

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 75 (2000)

Artikel: Lusternik-Schnirelman theory for closed 1-forms
Autor: Farber, Michael
DOI: <https://doi.org/10.5169/seals-56612>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 07.02.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Lusternik–Schnirelman theory for closed 1-forms

Michael Farber

Dedicated to S.P. Novikov on the occasion of his 60th birthday

Abstract. S. P. Novikov developed an analog of the Morse theory for closed 1-forms. In this paper we suggest an analog of the Lusternik - Schnirelman theory for closed 1-forms. For any cohomology class $\xi \in H^1(M, \mathbf{R})$ we define an integer $\text{cl}(\xi)$ (*the cup-length associated with ξ*); we prove that any closed 1-form representing ξ has at least $\text{cl}(\xi) - 1$ critical points. The number $\text{cl}(\xi)$ is defined using cup-products in cohomology of some flat line bundles, such that their monodromy is described by complex numbers, which are not Dirichlet units.

Mathematics Subject Classification (1991). 58E05.

Keywords. Lusternik–Schnirelman theory, closed 1-forms.

§1. The main result

1.1. Let M be a closed manifold and let $\xi \in H^1(M; \mathbf{R})$ be a nonzero cohomology class. The Novikov inequalities [N1], [N2], [N3] estimate the numbers of zeros $c_i(\omega)$ of different indices of any closed 1-form ω with Morse type singularities on M lying in the class ξ .

Novikov type inequalities were constructed in [BF1] for closed 1-forms with slightly more general singularities (non-degenerate in the sense of Bott [B]). In [BF2] an equivariant generalization of the Novikov inequalities was found.

In this paper we will consider the problem of estimating the number of critical points of closed 1-forms ω with no non-degeneracy assumption. We suggest here a version of the Lusternik - Schnirelman theory for closed 1-forms.

An announcement [F1] describes some results of this paper.

My recent preprint [F2] suggests a different approach to the Lusternik - Schnirelman theory of closed 1-forms; it uses untwisted cohomology and Massey products. Examples computed in [F2], show that the results of [F2] and of the present paper

The research was supported by a grant from the Israel Academy of Sciences and Humanities and by the Herman Minkowski Center for Geometry

are independent.

1.2 Let $\xi \in H^1(M; \mathbf{Z})$ be an integral cohomology class. We will define below a nonnegative integer $\text{cl}(\xi)$, which we will call *the cup-length associated with ξ* .

Recall, that a complex flat vector bundle E over M is determined by its monodromy, a linear representation of the fundamental group $\pi_1(M, x_0)$ in $\text{GL}_{\mathbf{C}}(E_0)$, where E_0 is the fiber over the base point $x_0 \in M$; this representation is given by the parallel transport of vectors along loops. For example, a flat line bundle is determined by a homomorphism $H_1(M; \mathbf{Z}) \rightarrow \mathbf{C}^*$, where \mathbf{C}^* is considered as a multiplicative abelian group.

Given class ξ as above and a nonzero complex number $a \in \mathbf{C}^*$, we have the complex flat line bundle over M with the following property: the monodromy along any loop $\gamma \in \pi_1(M)$ is the multiplication by $a^{\langle \xi, \gamma \rangle}$. We will denote this bundle by a^ξ . If $a, b \in \mathbf{C}^*$, we have the canonical isomorphism of flat line bundles

$$a^\xi \otimes b^\xi \simeq ab^\xi.$$

A lattice $\mathcal{L} \subset V$ in a finite dimensional vector space V is a finitely generated subgroup with $\text{rank } \mathcal{L} = \dim_{\mathbf{C}} V$. We will say that a complex flat bundle $E \rightarrow M$ of rank m *admits an integral lattice* if its monodromy representation $\pi_1(M, x_0) \rightarrow \text{GL}_{\mathbf{C}}(E_0)$ is conjugate to a homomorphism $\pi_1(M, x_0) \rightarrow \text{GL}_{\mathbf{Z}}(\mathcal{L}_0)$, where $\mathcal{L}_0 \subset E_0$ is a lattice in the fiber. This condition is equivalent to the assumption that E is obtained from a local system \tilde{E} of finitely generated free abelian groups over M by tensoring on \mathbf{C} .

1.3. Definition. *The cup-length $\text{cl}(\xi)$ is the largest integer k such that there exists a nontrivial k -fold cup product*

$$H^{d_1}(M; E_1) \otimes H^{d_2}(M; E_2) \otimes \cdots \otimes H^{d_k}(M; E_k) \rightarrow H^d(M; E), \quad (1-1)$$

where $d = d_1 + \cdots + d_k$, $E = E_1 \otimes E_2 \otimes \cdots \otimes E_k$, $d_1 > 0, \dots, d_k > 0$, and the first two flat bundles E_1 and E_2 have the following property: there exist nonzero complex numbers $a_1, a_2 \in \mathbf{C}^*$, and complex flat bundles F_1 and F_2 over M , admitting integral lattices, so that

$$E_i \simeq a_i^\xi \otimes F_i, \quad \text{for } i = 1, 2, \quad (1-2)$$

and both numbers a_1 and a_2 are not Dirichlet units.

Recall that a *Dirichlet unit* is defined as a complex number $b \neq 0$ such that b and its inverse b^{-1} are algebraic integers. In other words, Dirichlet units can be characterized as roots of polynomial equations

$$b^n + \gamma_1 b^{n-1} + \cdots + \gamma_{n-1} b + \gamma_n = 0,$$

where all γ_i are integers and $\gamma_n = \pm 1$.

Note that the cup-length $\text{cl}(\xi)$, defined by 1.3, satisfies $0 \leq \text{cl}(\xi) \leq \dim M$. We will see examples below showing that $\text{cl}(\xi) = \dim M$ is possible.

The definition of the cup-length $\text{cl}(\xi)$ above is slightly different from the one given in [F1]; following the present definition, we may have a larger cup-length $\text{cl}(\xi)$.

Theorem 1. *Let ω be a closed 1-form on M lying in an integral cohomology class $\xi \in H^1(M; \mathbf{Z})$. Let $S(\omega)$ denote the set of zeros of ω , i.e. the set of points $p \in M$ such that $\omega_p = 0$. Then the Lusternik - Schnirelman category of $S(\omega)$ satisfies*

$$\text{cat}(S(\omega)) \geq \text{cl}(\xi) - 1. \quad (1-3)$$

In particular, if the set of zeros $S(\omega)$ is finite, then for the total number $|S(\omega)|$ of zeros

$$|S(\omega)| \geq \text{cl}(\xi) - 1. \quad (1-4)$$

Here $\text{cat}(S)$ denotes the Lusternik - Schnirelman category of $S = S(\omega)$, i.e. the least number k , so that S can be covered by k closed subsets $A_1 \cup \dots \cup A_k$ such that each inclusion $A_j \rightarrow S$ is null-homotopic.

Proof of Theorem 1 is given in §2.

1.4. Corollary ([F1]). *Suppose that there exist complex numbers $a_1, a_2, \dots, a_m \in \mathbf{C}^*$, not all Dirichlet units, such that a cup product*

$$H^{d_1}(M; a_1^\xi) \otimes H^{d_2}(M; a_2^\xi) \otimes \dots \otimes H^{d_k}(M; a_k^\xi) \rightarrow H^d(M; a^\xi),$$

with $d_j > 0$, $j = 1, 2, \dots, k$, is nontrivial. Then for any closed 1-form ω on manifold M , lying in class $\xi \in H^1(M; \mathbf{Z})$, holds $\text{cat}(S(\omega)) \geq k - 1$.

Proof. We may assume that $\xi \neq 0$; otherwise the statement follows from the Lusternik - Schnirelman theory for functions.

Corollary 1.4 directly follows from Theorem 1, if there are at least two non Dirichlet units among a_1, a_2, \dots, a_k . Suppose that there is precisely one non Dirichlet unit. Denote $a = a_1 a_2 \dots a_k$. Then a is not a Dirichlet unit, and, in particular, $a \neq 1$. Hence $H^n(M; a^\xi) = 0$. Therefore, the dimension of the nontrivial cup-product above $d = d_1 + d_2 + \dots + d_k < n = \dim M$ is less than n . By the Poincaré duality, the cup-product pairing

$$H^d(M; a^\xi) \otimes H^{n-d}(M; a^{-\xi} \otimes \mathcal{L}_M) \rightarrow H^n(M; \mathcal{L}_M)$$

is non-degenerate. Here \mathcal{L}_M denotes the orientation flat line bundle of M . The monodromy of \mathcal{L}_M along any loop γ equals ± 1 depending on whether the orientation of M is preserved or reversed by γ . Note that \mathcal{L}_M admits an integral lattice.

Hence, we may find a nontrivial cup-product of length $k + 1$ with an extra factor in $H^{n-d}(M; a^{-\xi} \otimes \mathcal{L}_M)$. Now, Theorem 1 applies and gives $\text{cat}(S(\omega)) \geq k$. \square

1.5. It is clear that Corollary 1.4 becomes false if we remove the requirement that one of the numbers a_i are not Dirichlet units. The simplest example is provided by the torus T^n ; any cohomology class $\xi \in H^1(T^n; \mathbf{R})$ of the torus $M = T^n$ contains a closed 1-form without zeros, but the cup-length of T^n is n .

1.6. Remark. A crude estimate for the cup-length $\text{cl}(\xi)$ can be obtained by taking the maximal length of a non-trivial product (1-1) with $E_j = a_j^\xi$ and $a_j \in \mathbf{C}^*$ being *transcendental*, $j = 1, 2, \dots, k$. We will give an example (cf. 1.10, example 3) showing that this estimate can be really worse than the one provided by Theorem 1.

1.7. Remark. In the longest nontrivial product (1-1) the number d must be equal the dimension of the manifold $n = \dim M$. Indeed, any nontrivial cup-product (1-1) with $d < n$ can be made longer by using the Poincaré duality.

1.8. Forms with non-integral periods. In general, the cohomology class determined by a closed 1-form ω belongs to $H^1(M, \mathbf{R})$, i.e. it has real coefficients. It is clear that multiplying ω by a non-zero constant $\lambda \neq 0$ does not change the set of critical points $S(\omega)$ and multiplies the cohomology class by λ . Hence Theorem 1 also gives estimates in the case of *cohomology classes* $\xi \in H^1(M, \mathbf{R})$ of *rank 1* (i.e. for classes, which are real multiples of integral classes) if we define the associated cup-length $\text{cl}(\xi)$ as follows

$$\text{cl}(\lambda\xi) = \text{cl}(\xi), \quad \lambda \in \mathbf{R}, \quad \lambda \neq 0, \quad \xi \in H^1(M, \mathbf{Z}).$$

Recall, that given a cohomology class $\xi \in H^1(M, \mathbf{R})$, its *rank* is defined as the rank of the abelian group, which is the image of the homomorphism $H_1(M, \mathbf{Z}) \rightarrow \mathbf{R}$, determined by ξ . Note that the cohomology classes of rank 1 are dense in $H^1(M, \mathbf{R})$. Therefore the following definition makes sense.

Definition. Given a class $\xi \in H^1(M, \mathbf{R})$ of rank > 1 , we define $\text{cl}(\xi)$ as the largest number k , such that there exists a sequence of rank 1 classes $\xi_m \in H^1(M, \mathbf{R})$ with

$$\text{cl}(\xi_m) \geq k, \quad \lim_{m \rightarrow \infty} \xi_m = \xi, \quad (1-5)$$

and each ξ_m , considered as a homomorphism $H_1(M; \mathbf{Z}) \rightarrow \mathbf{R}$, vanishes on the kernel of the homomorphism $\xi : H_1(M; \mathbf{Z}) \rightarrow \mathbf{R}$.

Theorem 2. Let ω be a closed 1-form on M lying in a cohomology class $\xi \in H^1(M; \mathbf{R})$. Let $S(\omega)$ denote the set of zeros of ω . Then the Lusternik - Schnirelman category of $S(\omega)$ satisfies

$$\text{cat}(S(\omega)) \geq \text{cl}(\xi) - 1. \quad (1-6)$$

In particular, if the set of critical points $S(\omega)$ is finite then for the total number $|S(\omega)|$ of the critical points,

$$|S(\omega)| \geq \text{cl}(\xi) - 1. \quad (1-7)$$

For the proof see §3.

1.9. Connected sums. Let M_1 and M_2 be two closed n -dimensional manifolds. Assume for simplicity, that $n > 2$. We will denote by $M_1 \# M_2$ the connected sum of M_1 and M_2 . Given cohomology classes $\xi_\nu \in H^1(M_\nu; \mathbf{R})$, where $\nu = 1, 2$, the class $\xi_1 \# \xi_2 \in H^1(M_1 \# M_2; \mathbf{R})$ is well defined, in an obvious way.

In the description of examples (cf. 1.10) we will use the following statement:

Proposition 1. *In the situation described above,*

$$\text{cl}(\xi_1 \# \xi_2) = \max\{\text{cl}(\xi_1), \text{cl}(\xi_2)\}. \quad (1-8)$$

Proof is given in §3.

1.10. Examples. 1. In the notations of the previous subsection, let $\xi_1 = 0$ and suppose that $\xi_2 \neq 0$ can be realized by a closed 1-form with no critical points (for example, fibration over the circle). Then we obtain from Proposition 1 that $\text{cl}(\xi_1 \# \xi_2) = \text{cl}(\xi_1)$. Since $\xi_1 = 0$, the cup-length $\text{cl}(\xi_1)$ can be estimated from below by the usual cup-length of the manifold M_1 with complex coefficients.

To have a specific example, let us take $M_1 = T^n$, $M_2 = S^1 \times S^{n-1}$, $\xi_1 = 0$ and $\xi_2 \in H^1(M_2; \mathbf{Z})$ being a generator, where $n > 2$. Then we have for $\xi = \xi_1 \# \xi_2 \in H^1(M_1 \# M_2; \mathbf{R})$

$$\text{cl}(\xi_1 \# \xi_2) = n. \quad (1-9)$$

Therefore, by Theorem 1, any closed 1-form ω on $M_1 \# M_2$ lying in class ξ has a least $n - 1$ critical points.

2. In a similar way one may construct examples of cohomology classes of higher rank with many critical points. Namely, suppose that $M_1 = T^n$, where $n > 2$ and $\xi_1 = 0$; take for M_2 arbitrary closed manifold of dimension n with a cohomology class $\xi_2 \in H^1(M_2; \mathbf{R})$ of rank q . Then for the class $\xi = \xi_1 \# \xi_2 \in H^1(M_1 \# M_2; \mathbf{R})$ (having rank q) we again obtain $\text{cl}(\xi) = n$ (by Proposition 1).

One may take, for example, $M_2 = T^q \times S^{n-q}$ with ξ_2 induced from a maximally irrational class on the torus T^q .

3. Let M be a 3-dimensional manifold obtained by 0-framed surgery on the knot 5₂:

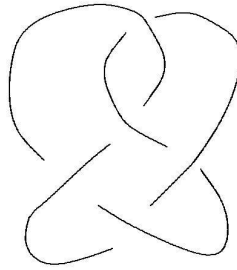


Figure 1.

This knot has Alexander polynomial $\Delta(\tau) = 2 - 3\tau + 2\tau^2$. Then $H^1(M; \mathbf{Z}) = \mathbf{Z}$ and taking $\xi \in H^1(M; \mathbf{Z})$ to be a generator we find that $H^1(M; a^\xi)$ is trivial for all $a \in \mathbf{C}^*$, which are not the roots of the Alexander polynomial. It is easy to check that if a is one of the roots of $2 - 3a + 2a^2 = 0$ then $H^1(M; a^\xi) \neq 0$. Note that the roots of $2 - 3a + 2a^2 = 0$ are not Dirichlet units. Hence we obtain that all Novikov Betti numbers are trivial (since, as it is known [N3], that the Novikov Betti numbers equal to $\dim H^*(M; a^\xi)$ for generic $a \in \mathbf{C}$). However by Corollary 1.4 we obtain that any closed 1-forms in class ξ has at least 1 critical point.

§2. Proof of Theorem 1

2.1. Since we assume that the cohomology class ξ of ω is integral, $\xi \in H^1(M; \mathbf{Z})$, there exists a smooth map $f : M \rightarrow S^1$, such that $\omega = f^*(d\theta)$, where $d\theta$ is the standard angular form on the circle $S^1 \subset \mathbf{C}$, $S^1 = \{z; |z| = 1\}$.

Denote $f^{-1}(b)$ by $V \subset M$, where $b \in S^1$ is a regular value; it is a codimension one submanifold. Let N denote the manifold obtained by cutting M along V . Note that N and V could be disconnected.

Each connected component of V yields two connected components of ∂N , the positive and the negative. In order to distinguish between the positive and the negative boundary components of ∂N , we use the orientation of the normal bundle to V in M , given by the form ω . The positive components are defined as those with the internal normal vector field to N being positive. The union of all positive (negative) boundary components of N will be denoted by $\partial_+ N$, or $\partial_- N$, correspondingly.

Let $p : N \rightarrow M$ denotes the natural projection. Then $p^*\omega = dg$, where $g : N \rightarrow \mathbf{R}$ is a smooth function, determined up to a constant on each connected component of N . It is clear that g is constant on each connected component of ∂N . The points of $\partial_+ N$ are points of local minimum of g ; the points of $\partial_- N$ are points of local maximum of g . The map g sends the set $S(g)$ of critical points of g diffeomorphically onto the set $S(\omega)$.

2.2. Relative Lusternik - Schnirelman category. We will use the well-known notion of relative Lusternik - Schnirelman category, cf. [Fa], [Fo], [S]. Let's recall it.

For any subset $X \subset N$ containing $\partial_+ N$ we will denote by $\text{cat}_{(N, \partial_+ N)}(X)$ the minimal number k such that X can be covered by $k + 1$ closed subsets

$$X \subset A_0 \cup A_1 \cup A_2 \cup \dots \cup A_k \subset N$$

with the following properties:

- (1) A_0 contains $\partial_+ N$ and the inclusion $A_0 \rightarrow N$ is homotopic to a map $A_0 \rightarrow \partial_+ N$ keeping the points of $\partial_+ N \subset A$ fixed;
- (2) for $j = 1, 2, \dots, k$, each inclusion $A_j \rightarrow N$ is null-homotopic.

We claim, that

$$\text{cat } S(\omega) = \text{cat } S(g) \geq \text{cat}_{(N, \partial_+ N)}(N). \quad (2-1)$$

This follows from known results, cf., for example, [Fo], Th. 4.2. We apply Theorem 4.2 of [Fo] to each of the connected components of N and to the restriction of function g on it; we use the additivity of the relative Lusternik - Schnirelman category with respect to disjoint union, cf. [Fo], Prop. 2.8.

Our next purpose will be to prove the inequality

$$\text{cat}_{(N, \partial_+ N)}(N) \geq \text{cl}(\xi) - 1. \quad (2-2)$$

Together with (2-1) this will complete the proof of the Theorem.

2.3. The deformation complex. The proof of (2-2) will consist of building a *polynomial deformation*, a finitely generated free cochain complex C^* over the ring $P = \mathbf{Z}[\tau]$ of polynomials with integral coefficients, having properties (a), (b) described below. With the help of the deformation complex we will prove the Lifting Property, cf. Corollary 2.6, playing a crucial role in the proof.

In [F3] we show how the deformation complex leads to inequalities, which are stronger than the Novikov inequalities.

The construction of the deformation complex is similar to [F2]; the difference is that in the present paper we will work over the integers, and in [F2] over a field.

Claim. *Let $E \rightarrow M$ be a flat vector bundle over M , admitting an integral lattice, and let \tilde{E} be a local system of free abelian groups over M such that $\tilde{E} \otimes \mathbf{C} \simeq E$. Denote by $\tilde{E}_0 = p^*(\tilde{E})$; it is a local system over N . There exists a free finitely generated cochain complex C^* over the ring $P = \mathbf{Z}[\tau]$ having the following properties:*

- (a) *for any nonzero complex number $a \in \mathbf{C}^*$ there is a canonical isomorphism*

$$H^q(C^* \otimes_P \mathbf{C}_a) \xrightarrow{\simeq} H^q(M; a^{-\xi} \otimes E). \quad (2-3)$$

Here \mathbf{C}_a is \mathbf{C} , which is viewed as a P -module with the following structure: $\tau x = ax$ for $x \in \mathbf{C}$.

(b) for $a = 0$ there is a canonical evaluation isomorphism

$$H^q(C^* \otimes_P \mathbf{Z}_0) \rightarrow H^q(N, \partial_+ N; \tilde{E}_0), \quad (2-4)$$

where \mathbf{Z}_0 is \mathbf{Z} with the following P -module structure: $\tau x = 0$ for any $x \in \mathbf{Z}$.

To construct C^* , we shall assume that N is triangulated and ∂N is a subcomplex. Let $i_{\pm} : V \rightarrow N$ be the inclusions, which identify V with $\partial_{\pm} N$ correspondingly. \tilde{E} determines also an isomorphism of local systems $\sigma : i_+^* \tilde{E}_0 \rightarrow i_-^* \tilde{E}_0$ over V .

Denote by $C^q(N)$ and $C^q(V)$ the free abelian groups of \tilde{E}_0 -valued cochains; $\delta_N : C^q(N) \rightarrow C^{q+1}(N)$ and $\delta_V : C^q(V) \rightarrow C^{q+1}(V)$ will denote the corresponding coboundary homomorphisms.

Let $C^q(N)[\tau]$ and $C^{q-1}(V)[\tau]$ denote the free P -modules formed by polynomials with coefficients in the corresponding abelian groups; for example, an element $c \in C^q(N)[\tau]$ is a formal sum $c = \sum_{i \geq 0} c_i \tau^i$ with $c_i \in C^q(N)$ and only finitely many c_i 's are nonzero. The P -module structure is given as follows: $\tau \cdot c = \sum_{i \geq 0} c_i \tau^{i+1}$. It is clear that $C^q(N)[\tau]$ and $C^{q-1}(V)[\tau]$ are free finitely generated P -modules.

The natural P -module extensions

$$\delta_N : C^q(N)[\tau] \rightarrow C^{q+1}(N)[\tau], \quad \text{and} \quad \delta_V : C^q(V)[\tau] \rightarrow C^{q+1}(V)[\tau]. \quad (2-5)$$

of the boundary homomorphisms act coefficientwise, so that δ_N and δ_V are P -homomorphisms. If $\alpha = \sum_{i \geq 0} \alpha_i \tau^i \in C^q(N)[\tau]$, then $\delta_N(\alpha) = \sum_{i \geq 0} \delta_N(\alpha_i) \tau^i$.

Define a finitely generated free cochain complex C^* over $P = \mathbf{Z}[\tau]$ (the deformation complex) as follows: $C^* = \oplus C^q$, where

$$C^q = C^q(N)[\tau] \oplus C^{q-1}(V)[\tau].$$

Elements of chain complex C^q will be denoted as pairs (α, β) , where $\alpha \in C^q(N)[\tau]$ and $\beta \in C^{q-1}(V)[\tau]$. The differential $\delta : C^q \rightarrow C^{q+1}$ is given by the following formula

$$\delta(\alpha, \beta) = (\delta_N(\alpha), (\sigma \otimes i_+^* - \tau i_-^*)(\alpha) - \delta_V(\beta)), \quad (2-6)$$

where $\alpha \in C^q(N)[\tau]$ and $\beta \in C^{q-1}(V)[\tau]$. Obviously, C^* is the cylinder of the chain map $\sigma \otimes i_+^* - \tau i_-^*$ with a shifted grading.

To show (a) we note that M is obtained from N by identifying all points $i_+(v)$ with $i_-(v)$, where $v \in V$; the flat bundle E over M is obtained from the flat bundle \tilde{E} over N by identifying the vectors $e_+ \in \tilde{E}|_{\partial_+ N}$ and $e_- \in \tilde{E}|_{\partial_- N}$ with $\sigma i_+^*(e_+) = ai_-^*(e_-)$. Hence $H^q(M; a^{-\xi} \otimes E)$ can be identified with the cohomology of complex $C^*(M; a^{-\xi} \otimes E)$, consisting of cochains $\alpha \in C^q(N)$ satisfying the boundary conditions

$$ai_-^*(\alpha) = \sigma \otimes i_+^*(\alpha) \in C^q(V).$$

The complex $C^q \otimes_P \mathbf{C}_a = C^q(N) \oplus C^{q-1}(V)$ has the differential given by

$$\delta(\alpha, \beta) = (\delta_N(\alpha), (\sigma \otimes i_+^* - ai_-^*)(\alpha) - \delta_V(\beta)), \quad (2-7)$$

where $\alpha \in C^q(N)$ and $\beta \in C^{q-1}(V)$. It is clear that there is a chain homomorphism $C^*(M; a^{-\xi} \otimes E) \rightarrow C^* \otimes_P \mathbf{C}_a$ (acting by $\alpha \mapsto (\alpha, 0)$). It is easy to see that it induces an isomorphism on the cohomology. Indeed, suppose that a cocycle $\alpha \in C^q(M; a^{-\xi} \otimes E)$ bounds in the complex $C^* \otimes_P \mathbf{C}_a$. Then there are $\alpha_1 \in C^{q-1}(N)$, $\beta_1 \in C^{q-2}(V)$ such that $\alpha = \delta_N(\alpha_1)$, $\sigma \otimes i_+^*(\alpha_1) - ai_-^*(\alpha_1) - \delta_V(\beta_1) = 0$. We may find a cochain $\beta_2 \in C^{q-2}(N)$ such that $\sigma i_+^*(\beta_2) = \beta_1$ and $i_-^*(\beta_2) = 0$ (by extending β_1 into a neighborhood of $\partial_+ N$). Then setting $\alpha_2 = \alpha_1 - \delta_N(\beta_2)$ we have

$$\alpha = \delta_N(\alpha_2), \quad \sigma i_+^*(\alpha_2) - ai_-^*(\alpha_2) = 0, \quad (2-8)$$

which means that α also bounds in $C^q(M; a^{-\xi} \otimes E)$.

Similarly, suppose that (α, β) is a cocycle of complex $C^* \otimes_P \mathbf{C}_a$. As above we may find a cochain $\beta' \in C^{q-1}(N)$ with $i_+^*(\beta') = \beta$ and $i_-^*(\beta') = 0$. Then $(\alpha - \delta_N(\beta'), 0)$ is a cocycle of $C^*(M; a^{-\xi} \otimes E)$ and it is cohomologous to the initial cocycle (α, β) . This proves (a).

(b) follows similarly. \square

2.4. Relative deformation complex. We will define now a relative version of the deformation complex C^* .

Let $A \subset N$ be a simplicial subcomplex. We will assume that A is disjoint from $\partial_+ N$. Let $C^q(N, A)$ denote the free abelian group of \tilde{E}_0 -valued cochains on N which vanish on A . Let $C^q(N, A)[\tau]$ be constructed similarly to $C^q(N)[\tau]$, cf. above. We define the complex C_A^* as follows:

$$C_A^q = C^q(N, A)[\tau] \oplus C^{q-1}(V)[\tau]. \quad (2-9)$$

The differential $\delta : C_A^q \rightarrow C_A^{q+1}$ is defined by the following formula:

$$\delta(\alpha, \beta) = (\delta_{N,A}(\alpha), (\sigma i_+^* - \tau i_-^*)(\alpha) - \delta_V(\beta)), \quad (2-10)$$

where $\alpha \in C^q(N, A)[\tau]$ and $\beta \in C^{q-1}(V)[\tau]$. Here $\delta_{N,A} : C^q(N, A) \rightarrow C^{q+1}(N, A)$ and $\delta_V : C^q(V) \rightarrow C^{q+1}(V)$ denote the coboundary homomorphisms and also their P -module extension. $i_{\pm}^* : C^q(N, A) \rightarrow C^q(V)$ denote the restriction maps of cochains, and the same symbols denote also their polynomial extensions $i_{\pm}^* : C^q(N, A)[\tau] \rightarrow C^q(V)[\tau]$.

Similarly to (a) and (b) in 2.3 we have:

(a') for any $a \in \mathbf{C}^*$ there is a natural isomorphism

$$H^i(C_A^* \otimes_P \mathbf{C}_a) \simeq H^i(M, p(A); a^{-\xi} \otimes E), \quad (2-11)$$

where $p : N \rightarrow M$ is the identification map, cf. 2.1;

(b') also,

$$H^i(C_A^* \otimes_P \mathbf{Z}_0) \simeq H^i(N, A \cup \partial_+ N; \tilde{E}_0). \quad (2-12)$$

2.5. Algebraic integers and lifting. In this section it will become clear why our definition of the cup-length $\text{cl}(\xi)$ involves the condition of not being a Dirichlet unit.

Proposition 2. *Suppose that $A \subset N$ is a subcomplex, disjoint from $\partial_+ N$, such that the inclusion $A \rightarrow N$ is homotopic to a map $A \rightarrow \partial_+ N$. Let $a \in \mathbf{C}^*$ be a complex number, such that a^{-1} is not an algebraic integer. Then the homomorphism $C_A^* \rightarrow C^*$ induces an epimorphism on the cohomology*

$$H^i(C_A^* \otimes_P \mathbf{C}_a) \rightarrow H^i(C^* \otimes_P \mathbf{C}_a), \quad i = 0, 1, 2, \dots \quad (2-13)$$

Proof. Let \mathbf{Z}_0 denote the group \mathbf{Z} considered as a P -module with the trivial τ action, i.e. $\mathbf{Z}_0 = P/\tau P$. We will show first that

$$H^i(C_A^* \otimes_P \mathbf{Z}_0) \rightarrow H^i(C^* \otimes_P \mathbf{Z}_0) \quad (2-14)$$

is an epimorphism. We know from (2-4) and (2-12) that

$$H^i(C_A^* \otimes_P \mathbf{Z}_0) \simeq H^i(N, A \cup \partial_+ N; \tilde{E}_0) \quad \text{and} \quad H^i(C^* \otimes_P \mathbf{Z}_0) \simeq H^i(N, \partial_+ N; \tilde{E}_0).$$

In the exact sequence

$$\dots \rightarrow H^i(N, A \cup \partial_+ N; \tilde{E}_0) \rightarrow H^i(N, \partial_+ N; \tilde{E}_0) \xrightarrow{j^*} H^i(A \cup \partial_+ N, \partial_+ N; \tilde{E}_0) \rightarrow \dots$$

j^* acts trivially (since the inclusion $(A \cup \partial_+ N, \partial_+ N) \rightarrow (N, \partial_+ N)$ is null-homotopic) and hence $H^i(N, A \cup \partial_+ N; \tilde{E}_0) \rightarrow H^i(N, \partial_+ N; \tilde{E}_0)$ is an epimorphism. This proves that (2-14) is an epimorphism. Now, Proposition 2 follows from Proposition 3 below. \square

Proposition 3. *Let C and D be chain complexes of free finitely generated $P = \mathbf{Z}[\tau]$ -modules and let $f : C \rightarrow D$ be a chain map. Suppose that for some q the induced map $f_* : H_q(C \otimes_P \mathbf{Z}_0) \rightarrow H_q(D \otimes_P \mathbf{Z}_0)$ is an epimorphism; here \mathbf{Z}_0 is \mathbf{Z} considered with the trivial P -action: $\mathbf{Z}_0 = P/\tau P$. Then for any complex number $a \in \mathbf{C}^*$, such that a^{-1} is not an algebraic integer, the homomorphism*

$$f_* : H_q(C \otimes_P \mathbf{C}_a) \rightarrow H_q(D \otimes_P \mathbf{C}_a) \quad (2-15)$$

is an epimorphism; here \mathbf{C}_a denotes \mathbf{C} with τ acting as the multiplication by a .

Proof. Denote by $Z_q(C), Z_q(D)$ the sets of cycles of C and D and by $B_q(C)$ and $B_q(D)$ the sets of their boundaries. Recall that the homological dimension of P is 2. We have the exact sequence

$$0 \rightarrow Z_q(C) \rightarrow C_q \rightarrow B_{q-1}(C) \rightarrow 0$$

and hence $Z_q(C)$ is a free P -module (since $B_{q-1}(C)$ is a submodule of a free module and so has a homological dimension ≤ 1). Similarly $Z_q(D)$ is free.

Choose free bases for $Z_q(C), Z_q(D)$ and D_{q+1} , and express in terms of these bases the map

$$f \oplus d : Z_q(C) \oplus D_{q+1} \rightarrow Z_q(D). \quad (2-16)$$

The resulting matrix \mathcal{G} is rectangular, with entries in P .

We claim: *there exist integers $b_j \in \mathbf{Z}$ and minors $A_j(\tau) \in P$ of the matrix \mathcal{G} of size $\text{rk } Z_q(D) \times \text{rk } Z_q(D)$, such that the polynomial with integer coefficients*

$$p(\tau) = \sum_j b_j A_j(\tau) \quad (2-17)$$

satisfies

$$p(0) = 1. \quad (2-18)$$

In fact, we will show that our claim is *equivalent* to the requirement that $f_* : H_q(C \otimes_P \mathbf{Z}_0) \rightarrow H_q(D \otimes_P \mathbf{Z}_0)$ is an epimorphism. Namely, using the resolvent $0 \rightarrow P \xrightarrow{\tau} P \rightarrow \mathbf{Z}_0 \rightarrow 0$ it is easy to see that $\text{Tor}_1^P(B_{q-1}(C), \mathbf{Z}_0) = 0$ (since $B_{q-1}(C)$ is a submodule of a free module). Hence we have the exact sequence

$$0 \rightarrow Z_q(C) \otimes_P \mathbf{Z}_0 \rightarrow C_q \otimes_P \mathbf{Z}_0 \rightarrow B_{q-1}(C) \otimes_P \mathbf{Z}_0 \rightarrow 0.$$

This means that $Z_q(C) \otimes_P \mathbf{Z}_0 = Z_q(C \otimes_P \mathbf{Z}_0)$, and $B_{q-1}(C) \otimes_P \mathbf{Z}_0 = B_{q-1}(C \otimes_P \mathbf{Z}_0)$. Hence, the hypothesis of the Proposition, the homomorphism

$$f \oplus d : (Z_q(C) \otimes_P \mathbf{Z}_0) \oplus (D_{q+1} \otimes_P \mathbf{Z}_0) \rightarrow Z_q(D) \otimes_P \mathbf{Z}_0$$

is an epimorphism. This epimorphism is described by the matrix $\mathcal{G}(0)$, where we substitute $\tau = 0$ into \mathcal{G} . Therefore, there are minors $A_j(\tau)$ of \mathcal{G} of size $\text{rk } Z_q(D) \times \text{rk } Z_q(D)$, so that the ideal in \mathbf{Z} , generated by the integers $A_j(0)$ contains 1. This proves (2-18).

Since $p(\tau)$ is an integral polynomial with $p(0) = 1$ and a^{-1} is not an algebraic integer it follows that

$$p(a) \neq 0. \quad (2-19)$$

Let us show that (2-19) is equivalent to the statement that (2-15) is an epimorphism. We have the exact sequence

$$0 \rightarrow Z_q(C) \otimes_P \mathbf{C}_a \rightarrow C_q \otimes_P \mathbf{C}_a \rightarrow B_{q-1} \otimes \mathbf{C}_a \rightarrow 0$$

(here we may work over $\mathbf{C}[\tau]$ which is a PID). Hence, similarly to the arguments above, we obtain that the map

$$f \oplus d : (Z_q(C) \otimes_P \mathbf{C}_a) \oplus (D_{q+1} \otimes_P \mathbf{C}_a) \rightarrow Z_q(D) \otimes_P \mathbf{C}_a \quad (2-20)$$

is described by the matrix \mathcal{G} with substitution $\tau = a$. We conclude that at least one of the $\text{rk } Z_q(D) \times \text{rk } Z_q(D)$ minors $A_j(a)$ is nonzero because of (2-19), and hence (2-20) and (2-15) are epimorphisms. \square

2.6. Corollary (Lifting Property). *Let $E \rightarrow M$ be a flat vector bundle admitting an integral lattice. Let $a \in \mathbf{C}^*$ be a complex number, not an algebraic integer. Let $A \subset M$ be a closed subset such that $A = p(A')$, where $A' \subset N - \partial_+ N$ is a closed polyhedral subset such that the inclusion $A' \rightarrow N$ is homotopic to a map with values in $\partial_+ N$. Then the restriction map*

$$H^q(M, A; a^\xi \otimes E) \rightarrow H^q(M; a^\xi \otimes E) \quad (2-21)$$

is an epimorphism.

Proof. We just combine the isomorphisms (2-3) and (2-11) and Proposition 2. \square

2.7. End of proof of Theorem 1. We need to establish inequality (2-2). In other words, we want to prove the triviality of any cup-product

$$v_0 \cup v_1 \cup v_2 \cup \dots \cup v_{m+1} = 0, \quad \text{where } v_j \in H^{d_j}(M; E_j), \quad (2-22)$$

(where m denotes $m = \text{cat}_{(N, \partial_+ N)}(N)$) assuming that $d_j > 0$ for $j = 0, 1, 2, \dots, m+1$, and the bundles E_0 and E_1 are of the form $a_i^\xi \otimes F_i$, where $i = 0, 1$, with the numbers $a_0, a_1 \in \mathbf{C}$ not Dirichlet units, and the bundles F_0 and F_1 admitting integral lattices.

Moreover, we will assume that one of the numbers a_0 and a_1 is not an algebraic integer. In the case when both a_0 and a_1 are algebraic integers, the inverse numbers a_0^{-1} and a_1^{-1} are not algebraic integers, and we shall apply the arguments following below to the form $-\omega$ (representing the cohomology class $-\xi$), which obviously has the same set of critical points.)

Since we may always rename the numbers a_0 and a_1 , we will assume below that a_0 is not an algebraic integer.

Suppose that N can be covered by closed subsets $A_0, A_1 \cup \dots \cup A_m = N$ so that A_0 contains $\partial_+ N$ and the inclusion $A_0 \rightarrow N$ is homotopic to a map into $\partial_+ N$ keeping the points of $\partial_+ N$ fixed, (cf. 2.2), and for $j = 1, 2, \dots, m$ the subset A_j is null-homotopic in N . Without loss of generality we may assume that all A_j are polyhedral.

Let U_\pm be a small cylindrical neighborhood of $\partial_\pm N$ in N . We observe that for $j = 2, 3, \dots, m+1$ we may lift the class v_j to a relative cohomology class lying in

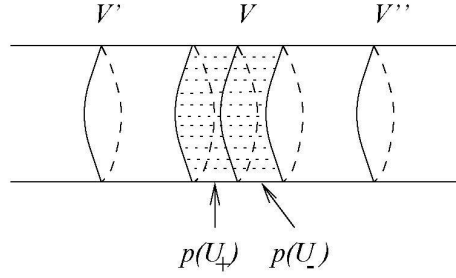


Figure 2.

$\tilde{v}_j \in H^{d_j}(M, B_j; E_j)$, where $B_j = p(A_{j-1} - U_+)$, since B_j is null-homotopic in M and $d_j > 0$. Recall that $p : N \rightarrow M$ denotes the natural identification map.

Applying Corollary 2.6, class v_0 can be lifted to a class $\tilde{v}_0 \in H^{d_0}(M, B_0; E_0)$, where $B_0 = p(A_0 - U_+)$.

Let B_1 be a closed cylindrical neighborhood of V in M containing $\overline{p(U_-)} \cup \overline{p(U_+)}$. We claim that we may lift the class $v_1 \in H^{d_1}(M; E_1)$ to a class $\tilde{v}_1 \in H^{d_1}(M, B_1; E_1)$. We will use Corollary 2.6. First, find two shifts of V into $M - B_1$, one (denoted V') in the positive normal direction and the other (denoted V'') in the negative normal direction (cf. Figure 2). If the number a_1 is not an algebraic integer we may apply Corollary 2.6 to the cut V'' . If the number a_1^{-1} is not an algebraic integer we may apply Corollary 2.6 to the cut V' .

Now, it is clear that the product $v_0 \cup \dots \cup v_{m+1}$ must be trivial since it is obtained from the product $\tilde{v}_0 \cup \dots \cup \tilde{v}_{m+1}$ (lying in $H^d(M, \cup_{j=0}^{m+1} B_j; E)$, where $E = \otimes_{j=0}^{m+1} E_j$) by restricting onto M , and the group $H^d(M, \cup_{j=0}^{m+1} B_j; E)$ vanishes, since $M = \cup_{j=0}^{m+1} B_j$. \square

§3. Proofs of Theorem 2 and Proposition 1

3.1. Proof of Theorem 2. Let ω be a closed 1-form lying in a cohomology class $\xi \in H^1(M; \mathbf{R})$ of rank $r > 1$. Let $S = S(\omega)$ denote the set of zeros of ω . It is clear that $\xi|_S = 0$.

Let r be the rank of ξ and let $\xi_1, \dots, \xi_r \in H^1(M; \mathbf{Z})$ be a basis of the free abelian group $\text{Hom}(H_1(M)/\ker(\xi); \mathbf{Z})$. We may write $\xi = \sum_{i=1}^r \alpha_i \xi_i$, and the coefficients are real $\alpha_i \in \mathbf{R}$.

Suppose that ξ_m is a sequence of rank 1 classes with $\text{cl}(\xi_m) \geq \text{cl}(\xi)$, which converges to ξ as $m \rightarrow \infty$, and each of the classes ξ_m vanishes on $\ker(\xi)$. Then we have $\xi_m = \sum_i \alpha_{i,m} \xi_i$, where $\alpha_{i,m} = \lambda_m \cdot n_{i,m}$, $\lambda_m \in \mathbf{R}$, and $n_{i,m} \in \mathbf{Z}$ for $i = 1, 2, \dots, r$. Each sequence $\alpha_{i,m}$ converges to α_i as m tends to ∞ .

Choose a closed 1-form ω_i in the class ξ_i for $i = 1, \dots, r$; since $\xi_i|_S = 0$ we may choose it so that it vanishes identically on a neighborhood of S . Define the

following sequence of closed 1-forms

$$\omega_m = \omega - \sum_{i=1}^r (\alpha_i - \alpha_{i,m}) \omega_i.$$

It is clear that ω_m has rank 1 and for m large enough $S(\omega_m) = S(\omega)$. The cohomology class of ω_m is ξ_m . By Theorem 1 we have $\text{cat}(S(\omega)) \geq \text{cl}(\xi_m) - 1$. Hence we obtain $\text{cat}(S(\omega)) \geq \text{cl}(\xi) - 1$. \square

3.2. Proof of Proposition 1. It is clear that it is enough to prove (1-8) assuming that the classes ξ_1 and ξ_2 are integral $\xi_\nu \in H^1(M_\nu; \mathbf{Z})$ for $\nu = 1, 2$. The general statement then follows automatically due to the nature of our definition of $\text{cl}(\xi)$ for general ξ , cf. 1.8. One may use here an equivalent definition of the cup-length $\text{cl}(\xi)$ for $\text{rk}(\xi) > 1$, which can be obtained from the definition given in 1.8 if in (1-5) we will additionally require that the approximating rank 1 classes ξ_m belong to $H^1(M; \mathbf{Q})$.

Position M_1 and M_2 so that their intersection is a small n -dimensional disk D^n , where $n = \dim M_1 = \dim M_2$, and then the connected sum $M_1 \# M_2$ is obtained from the union $M_1 \cup M_2$ by removing the interior of D^n . Let E be a flat bundle over the connected sum $M_1 \# M_2$ and let E_ν be a flat bundle over M_ν so that

$$E|_{M_\nu - D^n} \simeq E_\nu|_{M_\nu - D^n}, \quad (3-1)$$

for $\nu = 1, 2$. Here we use the assumption that $n > 2$ and so the sphere S^{n-1} is simply connected.

As follows from the Mayer - Vietoris sequence, there is a canonical isomorphism

$$\psi : H^q(M_1; E_1) \oplus H^q(M_2; E_2) \rightarrow H^q(M_1 \# M_2; E)$$

for $0 < q < n$. It will be clear from the rest of the proof that we do not need to worry about the case $q = n$. ψ is multiplicative in the following sense. Suppose that we have another flat bundle F over the connected sum $M_1 \# M_2$ and let F_ν be flat bundles over M_ν , $\nu = 1, 2$, satisfying condition (3-1). Then for any $v \in H^i(M_1; E_1)$ and $w \in H^j(M_1; F_1)$ with $0 < i$, $0 < j$, and $i + j < d$, holds $\psi(v \cup w, 0) = \psi(v, 0) \cup \psi(w, 0)$. Similar property holds with respect to the other variable.

Suppose now that $k = \text{cl}(\xi_1)$ and we have cohomology classes $v_j \in H^{d_j}(M_1; E_j)$, where $j = 1, 2, \dots, k$, satisfying all the properties of Definition 1.3; in particular, their product $v_1 \cup \dots \cup v_k$ is non-trivial. Then $\sum d_j = n$ (cf. 1.7). Extend each flat bundle E_j to a flat bundle \tilde{E}_j over M ; for $j = 1, 2$ we will make this extension so, that \tilde{E}_1 and \tilde{E}_2 will still satisfy condition (1-2).

We will first assume that $k > 2$. Then the classes

$$u_j = \psi(v_j, 0) \in H^{d_j}(M; \tilde{E}_j), \quad j = 1, 2, \dots, k-1,$$

have non-trivial cup product $u_1 \cup \cdots \cup u_{k-1}$ and satisfy all the properties of Definition 1.3. Using the Poincaré duality (as in the proof of Corollary 1.4), we may find a non-trivial cup product $u_1 \cup \cdots \cup u_{k-1} \cup u$, where $u \in H^{d_k}(M; E^* \otimes \mathcal{L}_M)$, $E = \otimes_{j=1}^{k-1} \tilde{E}_j$, and \mathcal{L}_M is the orientation flat line bundle of M .

In case, when $k = 2$ by the same reasons we will have a non-trivial cup-product $u_1 \cup u$, where $u \in H^{d_2}(M; \tilde{E}_1^* \otimes \mathcal{L}_M)$ and the bundle $\tilde{E}_1^* \otimes \mathcal{L}_M$ satisfies (1-2) assuming that E_1 does.

This proves inequality $\text{cl}(\xi) \geq \text{cl}(\xi_1)$. Therefore $\text{cl}(\xi) \geq \max\{\text{cl}(\xi_1), \text{cl}(\xi_2)\}$.

The inverse inequality follows similarly, using the properties of the map ψ mentioned above. \square

References

- [B] R. Bott, Non degenerate critical manifolds, *Ann. of Math.* **60** (1954), 248–261.
- [BF1] M. Braverman, M. Farber, Novikov type inequalities for differential forms with non-isolated zeros, *Mathematical Proceedings of the Cambridge Philosophical Society* **122** (1997), 357–375.
- [BF2] M. Braverman, M. Farber, Equivariant Novikov inequalities, *J. of K-theory* **12** (1997), 293–318.
- [DNF] B. A. Dubrovin, S. P. Novikov, A. T. Fomenko, *Modern Geometry; Methods of homology theory* (in Russian) 1984.
- [Fa] E. Fadell, Cohomological methods in non-free G -spaces with applications to general Borsuk - Ulam theorems and critical point theorems for invariant functionals. In: S.P. Singh (ed.), *Nonlinear functional Analysis and its applications* 1986, pp. 1–45.
- [Fo] G. Fournier and M. Willem, *The mountain circle theorem* Lect. Notes in Math. Vol **1475** 1991, pp 147–160.
- [F] M. S. Farber, Exactness of the Novikov inequalities, *Functional Analysis and its Applications* **19**(1) (1985), 40–49.
- [F1] M. Farber, Dirichlet units and critical points of closed 1-forms, *C.R. Acad. Sci. Paris* **328** (1999), 695–700.
- [F2] M. Farber, Topology of closed 1-forms and their critical points. Preprint math.DG/9811173.
- [F3] M. Farber, Counting zeros of closed 1-forms. Preprint math.DG/9903133, to appear in *Topology*.
- [N1] S. P. Novikov, Multivalued functions and functionals. An analogue of the Morse theory, *Soviet Math. Dokl.* **24** (1981), 222–226.
- [N2] S. P. Novikov, The Hamiltonian formalism and a multivalued analogue of Morse theory, *Russian Math. Surveys* **37** (1982), 1–56.
- [N3] S. P. Novikov, Bloch homology, critical points of functions and closed 1-forms, *Soviet Math. Dokl.* **33** (1986), 551–555.
- [S] A. Szulkin, A relative category and applications to critical point theory for strongly indefinite functionals, *Nonlinear Anal* **15** (1990), 725–739.

Michael Farber
 School of Mathematical Sciences
 Tel-Aviv University
 Ramat-Aviv 69978
 Israel
 e-mail: farber@math.tau.ac.il

(Received: May 20, 1998)