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Complexity of degenerations of modules

R. Aehle, Ch. Riedtmann and G. Zwara

Abstract. A module M over an associative algebra A over an algebraically closed field k is said to degenerate to a module N if N belongs to the closure of the isomorphism class of M in the algebraic variety of d -dimensional A -modules, $d \in \mathbb{N}$. We associate a non-negative integer to a degeneration $M \leq_{\text{deg}} N$, its complexity, and study its properties.

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1. Introduction

Let k be an algebraically closed field, A a finite dimensional associative k -algebra with a unit and $\text{mod } A$ the category of finite dimensional left A -modules. Let $\mathbb{M}_d(k)$ denote the k -algebra of the $d \times d$ -matrices with coefficients in k . We view A as a quotient of a free associative algebra $k\langle X_1, \dots, X_r \rangle$ by a two-sided ideal I . We define the affine variety $\text{mod}_A^d(k)$ as the set of r -tuples (m_1, \dots, m_r) such that $m_i \in \mathbb{M}_d(k)$ and $\rho(m_1, \dots, m_r)$ is the zero matrix for any $\rho \in I$. The general linear group $GL_d(k)$ acts on $\text{mod}_A^d(k)$ by conjugation.

As an ordinary set, $\text{mod}_A^d(k)$ is just the set $\text{Hom}_{k\text{-alg}}(A, \mathbb{M}_d(k))$ and hence $GL_d(k)$ -orbits in $\text{mod}_A^d(k)$ correspond bijectively to isomorphism classes of d -dimensional left A -modules.

Let M and N be two d -dimensional A -modules. By definition, M degenerates to N , noted $M \leq_{\text{deg}} N$, if N lies in the closure of the $GL_d(k)$ -orbit of M in $\text{mod}_A^d(k)$, with respect to the Zariski topology. This defines a partial order on the set of isomorphism classes of d -dimensional A -modules.

Denote by Q the quiver

$$Q = 1 \begin{array}{c} \xrightarrow{a_1} \\ \xleftarrow{b_1} \end{array} 2 \begin{array}{c} \xrightarrow{a_2} \\ \xleftarrow{b_2} \end{array} 3 \dots$$

with vertex set $Q_0 = \mathbb{N} \setminus \{0\}$ and arrows $a_i : i \rightarrow i+1$, $b_i : i+1 \rightarrow i$ for every $i \in Q_0$.

We call a representation

$$T = N_1 \begin{smallmatrix} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{smallmatrix} N_2 \cdots N_i \begin{smallmatrix} \xrightarrow{\alpha_i} \\ \xleftarrow{\beta_i} \end{smallmatrix} N_{i+1} \cdots$$

of Q in $\text{mod } A$, the category of finite dimensional A -modules, an *exact tube* if the sequence

$$0 \rightarrow N_i \xrightarrow{\begin{pmatrix} \beta_{i-1} \\ \alpha_i \end{pmatrix}} N_{i-1} \oplus N_{i+1} \xrightarrow{(-\alpha_{i-1}, \beta_i)} N_i \rightarrow 0$$

or equivalently the square

$$\begin{array}{ccc} N_i & \xrightarrow{\alpha_i} & N_{i+1} \\ \downarrow \beta_{i-1} & & \downarrow \beta_i \\ N_{i-1} & \xrightarrow{\alpha_{i-1}} & N_i \end{array}$$

is exact for all $i \geq 1$. Here we set $N_0 = 0$. Note that N_i is an A -module, that α_i, β_i are A -linear and that α_i is injective, β_i is surjective, for all $i \geq 1$. We say that T is an (M, N) -tube if there is a natural number h such that

- (i) $N_1 \xrightarrow[A]{\sim} N$,
- (ii) $N_{h+j+1} \xrightarrow[A]{\sim} N_{h+j} \oplus M$, for all $j \in \mathbb{N}$.

We call the smallest such number h the complexity $\text{cpl}(T)$ of the tube.

Let T be an (M, N) -tube. Note that the sequence

$$0 \rightarrow N_k \xrightarrow{\alpha_k} N_{k+1} \xrightarrow{\beta_1 \cdots \beta_k} N_1 \rightarrow 0$$

is exact for any k . As N_{k+1} is isomorphic to $N_k \oplus M$ for $k \geq \text{cpl}(T)$, there is an exact sequence

$$0 \rightarrow N_k \rightarrow N_k \oplus M \rightarrow N \rightarrow 0,$$

and therefore M degenerates to N [5].

Conversely, whenever M degenerates to N , there exists an (M, N) -tube: Indeed, the third author showed in [7] that there is a short exact sequence

$$0 \rightarrow Z \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} Z \oplus M \rightarrow N \rightarrow 0, \quad (1.1)$$

and in [6] he associated an exact tube $T_{f,g}$ with such a sequence (see also Section 4). In fact, $T_{f,g}$ is the cokernel of the injection $\varphi : X \rightarrow X'$ between the following representations of Q :

$$\begin{array}{ccccccc} X : & & Z & \xrightleftharpoons[1]{f} & Z & \xrightleftharpoons[1]{f} & \cdots & \xrightleftharpoons[1]{f} & Z & \xrightleftharpoons[1]{f} & Z & \cdots \\ \varphi \downarrow & & \varphi_1 \downarrow & & \varphi_2 \downarrow & & & & \varphi_i \downarrow & & \varphi_{i+1} \downarrow & \\ X' : & & Z \oplus M & \xrightleftharpoons[\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}]{(10)} & Z \oplus M^2 & \xrightleftharpoons[\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}]{(10)} & \cdots & \xrightleftharpoons[\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}]{(10)} & Z \oplus M^i & \xrightleftharpoons[\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}]{(10)} & Z \oplus M^{i+1} & \cdots \end{array}$$

with $\varphi_i = (f^i, gf^{i-1}, \dots, g)^t : Z \rightarrow Z \oplus M^i$. Both X and X' are almost exact tubes: they satisfy all requirements except those related to a_1 and b_1 . The only condition left to be checked for $T_{f,g}$ is the exactness of the bottom row in the commutative diagram (Figure 1) with exact columns. This is done by diagram chasing.

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & Z & \xrightarrow{1} & Z & \xrightarrow{f} & Z & \\
 & \downarrow \binom{f}{g} & & \downarrow \binom{f^2}{gf} & & \downarrow \binom{f}{g} & \\
 & Z \oplus M & \xrightarrow{\binom{f \ 0}{g \ 0}} & Z \oplus M^2 & \xrightarrow{\binom{1 \ 0 \ 0}{0 \ 1 \ 0}} & Z \oplus M & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \longrightarrow & N_1 & \xrightarrow{\alpha_1} & N_2 & \xrightarrow{\beta_1} & N_1 & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

Figure 1

By construction, $N_1 = \operatorname{coker} \varphi_1$ is isomorphic to N . Using Fitting's lemma in order to replace Z by a direct summand if necessary in the exact sequence (1.1), we may assume that f is nilpotent, say $f^h = 0$. Then φ_{h+j} has the form $\varphi_{h+j} = (0, \dots, 0, gf^{h-1}, \dots, g)^t : Z \rightarrow Z \oplus M^{h+j}$, and its cokernel N_{h+j} is isomorphic to $M^j \oplus N_h$ for $j \geq 0$. We conclude that $T_{f,g}$ is an (M, N) -tube of complexity at most h . In fact, $T_{f,g}$ is an (M, N) -tube even if f is not nilpotent (compare with Proposition 4.2).

We define the *complexity* of a degeneration $M \leq_{\deg} N$ to be

$$\operatorname{cpl}(M, N) = \min \operatorname{cpl}(T),$$

where T ranges over all (M, N) -tubes. This seems to be a good way to measure how “complicated” a degeneration is.

Indeed, we will prove in Sections 3 and 4 that a degeneration $M \leq_{\deg} N$ is of complexity 1 if and only if there exists a non-split exact sequence

$$0 \rightarrow N' \rightarrow M \rightarrow N'' \rightarrow 0$$

with $N \xrightarrow{\sim} N' \oplus N''$. So these are the “simplest” degenerations. In particular, any degeneration to an indecomposable N must have complexity at least 2.

It is quite difficult to compute the complexity of a degeneration. The construc-

tion described before gives an estimate from above: if

$$0 \rightarrow Z \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} Z \oplus M \rightarrow N \rightarrow 0$$

is an exact sequence and $f^h = 0$, then $\text{cpl}(M, N) \leq h$. Conversely, it is easy to show that

$$\text{cpl}(M, N) \geq \frac{\ell\ell(M)}{\ell\ell(N)} - 1,$$

where $\ell\ell(X)$ is the Loewy length of X ; i.e., the smallest number r for which $(\text{rad } A)^r \cdot X = 0$ (see Proposition 3.5). Both bounds are sharp, but in general the complexity differs from both.

The complexity of a degeneration $M \leq_{\text{deg}} N$ obtained from two degenerations $M \leq_{\text{deg}} P \leq_{\text{deg}} N$ seems to be quite unrelated to the sum of the complexities of $M \leq_{\text{deg}} P$ and $P \leq_{\text{deg}} N$. For instance, if we take non-split exact sequences

$$0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0, \quad i = 1, \dots, r,$$

then there is a sequence of degenerations

$$\begin{aligned} \bigoplus_{i=1}^r B_i &\leq_{\text{deg}} \left(\bigoplus_{i=1}^{r-1} B_i \right) \oplus A_r \oplus C_r \leq_{\text{deg}} \dots \leq_{\text{deg}} \left(\bigoplus_{i=1}^s B_i \right) \oplus \bigoplus_{i=s+1}^r (A_i \oplus C_i) \\ &\leq_{\text{deg}} \dots \leq_{\text{deg}} \bigoplus_{i=1}^r (A_i \oplus C_i), \end{aligned}$$

but the complexity of

$$\bigoplus_{i=1}^r B_i \leq_{\text{deg}} \bigoplus_{i=1}^r (A_i \oplus C_i)$$

is 1. On the other hand, we give an example of a chain of degenerations $M \leq_{\text{deg}} P \leq_{\text{deg}} N$ in Section 5.1 for which $\text{cpl}(M, P) + \text{cpl}(P, N) < \text{cpl}(M, N)$. By Proposition 5.1, a minimal degeneration can have arbitrarily high complexity. A degeneration $M \leq_{\text{deg}} N$ is called minimal if M is not isomorphic to N and moreover $M \leq_{\text{deg}} P \leq_{\text{deg}} N$ implies that P is isomorphic to either M or N .

2. Degenerations, bimodules and exact tubes

The following construction is explained in detail in [7] (compare also [2] and [3], pp. 176–177): If $M \leq_{\text{deg}} N$ is a degeneration, there exists a discrete valuation k -algebra R with maximal ideal \mathfrak{m} and residue class field k and an A - R -bimodule \mathcal{Y} , which is free of rank d over R , such that

- i) $\mathcal{Y}/\mathfrak{m} \cdot \mathcal{Y} \xrightarrow[A]{\sim} N$
- ii) \mathcal{Y} contains $R \otimes_k M$ as an A - R -submodule.

These data are related to mapping a curve c to $\text{mod}_A^d(k)$ in such a way that its image lies generically in the orbit of M and intersects the orbit of N . Assuming c to be non-singular and passing to the completion, we may assume that $R = k[[t]]$. The representation $T = (N_i, \alpha_i, \beta_i)$ defined by the setting

$$N_i = \mathcal{Y}/(t^i) \cdot \mathcal{Y}$$

and letting $\alpha_i : N_i \rightarrow N_{i+1}$ and $\beta_i : N_{i+1} \rightarrow N_i$ be induced by multiplication by t and the identity, respectively, is easily seen to be an exact tube, and by [7] it is moreover an (M, N) -tube.

This construction associating an exact tube with a bimodule is an equivalence:

Proposition 2.1. *The category \mathcal{T} of exact tubes is equivalent to the category $\text{mod } A\text{-}k[[t]]$ of $A\text{-}k[[t]]$ -bimodules which are free of finite rank over $k[[t]]$.*

Proof. We just describe a quasi-inverse functor. For an exact tube $T = (N_i, \alpha_i, \beta_i)$ we set

$$\mathcal{Y} = \varprojlim (N_i, \beta_i),$$

and we put

$$t \cdot (n_1, n_2, \dots) = (0, \alpha_1(n_1), \alpha_2(n_2), \dots)$$

for any infinite sequence (n_1, n_2, \dots) with $n_i \in N_i$ and $\beta_i(n_i) = n_{i-1}$ representing an element of \mathcal{Y} . As T is an exact tube, this defines an $A\text{-}k[[t]]$ -bimodule structure on \mathcal{Y} . As t acts without torsion, \mathcal{Y} is free as a $k[[t]]$ -module, and its rank equals $\dim_k N_1$, since clearly $\mathcal{Y}/(t) \cdot \mathcal{Y}$ is isomorphic to N_1 . \square

We give a direct construction of the bimodule corresponding to $T_{f,g}$ for an exact sequence

$$0 \rightarrow Z \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} Z \oplus M \rightarrow N \rightarrow 0 \quad (2.1)$$

with a nilpotent map f . Set

$$\mathcal{Y}_{f,g} = k[[t]] \otimes_k M \oplus Z$$

as an A -module and define the action of t on Z by

$$t \cdot (0, z) = (1 \otimes g(z), f(z)).$$

Clearly, this action of t is torsion free, and $\mathcal{Y}_{f,g}/(t)\mathcal{Y}_{f,g}$ is isomorphic to N , so that $\mathcal{Y}_{f,g}$ actually belongs to $\text{mod } A\text{-}k[[t]]$. It is easy to see that the exact tube associated with $\mathcal{Y}_{f,g}$ is $T_{f,g}$.

We will need the following truncated version of an exact tube:

Definition 2.2. For $m \geq 1$, an exact tube of height m is a representation in $\text{mod } A$

$$N_1 \begin{array}{c} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{array} N_2 \dots N_{m-1} \begin{array}{c} \xrightarrow{\alpha_{m-1}} \\ \xleftarrow{\beta_{m-1}} \end{array} N_m$$

of the full subquiver Q_m of Q whose vertices are $1, 2, \dots, m$, such that the square

$$\begin{array}{ccc} N_i & \xrightarrow{\alpha_i} & N_{i+1} \\ \downarrow \beta_{i-1} & & \downarrow \beta_i \\ N_{i-1} & \xrightarrow{\alpha_{i-1}} & N_i \end{array}$$

is exact for $i = 1, \dots, m-1$. Again we set $N_0 = 0$.

The category of exact tubes of height m is equivalent to the category of A - $k[t]/(t^m)$ -bimodules which are free of finite rank over $k[t]/(t^m)$.

Obviously, an exact tube T restricts to an exact tube $T_{\leq m}$ of height m for all m . We will see in Section 4 that an M -extendible tube $T = (N_i, \alpha_i, \beta_i)$ of height $h \geq 1$ (see next definition) is always the restriction of an (M, N_1) -tube.

Definition 2.3. A tube $T = (N_i, \alpha_i, \beta_i)$ of height h is called M -extendible if there is a decomposition $N_h = Z \oplus Z'$ and an exact sequence

$$0 \rightarrow Z \xrightarrow{\begin{pmatrix} a \\ b \end{pmatrix}} N_{h-1} \oplus M \xrightarrow{(c \ d)} Z' \rightarrow 0$$

such that $a = \beta_{h-1}|_Z$ and $c = pr_{Z'} \circ \alpha_{h-1}$, where $pr_{Z'} : Z \oplus Z' \rightarrow Z'$ is the natural projection.

We end this section with some questions. We do not know how to describe the full subcategory of $\text{mod } A\text{-}k[[t]]$ corresponding to (M, N) -tubes. Conceivably, its objects are just those bimodules \mathcal{Y} which contain $k[[t]] \otimes_k M$ as a subbimodule. This would follow if we knew that any (M, N) -tube is of the form $\mathcal{Y}_{f,g}$ for some exact sequence (2.1).

3. Complexity

Definition 3.1. We call a map

$$\begin{pmatrix} f \\ g \end{pmatrix} : Z \rightarrow Z \oplus M$$

an (M, N) -monomorphism provided N is isomorphic to $\text{coker} \begin{pmatrix} f \\ g \end{pmatrix}$.

Recall that, for a degeneration $M \leq_{\text{deg}} N$, we defined the complexity as

$$\text{cpl}(M, N) = \min \text{cpl}(T),$$

where T ranges over all (M, N) -tubes. There always are (M, N) -tubes with different complexities. For instance, if $(f, g)^t : Z \rightarrow Z \oplus M$ is an (M, N) -monomorphism and we set

$$f' = \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix} : Z^2 \longrightarrow Z^2, \quad g' = (g \ 0) : Z^2 \longrightarrow M,$$

the map $(f', g')^t$ will be an (M, N) -monomorphism, too, and it is easy to see that

$$\text{cpl}(T_{f', g'}) = 2 \text{cpl}(T_{f, g}).$$

Theorem 3.2. *Let $h \geq 1$ be a natural number and $M \leq_{\text{deg}} N$ a degeneration. The following conditions are equivalent:*

- (i) $\text{cpl}(M, N) \leq h$
- (ii) *There is an exact sequence*

$$0 \longrightarrow Z \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} Z \oplus M \longrightarrow N \longrightarrow 0$$

such that $\text{cpl}(T_{f, g}) \leq h$.

- (iii) *There exists an exact tube $T = (N_i, \alpha_i, \beta_i)$ of height $2h + 1$ with $N \xrightarrow[A]{} N_1$ and such that*

$$N_{h+j+1} \xrightarrow[A]{} N_{h+j} \oplus M$$

for $j = 0, \dots, h$.

- (iv) *There exists an M -extendible exact tube $T = (N_i, \alpha_i, \beta_i)$ of height h with $N \xrightarrow[A]{} N_1$.*

Proof. Most ingredients for the proof will be given in Section 4. Here we indicate how they fit together: The implications (ii) \Rightarrow (i) \Rightarrow (iii) are obvious. The results of Section 4 up to Proposition 4.6 give that (ii) implies (iv), and Proposition 4.8 shows (iv) \Rightarrow (ii). Finally, the implication (iii) \Rightarrow (ii) follows from Proposition 4.9 and the next lemma. \square

Lemma 3.3. *Let $T = (N_i, \alpha_i, \beta_i)$ be an (M, N) -tube, and assume that $N_{h+1} \xrightarrow[A]{} N_h \oplus M$ for some $h \geq 1$. Then $\text{cpl}(T) \leq h$.*

Proof. As T is an (M, N) -tube, there exists a natural number $j \geq h$ such that $N_{i+1} \xrightarrow[A]{} N_i \oplus M$ for all $i \geq j$. Take an integer i with $h < i < j$, and consider the

two exact squares

$$\begin{array}{ccccccc}
 N_h \oplus M & \xrightarrow{\sim} & N_{h+1} & \xrightarrow{\alpha_i \dots \alpha_{h+1}} & N_{i+1} & \xrightarrow{\alpha_j \dots \alpha_{i+1}} & N_{j+1} \xrightarrow{\sim} N_j \oplus M \\
 & & \downarrow \beta_h & & \downarrow \beta_i & & \downarrow \beta_j \\
 & & N_h & \xrightarrow{\alpha_{i-1} \dots \alpha_h} & N_i & \xrightarrow{\alpha_{j-1} \dots \alpha_i} & N_j.
 \end{array}$$

The big square splits, and therefore the two small squares split as well. We conclude that N_{i+1} is isomorphic to $N_i \oplus M$. \square

As $N_0 = 0$, our theorem takes the following simpler form for $h = 1, 2$:

Corollary 3.4. *Let $M \leq_{\text{deg}} N$ be a degeneration. Then*

i) $\text{cpl}(M, N) \leq 1$ *if and only if* $N = Z \oplus Z'$ *and there exists an exact sequence*

$$0 \rightarrow Z \rightarrow M \rightarrow Z' \rightarrow 0.$$

ii) $\text{cpl}(M, N) \leq 2$ *if and only if there exist two exact squares*

$$\begin{array}{ccc}
 Z & \xrightarrow{a} & N \\
 \downarrow & & \downarrow c \\
 M & \longrightarrow & Z'
 \end{array}
 \qquad
 \begin{array}{ccc}
 N & \longrightarrow & Z \\
 \downarrow c & & \downarrow a \\
 Z' & \longrightarrow & N
 \end{array}$$

Proposition 3.5. *For any degeneration $M \leq_{\text{deg}} N$ we have*

$$\text{cpl}(M, N) \geq \frac{\ell\ell(M)}{\ell\ell(N)} - 1,$$

where $\ell\ell(X)$ denotes the Loewy length of X ; i.e., the smallest integer r such that $(\text{rad } A)^r X = 0$.

Proof. Choose an (M, N) -tube $T = (N_i, \alpha_i, \beta_i)$ of complexity $h = \text{cpl}(M, N)$. Then M is a direct summand of N_{h+1} , and hence $\ell\ell(M) \leq \ell\ell(N_{h+1})$. We claim that, for all $i \geq 1$,

$$\ell\ell(N_i) \leq i \ell\ell(N_1).$$

In fact, for any exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

the relation

$$\ell\ell(B) \leq \ell\ell(A) + \ell\ell(C)$$

holds true. Our claim follows by induction, considering the exact sequences

$$0 \rightarrow N_{i-1} \rightarrow N_i \rightarrow N_1 \rightarrow 0. \quad \square$$

4. Exact tubes from monomorphisms

Throughout this section, $(f, g)^t : Z \rightarrow Z \oplus M$ denotes an (M, N) -monomorphism.

Definition 4.1. We call two exact tubes $T = (N_i, \alpha_i, \beta_i)$ and $T' = (N'_i, \alpha'_i, \beta'_i)$ *similar* if N_i is isomorphic to N'_i for all $i \geq 1$.

So we do not ask for any compatibility with the maps in the tubes. Note that the property of being an (M, N) -tube is preserved under similarity, and so is complexity.

Proposition 4.2. *There is a direct summand Z' of Z and an exact sequence*

$$0 \rightarrow Z' \xrightarrow{\begin{pmatrix} f|_{Z'} \\ g|_{Z'} \end{pmatrix}} Z' \oplus M \rightarrow N \rightarrow 0$$

such that $f|_{Z'}$ is nilpotent and $T_{f,g}$ is similar to $T_{f|_{Z'}, g|_{Z'}}$. As a consequence, $T_{f,g}$ is an (M, N) -tube.

Proof. By Fitting's lemma, there is a decomposition $Z = Z' \oplus Z''$ of Z as a direct sum which is preserved under f and such that $f' = f|_{Z'}$ is nilpotent and $f'' = f|_{Z''}$ is an automorphism of Z'' . Set $g' = g|_{Z'}$ and $g'' = g|_{Z''}$. Obviously the maps

$$\begin{pmatrix} f'^i & 0 & g' f'^{i-1} & \cdots & g' \\ 0 & f''^i & g'' f''^{i-1} & \cdots & g'' \end{pmatrix}^t : Z' \oplus Z'' \longrightarrow Z' \oplus Z'' \oplus M^i$$

and

$$(f'^i \quad g' f'^{i-1} \cdots g')^t : Z' \longrightarrow Z' \oplus M^i$$

have isomorphic cokernels as $(f'')^i$ is an isomorphism for $i \geq 1$. Since f' is nilpotent, $T_{f',g'}$ is an (M, N) -tube. \square

Remark 4.3. Suppose that $f^h = 0$. As

$$\varphi_{h+j} = (0, \dots, 0, g f^{h-1}, \dots, g)^t : Z \longrightarrow Z \oplus M^{h+j},$$

for $j \in \mathbb{N}$, the exact tube $T_{f,g}$ has the following particularly simple form:

$$N_{h+j} = Z \oplus M^j \oplus Z', \quad N_{h+j+1} = Z \oplus M^{j+1} \oplus Z',$$

$$\alpha_{h+j} = \begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix} : Z \oplus (M^j \oplus Z') \rightarrow Z \oplus M \oplus (M^j \oplus Z'),$$

$$\beta_{h+j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & l \end{pmatrix} : (Z \oplus M^j) \oplus M \oplus Z' \rightarrow (Z \oplus M^j) \oplus Z',$$

for $j \in \mathbb{N}$, where Z' is a cokernel of

$$\psi = (g \circ (f^{h-1}, \dots, f, 1))^t : Z \longrightarrow M^h$$

and

$$(k, \quad l) : M \oplus Z' \longrightarrow Z'$$

is obtained from the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Z & \xrightarrow{\varphi_{h+1}} & Z \oplus M^{h+1} & \longrightarrow & N_{h+1} = Z \oplus M \oplus Z' \longrightarrow 0 \\
 & & \downarrow f & & \downarrow (1 \ 0) & & \downarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & l \end{pmatrix} \\
 0 & \longrightarrow & Z & \xrightarrow{\varphi_h} & Z \oplus M^h & \longrightarrow & N_h = Z \oplus Z' \longrightarrow 0
 \end{array}$$

with exact rows.

Our next goal is to show that, up to similarity, we may choose $g \in \text{rad}(Z, M)$. We start with an auxiliary result:

Lemma 4.4. *The tube $T_{f,g}$ is similar to $T_{f',g}$ with $f' = f - hg$, where $h : M \rightarrow Z$ is any homomorphism.*

Proof. It suffices to check the identity $\psi_i \circ \varphi'_i = \varphi_i$, for $i \geq 1$, where

$$\begin{aligned}
 \varphi_i &= (f^i, g f^{i-1}, \dots, g)^t : Z \longrightarrow Z \oplus M^i, \\
 \varphi'_i &= (f'^i, g f'^{i-1}, \dots, g)^t : Z \longrightarrow Z \oplus M^i
 \end{aligned}$$

and

$$\psi_i := \begin{pmatrix} 1 & h & fh & f^2h & \dots & f^{i-1}h \\ 0 & 1 & gh & gfh & \dots & g f^{i-2}h \\ 0 & 0 & 1 & gh & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & gh \\ \vdots & & \ddots & \ddots & \ddots & gh \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix} : Z \oplus M^i \rightarrow Z \oplus M^i.$$

The key is the equation

$$f^r = f'^r + \sum_{s=0}^{r-1} f^s (hg) f'^{r-1-s}, \quad r \geq 1,$$

which is proved by induction. \square

Proposition 4.5. *There exists a direct summand Z' of Z and an exact sequence*

$$0 \rightarrow Z' \xrightarrow{\begin{pmatrix} f' \\ g' \end{pmatrix}} Z' \oplus M \rightarrow N \rightarrow 0 \quad (4.1)$$

with $g' \in \text{rad}(Z', M)$ and such that $T_{f,g}$ is similar to $T_{f',g'}$.

Proof. If $g \in \text{rad}(Z, M)$, there is nothing to be proved. Otherwise, we prove that a sequence (4.1) exists such that $T_{f,g}$ is similar to $T_{f',g'}$ and $\dim Z' < \dim Z$ and then proceed by induction on $\dim Z$. We choose a non-zero direct summand Z_2 of Z for which $g|_{Z_2}$ is a section. Replacing Z by an isomorphic module if

necessary, which leads to an isomorphic tube, we may assume that $Z = Z_1 \oplus Z_2$, $M = M_1 \oplus Z_2$,

$$g = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Applying the preceding lemma for

$$h = \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix},$$

we obtain a monomorphism $\begin{pmatrix} f'' \\ g \end{pmatrix}$ of the form

$$\begin{pmatrix} f'' \\ g \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 0 \\ q & 0 \\ 0 & 1 \end{pmatrix} : Z_1 \oplus Z_2 \rightarrow Z_1 \oplus Z_2 \oplus M_1 \oplus Z_2.$$

Now we may take $Z' = Z_1$, $M = Z_2 \oplus M_1$, $f' = a$ and $g' = \begin{pmatrix} b \\ q \end{pmatrix}$. \square

Proposition 4.6. *Set $h = \text{cpl}(T_{f,g})$, and suppose that $g \in \text{rad}(Z, M)$ and that f is nilpotent. Then $(T_{f,g})_{\leq h}$ is M -extendible.*

Proof. Our assumptions on f and g imply that, for some i , the restriction $\psi|_Z$ of the composition

$$\psi = \begin{pmatrix} \varphi_i & 0 \\ 0 & 1_{M^h} \end{pmatrix} : Z \oplus M^h \rightarrow Z \oplus M^i \oplus M^h$$

of the maps

$$Z \oplus M^h \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} Z \oplus M^{1+h} \rightarrow \dots \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} Z \oplus M^{i+h}$$

belongs to $\text{rad}(Z, Z \oplus M^{i+h})$. By construction of $T_{f,g}$, the square

$$\begin{array}{ccc} Z \oplus M^h & \xrightarrow{\psi} & Z \oplus M^i \oplus M^h \\ \downarrow \pi_h & & \downarrow \pi_{h+i} \\ N_h & \xrightarrow{\alpha_{i+h-1} \cdots \alpha_h} & N_{i+h} \end{array}$$

is exact, where $\pi_j : Z \oplus M^j \rightarrow N_j$ is the projection to the cokernel of $\varphi_j : Z \rightarrow Z \oplus M^j$, and it splits, since $h = \text{cpl}(T_{f,g})$. Therefore, $\pi_h|_Z$ is a section, and replacing N_h by an isomorphic module, we may assume that

$$N_h = Z \oplus Z', \quad \pi_h = \begin{pmatrix} 1 & * & * \\ 0 & d & * \end{pmatrix} : Z \oplus M \oplus M^{h-1} \longrightarrow Z \oplus Z',$$

where $*$ is an arbitrary map.

Now consider the exact squares

$$\begin{array}{ccc} Z \oplus M^{h-1} & \xrightleftharpoons[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}]{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} & Z \oplus M \oplus M^{h-1} \\ \downarrow \pi_{h-1} & & \downarrow \pi_h = \begin{pmatrix} 1 & * & * \\ 0 & d & * \end{pmatrix} \\ N_{h-1} & \xrightleftharpoons[\beta_{h-1}]{\alpha_{h-1}} & Z \oplus Z' \end{array}$$

It is easy to see that the square

$$\begin{array}{ccc} Z & \xrightarrow{g} & M \\ \downarrow \pi_{h-1}|_Z & & \downarrow d \\ N_{h-1} & \xrightarrow{\text{pr}_{z'} \circ \alpha_{h-1}} & Z' \end{array}$$

is exact as well. Moreover, we have

$$\pi_{h-1}|_Z = \beta_{h-1}|_Z.$$

□

Next we recall a different construction for $T_{f,g}$, which has been presented for the most part in [6]. From $(f, g)^t$ we obtain the commutative diagram (Figure 2) with exact rows and $(k_i, l_i) = \beta_i(k_{i+1}, l_{i+1})$ for $i \leq m-1$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z & \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} & Z \oplus M & \xrightarrow{(k_1, l_1)} & N_1 = N \longrightarrow 0 \\ & & \downarrow k_{m-1} & & \downarrow (k_m, l_m) & & \parallel \\ 0 & \longrightarrow & N_{m-1} & \xrightarrow{\alpha_{m-1}} & N_m & \xrightarrow{\beta_1 \dots \beta_{m-1}} & N_1 \longrightarrow 0 \\ & & \downarrow \beta_{m-2} & & \downarrow \beta_{m-1} & & \parallel \\ 0 & \longrightarrow & N_{m-2} & \xrightarrow{\alpha_{m-2}} & N_{m-1} & \xrightarrow{\beta_1 \dots \beta_{m-2}} & N_1 \longrightarrow 0 \\ & & \vdots & & \vdots & & \vdots \\ 0 & \longrightarrow & N_2 & \xrightarrow{\alpha_2} & N_3 & \xrightarrow{\beta_1 \beta_2} & N_1 \longrightarrow 0 \\ & & \downarrow \beta_1 & & \downarrow \beta_2 & & \parallel \\ 0 & \longrightarrow & N_1 & \xrightarrow{\alpha_1} & N_2 & \xrightarrow{\beta_1} & N_1 \longrightarrow 0 \end{array}$$

Figure 2

The next step is always obtained by squeezing the push-out of the top sequence by k_m between the two top rows.

We claim that the exact tube (N_i, α_i, β_i) of height m thus obtained is isomorphic to the restriction $(T_{f,g})_{\leq m}$ of $T_{f,g}$.

By induction, we obtain the following series of exact squares:

$$\begin{array}{ccccccc}
 Z & \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \end{pmatrix}} & Z \oplus M & \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} & Z \oplus M^2 & \cdots & Z \oplus M^{m-1} \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} & Z \oplus M^m \\
 \downarrow & & \downarrow \psi_1 & & \downarrow \psi_2 & & \downarrow \psi_{m-1} & & \downarrow \psi_m \\
 0 & \longrightarrow & N_1 & \xrightarrow{\alpha_1} & N_2 & \cdots & N_{m-1} & \xrightarrow{\alpha_{m-1}} & N_m
 \end{array}$$

with $\psi_i = (k_i, l_i, \alpha_{i-1}l_{i-1}, \dots, \alpha_{i-1} \dots \alpha_1 l_1) : Z \oplus M^i \rightarrow N_i$.

Note that the composition of the first i maps of the top row is just $\varphi_i : Z \rightarrow Z \oplus M^i$ and that the sequence

$$0 \rightarrow Z \xrightarrow{\varphi_i} Z \oplus M^i \xrightarrow{\psi_i} N_i \rightarrow 0$$

is exact for $i = 1, \dots, m$. So $N_i \xrightarrow{\sim} \text{coker } \varphi_i$, and the maps α_i are the ones we claim. As for β_i , it suffices to show that

$$\psi_i \circ (1 \ 0) = \beta_i \circ \psi_{i+1}.$$

This follows easily from the explicit formulas for ψ_i , ψ_{i+1} , the equation

$$(k_i, l_i) = \beta_i(k_{i+1}, l_{i+1})$$

and the fact that (N_i, α_i, β_i) is an exact tube of height m . As a consequence we have:

Remark 4.7. Let $(f, g)^t$ be an (M, N) -monomorphism and $T' = (N'_i, \alpha'_i, \beta'_i)$ an exact tube of height m . Then T' is isomorphic to $(T_{f,g})_{\leq m}$ if and only if there exists an exact square

$$\begin{array}{ccc}
 Z & \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} & Z \oplus M \\
 \downarrow \beta'_{m-1} \circ k & & \downarrow (k, l) \\
 N'_{m-1} & \xrightarrow{\alpha'_{m-1}} & N'_m
 \end{array}$$

Proposition 4.8. Any M -extendible exact tube $T = (N_i, \alpha_i, \beta_i)$ of height m with $N_1 \xrightarrow[A]{\sim} N$ is the restriction of the exact tube $T_{f,g}$ to Q_m for some (M, N) -monomorphism $(f, g)^t$.

Proof. Let

$$0 \rightarrow Z \xrightarrow{\begin{pmatrix} a \\ b \end{pmatrix}} N_{m-1} \oplus M \xrightarrow{(c \ d)} Z' \rightarrow 0$$

be an exact sequence with $N_m = Z \oplus Z'$, $a = \beta_{m-1}|_Z$ and $c = \text{pr}_{Z'} \circ \alpha_{m-1}$. The square

$$\begin{array}{ccc} N_m = Z \oplus Z' & \xrightarrow{\begin{pmatrix} c'a & c'a' \\ b & 0 \\ 0 & 1 \end{pmatrix}} & Z \oplus M \oplus Z' \\ \downarrow (a,a')=\beta_{m-1} & & \downarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -d & ca' \end{pmatrix} \\ N_{m-1} & \xrightarrow{(c')=\alpha_{m-1}} & N_m = Z \oplus Z' \end{array}$$

is exact. Setting

$$N_{m+1} = Z \oplus M \oplus Z', \quad \alpha_m = \begin{pmatrix} c'a & c'a' \\ b & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -d & ca' \end{pmatrix}$$

we may extend T to an exact tube of height $m+1$. By construction, the map

$$\begin{pmatrix} c'a \\ b \end{pmatrix} : Z \longrightarrow Z \oplus M$$

is an (M, N) -monomorphism, and the square

$$\begin{array}{ccc} Z & \xrightarrow{\begin{pmatrix} c'a \\ b \end{pmatrix}} & Z \oplus M \\ \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \\ N_m = Z \oplus Z' & \xrightarrow{\alpha_m} & N_{m+1} = Z \oplus M \oplus Z' \end{array}$$

is exact with

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \beta_m \circ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The result now follows from Remark 4.7. \square

Proposition 4.9. *Let $T = (N_i, \alpha_i, \beta_i)$ be an exact tube of height $h+m$, for some $h \geq 1$ and $m \geq 1$. Suppose that*

$$N_{h+j+1} \xrightarrow[A]{\sim} N_{h+j} \oplus M$$

for $j \in \{0, \dots, m-1\}$. Then there is an (M, N) -monomorphism

$$\begin{pmatrix} f \\ g \end{pmatrix} : N_{h+m-1} \longrightarrow N_{h+m-1} \oplus M$$

such that the restrictions $T_{\leq m}$ and $(T_{f,g})_{\leq m}$ are isomorphic.

Proof. We wish to choose

$$\begin{pmatrix} f \\ g \end{pmatrix} = \chi \circ \alpha_{h+m-1}$$

for a suitable isomorphism $\chi : N_{h+m} \rightarrow N_{h+m-1} \oplus M$ and to apply Remark 4.7 to the diagram

$$\begin{array}{ccccc} N_{h+m-1} & \xrightarrow{\alpha_{h+m-1}} & N_{h+m} & \xrightarrow[\chi]{\sim} & N_{h+m-1} \oplus M \\ \beta = \beta_{m-1} \cdots \beta_{h+m-2} \downarrow & & \beta_m \cdots \beta_{h+m-1} \downarrow & & \swarrow \\ N_{m-1} & \xrightarrow{\alpha_{m-1}} & N_m & & \end{array}$$

In order to do this, we only need to construct a section

$$s : N_{h+m-1} \longrightarrow N_{h+m}$$

satisfying

$$\beta_{m-1} \beta_m \cdots \beta_{h+m-1} s = \beta \beta_{h+m-1} s = \beta.$$

By our hypothesis, the square

$$\begin{array}{ccc} N_{h+1} & \xrightarrow{\alpha} & N_{h+m} \\ \downarrow \beta_h & & \downarrow \beta_{h+m-1} \\ N_h & \xrightarrow{\alpha'} & N_{h+m-1} \end{array}$$

splits, where $\alpha = \alpha_{h+m-1} \cdots \alpha_{h+1}$ and $\alpha' = \alpha_{h+m-2} \cdots \alpha_h$. Choose a maximal direct summand A of N_{h+m} for which $\beta_{h+m-1}|_A$ is a section. Replacing T by an isomorphic exact tube, we may assume that we have

$$\beta_{h+m-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & \delta \end{pmatrix} : A \oplus B \oplus M \longrightarrow A \oplus B,$$

$$\alpha' = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} : C \oplus B \longrightarrow A \oplus B$$

for some maps $\gamma, \delta, \varepsilon$. Setting

$$s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} : A \oplus B \longrightarrow A \oplus B \oplus M,$$

we obtain

$$1_{A \oplus B} - \beta_{h+m-1} s = \begin{pmatrix} 0 & 0 \\ 0 & 1 - \gamma \end{pmatrix} : A \oplus B \longrightarrow A \oplus B,$$

which factors through α' . But the sequence

$$0 \longrightarrow N_h \xrightarrow{\alpha'} N_{h+m-1} \xrightarrow{\beta} N_{m-1} \longrightarrow 0$$

is exact, which implies $\beta = \beta \beta_{h+m-1} s$ as required. \square

5. Examples

All our examples are representations of quivers with relations. Let Q be a quiver with vertex set $Q_0 = \{1, \dots, n\}$, I an admissible two-sided ideal in the quiver algebra kQ , $\underline{d} = (d_1, \dots, d_n)$ a vector in \mathbb{N}^n , and denote by

$$\text{Rep}(Q, I, \underline{d})$$

the affine algebraic variety of representations X of (Q, I) with $X(i) = k^{d_i}$, $i \in Q_0$. The dimension vector of X in $\text{Rep}(Q, I, \underline{d})$ is \underline{d} . The group $G(\underline{d}) = \prod_{i=1}^n GL(d_i)$ acts on $\text{Rep}(Q, I, \underline{d})$ by

$$(g \cdot X)(\alpha) = g_j \circ X(\alpha) \circ g_i^{-1}$$

for an arrow $\alpha : i \rightarrow j$ and $g = (g_1, \dots, g_n) \in G(\underline{d})$.

If we view M, N in $\text{Rep}(Q, I, \underline{d})$ as modules over kQ/I of dimension $d = \sum_{i=1}^n d_i$, then M degenerates to N if and only if the representation N belongs to the closure of the orbit $G(\underline{d}) \cdot M$ of M in $\text{Rep}(Q, I, \underline{d})$ [1]. This allows us to work with the smaller group $G(\underline{d})$.

5.1. We begin with an example of a degeneration whose complexity is easy to compute: Choose a natural number $n \geq 2$ and let \vec{A}_n be the equioriented quiver with underlying graph A_n :

$$\vec{A}_n = 1 \xleftarrow{\gamma_1} 2 \xleftarrow{\gamma_2} \dots \xleftarrow{\gamma_{n-1}} n.$$

Denote by X_i the indecomposable representation of \vec{A}_n given by

$$X_i(j) = \begin{cases} k & j \leq i, \\ 0 & j > i, \end{cases}$$

$$X_i(\gamma_j) = \begin{cases} 1 & j < i, \\ 0 & j \geq i. \end{cases}$$

Then $M = X_n$ has a filtration

$$M = X_n \supset X_{n-1} \supset \dots \supset X_2 \supset X_1,$$

and it is well-known that M degenerates to the associated graded module

$$N = \bigoplus_{i=1}^n X_i / X_{i-1},$$

where we set $X_0 = 0$. We wish to compute the complexity $\text{cpl}(M, N)$, thereby

showing again that M actually degenerates to N . Set

$$Z = \bigoplus_{i=1}^{n-1} X_i,$$

$$f = \begin{pmatrix} 0 & & & 0 \\ \iota_2 & \ddots & & \\ & \ddots & \ddots & \\ & & \iota_{n-1} & 0 \end{pmatrix} : Z \longrightarrow Z \quad \text{and}$$

$$g = (0 \cdots 0 \ \iota_n) : Z \longrightarrow M = X_n,$$

where $\iota_i : X_{i-1} \rightarrow X_i$ is the inclusion. It is easy to check that $(f, g)^t$ is an (M, N) -monomorphism. Moreover, $f^{n-1} = 0$, and thus

$$\text{cpl}(M, N) \leq n - 1.$$

On the other hand, the Loewy lengths of M and N are n and 1, respectively, which implies

$$\text{cpl}(M, N) \geq \frac{\ell\ell(M)}{\ell\ell(N)} - 1 = n - 1$$

by Proposition 3.5. This example shows that there are degenerations of arbitrary complexity.

Note that for $n = 4$ we obtain the following chain of degenerations:

$$M = k \xleftarrow{1} k \xleftarrow{1} k \xleftarrow{1} k \leq_{\deg} P = k \xleftarrow{1} k \xleftarrow{0} k \xleftarrow{1} k$$

$$\leq_{\deg} N = k \xleftarrow{0} k \xleftarrow{0} k \xleftarrow{0} k.$$

The complexities are

$$\text{cpl}(M, P) = 1 = \text{cpl}(P, N) \quad \text{and}$$

$$\text{cpl}(M, N) = 3 > \text{cpl}(M, P) + \text{cpl}(P, N).$$

Comparing with the example given in the introduction, we see that $\text{cpl}(M, P) + \text{cpl}(P, N)$ can be either smaller or greater than $\text{cpl}(M, N)$ for a chain

$$M \leq_{\deg} P \leq_{\deg} N.$$

5.2. Next we give an example of a minimal degeneration of arbitrary complexity: Let Q be the quiver

$$Q = 1 \xrightarrow{\alpha} 2 \begin{array}{c} \curvearrowright \\ \beta \end{array},$$

choose a natural number $n \geq 2$, and let I be the ideal generated by β^n . Define M and N to be the representations of dimension vector $(1, n)$ given by

$$M(\alpha) = e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad N(\alpha) = e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$M(\beta) = N(\beta) = J_n, \text{ respectively,}$$

where e_1, \dots, e_n is the standard basis of k^n and J_m is the Jordan block

$$J_m = \begin{pmatrix} 0 & & 0 \\ 1 & \ddots & \\ & \ddots & \ddots \\ 0 & & 1 & 0 \end{pmatrix}$$

in $\mathbb{M}_m(k)$, for $m \in \mathbb{N}$.

Proposition 5.1. *There is a degeneration $M \leq_{\deg} N$, which is minimal, and $\text{cpl}(M, N) = n$.*

Proof. Denote by Z the indecomposable representation with dimension vector $(0, n)$, given by $Z(\beta) = J_n$, and let $(f, g)^t : Z \rightarrow Z \oplus M$ be given by

$$f = (0, J_n), \quad g = (0, 1).$$

It is easy to see that $(f, g)^t$ is an (M, N) -monomorphism, so M degenerates to N . Moreover, we have $f^n = 0$, and therefore $\text{cpl}(M, N) \leq n$. As

$$\dim \text{End } M = 1 \text{ and } \dim \text{End } N = 2,$$

the orbit of N has codimension 1 in the closure of the orbit of M , which implies that the degeneration is minimal.

Suppose $\text{cpl}(M, N) \leq n - 1$, and choose an (M, N) -tube $T = (N_i, \alpha_i, \beta_i)$ with $N_n \xrightarrow{\sim} N_{n-1} \oplus M$. Let $\psi_n : N_n \rightarrow M$ be the surjection obtained from this decomposition.

Claim. *For $i = 1, \dots, n$, there exists a surjection*

$$\psi_i : N_i \rightarrow M^{(i)},$$

where $M^{(i)}$ has dimension vector $(1, i)$ and is given by

$$M^{(i)}(\alpha) = (1, 0, \dots, 0)^t, \quad M^{(i)}(\beta) = J_i.$$

Using the claim for $i = 1$, we obtain a surjection $\psi_1 : N_1 = N \rightarrow M^{(1)}$, which is impossible.

We prove the claim by descending induction on i . Observe that any map from N to $M^{(i)}$ factors through the socle $\text{soc } M^{(i)}$ and that $M^{(i)}/\text{soc } M^{(i)} \xrightarrow{\sim} M^{(i-1)}$. Writing this factorization for $\psi_i \circ \alpha_{i-1} \cdots \alpha_1$, we obtain $\psi_{i-1} : N_{i-1} \rightarrow M^{(i-1)}$ from the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & N & \xrightarrow{\alpha_{i-1} \cdots \alpha_1} & N_i & \xrightarrow{\beta_{i-1}} & N_{i-1} \longrightarrow 0 \\
& & \downarrow & & \downarrow \psi_i & & \downarrow \psi_{i-1} \\
0 & \longrightarrow & \text{soc} M^{(i)} & \longrightarrow & M^{(i)} & \longrightarrow & M^{(i-1)} \longrightarrow 0.
\end{array}$$

As ψ_i is surjective, ψ_{i-1} is as well. \square

A version of this argument implies the following result, which we will not use:

$$\text{cpl}(M, M_i) = \left\lfloor \frac{n-1}{i-1} \right\rfloor + 1, \quad i \geq 2.$$

The representation M_i is given by

$$M_i(\alpha) = e_i, \quad M_i(\beta) = J_n.$$

5.3. We now exhibit a degeneration $M \leq_{\text{deg}} N$ of complexity 2 with the property that $f^2 \neq 0$ for all (M, N) -monomorphisms $(f, g)^t$. Therefore the complexity can be strictly less than the “index of nilpotence of M and N ”; i.e., the number

$$\min\{r : f^r = 0\},$$

where the minimum is taken over all (M, N) -monomorphisms $(f, g)^t$. We stay with the same quiver Q , and we choose I to be generated by β^3 ; i.e., we set $n = 3$ in the preceding example. Note that kQ/I is representation-finite: it admits 29 indecomposables [4].

We let M and N be given by

$$M(\alpha) = e_2, \quad N(\alpha) = e_3, \quad M(\beta) = N(\beta) = J_3,$$

where e_1, e_2, e_3 is the standard basis of k^3 . Choose

$$\begin{array}{c}
Z' = 0 \longrightarrow k^3 \xrightarrow{J_3} k^3 \\
f' = (0, J_3) : Z' \longrightarrow Z' \quad \text{and} \\
g' = (0, 1) : Z' \longrightarrow M.
\end{array}$$

Then $(f', g')^t$ is an (M, N) -monomorphism. As f'^2 factors through g' , the cokernel N_3 of the map

$$\varphi_3 = (f'^3, g'f'^2, g'f', g')^t : Z' \longrightarrow Z' \oplus M^3$$

used to define the tube $T_{f', g'} = (N_i, \alpha_i, \beta_i)$ is isomorphic to the cokernel of

$$(f'^2, 0, g'f', g')^t : Z' \longrightarrow Z' \oplus M^3$$

and thus isomorphic to $M \oplus N_2$. By Lemma 3.3, we know that

$$\text{cpl}(M, N) \leq 2.$$

On the other hand, as N is indecomposable, the complexity must exceed 1, so

$$\text{cpl}(M, N) = 2.$$

Claim. *For any (M, N) -monomorphism*

$$(f, g)^t : Z \longrightarrow Z \oplus M,$$

we have $f^2 \neq 0$.

First we show:

Lemma 5.2. *For any (M, N) -monomorphism*

$$(f, g)^t : Z \longrightarrow Z \oplus M,$$

Z' is a direct summand of Z .

Proof. Consider the exact sequences

$$\Sigma' : 0 \longrightarrow Z' \xrightarrow{\begin{pmatrix} f' \\ g' \end{pmatrix}} Z' \oplus M \xrightarrow{(k', l')} N \longrightarrow 0$$

and

$$\Sigma : 0 \longrightarrow Z \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} Z \oplus M \xrightarrow{(k, l)} N \longrightarrow 0.$$

It is easy to check that

$$\begin{aligned} \dim \text{Hom}(Z', M) &= \dim \text{Hom}(Z', N) = 3, \\ \dim \text{End } M &= \dim \text{Hom}(M, N) = 2. \end{aligned}$$

Therefore the sequence of vector spaces

$$0 \longrightarrow \text{Hom}(Z' \oplus M, Z) \longrightarrow \text{Hom}(Z' \oplus M, Z \oplus M) \longrightarrow \text{Hom}(Z' \oplus M, N) \longrightarrow 0$$

obtained from mapping $Z' \oplus M$ into Σ is exact. In particular, $(k', l') : Z' \oplus M \rightarrow N$ factors through $(k, l) : Z \oplus M \rightarrow N$, and hence we have the following commutative diagram (Figure 3) with exact rows and columns.

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
 \Sigma : & 0 \longrightarrow & Z & \longrightarrow & Z \oplus M & \xrightarrow{(k,l)} & N \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow (k',l') \\
 & 0 \longrightarrow & Z & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & Z \oplus (Z' \oplus M) & \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & Z' \oplus M \longrightarrow 0 \\
 & & & & \uparrow \begin{pmatrix} s' \\ f' \\ g' \end{pmatrix} & & \uparrow \begin{pmatrix} f' \\ g' \end{pmatrix} \\
 & & & & Z' & \xlongequal{\quad} & Z' \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & & 0
 \end{array}$$

Figure 3

So the middle column splits as well, and since by construction f', g' lie in the radical,

$$s : Z' \longrightarrow Z$$

must be a section. \square

Let $(f, g)^t : Z \rightarrow Z \oplus M$ be an (M, N) -monomorphism, suppose $f^2 = 0$, and consider the commutative diagram (Figure 4) with exact rows and columns.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 \longrightarrow & \ker f & \longrightarrow & M & \longrightarrow & X & \longrightarrow 0 \\
 & \downarrow & & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \downarrow & \\
 0 \longrightarrow & Z & \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} & Z \oplus M & \xrightarrow{(k,l)} & N & \longrightarrow 0 \\
 & \downarrow & & \downarrow \begin{pmatrix} 1, 0 \end{pmatrix} & & \downarrow & \\
 0 \longrightarrow & \operatorname{im} f & \longrightarrow & Z & \longrightarrow & \operatorname{coker} f & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

Figure 4

Then X must be a quotient of M and a submodule of N , which is possible in

exactly two ways:

$$(i) \quad X = k \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} k^2 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} J_2$$

$$(ii) \quad X = 0 \longrightarrow k \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} 0$$

In the first case, we have

$$\ker f = 0 \longrightarrow k \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} 0,$$

and our assumption $f^2 = 0$ implies that $\dim Z(2) \leq 2$. But then Z cannot contain Z' as a direct summand.

In the second case, we see that

$$\ker f = k \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} k^2 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} J_2.$$

Now $f^2 = 0$ implies that $\dim Z(2) \leq 4$. But then necessarily $Z(\beta\alpha) = 0$, since Z' must be a direct summand of Z , and Z cannot contain $\ker f$ as a submodule.

5.4. As our last example, we find a degeneration $M \leq_{\text{deg}} N$ of complexity 2 for which there exists an exact sequence

$$\Sigma : 0 \longrightarrow N \xrightarrow{\alpha_1 = \begin{pmatrix} f \\ g \end{pmatrix}} N \oplus M \xrightarrow{\beta_1 = (f, -l)} N \longrightarrow 0.$$

So we have an exact tube

$$T = (N_1 = N, N_2 = N \oplus M, \alpha_1, \beta_1)$$

of height 2. If this tube were the restriction of an (infinite) exact tube, the complexity $\text{cpl}(M, N)$ would have to equal 1. So the number $2h + 1$ in condition (iii) of our main theorem cannot be replaced by $2h$.

Choose $A = k[\alpha, \beta]/(\alpha^2, \beta^2)$, let M and N be 4-dimensional with

$$M(\alpha) = N(\alpha) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad M(\beta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad N(\beta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and set

$$f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad l = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is easy to check that the sequence Σ obtained from these choices is exact. So M degenerates to N . As N is indecomposable and f^2 factors through g , the same argument as in Section 5.3 implies that $\text{cpl}(M, N) = 2$.

This example has another surprising feature: For any degeneration $M \leq_{\text{deg}} N$ we obtain

$$\text{cpl}(M^r, N^r) \leq \text{cpl}(M, N), \quad r \geq 1,$$

by taking for $M^r \leq_{\text{deg}} N^r$ the direct sum of r copies of an (M, N) -tube of minimal complexity. In our example, we have

$$\text{cpl}(M^2, N^2) = 1 < \text{cpl}(M, N) = 2.$$

Indeed, M^2 is a projective cover for N , and the kernel of an epimorphism $M^2 \rightarrow N$ is N again. So there is an exact sequence

$$0 \longrightarrow N \longrightarrow M^2 \longrightarrow N \longrightarrow 0.$$

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