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# Complexity of degenerations of modules

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**Abstract.** A module $M$ over an associative algebra $A$ over an algebraically closed field $k$ is said to degenerate to a module $N$ if $N$ belongs to the closure of the isomorphism class of $M$ in the algebraic variety of $d$-dimensional $A$-modules, $d \in \mathbb{N}$. We associate a non-negative integer to a degeneration $M \leq_{\deg} N$, its complexity, and study its properties.

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## 1. Introduction

Let $k$ be an algebraically closed field, $A$ a finite dimensional associative $k$-algebra with a unit and $\operatorname{mod} A$ the category of finite dimensional left $A$-modules. Let $\mathbb{M}_d(k)$ denote the $k$-algebra of the $d \times d$-matrices with coefficients in $k$. We view $A$ as a quotient of a free associative algebra $k\langle X_1, \ldots, X_r\rangle$ by a two-sided ideal $I$. We define the affine variety $\operatorname{mod}_A^d(k)$ as the set of $r$-tuples $(m_1, \ldots, m_r)$ such that $m_i \in \mathbb{M}_d(k)$ and $\rho(m_1, \ldots, m_r)$ is the zero matrix for any $\rho \in I$. The general linear group $Gl_d(k)$ acts on $\operatorname{mod}_A^d(k)$ by conjugation.

As an ordinary set, $\operatorname{mod}_A^d(k)$ is just the set $\operatorname{Hom}_{k-alg}(A, \mathbb{M}_d(k))$ and hence $Gl_d(k)$-orbits in $\operatorname{mod}_A^d(k)$ correspond bijectively to isomorphism classes of $d$-dimensional left $A$-modules.

Let $M$ and $N$ be two $d$-dimensional $A$-modules. By definition, $M$ degenerates to $N$, noted $M \leq_{\deg} N$, if $N$ lies in the closure of the $Gl_d(k)$-orbit of $M$ in $\operatorname{mod}_A^d(k)$, with respect to the Zariski topology. This defines a partial order on the set of isomorphism classes of $d$-dimensional $A$-modules.

Denote by $Q$ the quiver

$$Q = 1 \underset{b_1}{\overset{a_1}{\rightleftarrows}} 2 \underset{b_2}{\overset{a_2}{\rightleftarrows}} 3 \cdots$$

with vertex set $Q_0 = \mathbb{N} \setminus \{0\}$ and arrows $a_i : i \to i+1,\ b_i : i+1 \to i$ for every $i \in Q_0$.

We call a representation

$$T = N_1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} N_2 \cdots N_i \underset{\beta_i}{\overset{\alpha_i}{\rightleftarrows}} N_{i+1} \cdots$$

of $Q$ in mod$A$, the category of finite dimensional $A$-modules, an *exact tube* if the sequence

$$0 \to N_i \xrightarrow{\binom{\beta_{i-1}}{\alpha_i}} N_{i-1} \oplus N_{i+1} \xrightarrow{(-\alpha_{i-1}, \beta_i)} N_i \to 0$$

or equivalently the square

$$\begin{tikzcd}
N_i \arrow[r, "\alpha_i"] \arrow[d, "\beta_{i-1}"] & N_{i+1} \arrow[d, "\beta_i"] \\
N_{i-1} \arrow[r, "\alpha_{i-1}"] & N_i
\end{tikzcd}$$

is exact for all $i \geq 1$. Here we set $N_0 = 0$. Note that $N_i$ is an $A$-module, that $\alpha_i, \beta_i$ are $A$-linear and that $\alpha_i$ is injective, $\beta_i$ is surjective, for all $i \geq 1$. We say that $T$ is an $(M, N)$-tube if there is a natural number $h$ such that

(i) $N_1 \underset{A}{\xrightarrow{\sim}} N$,

(ii) $N_{h+j+1} \underset{A}{\xrightarrow{\sim}} N_{h+j} \oplus M$, for all $j \in \mathbb{N}$.

We call the smallest such number $h$ the complexity $\mathrm{cpl}(T)$ of the tube.

Let $T$ be an $(M, N)$-tube. Note that the sequence

$$0 \to N_k \xrightarrow{\alpha_k} N_{k+1} \xrightarrow{\beta_1 \cdots \beta_k} N_1 \to 0$$

is exact for any $k$. As $N_{k+1}$ is isomorphic to $N_k \oplus M$ for $k \geq \mathrm{cpl}(T)$, there is an exact sequence

$$0 \to N_k \longrightarrow N_k \oplus M \longrightarrow N \to 0,$$

and therefore $M$ degenerates to $N$ [5].

Conversely, whenever $M$ degenerates to $N$, there exists an $(M, N)$-tube: Indeed, the third author showed in [7] that there is a short exact sequence

$$0 \to Z \xrightarrow{\binom{f}{g}} Z \oplus M \to N \to 0, \tag{1.1}$$

and in [6] he associated an exact tube $T_{f,g}$ with such a sequence (see also Section 4). In fact, $T_{f,g}$ is the cokernel of the injection $\varphi : X \to X'$ between the following representations of $Q$:

$$\begin{tikzcd}
X: \arrow[d, "\varphi"] & Z \arrow[r, shift right, "1"'] \arrow[d, "\varphi_1"] & Z \arrow[l, shift right, "f"'] \arrow[r, shift right, "1"'] \arrow[d, "\varphi_2"] & \cdots \arrow[l, shift right, "f"'] \arrow[r, shift right, "1"'] & Z \arrow[l, shift right, "f"'] \arrow[r, shift right, "1"'] \arrow[d, "\varphi_i"] & Z \arrow[l, shift right, "f"'] \arrow[d, "\varphi_{i+1}"] & \cdots \\
X': & Z \oplus M \arrow[r, shift right, "\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}"'] & Z \oplus M^2 \arrow[l, shift right, "(1\,0)"'] \arrow[r, shift right, "\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}"'] & \cdots \arrow[l, shift right, "(1\,0)"'] \arrow[r, shift right, "\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}"'] & Z \oplus M^i \arrow[l, shift right, "(1\,0)"'] \arrow[r, shift right, "\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}"'] & Z \oplus M^{i+1} \arrow[l, shift right, "(1\,0)"'] & \cdots
\end{tikzcd}$$

with $\varphi_i = (f^i, gf^{i-1}, \dots, g)^t : Z \to Z \oplus M^i$. Both $X$ and $X'$ are almost exact tubes: they satisfy all requirements except those related to $a_1$ and $b_1$. The only condition left to be checked for $T_{f,g}$ is the exactness of the bottom row in the commutative diagram (Figure 1) with exact columns. This is done by diagram chasing.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & Z \arrow[r, "1"] \arrow[d, "\binom{f}{g}"] & Z \arrow[r, "f"] \arrow[d, "\begin{pmatrix} f^2 \\ gf \\ g \end{pmatrix}"] & Z \arrow[d, "\binom{f}{g}"] & \\
 & Z \oplus M \arrow[r, hook, "\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}"'] \arrow[d] & Z \oplus M^2 \arrow[r, two heads, "\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}"] \arrow[d] & Z \oplus M \arrow[d] & \\
0 \arrow[r] & N_1 \arrow[r, "\alpha_1"'] \arrow[d] & N_2 \arrow[r, "\beta_1"'] \arrow[d] & N_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Figure 1

By construction, $N_1 = \operatorname{coker} \varphi_1$ is isomorphic to $N$. Using Fitting's lemma in order to replace $Z$ by a direct summand if necessary in the exact sequence (1.1), we may assume that $f$ is nilpotent, say $f^h = 0$. Then $\varphi_{h+j}$ has the form $\varphi_{h+j} = (0, \dots, 0, gf^{h-1}, \dots, g)^t : Z \to Z \oplus M^{h+j}$, and its cokernel $N_{h+j}$ is isomorphic to $M^j \oplus N_h$ for $j \geq 0$. We conclude that $T_{f,g}$ is an $(M,N)$-tube of complexity at most $h$. In fact, $T_{f,g}$ is an $(M,N)$-tube even if $f$ is not nilpotent (compare with Proposition 4.2).

We define the *complexity* of a degeneration $M \leq_{\deg} N$ to be

$$\operatorname{cpl}(M,N) = \min \operatorname{cpl}(T),$$

where $T$ ranges over all $(M,N)$-tubes. This seems to be a good way to measure how "complicated" a degeneration is.

Indeed, we will prove in Sections 3 and 4 that a degeneration $M \leq_{\deg} N$ is of complexity 1 if and only if there exists a non-split exact sequence

$$0 \to N' \to M \to N'' \to 0$$

with $N \xrightarrow{\sim} N' \oplus N''$. So these are the "simplest" degenerations. In particular, any degeneration to an indecomposable $N$ must have complexity at least 2.

It is quite difficult to compute the complexity of a degeneration. The construction described before gives an estimate from above: if

$$0 \to Z \xrightarrow{\binom{f}{g}} Z \oplus M \to N \to 0$$

is an exact sequence and $f^h = 0$, then $\mathrm{cpl}(M, N) \leq h$. Conversely, it is easy to show that

$$\mathrm{cpl}(M, N) \geq \frac{\ell\ell(M)}{\ell\ell(N)} - 1,$$

where $\ell\ell(X)$ is the Loewy length of $X$; i.e., the smallest number $r$ for which $(\mathrm{rad}\, A)^r \cdot X = 0$ (see Proposition 3.5). Both bounds are sharp, but in general the complexity differs from both.

The complexity of a degeneration $M \leq_{\deg} N$ obtained from two degenerations $M \leq_{\deg} P \leq_{\deg} N$ seems to be quite unrelated to the sum of the complexities of $M \leq_{\deg} P$ and $P \leq_{\deg} N$. For instance, if we take non-split exact sequences

$$0 \to A_i \to B_i \to C_i \to 0, \quad i = 1, \ldots, r,$$

then there is a sequence of degenerations

$$\bigoplus_{i=1}^{r} B_i \leq_{\deg} \left(\bigoplus_{i=1}^{r-1} B_i\right) \oplus A_r \oplus C_r \leq_{\deg} \ldots \leq_{\deg} \left(\bigoplus_{i=1}^{s} B_i\right) \oplus \bigoplus_{i=s+1}^{r} (A_i \oplus C_i)$$

$$\leq_{\deg} \ldots \leq_{\deg} \bigoplus_{i=1}^{r} (A_i \oplus C_i),$$

but the complexity of

$$\bigoplus_{i=1}^{r} B_i \leq_{\deg} \bigoplus_{i=1}^{r} (A_i \oplus C_i)$$

is 1. On the other hand, we give an example of a chain of degenerations $M \leq_{\deg} P \leq_{\deg} N$ in Section 5.1 for which $\mathrm{cpl}(M, P) + \mathrm{cpl}(P, N) < \mathrm{cpl}(M, N)$. By Proposition 5.1, a minimal degeneration can have arbitrarily high complexity. A degeneration $M \leq_{\deg} N$ is called minimal if $M$ is not isomorphic to $N$ and moreover $M \leq_{\deg} P \leq_{\deg} N$ implies that $P$ is isomorphic to either $M$ or $N$.

## 2. Degenerations, bimodules and exact tubes

The following construction is explained in detail in [7] (compare also [2] and [3], pp. 176–177): If $M \leq_{\deg} N$ is a degeneration, there exists a discrete valuation $k$-algebra $R$ with maximal ideal $\mathfrak{m}$ and residue class field $k$ and an $A$-$R$-bimodule $\mathcal{Y}$, which is free of rank $d$ over $R$, such that

i) $\mathcal{Y}/\mathfrak{m} \cdot \mathcal{Y} \underset{A}{\xrightarrow{\sim}} N$

ii) $\mathcal{Y}$ contains $R \otimes_k M$ as an $A$-$R$-submodule.

These data are related to mapping a curve $c$ to $\operatorname{mod}_A^d(k)$ in such a way that its image lies generically in the orbit of $M$ and intersects the orbit of $N$. Assuming $c$ to be non-singular and passing to the completion, we may assume that $R = k[[t]]$. The representation $T = (N_i, \alpha_i, \beta_i)$ defined by the setting

$$N_i = \mathcal{Y}/(t^i) \cdot \mathcal{Y}$$

and letting $\alpha_i : N_i \to N_{i+1}$ and $\beta_i : N_{i+1} \to N_i$ be induced by multiplication by $t$ and the identity, respectively, is easily seen to be an exact tube, and by [7] it is moreover an $(M, N)$-tube.

This construction associating an exact tube with a bimodule is an equivalence:

**Proposition 2.1.** *The category $\mathcal{T}$ of exact tubes is equivalent to the category* $\operatorname{mod}f\ A\text{-}k[[t]]$ *of $A$-$k[[t]]$-bimodules which are free of finite rank over $k[[t]]$.*

*Proof.* We just describe a quasi-inverse functor. For an exact tube $T = (N_i, \alpha_i, \beta_i)$ we set

$$\mathcal{Y} = \varprojlim\,(N_i, \beta_i),$$

and we put

$$t \cdot (n_1, n_2, \ldots) = (0, \alpha_1(n_1), \alpha_2(n_2), \ldots)$$

for any infinite sequence $(n_1, n_2, \ldots)$ with $n_i \in N_i$ and $\beta_i(n_i) = n_{i-1}$ representing an element of $\mathcal{Y}$. As $T$ is an exact tube, this defines an $A$-$k[[t]]$-bimodule structure on $\mathcal{Y}$. As $t$ acts without torsion, $\mathcal{Y}$ is free as a $k[[t]]$-module, and its rank equals $\dim_k N_1$, since clearly $\mathcal{Y}/(t) \cdot \mathcal{Y}$ is isomorphic to $N_1$. $\square$

We give a direct construction of the bimodule corresponding to $T_{f,g}$ for an exact sequence

$$0 \to Z \xrightarrow{\binom{f}{g}} Z \oplus M \to N \to 0 \tag{2.1}$$

with a nilpotent map $f$. Set

$$\mathcal{Y}_{f,g} = k[[t]] \otimes_k M \oplus Z$$

as an $A$-module and define the action of $t$ on $Z$ by

$$t \cdot (0, z) = (1 \otimes g(z), f(z)).$$

Clearly, this action of $t$ is torsion free, and $\mathcal{Y}_{f,g}/(t)\mathcal{Y}_{f,g}$ is isomorphic to $N$, so that $\mathcal{Y}_{f,g}$ actually belongs to $\operatorname{mod}f\ A\text{-}k[[t]]$. It is easy to see that the exact tube associated with $\mathcal{Y}_{f,g}$ is $T_{f,g}$.

We will need the following truncated version of an exact tube:

**Definition 2.2.** For $m \geq 1$, an exact tube of height $m$ is a representation in $\mathrm{mod} A$

$$N_1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} N_2 \dots N_{m-1} \underset{\beta_{m-1}}{\overset{\alpha_{m-1}}{\rightleftarrows}} N_m$$

of the full subquiver $Q_m$ of $Q$ whose vertices are $1, 2, \dots, m$, such that the square

$$\begin{tikzcd} N_i \arrow[r, "\alpha_i"] \arrow[d, "\beta_{i-1}"] & N_{i+1} \arrow[d, "\beta_i"] \\ N_{i-1} \arrow[r, "\alpha_{i-1}"] & N_i \end{tikzcd}$$

is exact for $i = 1, \dots m-1$. Again we set $N_0 = 0$.

The category of exact tubes of height $m$ is equivalent to the category of $A$-$k[t]/(t^m)$-bimodules which are free of finite rank over $k[t]/(t^m)$.

Obviously, an exact tube $T$ restricts to an exact tube $T_{\leq m}$ of height $m$ for all $m$. We will see in Section 4 that an $M$-extendible tube $T = (N_i, \alpha_i, \beta_i)$ of height $h \geq 1$ (see next definition) is always the restriction of an $(M, N_1)$-tube.

**Definition 2.3.** A tube $T = (N_i, \alpha_i, \beta_i)$ of height $h$ is called $M$-extendible if there is a decomposition $N_h = Z \oplus Z'$ and an exact sequence

$$0 \to Z \xrightarrow{\binom{a}{b}} N_{h-1} \oplus M \xrightarrow{(c\, d)} Z' \to 0$$

such that $a = \beta_{h-1}|_Z$ and $c = pr_{Z'} \circ \alpha_{h-1}$, where $pr_{Z'} : Z \oplus Z' \to Z'$ is the natural projection.

We end this section with some questions. We do not know how to describe the full subcategory of $\mathrm{mod}f$ $A$-$k[[t]]$ corresponding to $(M, N)$-tubes. Conceivably, its objects are just those bimodules $\mathcal{Y}$ which contain $k[[t]] \otimes_k M$ as a subbimodule. This would follow if we knew that any $(M, N)$-tube is of the form $\mathcal{Y}_{f,g}$ for some exact sequence (2.1).

## 3. Complexity

**Definition 3.1.** We call a map

$$\begin{pmatrix} f \\ g \end{pmatrix} : Z \to Z \oplus M$$

an $(M, N)$-monomorphism provided $N$ is isomorphic to coker $\binom{f}{g}$.

Recall that, for a degeneration $M \leq_{\deg} N$, we defined the complexity as

$$\operatorname{cpl}(M, N) = \min \operatorname{cpl}(T),$$

where $T$ ranges over all $(M, N)$-tubes. There always are $(M, N)$-tubes with different complexities. For instance, if $(f,\, g)^t : Z \to Z \oplus M$ is an $(M, N)$-monomorphism and we set

$$f' = \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix} : Z^2 \longrightarrow Z^2, \quad g' = (g\ 0) : Z^2 \longrightarrow M,$$

the map $(f',\, g')^t$ will be an $(M, N)$-monomorphism, too, and it is easy to see that

$$\operatorname{cpl}(T_{f',g'}) = 2 \operatorname{cpl}(T_{f,g}).$$

**Theorem 3.2.** *Let $h \geq 1$ be a natural number and $M \leq_{\deg} N$ a degeneration. The following conditions are equivalent:*

(i) $\operatorname{cpl}(M, N) \leq h$

(ii) *There is an exact sequence*

$$0 \longrightarrow Z \xrightarrow{\binom{f}{g}} Z \oplus M \longrightarrow N \longrightarrow 0$$

*such that* $\operatorname{cpl}(T_{f,g}) \leq h$.

(iii) *There exists an exact tube $T = (N_i, \alpha_i, \beta_i)$ of height $2h + 1$ with $N \xrightarrow[A]{\sim} N_1$ and such that*

$$N_{h+j+1} \xrightarrow[A]{\sim} N_{h+j} \oplus M$$

*for $j = 0, \ldots, h$.*

(iv) *There exists an $M$-extendible exact tube $T = (N_i, \alpha_i, \beta_i)$ of height $h$ with $N \xrightarrow[A]{\sim} N_1$.*

*Proof.* Most ingredients for the proof will be given in Section 4. Here we indicate how they fit together: The implications (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) are obvious. The results of Section 4 up to Proposition 4.6 give that (ii) implies (iv), and Proposition 4.8 shows (iv) $\Rightarrow$ (ii). Finally, the implication (iii) $\Rightarrow$ (ii) follows from Proposition 4.9 and the next lemma. □

**Lemma 3.3.** *Let $T = (N_i, \alpha_i, \beta_i)$ be an $(M, N)$-tube, and assume that $N_{h+1} \xrightarrow{\sim} N_h \oplus M$ for some $h \geq 1$. Then* $\operatorname{cpl}(T) \leq h$.

*Proof.* As $T$ is an $(M, N)$-tube, there exists a natural number $j \geq h$ such that $N_{i+1} \xrightarrow{\sim} N_i \oplus M$ for all $i \geq j$. Take an integer $i$ with $h < i < j$, and consider the two exact squares

$$\begin{tikzcd}
N_h \oplus M \arrow[r, "\sim"] & N_{h+1} \arrow[r, "\alpha_i \cdots \alpha_{h+1}"] \arrow[d, "\beta_h"] & N_{i+1} \arrow[r, "\alpha_j \cdots \alpha_{i+1}"] \arrow[d, "\beta_i"] & N_{j+1} \arrow[r, "\sim"] \arrow[d, "\beta_j"] & N_j \oplus M \\
& N_h \arrow[r, "\alpha_{i-1} \cdots \alpha_h"] & N_i \arrow[r, "\alpha_{j-1} \cdots \alpha_i"] & N_j .
\end{tikzcd}$$

The big square splits, and therefore the two small squares split as well. We conclude that $N_{i+1}$ is isomorphic to $N_i \oplus M$. □

As $N_0 = 0$, our theorem takes the following simpler form for $h = 1, 2$:

**Corollary 3.4.** *Let $M \leq_{\mathrm{deg}} N$ be a degeneration. Then*

i) $\mathrm{cpl}(M,N) \leq 1$ *if and only if* $N = Z \oplus Z'$ *and there exists an exact sequence*

$$0 \to Z \to M \to Z' \to 0.$$

ii) $\mathrm{cpl}(M,N) \leq 2$ *if and only if there exist two exact squares*

$$\begin{tikzcd}
Z \arrow[r, "a"] \arrow[d] & N \arrow[d, "c"] \\
M \arrow[r] & Z'
\end{tikzcd} \qquad \begin{tikzcd}
N \arrow[r] \arrow[d, "c"] & Z \arrow[d, "a"] \\
Z' \arrow[r] & N
\end{tikzcd}$$

**Proposition 3.5.** *For any degeneration $M \leq_{\mathrm{deg}} N$ we have*

$$\mathrm{cpl}(M,N) \geq \frac{\ell\ell(M)}{\ell\ell(N)} - 1,$$

*where $\ell\ell(X)$ denotes the Loewy length of $X$; i.e., the smallest integer $r$ such that* $(\mathrm{rad}\, A)^r X = 0$.

*Proof.* Choose an $(M,N)$-tube $T = (N_i, \alpha_i, \beta_i)$ of complexity $h = \mathrm{cpl}(M,N)$. Then $M$ is a direct summand of $N_{h+1}$, and hence $\ell\ell(M) \leq \ell\ell(N_{h+1})$. We claim that, for all $i \geq 1$,

$$\ell\ell(N_i) \leq i\ \ell\ell(N_1).$$

In fact, for any exact sequence

$$0 \to A \to B \to C \to 0$$

the relation

$$\ell\ell(B) \leq \ell\ell(A) + \ell\ell(C)$$

holds true. Our claim follows by induction, considering the exact sequences

$$0 \to N_{i-1} \to N_i \to N_1 \to 0.$$

□

## 4. Exact tubes from monomorphisms

Throughout this section, $(f,g)^t : Z \to Z \oplus M$ denotes an $(M,N)$-monomorphism.

**Definition 4.1.** We call two exact tubes $T = (N_i, \alpha_i, \beta_i)$ and $T' = (N_i', \alpha_i', \beta_i')$ *similar* if $N_i$ is isomorphic to $N_i'$ for all $i \geq 1$.

So we do not ask for any compatibility with the maps in the tubes. Note that the property of being an $(M,N)$-tube is preserved under similarity, and so is complexity.

**Proposition 4.2.** *There is a direct summand $Z'$ of $Z$ and an exact sequence*

$$0 \to Z' \xrightarrow{\binom{f|_{Z'}}{g|_{Z'}}} Z' \oplus M \to N \to 0$$

*such that $f|_{Z'}$ is nilpotent and $T_{f,g}$ is similar to $T_{f|_{Z'}, g|_{Z'}}$. As a consequence, $T_{f,g}$ is an $(M,N)$-tube.*

*Proof.* By Fitting's lemma, there is a decomposition $Z = Z' \oplus Z''$ of $Z$ as a direct sum which is preserved under $f$ and such that $f' = f|_{Z'}$ is nilpotent and $f'' = f|_{Z''}$ is an automorphism of $Z''$. Set $g' = g|_{Z'}$ and $g'' = g|_{Z''}$. Obviously the maps

$$\begin{pmatrix} f'^i & 0 & g'f'^{i-1} & \cdots g' \\ 0 & f''^i & g''f''^{i-1} & \cdots g'' \end{pmatrix}^t : Z' \oplus Z'' \longrightarrow Z' \oplus Z'' \oplus M^i$$

and

$$(f'^i \quad g'f'^{i-1} \cdots g')^t : Z' \longrightarrow Z' \oplus M^i$$

have isomorphic cokernels as $(f'')^i$ is an isomorphism for $i \geq 1$. Since $f'$ is nilpotent, $T_{f',g'}$ is an $(M,N)$-tube. □

**Remark 4.3.** Suppose that $f^h = 0$. As

$$\varphi_{h+j} = (0, \ldots, 0, gf^{h-1}, \ldots, g)^t : Z \longrightarrow Z \oplus M^{h+j},$$

for $j \in \mathbb{N}$, the exact tube $T_{f,g}$ has the following particularly simple form:

$$N_{h+j} = Z \oplus M^j \oplus Z', \quad N_{h+j+1} = Z \oplus M^{j+1} \oplus Z',$$

$$\alpha_{h+j} = \begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix} : Z \oplus (M^j \oplus Z') \to Z \oplus M \oplus (M^j \oplus Z'),$$

$$\beta_{h+j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & l \end{pmatrix} : (Z \oplus M^j) \oplus M \oplus Z' \to (Z \oplus M^j) \oplus Z',$$

for $j \in \mathbb{N}$, where $Z'$ is a cokernel of

$$\psi = (g \circ (f^{h-1}, \ldots, f, 1))^t : Z \longrightarrow M^h$$

and

$$(k, \ \ l) : M \oplus Z' \longrightarrow Z'$$

is obtained from the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r, "\varphi_{h+1}"] \arrow[d, "f"] & Z \oplus M^{h+1} \arrow[r] \arrow[d, "(1\,0)"] & N_{h+1} = Z \oplus M \oplus Z' \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & k & l \end{smallmatrix}\right)"] & 0 \\
0 \arrow[r] & Z \arrow[r, "\varphi_h"'] & Z \oplus M^h \arrow[r] & N_h = Z \oplus Z' \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

Our next goal is to show that, up to similarity, we may choose $g \in \mathrm{rad}(Z, M)$. We start with an auxiliary result:

**Lemma 4.4.** *The tube $T_{f,g}$ is similar to $T_{f',g}$ with $f' = f - hg$, where $h : M \to Z$ is any homomorphism.*

*Proof.* It suffices to check the identity $\psi_i \circ {\varphi'}_i = \varphi_i$, for $i \geq 1$, where

$$\begin{aligned}
\varphi_i &= (f^i, gf^{i-1}, \ldots, g)^t : Z \longrightarrow Z \oplus M^i, \\
\varphi'_i &= ({f'}^i, g{f'}^{i-1}, \ldots, g)^t : Z \longrightarrow Z \oplus M^i
\end{aligned}$$

and

$$\psi_i := \begin{pmatrix}
1 & h & fh & f^2h & \cdots & f^{i-1}h \\
0 & 1 & gh & gfh & \cdots & gf^{i-2}h \\
0 & 0 & 1 & gh & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & gfh \\
\vdots & & \ddots & \ddots & \ddots & gh \\
0 & \cdots & \cdots & \cdots & 0 & 1
\end{pmatrix} : Z \oplus M^i \to Z \oplus M^i.$$

The key is the equation

$$f^r = {f'}^r + \sum_{s=0}^{r-1} f^s (hg) {f'}^{r-1-s}, \qquad r \geq 1,$$

which is proved by induction. □

**Proposition 4.5.** *There exists a direct summand $Z'$ of $Z$ and an exact sequence*

$$0 \to Z' \xrightarrow{\binom{f'}{g'}} Z' \oplus M \to N \to 0 \tag{4.1}$$

*with $g' \in \mathrm{rad}(Z', M)$ and such that $T_{f,g}$ is similar to $T_{f',g'}$.*

*Proof.* If $g \in \mathrm{rad}(Z, M)$, there is nothing to be proved. Otherwise, we prove that a sequence (4.1) exists such that $T_{f,g}$ is similar to $T_{f',g'}$ and $\dim Z' < \dim Z$ and then proceed by induction on $\dim Z$. We choose a non-zero direct summand $Z_2$ of $Z$ for which $g|_{Z_2}$ is a section. Replacing $Z$ by an isomorphic module if necessary, which leads to an isomorphic tube, we may assume that $Z = Z_1 \oplus Z_2$, $M = M_1 \oplus Z_2$,

$$g = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Applying the preceding lemma for

$$h = \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix},$$

we obtain a monomorphism $\binom{f''}{g}$ of the form

$$\begin{pmatrix} f'' \\ g \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 0 \\ q & 0 \\ 0 & 1 \end{pmatrix} : Z_1 \oplus Z_2 \to Z_1 \oplus Z_2 \oplus M_1 \oplus Z_2.$$

Now we may take $Z' = Z_1$, $M = Z_2 \oplus M_1$, $f' = a$ and $g' = \binom{b}{q}$. □

**Proposition 4.6.** *Set $h = \mathrm{cpl}(T_{f,g})$, and suppose that $g \in \mathrm{rad}(Z, M)$ and that $f$ is nilpotent. Then $(T_{f,g})_{\leq h}$ is $M$-extendible.*

*Proof.* Our assumptions on $f$ and $g$ imply that, for some $i$, the restriction $\psi|_Z$ of the composition

$$\psi = \begin{pmatrix} \varphi_i & 0 \\ 0 & 1_{M^h} \end{pmatrix} : Z \oplus M^h \to Z \oplus M^i \oplus M^h$$

of the maps

$$Z \oplus M^h \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{smallmatrix}\right)} Z \oplus M^{1+h} \to \cdots \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{smallmatrix}\right)} Z \oplus M^{i+h}$$

belongs to $\mathrm{rad}(Z, Z \oplus M^{i+h})$. By construction of $T_{f,g}$, the square

$$\begin{tikzcd} Z \oplus M^h \arrow[r, "\psi"] \arrow[d, "\pi_h"] & Z \oplus M^i \oplus M^h \arrow[d, "\pi_{h+i}"] \\ N_h \arrow[r, "\alpha_{i+h-1} \cdots \alpha_h"] & N_{i+h} \end{tikzcd}$$

is exact, where $\pi_j : Z \oplus M^j \to N_j$ is the projection to the cokernel of $\varphi_j : Z \to Z \oplus M^j$, and it splits, since $h = \mathrm{cpl}(T_{f,g})$. Therefore, $\pi_h|_Z$ is a section, and replacing $N_h$ by an isomorphic module, we may assume that

$$N_h = Z \oplus Z', \quad \pi_h = \begin{pmatrix} 1 & * & * \\ 0 & d & * \end{pmatrix} : Z \oplus M \oplus M^{h-1} \longrightarrow Z \oplus Z',$$

where $*$ is an arbitrary map.

Now consider the exact squares

$$\begin{tikzcd}[column sep=huge]
Z \oplus M^{h-1} \arrow[r, shift left, "\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[d, "\pi_{h-1}"] & Z \oplus M \oplus M^{h-1} \arrow[l, shift left, "\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}"] \arrow[d, "\pi_h = \begin{pmatrix} 1 & * & * \\ 0 & d & * \end{pmatrix}"] \\
N_{h-1} \arrow[r, shift left, "\alpha_{h-1}"] & Z \oplus Z'. \arrow[l, shift left, "\beta_{h-1}"]
\end{tikzcd}$$

It is easy to see that the square

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "\pi_{h-1}|_Z"] & M \arrow[d, "d"] \\
N_{h-1} \arrow[r, "pr_{Z'} \circ \alpha_{h-1}"] & Z'
\end{tikzcd}$$

is exact as well. Moreover, we have

$$\pi_{h-1}|_Z = \beta_{h-1}|_Z. \qquad \square$$

Next we recall a different construction for $T_{f,g}$, which has been presented for the most part in [6]. From $(f,\, g)^t$ we obtain the commutative diagram (Figure 2) with exact rows and $(k_i, l_i) = \beta_i(k_{i+1}, l_{i+1})$ for $i \leq m-1$.

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r, "\binom{f}{g}"] \arrow[d, "k_{m-1}"] & Z \oplus M \arrow[r, "{(k_1,l_1)}"] \arrow[d, "{(k_m,l_m)}"] & N_1 = N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N_{m-1} \arrow[r, "\alpha_{m-1}"] \arrow[d, "\beta_{m-2}"] & N_m \arrow[r, "\beta_1 \cdots \beta_{m-1}"] \arrow[d, "\beta_{m-1}"] & N_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N_{m-2} \arrow[r, "\alpha_{m-2}"] \arrow[d, dotted, no head] & N_{m-1} \arrow[r, "\beta_1 \cdots \beta_{m-2}"] \arrow[d, dotted, no head] & N_1 \arrow[r] \arrow[d, dotted, no head] & 0 \\
0 \arrow[r] & N_2 \arrow[r, "\alpha_2"] \arrow[d, "\beta_1"] & N_3 \arrow[r, "\beta_1 \beta_2"] \arrow[d, "\beta_2"] & N_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N_1 \arrow[r, "\alpha_1"] & N_2 \arrow[r, "\beta_1"] & N_1 \arrow[r] & 0
\end{tikzcd}$$

Figure 2

The next step is always obtained by squeezing the push-out of the top sequence by $k_m$ between the two top rows.

We claim that the exact tube $(N_i,\ \alpha_i,\ \beta_i)$ of height $m$ thus obtained is isomorphic to the restriction $(T_{f,g})_{\leq m}$ of $T_{f,g}$.

By induction, we obtain the following series of exact squares:

$$\begin{tikzcd}
Z \arrow[r, "\binom{f}{g}"] \arrow[d] & Z \oplus M \arrow[r, "\left(\begin{smallmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "\psi_1"] & Z \oplus M^2 \arrow[r, dotted, no head] \arrow[d, "\psi_2"] & Z \oplus M^{m-1} \arrow[r, "\left(\begin{smallmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "\psi_{m-1}"] & Z \oplus M^m \arrow[d, "\psi_m"] \\
0 \arrow[r] & N_1 \arrow[r, "\alpha_1"] & N_2 \arrow[r, dotted, no head] & N_{m-1} \arrow[r, "\alpha_{m-1}"] & N_m
\end{tikzcd}$$

with $\psi_i = (k_i,\ l_i,\ \alpha_{i-1}l_{i-1}, \ldots,\ \alpha_{i-1}\ldots\alpha_1 l_1) : Z \oplus M^i \to N_i$.

Note that the composition of the first $i$ maps of the top row is just $\varphi_i : Z \to Z \oplus M^i$ and that the sequence

$$0 \to Z \xrightarrow{\varphi_i} Z \oplus M^i \xrightarrow{\psi_i} N_i \to 0$$

is exact for $i = 1, \ldots, m$. So $N_i \xrightarrow{\sim} \operatorname{coker} \varphi_i$, and the maps $\alpha_i$ are the ones we claim. As for $\beta_i$, it suffices to show that

$$\psi_i \circ (1\ 0) = \beta_i \circ \psi_{i+1}.$$

This follows easily from the explicit formulas for $\psi_i,\ \psi_{i+1}$, the equation

$$(k_i,\ l_i) = \beta_i(k_{i+1},\ l_{i+1})$$

and the fact that $(N_i,\ \alpha_i,\ \beta_i)$ is an exact tube of height $m$. As a consequence we have:

**Remark 4.7.** Let $(f, g)^t$ be an $(M, N)$-monomorphism and $T' = (N'_i,\ \alpha'_i,\ \beta'_i)$ an exact tube of height $m$. Then $T'$ is isomorphic to $(T_{f,g})_{\leq m}$ if and only if there exists an exact square

$$\begin{tikzcd}
Z \arrow[r, "\binom{f}{g}"] \arrow[d, "\beta'_{m-1}\circ k"] & Z \oplus M \arrow[d, "{(k,l)}"] \\
N'_{m-1} \arrow[r, "\alpha'_{m-1}"] & N'_m
\end{tikzcd}\ .$$

**Proposition 4.8.** *Any $M$-extendible exact tube $T = (N_i,\ \alpha_i,\ \beta_i)$ of height $m$ with $N_1 \underset{A}{\xrightarrow{\sim}} N$ is the restriction of the exact tube $T_{f,g}$ to $Q_m$ for some $(M, N)$-monomorphism $(f, g)^t$.*

*Proof.* Let

$$0 \to Z \xrightarrow{\binom{a}{b}} N_{m-1} \oplus M \xrightarrow{(c\,d)} Z' \to 0$$

be an exact sequence with $N_m = Z \oplus Z'$, $a = \beta_{m-1}|_Z$ and $c = \mathrm{pr}_{Z'} \circ \alpha_{m-1}$. The square

$$\begin{tikzcd} N_m = Z \oplus Z' \arrow[r, "\left(\begin{smallmatrix} c'a & c'a' \\ b & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "(a, a') = \beta_{m-1}"'] & Z \oplus M \oplus Z' \arrow[d, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -d & ca' \end{smallmatrix}\right)"] \\ N_{m-1} \arrow[r, "\left(\begin{smallmatrix} c' \\ c \end{smallmatrix}\right) = \alpha_{m-1}"] & N_m = Z \oplus Z' \end{tikzcd}$$

is exact. Setting

$$N_{m+1} = Z \oplus M \oplus Z', \quad \alpha_m = \begin{pmatrix} c'a & c'a' \\ b & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -d & ca' \end{pmatrix}$$

we may extend $T$ to an exact tube of height $m+1$. By construction, the map

$$\begin{pmatrix} c'a \\ b \end{pmatrix} : Z \longrightarrow Z \oplus M$$

is an $(M, N)$-monomorphism, and the square

$$\begin{tikzcd} Z \arrow[r, "\left(\begin{smallmatrix} c'a \\ b \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & Z \oplus M \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)"] \\ N_m = Z \oplus Z' \arrow[r, "\alpha_m"] & N_{m+1} = Z \oplus M \oplus Z' \end{tikzcd}$$

is exact with

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \beta_m \circ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The result now follows from Remark 4.7. $\square$

**Proposition 4.9.** *Let $T = (N_i,\ \alpha_i,\ \beta_i)$ be an exact tube of height $h+m$, for some $h \geq 1$ and $m \geq 1$. Suppose that*

$$N_{h+j+1} \xrightarrow[A]{\sim} N_{h+j} \oplus M$$

*for $j \in \{0, \dots, m-1\}$. Then there is an $(M, N)$-monomorphism*

$$\begin{pmatrix} f \\ g \end{pmatrix} : N_{h+m-1} \longrightarrow N_{h+m-1} \oplus M$$

*such that the restrictions $T_{\leq m}$ and $(T_{f,g})_{\leq m}$ are isomorphic.*

*Proof.* We wish to choose

$$\begin{pmatrix} f \\ g \end{pmatrix} = \chi \circ \alpha_{h+m-1}$$

for a suitable isomorphism $\chi : N_{h+m} \to N_{h+m-1} \oplus M$ and to apply Remark 4.7 to the diagram

$$\begin{tikzcd}
N_{h+m-1} \arrow[r, "\alpha_{h+m-1}"] \arrow[d, "\beta=\beta_{m-1}\cdots\beta_{h+m-2}"'] & N_{h+m} \arrow[r, "\sim", "\chi"'] \arrow[d, "\beta_m\cdots\beta_{h+m-1}"'] & N_{h+m-1} \oplus M . \arrow[dl] \\
N_{m-1} \arrow[r, "\alpha_{m-1}"'] & N_m &
\end{tikzcd}$$

In order to do this, we only need to construct a section

$$s : N_{h+m-1} \longrightarrow N_{h+m}$$

satisfying

$$\beta_{m-1}\beta_m \cdots \beta_{h+m-1}s = \beta\beta_{h+m-1}s = \beta.$$

By our hypothesis, the square

$$\begin{tikzcd}
N_{h+1} \arrow[r, "\alpha"] \arrow[d, "\beta_h"] & N_{h+m} \arrow[d, "\beta_{h+m-1}"] \\
N_h \arrow[r, "\alpha'"] & N_{h+m-1}
\end{tikzcd}$$

splits, where $\alpha = \alpha_{h+m-1} \dots \alpha_{h+1}$ and $\alpha' = \alpha_{h+m-2} \dots \alpha_h$. Choose a maximal direct summand $A$ of $N_{h+m}$ for which $\beta_{h+m-1}|_A$ is a section. Replacing $T$ by an isomorphic exact tube, we may assume that we have

$$\beta_{h+m-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & \delta \end{pmatrix} : A \oplus B \oplus M \longrightarrow A \oplus B,$$

$$\alpha' = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} : C \oplus B \longrightarrow A \oplus B$$

for some maps $\gamma$, $\delta$, $\varepsilon$. Setting

$$s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} : A \oplus B \longrightarrow A \oplus B \oplus M,$$

we obtain

$$1_{A\oplus B} - \beta_{h+m-1}s = \begin{pmatrix} 0 & 0 \\ 0 & 1-\gamma \end{pmatrix} : A \oplus B \longrightarrow A \oplus B,$$

which factors through $\alpha'$. But the sequence

$$0 \longrightarrow N_h \xrightarrow{\alpha'} N_{h+m-1} \xrightarrow{\beta} N_{m-1} \longrightarrow 0$$

is exact, which implies $\beta = \beta\beta_{h+m-1}s$ as required. $\square$

## 5. Examples

All our examples are representations of quivers with relations. Let $Q$ be a quiver with vertex set $Q_0 = \{1, \ldots, n\}$, $I$ an admissible two-sided ideal in the quiver algebra $kQ$, $\underline{d} = (d_1, \ldots d_n)$ a vector in $\mathbb{N}^n$, and denote by

$$\mathrm{Rep}(Q, I, \underline{d})$$

the affine algebraic variety of representations $X$ of $(Q, I)$ with $X(i) = k^{d_i}$, $i \in Q_0$. The dimension vector of $X$ in $\mathrm{Rep}(Q, I, \underline{d})$ is $\underline{d}$. The group $G(\underline{d}) = \prod_{i=1}^{n} GL(d_i)$ acts on $\mathrm{Rep}(Q, I, \underline{d})$ by

$$(g \cdot X)(\alpha) = g_j \circ X(\alpha) \circ g_i^{-1}$$

for an arrow $\alpha : i \to j$ and $g = (g_1, \ldots, g_n) \in G(\underline{d})$.

If we view $M, N$ in $\mathrm{Rep}(Q, I, \underline{d})$ as modules over $kQ/I$ of dimension $d = \sum_{i=1}^{n} d_i$, then $M$ degenerates to $N$ if and only if the representation $N$ belongs to the closure of the orbit $G(\underline{d}) \cdot M$ of $M$ in $\mathrm{Rep}(Q, I, \underline{d})$ [1]. This allows us to work with the smaller group $G(\underline{d})$.

**5.1.** We begin with an example of a degeneration whose complexity is easy to compute: Choose a natural number $n \geq 2$ and let $\vec{A_n}$ be the equioriented quiver with underlying graph $A_n$:

$$\vec{A_n} = 1 \xleftarrow{\gamma_1} 2 \longleftarrow \cdots \longleftarrow n-1 \xleftarrow{\gamma_{n-1}} n.$$

Denote by $X_i$ the indecomposable representation of $\vec{A_n}$ given by

$$X_i(j) = \begin{cases} k & j \leq i, \\ 0 & j > i, \end{cases}$$
$$X_i(\gamma_j) = \begin{cases} 1 & j < i, \\ 0 & j \geq i. \end{cases}$$

Then $M = X_n$ has a filtration

$$M = X_n \supset X_{n-1} \supset \cdots \supset X_2 \supset X_1,$$

and it is well-known that $M$ degenerates to the associated graded module

$$N = \bigoplus_{i=1}^{n} X_i / X_{i-1},$$

where we set $X_0 = 0$. We wish to compute the complexity $\mathrm{cpl}(M, N)$, thereby showing again that $M$ actually degenerates to $N$. Set

$$Z = \bigoplus_{i=1}^{n-1} X_i,$$

$$f = \begin{pmatrix} 0 & & & 0 \\ \iota_2 & \ddots & & \\ & \ddots & \ddots & \\ & & \iota_{n-1} & 0 \end{pmatrix} : Z \longrightarrow Z \quad \text{and}$$

$$g = (0 \cdots 0 \ \ \iota_n) : Z \longrightarrow M = X_n,$$

where $\iota_i : X_{i-1} \to X_i$ is the inclusion. It is easy to check that $(f,g)^t$ is an $(M,N)$-monomorphism. Moreover, $f^{n-1} = 0$, and thus

$$\mathrm{cpl}(M,N) \leq n-1.$$

On the other hand, the Loewy lengths of $M$ and $N$ are $n$ and 1, respectively, which implies

$$\mathrm{cpl}(M,N) \geq \frac{\ell\ell(M)}{\ell\ell(N)} - 1 = n-1$$

by Proposition 3.5. This example shows that there are degenerations of arbitrary complexity.

Note that for $n = 4$ we obtain the following chain of degenerations:

$$M = k \xleftarrow{1} k \xleftarrow{1} k \xleftarrow{1} k \leq_{\mathrm{deg}} P = k \xleftarrow{1} k \xleftarrow{0} k \xleftarrow{1} k$$
$$\leq_{\mathrm{deg}} N = k \xleftarrow{0} k \xleftarrow{0} k \xleftarrow{0} k.$$

The complexities are

$$\mathrm{cpl}(M,P) = 1 = \mathrm{cpl}(P,N) \qquad \text{and}$$
$$\mathrm{cpl}(M,N) = 3 > \mathrm{cpl}(M,P) + \mathrm{cpl}(P,N).$$

Comparing with the example given in the introduction, we see that $\mathrm{cpl}(M,P) + \mathrm{cpl}(P,N)$ can be either smaller or greater than $\mathrm{cpl}(M,N)$ for a chain

$$M \leq_{\mathrm{deg}} P \leq_{\mathrm{deg}} N.$$

**5.2.** Next we give an example of a minimal degeneration of arbitrary complexity: Let $Q$ be the quiver

$$Q = 1 \xrightarrow{\alpha} 2 \circlearrowleft \beta\ ,$$

choose a natural number $n \geq 2$, and let $I$ be the ideal generated by $\beta^n$. Define $M$ and $N$ to be the representations of dimension vector $(1,n)$ given by

$$M(\alpha) = e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad N(\alpha) = e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$M(\beta) = N(\beta) = J_n, \quad \text{respectively,}$$

where $e_1, \ldots, e_n$ is the standard basis of $k^n$ and $J_m$ is the Jordan block

$$J_m = \begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}$$

in $\mathbb{M}_m(k)$, for $m \in \mathbb{N}$.

**Proposition 5.1.** *There is a degeneration $M \leq_{\deg} N$, which is minimal, and $\mathrm{cpl}(M,N) = n$.*

*Proof.* Denote by $Z$ the indecomposable representation with dimension vector $(0,n)$, given by $Z(\beta) = J_n$, and let $(f,g)^t : Z \longrightarrow Z \oplus M$ be given by

$$f = (0, J_n), \quad g = (0, 1).$$

It is easy to see that $(f,g)^t$ is an $(M,N)$-monomorphism, so $M$ degenerates to $N$. Moreover, we have $f^n = 0$, and therefore $\mathrm{cpl}(M,N) \leq n$. As

$$\dim \mathrm{End}\, M = 1 \text{ and } \dim \mathrm{End}\, N = 2,$$

the orbit of $N$ has codimension 1 in the closure of the orbit of $M$, which implies that the degeneration is minimal.

Suppose $\mathrm{cpl}(M,N) \leq n-1$, and choose an $(M,N)$-tube $T = (N_i, \alpha_i, \beta_i)$ with $N_n \xrightarrow{\sim} N_{n-1} \oplus M$. Let $\psi_n : N_n \to M$ be the surjection obtained from this decomposition.

**Claim.** *For $i = 1, \ldots, n$, there exists a surjection*

$$\psi_i : N_i \longrightarrow M^{(i)},$$

*where $M^{(i)}$ has dimension vector $(1,i)$ and is given by*

$$M^{(i)}(\alpha) = (1, 0, \ldots, 0)^t, \quad M^{(i)}(\beta) = J_i.$$

Using the claim for $i = 1$, we obtain a surjection $\psi_1 : N_1 = N \longrightarrow M^{(1)}$, which is impossible.

We prove the claim by descending induction on $i$. Observe that any map from $N$ to $M^{(i)}$ factors through the socle $\mathrm{soc} M^{(i)}$ and that $M^{(i)}/\mathrm{soc} M^{(i)} \xrightarrow{\sim} M^{(i-1)}$. Writing this factorization for $\psi_i \circ \alpha_{i-1} \cdots \alpha_1$, we obtain $\psi_{i-1} : N_{i-1} \to M^{(i-1)}$ from the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\alpha_{i-1}\cdots\alpha_1"] \arrow[d] & N_i \arrow[r, "\beta_{i-1}"] \arrow[d, "\psi_i"] & N_{i-1} \arrow[r] \arrow[d, dotted, "\psi_{i-1}"] & 0 \\ 0 \arrow[r] & \mathrm{soc} M^{(i)} \arrow[r] & M^{(i)} \arrow[r] & M^{(i-1)} \arrow[r] & 0. \end{tikzcd}$$

As $\psi_i$ is surjective, $\psi_{i-1}$ is as well. ☐

A version of this argument implies the following result, which we will not use:

$$\mathrm{cpl}(M, M_i) = \left[\frac{n-1}{i-1}\right] + 1, \quad i \geq 2.$$

The representation $M_i$ is given by

$$M_i(\alpha) = e_i, \quad M_i(\beta) = J_n.$$

**5.3.** We now exhibit a degeneration $M \leq_{\deg} N$ of complexity 2 with the property that $f^2 \neq 0$ for all $(M, N)$-monomorphisms $(f, g)^t$. Therefore the complexity can be strictly less than the "index of nilpotence of $M$ and $N$"; i.e., the number

$$\min\{r : f^r = 0\},$$

where the minimum is taken over all $(M, N)$-monomorphisms $(f, g)^t$. We stay with the same quiver $Q$, and we choose $I$ to be generated by $\beta^3$; i.e., we set $n = 3$ in the preceding example. Note that $kQ/I$ is representation-finite: it admits 29 indecomposables [4].

We let $M$ and $N$ be given by

$$M(\alpha) = e_2, \ \ N(\alpha) = e_3, \ \ M(\beta) = N(\beta) = J_3,$$

where $e_1$, $e_2$, $e_3$ is the standard basis of $k^3$. Choose

$$\begin{tikzcd} Z' = 0 \arrow[r] & k^3 \arrow[loop right, "J_3"] \end{tikzcd}$$

$$\begin{aligned} f' = (0,\, J_3) &: Z' \longrightarrow Z' \quad \text{and} \\ g' = (0,\, 1) &: Z' \longrightarrow M. \end{aligned}$$

Then $(f', g')^t$ is an $(M, N)$-monomorphism. As ${f'}^2$ factors through $g'$, the cokernel $N_3$ of the map

$$\varphi_3 = ({f'}^3,\, g'{f'}^2,\, g'f',\, g')^t : Z' \longrightarrow Z' \oplus M^3$$

used to define the tube $T_{f',g'} = (N_i, \alpha_i, \beta_i)$ is isomorphic to the cokernel of

$$({f'}^2,\, 0,\, g'f',\, g')^t : Z' \longrightarrow Z' \oplus M^3$$

and thus isomorphic to $M \oplus N_2$. By Lemma 3.3, we know that

$$\mathrm{cpl}(M, N) \leq 2.$$

On the other hand, as $N$ is indecomposable, the complexity must exceed 1, so

$$\operatorname{cpl}(M,N)=2.$$

**Claim.** *For any $(M,N)$-monomorphism*

$$(f,g)^t : Z \longrightarrow Z\oplus M,$$

*we have $f^2\neq 0$.*

First we show:

**Lemma 5.2.** *For any $(M,N)$-monomorphism*

$$(f,g)^t : Z \longrightarrow Z\oplus M,$$

*$Z'$ is a direct summand of $Z$.*

*Proof.* Consider the exact sequences

$$\Sigma' : 0 \longrightarrow Z' \xrightarrow{\binom{f'}{g'}} Z'\oplus M \xrightarrow{(k',l')} N \longrightarrow 0$$

and

$$\Sigma : 0 \longrightarrow Z \xrightarrow{\binom{f}{g}} Z\oplus M \xrightarrow{(k,l)} N \longrightarrow 0.$$

It is easy to check that

$$\begin{aligned} \dim\operatorname{Hom}(Z',M) &= \dim\operatorname{Hom}(Z',N)=3,\\ \dim\operatorname{End} M &= \dim\operatorname{Hom}(M,N)=2. \end{aligned}$$

Therefore the sequence of vector spaces

$$0\longrightarrow \operatorname{Hom}(Z'\oplus M,Z)\longrightarrow \operatorname{Hom}(Z'\oplus M,\, Z\oplus M)\longrightarrow \operatorname{Hom}(Z'\oplus M,N)\longrightarrow 0$$

obtained from mapping $Z'\oplus M$ into $\Sigma$ is exact. In particular, $(k',l') : Z'\oplus M\to N$ factors through $(k,l): Z\oplus M\to N$, and hence we have the following commutative diagram (Figure 3) with exact rows and columns.

$$\begin{tikzcd}
\Sigma: & & & 0 & 0 & \\
& 0 \arrow[r] & Z \arrow[r] & Z \oplus M \arrow[r, "{(k,l)}"] \arrow[u] & N \arrow[r] \arrow[u] & 0 \\
& 0 \arrow[r] & Z \arrow[r, "\binom{1}{0}"] \arrow[u, equal] & Z \oplus (Z' \oplus M) \arrow[r, "(0\ \ 1)"] \arrow[u] & Z' \oplus M \arrow[r] \arrow[u, "{(k',l')}"'] & 0 \\
& & & Z' \arrow[r, equal] \arrow[u, "\left(\begin{smallmatrix} s \\ f' \\ g' \end{smallmatrix}\right)"'] & Z' \arrow[u, "\binom{f'}{g'}"'] & \\
& & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Figure 3

So the middle column splits as well, and since by construction $f', g'$ lie in the radical,

$$s : Z' \longrightarrow Z$$

must be a section. $\square$

Let $(f,g)^t : Z \to Z \oplus M$ be an $(M,N)$-monomorphism, suppose $f^2 = 0$, and consider the commutative diagram (Figure 4) with exact rows and columns.

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker f \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\binom{0}{1}"] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z \arrow[r, "\binom{f}{g}"] \arrow[d] & Z \oplus M \arrow[r, "{(k,l)}"] \arrow[d, "{(1,0)}"] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{im} f \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \operatorname{coker} f \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Figure 4

Then $X$ must be a quotient of $M$ and a submodule of $N$, which is possible in exactly two ways:

(i) $X = k \xrightarrow{\binom{0}{1}} k^2 \circlearrowleft J_2$

(ii) $X = 0 \longrightarrow k \circlearrowleft 0$

In the first case, we have

$$\ker f = 0 \longrightarrow k \circlearrowleft 0 \ ,$$

and our assumption $f^2 = 0$ implies that $\dim Z(2) \leq 2$. But then $Z$ cannot contain $Z'$ as a direct summand.

In the second case, we see that

$$\ker f = k \xrightarrow{\binom{1}{0}} k^2 \circlearrowleft J_2 \ .$$

Now $f^2 = 0$ implies that $\dim Z(2) \leq 4$. But then necessarily $Z(\beta\alpha) = 0$, since $Z'$ must be a direct summand of $Z$, and $Z$ cannot contain $\ker f$ as a submodule.

**5.4.** As our last example, we find a degeneration $M \leq_{\text{deg}} N$ of complexity 2 for which there exists an exact sequence

$$\Sigma : 0 \longrightarrow N \xrightarrow{\alpha_1 = \binom{f}{g}} N \oplus M \xrightarrow{\beta_1 = (f, -l)} N \longrightarrow 0.$$

So we have an exact tube

$$T = (N_1 = N,\ N_2 = N \oplus M,\ \alpha_1,\ \beta_1)$$

of height 2. If this tube were the restriction of an (infinite) exact tube, the complexity $\operatorname{cpl}(M, N)$ would have to equal 1. So the number $2h + 1$ in condition (iii) of our main theorem cannot be replaced by $2h$.

Choose $A = k[\alpha, \beta]/(\alpha^2, \beta^2)$, let $M$ and $N$ be 4-dimensional with

$$M(\alpha) = N(\alpha) = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix},\ M(\beta) = \begin{pmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \end{pmatrix},\ N(\beta) = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix},$$

and set

$$f = \begin{pmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{pmatrix},\ g = \begin{pmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \end{pmatrix},\ l = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \end{pmatrix}.$$

It is easy to check that the sequence $\Sigma$ obtained from these choices is exact. So $M$ degenerates to $N$. As $N$ is indecomposable and $f^2$ factors through $g$, the same argument as in Section 5.3 implies that $\operatorname{cpl}(M, N) = 2$.

This example has another surprising feature: For any degeneration $M \leq_{\text{deg}} N$ we obtain

$$\operatorname{cpl}(M^r, N^r) \leq \operatorname{cpl}(M, N), \quad r \geq 1,$$

by taking for $M^r \leq_{\deg} N^r$ the direct sum of $r$ copies of an $(M, N)$-tube of minimal complexity. In our example, we have

$$\mathrm{cpl}(M^2, N^2) = 1 < \mathrm{cpl}(M, N) = 2.$$

Indeed, $M^2$ is a projective cover for $N$, and the kernel of an epimorphism $M^2 \to N$ is $N$ again. So there is an exact sequence

$$0 \longrightarrow N \longrightarrow M^2 \longrightarrow N \longrightarrow 0.$$

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