  $K$  be a positive knot or link in  $S^3$  and  $F$  be a decomposing sphere for  $K$ . We put  $K$  and  $F$  as the proof of Theorem 2.1 except that two points  $p_1$  and  $p_2$  of  $F \cap K$  are in  $\text{int } B^+$  or  $\text{int } B^-$ . Note that  $p_1$  and  $p_2$  can not be the ends of a single arc of  $F \cap D^\pm$  because the tangle  $(B^\pm, K^\pm)$  is trivial and  $F$  is a decomposing sphere. Hence, there are two arcs  $e_1$  and  $e_2$  of  $F \cap D^\pm$  whose ends contain  $p_1$  and  $p_2$  respectively. We deform  $F$  by an isotopy relative to  $K$  so that  $p(e_i) = p(p_i)$  ( $i = 1, 2$ ). We take the number of bridges  $n$  minimal.

Thus we have the following data in addition to the data in the proof of Theorem 2.1.

- $F \cap K = p_1 \cup p_2 \subset \text{int } B^\pm$ .
- $F \cap D^\pm \supset e_i \supset p_i$  ( $i = 1, 2$ ).
- $p(e_i) = p(p_i)$  ( $i = 1, 2$ ).

**Lemma 3.1.** *We may assume that:*

- (a)  $F \cap D^- \subset e_1 \cup e_2$ ,
- (b)  $F \cap B^-$  consists of disks,
- (c)  $F \cap D^+$  consists of arcs, and
- (d) any component of  $F \cap B^+ - D^+$  is a disk.

*Proof.* This can be done by an isotopy of  $F$  since Theorem 1.3 assures us that  $S^3 - K$  is irreducible.  $\square$

We take a 2-tuple lexicographically ordered complexity measure  $(|F \cap B^-|, |(F \cap D^+) - (e_1 \cup e_2)|)$  minimal. Then we obtain a connected graph  $G$  in  $F$  by regarding  $F \cap B^-$  and  $(F \cap D^+) - (e_1 \cup e_2)$  as vertices and edges respectively. Corners of each face of  $G$  may contain two points  $\partial e_1 - p_1$  and  $\partial e_2 - p_2$ . Note that the complexity measure is not  $(0, *)$ , otherwise  $F$  is not a decomposing sphere since  $(B^\pm, K^\pm)$  is a trivial tangle. If the complexity measure is  $(1, 0)$ , then  $F \cap S$  gives a desired loop since  $p(e_i) = p(p_i)$  ( $i = 1, 2$ ).

**Lemma 3.2.** *For any arc  $\alpha_j$  of  $(F \cap D^+) - (e_1 \cup e_2)$ ,  $\beta_j \cap p(K^-) \neq \emptyset$ .*

*Proof.* This can be done by the same argument to Lemma 2.3.  $\square$

Hereafter, we assume that  $\tilde{K}$  is prime.

**Lemma 3.3.** *There is no vertex of  $G$  with valency 1.*

*Proof.* Suppose that there is a vertex  $V$  with valency 1. Then only one edge  $\alpha$  incident to  $V$ , and hence exactly one of  $e_1$  and  $e_2$  is attached to  $V$  or contained in  $V$ . Thus  $\partial V$  intersects  $\tilde{K}$  in two points. Since  $\tilde{K}$  is prime,  $\partial V$  bounds a disk  $E$  in  $S$  which intersects  $p(K)$  in an unknotted arc. In the former case,  $p(K) \cap E$  lies under a subarc of  $K^+$  by the minimality of the number of bridges  $n$ . Then by an isotopy of  $F$  along the 3-ball which is bounded by  $V \cup E$ , we can reduce the complexity. See Figure 5. In the latter case,  $E$  intersects  $K$  in one point, and  $V \cup E$  bounds a pair of a 3-ball and an unknotted subarc of  $K^-$  by the minimality of  $n$ . Then an isotopy of  $F$  along the pair can reduce the complexity. See Figure 6. In both cases, there is a contradiction in the assumption the complexity is minimal.  $\square$

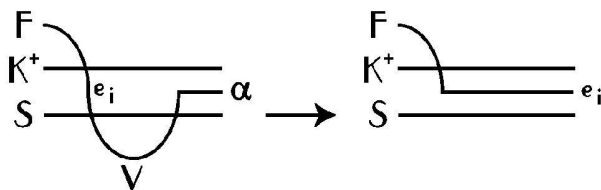


Figure 5. Isotopy of  $F$  along the 3-ball

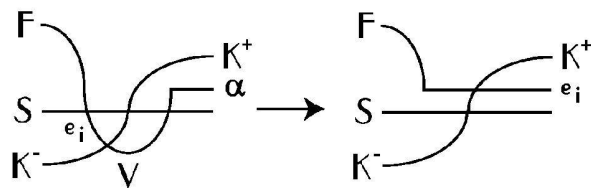


Figure 6. Isotopy of  $F$  along the pair

**Lemma 3.4.** *There is no face  $f$  of  $G$  in  $F$  such that  $\partial f$  is a loop of  $G$ .*



*Proof.* Suppose there exists a face  $f$  as Lemma 3.4. Then  $\partial f$  consists of an edge  $\alpha$  of  $G$  and a subarc  $\gamma$  of the boundary of a vertex  $V$  of  $G$ . By Lemma 3.2,  $p(\alpha)$  intersects  $p(K^-)$ . Moreover, since the loop  $\gamma \cup p(\alpha)$  bounds a disk  $E$  in  $S$ ,  $|p(\alpha) \cap p(K^-)| = 1$  and  $\gamma$  meets exactly one of  $e_1$  and  $e_2$ , say  $e_1$ . Thus a loop  $l = \partial N(\partial E; E) - \partial E$  intersects  $\tilde{K}$  in two points. Since  $\tilde{K}$  is prime,  $\text{int } E$  intersects  $p(K)$  in an embedded arc. Then, there are two possibilities for  $e_1$ ,  $e_1 \subset f$  or  $e_1 \subset V$ . In the former case,  $f \cup E$  bounds a pair of a 3-ball and an unknotted arc, and an isotopy of  $F$  along the pair eliminates  $\alpha$ . In the latter case,  $f \cup E$  bounds a 3-ball, and an isotopy of  $F$  along the 3-ball eliminates  $\alpha$ . These contradict the minimality of the complexity.  $\square$

Hence we have a condition that:

- $G$  has at least two vertices,
- every vertex has valency at least two, and
- all faces of  $G$  in  $F$  are disks.

Next, we pay attention to a face of  $G$  in  $F$ .

**Lemma 3.5.** *For any face  $f$ , the cycle  $\partial f$  can not be oriented.*

*Proof.* If all corners of  $f$  do not meet  $e_1 \cup e_2$ , then this is same to Lemma 2.4.

If exactly one corner of  $f$  meets  $e_1$  or  $e_2$  at one point, then  $f$  and some  $K_i^+$  have the intersection number  $\pm 1$ , or a vertex which meets  $f$  along the corner intersects some  $K_k^-$  in one point. Since  $p(\partial f)$  and  $p(K^-) \cap p(K_i^+)$  must have the intersection number zero,  $\partial f$  is bounded by a loop of  $G$  consisting of a vertex and an edge  $\alpha$ , and  $p(\alpha)$  intersects  $p(K^-)$  in one point. Then Lemma 3.4 gives the conclusion.

If some corners of  $f$  meet both  $e_1$  and  $e_2$ , then the corners of  $f$  have the intersection number zero with  $p(K)$  because  $F$  and  $K$  have the intersection number zero. In such a situation, we have a contradiction same as the proof of Lemma 2.4.  $\square$

By Lemma 3.5, starting a face  $f$  of  $G$  in  $F$  whose closure is a disk, we can get a loop  $l$  in  $F - K$  with  $|lk(l, K)| \geq 2$ . But this is impossible because any loop in  $F - K$  is null-homotopic in  $S^3 - K$  or has linking number  $\pm 1$  with  $K$ . This finishes the proof of Theorem 1.4.

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## References

- [1] R. J. Aumann, Asphericity of alternating knots, *Ann. of Math.* **64** (1956), 374–392.
- [2] P. R. Cromwell, Homogeneous links, *J. London Math. Soc.* (2) **39** (1989), 535–552.

- [3] P. R. Cromwell, Positive braids are visually prime, *Proc. London Math. Soc.* (3) **67** (1993), 384–424.
- [4] W. Menasco, Closed incompressible surfaces in alternating knot and link complements, *Topology* **23** (1984), 37–44.
- [5] M. Ozawa, Synchronism of an incompressible non-free Seifert surface for a knot and an algebraically split closed surface in the knot complement, *Proc. Amer. Math. Soc.* **128** (2000), 919–922.
- [6] T. Tsukamoto, A criterion for almost alternating links to be non-splittable, preprint.

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