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## Lusternik–Schnirelman theory for closed 1-forms

Michael Farber

*Dedicated to S.P. Novikov on the occasion of his 60<sup>th</sup> birthday*

**Abstract.** S. P. Novikov developed an analog of the Morse theory for closed 1-forms. In this paper we suggest an analog of the Lusternik - Schnirelman theory for closed 1-forms. For any cohomology class  $\xi \in H^1(M, \mathbf{R})$  we define an integer  $\text{cl}(\xi)$  (*the cup-length associated with  $\xi$* ); we prove that any closed 1-form representing  $\xi$  has at least  $\text{cl}(\xi) - 1$  critical points. The number  $\text{cl}(\xi)$  is defined using cup-products in cohomology of some flat line bundles, such that their monodromy is described by complex numbers, which are not Dirichlet units.

**Mathematics Subject Classification (1991).** 58E05.

**Keywords.** Lusternik–Schnirelman theory, closed 1-forms.

### §1. The main result

**1.1.** Let  $M$  be a closed manifold and let  $\xi \in H^1(M; \mathbf{R})$  be a nonzero cohomology class. The Novikov inequalities [N1], [N2], [N3] estimate the numbers of zeros  $c_i(\omega)$  of different indices of any closed 1-form  $\omega$  with Morse type singularities on  $M$  lying in the class  $\xi$ .

Novikov type inequalities were constructed in [BF1] for closed 1-forms with slightly more general singularities (non-degenerate in the sense of Bott [B]). In [BF2] an equivariant generalization of the Novikov inequalities was found.

In this paper we will consider the problem of estimating the number of critical points of closed 1-forms  $\omega$  with no non-degeneracy assumption. We suggest here a version of the Lusternik - Schnirelman theory for closed 1-forms.

An announcement [F1] describes some results of this paper.

My recent preprint [F2] suggests a different approach to the Lusternik - Schnirelman theory of closed 1-forms; it uses untwisted cohomology and Massey products. Examples computed in [F2], show that the results of [F2] and of the present paper

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are independent.

**1.2** Let  $\xi \in H^1(M; \mathbf{Z})$  be an integral cohomology class. We will define below a nonnegative integer  $\text{cl}(\xi)$ , which we will call *the cup-length associated with  $\xi$* .

Recall, that a complex flat vector bundle  $E$  over  $M$  is determined by its monodromy, a linear representation of the fundamental group  $\pi_1(M, x_0)$  in  $\text{GL}_{\mathbf{C}}(E_0)$ , where  $E_0$  is the fiber over the base point  $x_0 \in M$ ; this representation is given by the parallel transport of vectors along loops. For example, a flat line bundle is determined by a homomorphism  $H_1(M; \mathbf{Z}) \rightarrow \mathbf{C}^*$ , where  $\mathbf{C}^*$  is considered as a multiplicative abelian group.

Given class  $\xi$  as above and a nonzero complex number  $a \in \mathbf{C}^*$ , we have the complex flat line bundle over  $M$  with the following property: the monodromy along any loop  $\gamma \in \pi_1(M)$  is the multiplication by  $a^{\langle \xi, \gamma \rangle}$ . We will denote this bundle by  $a^\xi$ . If  $a, b \in \mathbf{C}^*$ , we have the canonical isomorphism of flat line bundles

$$a^\xi \otimes b^\xi \simeq ab^\xi.$$

A lattice  $\mathcal{L} \subset V$  in a finite dimensional vector space  $V$  is a finitely generated subgroup with  $\text{rank } \mathcal{L} = \dim_{\mathbf{C}} V$ . We will say that a complex flat bundle  $E \rightarrow M$  of rank  $m$  admits an integral lattice if its monodromy representation  $\pi_1(M, x_0) \rightarrow \text{GL}_{\mathbf{C}}(E_0)$  is conjugate to a homomorphism  $\pi_1(M, x_0) \rightarrow \text{GL}_{\mathbf{Z}}(\mathcal{L}_0)$ , where  $\mathcal{L}_0 \subset E_0$  is a lattice in the fiber. This condition is equivalent to the assumption that  $E$  is obtained from a local system  $\tilde{E}$  of finitely generated free abelian groups over  $M$  by tensoring on  $\mathbf{C}$ .

**1.3. Definition.** *The cup-length  $\text{cl}(\xi)$  is the largest integer  $k$  such that there exists a nontrivial  $k$ -fold cup product*

$$H^{d_1}(M; E_1) \otimes H^{d_2}(M; E_2) \otimes \cdots \otimes H^{d_k}(M; E_k) \rightarrow H^d(M; E), \quad (1-1)$$

where  $d = d_1 + \cdots + d_k$ ,  $E = E_1 \otimes E_2 \otimes \cdots \otimes E_k$ ,  $d_1 > 0, \dots, d_k > 0$ , and the first two flat bundles  $E_1$  and  $E_2$  have the following property: there exist nonzero complex numbers  $a_1, a_2 \in \mathbf{C}^*$ , and complex flat bundles  $F_1$  and  $F_2$  over  $M$ , admitting integral lattices, so that

$$E_i \simeq a_i^\xi \otimes F_i, \quad \text{for } i = 1, 2, \quad (1-2)$$

and both numbers  $a_1$  and  $a_2$  are not Dirichlet units.

Recall that a Dirichlet unit is defined as a complex number  $b \neq 0$  such that  $b$  and its inverse  $b^{-1}$  are algebraic integers. In other words, Dirichlet units can be characterized as roots of polynomial equations

$$b^n + \gamma_1 b^{n-1} + \cdots + \gamma_{n-1} b + \gamma_n = 0,$$

where all  $\gamma_i$  are integers and  $\gamma_n = \pm 1$ .

Note that the cup-length  $\text{cl}(\xi)$ , defined by 1.3, satisfies  $0 \leq \text{cl}(\xi) \leq \dim M$ . We will see examples below showing that  $\text{cl}(\xi) = \dim M$  is possible.

The definition of the cup-length  $\text{cl}(\xi)$  above is slightly different from the one given in [F1]; following the present definition, we may have a larger cup-length  $\text{cl}(\xi)$ .

**Theorem 1.** *Let  $\omega$  be a closed 1-form on  $M$  lying in an integral cohomology class  $\xi \in H^1(M; \mathbf{Z})$ . Let  $S(\omega)$  denote the set of zeros of  $\omega$ , i.e. the set of points  $p \in M$  such that  $\omega_p = 0$ . Then the Lusternik - Schnirelman category of  $S(\omega)$  satisfies*

$$\text{cat}(S(\omega)) \geq \text{cl}(\xi) - 1. \quad (1-3)$$

*In particular, if the set of zeros  $S(\omega)$  is finite, then for the total number  $|S(\omega)|$  of zeros*

$$|S(\omega)| \geq \text{cl}(\xi) - 1. \quad (1-4)$$

Here  $\text{cat}(S)$  denotes the Lusternik - Schnirelman category of  $S = S(\omega)$ , i.e. the least number  $k$ , so that  $S$  can be covered by  $k$  closed subsets  $A_1 \cup \dots \cup A_k$  such that each inclusion  $A_j \rightarrow S$  is null-homotopic.

Proof of Theorem 1 is given in §2.

**1.4. Corollary ([F1]).** *Suppose that there exist complex numbers  $a_1, a_2, \dots, a_m \in \mathbf{C}^*$ , not all Dirichlet units, such that a cup product*

$$H^{d_1}(M; a_1^\xi) \otimes H^{d_2}(M; a_2^\xi) \otimes \dots \otimes H^{d_k}(M; a_k^\xi) \rightarrow H^d(M; a^\xi),$$

*with  $d_j > 0$ ,  $j = 1, 2, \dots, k$ , is nontrivial. Then for any closed 1-form  $\omega$  on manifold  $M$ , lying in class  $\xi \in H^1(M; \mathbf{Z})$ , holds  $\text{cat}(S(\omega)) \geq k - 1$ .*

*Proof.* We may assume that  $\xi \neq 0$ ; otherwise the statement follows from the Lusternik - Schnirelman theory for functions.

Corollary 1.4 directly follows from Theorem 1, if there are at least two non Dirichlet units among  $a_1, a_2, \dots, a_k$ . Suppose that there is precisely one non Dirichlet unit. Denote  $a = a_1 a_2 \dots a_k$ . Then  $a$  is not a Dirichlet unit, and, in particular,  $a \neq 1$ . Hence  $H^n(M; a^\xi) = 0$ . Therefore, the dimension of the nontrivial cup-product above  $d = d_1 + d_2 + \dots + d_k < n = \dim M$  is less than  $n$ . By the Poincaré duality, the cup-product pairing

$$H^d(M; a^\xi) \otimes H^{n-d}(M; a^{-\xi} \otimes \mathcal{L}_M) \rightarrow H^n(M; \mathcal{L}_M)$$

is non-degenerate. Here  $\mathcal{L}_M$  denotes the orientation flat line bundle of  $M$ . The monodromy of  $\mathcal{L}_M$  along any loop  $\gamma$  equals  $\pm 1$  depending on whether the orientation of  $M$  is preserved or reversed by  $\gamma$ . Note that  $\mathcal{L}_M$  admits an integral lattice.

Hence, we may find a nontrivial cup-product of length  $k + 1$  with an extra factor in  $H^{n-d}(M; a^{-\xi} \otimes \mathcal{L}_M)$ . Now, Theorem 1 applies and gives  $\text{cat}(S(\omega)) \geq k$ .  $\square$

**1.5.** It is clear that Corollary 1.4 becomes false if we remove the requirement that one of the numbers  $a_i$  are not Dirichlet units. The simplest example is provided by the torus  $T^n$ ; any cohomology class  $\xi \in H^1(T^n; \mathbf{R})$  of the torus  $M = T^n$  contains a closed 1-form without zeros, but the cup-length of  $T^n$  is  $n$ .

**1.6. Remark.** A crude estimate for the cup-length  $\text{cl}(\xi)$  can be obtained by taking the maximal length of a non-trivial product (1-1) with  $E_j = a_j^\xi$  and  $a_j \in \mathbf{C}^*$  being *transcendental*,  $j = 1, 2, \dots, k$ . We will give an example (cf. 1.10, example 3) showing that this estimate can be really worse than the one provided by Theorem 1.

**1.7. Remark.** *In the longest nontrivial product (1-1) the number  $d$  must be equal the dimension of the manifold  $n = \dim M$ .* Indeed, any nontrivial cup-product (1-1) with  $d < n$  can be made longer by using the Poincaré duality.

**1.8. Forms with non-integral periods.** In general, the cohomology class determined by a closed 1-form  $\omega$  belongs to  $H^1(M, \mathbf{R})$ , i.e. it has real coefficients. It is clear that multiplying  $\omega$  by a non-zero constant  $\lambda \neq 0$  does not change the set of critical points  $S(\omega)$  and multiplies the cohomology class by  $\lambda$ . Hence Theorem 1 also gives estimates in the case of *cohomology classes*  $\xi \in H^1(M, \mathbf{R})$  of rank 1 (i.e. for classes, which are real multiples of integral classes) if we define the associated cup-length  $\text{cl}(\xi)$  as follows

$$\text{cl}(\lambda\xi) = \text{cl}(\xi), \quad \lambda \in \mathbf{R}, \quad \lambda \neq 0, \quad \xi \in H^1(M, \mathbf{Z}).$$

Recall, that given a cohomology class  $\xi \in H^1(M, \mathbf{R})$ , its *rank* is defined as the rank of the abelian group, which is the image of the homomorphism  $H_1(M, \mathbf{Z}) \rightarrow \mathbf{R}$ , determined by  $\xi$ . Note that the cohomology classes of rank 1 are dense in  $H^1(M, \mathbf{R})$ . Therefore the following definition makes sense.

**Definition.** Given a class  $\xi \in H^1(M, \mathbf{R})$  of rank  $> 1$ , we define  $\text{cl}(\xi)$  as the largest number  $k$ , such that there exists a sequence of rank 1 classes  $\xi_m \in H^1(M, \mathbf{R})$  with

$$\text{cl}(\xi_m) \geq k, \quad \lim_{m \rightarrow \infty} \xi_m = \xi, \quad (1-5)$$

and each  $\xi_m$ , considered as a homomorphism  $H_1(M; \mathbf{Z}) \rightarrow \mathbf{R}$ , vanishes on the kernel of the homomorphism  $\xi : H_1(M; \mathbf{Z}) \rightarrow \mathbf{R}$ .

**Theorem 2.** *Let  $\omega$  be a closed 1-form on  $M$  lying in a cohomology class  $\xi \in H^1(M; \mathbf{R})$ . Let  $S(\omega)$  denote the set of zeros of  $\omega$ . Then the Lusternik - Schnirelman category of  $S(\omega)$  satisfies*

$$\text{cat}(S(\omega)) \geq \text{cl}(\xi) - 1. \quad (1-6)$$

In particular, if the set of critical points  $S(\omega)$  is finite then for the total number  $|S(\omega)|$  of the critical points,

$$|S(\omega)| \geq \text{cl}(\xi) - 1. \quad (1-7)$$

For the proof see §3.

**1.9. Connected sums.** Let  $M_1$  and  $M_2$  be two closed  $n$ -dimensional manifolds. Assume for simplicity, that  $n > 2$ . We will denote by  $M_1 \# M_2$  the connected sum of  $M_1$  and  $M_2$ . Given cohomology classes  $\xi_\nu \in H^1(M_\nu; \mathbf{R})$ , where  $\nu = 1, 2$ , the class  $\xi_1 \# \xi_2 \in H^1(M_1 \# M_2; \mathbf{R})$  is well defined, in an obvious way.

In the description of examples (cf. 1.10) we will use the following statement:

**Proposition 1.** *In the situation described above,*

$$\text{cl}(\xi_1 \# \xi_2) = \max\{\text{cl}(\xi_1), \text{cl}(\xi_2)\}. \quad (1-8)$$

Proof is given in §3.

**1.10. Examples.** 1. In the notations of the previous subsection, let  $\xi_1 = 0$  and suppose that  $\xi_2 \neq 0$  can be realized by a closed 1-form with no critical points (for example, fibration over the circle). Then we obtain from Proposition 1 that  $\text{cl}(\xi_1 \# \xi_2) = \text{cl}(\xi_1)$ . Since  $\xi_1 = 0$ , the cup-length  $\text{cl}(\xi_1)$  can be estimated from below by the usual cup-length of the manifold  $M_1$  with complex coefficients.

To have a specific example, let us take  $M_1 = T^n$ ,  $M_2 = S^1 \times S^{n-1}$ ,  $\xi_1 = 0$  and  $\xi_2 \in H^1(M_2; \mathbf{Z})$  being a generator, where  $n > 2$ . Then we have for  $\xi = \xi_1 \# \xi_2 \in H^1(M_1 \# M_2; \mathbf{R})$

$$\text{cl}(\xi_1 \# \xi_2) = n. \quad (1-9)$$

Therefore, by Theorem 1, any closed 1-form  $\omega$  on  $M_1 \# M_2$  lying in class  $\xi$  has at least  $n - 1$  critical points.

2. In a similar way one may construct examples of cohomology classes of higher rank with many critical points. Namely, suppose that  $M_1 = T^n$ , where  $n > 2$  and  $\xi_1 = 0$ ; take for  $M_2$  arbitrary closed manifold of dimension  $n$  with a cohomology class  $\xi_2 \in H^1(M_2; \mathbf{R})$  of rank  $q$ . Then for the class  $\xi = \xi_1 \# \xi_2 \in H^1(M_1 \# M_2; \mathbf{R})$  (having rank  $q$ ) we again obtain  $\text{cl}(\xi) = n$  (by Proposition 1).

One may take, for example,  $M_2 = T^q \times S^{n-q}$  with  $\xi_2$  induced from a maximally irrational class on the torus  $T^q$ .

3. Let  $M$  be a 3-dimensional manifold obtained by 0-framed surgery on the knot  $5_2$ :

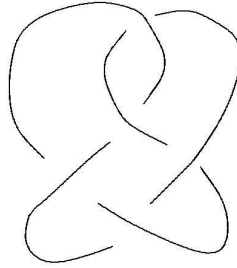


Figure 1.

This knot has Alexander polynomial  $\Delta(\tau) = 2 - 3\tau + 2\tau^2$ . Then  $H^1(M; \mathbf{Z}) = \mathbf{Z}$  and taking  $\xi \in H^1(M; \mathbf{Z})$  to be a generator we find that  $H^1(M; a^\xi)$  is trivial for all  $a \in \mathbf{C}^*$ , which are not the roots of the Alexander polynomial. It is easy to check that if  $a$  is one of the roots of  $2 - 3a + 2a^2 = 0$  then  $H^1(M; a^\xi) \neq 0$ . Note that the roots of  $2 - 3a + 2a^2 = 0$  are not Dirichlet units. Hence we obtain that all Novikov Betti numbers are trivial (since, as it is known [N3], that the Novikov Betti numbers equal to  $\dim H^*(M; a^\xi)$  for generic  $a \in \mathbf{C}$ ). However by Corollary 1.4 we obtain that any closed 1-forms in class  $\xi$  has at least 1 critical point.

## §2. Proof of Theorem 1

**2.1.** Since we assume that the cohomology class  $\xi$  of  $\omega$  is integral,  $\xi \in H^1(M, \mathbf{Z})$ , there exists a smooth map  $f : M \rightarrow S^1$ , such that  $\omega = f^*(d\theta)$ , where  $d\theta$  is the standard angular form on the circle  $S^1 \subset \mathbf{C}$ ,  $S^1 = \{z; |z| = 1\}$ .

Denote  $f^{-1}(b)$  by  $V \subset M$ , where  $b \in S^1$  is a regular value; it is a codimension one submanifold. Let  $N$  denote the manifold obtained by cutting  $M$  along  $V$ . Note that  $N$  and  $V$  could be disconnected.

Each connected component of  $V$  yields two connected components of  $\partial N$ , the positive and the negative. In order to distinguish between the positive and the negative boundary components of  $\partial N$ , we use the orientation of the normal bundle to  $V$  in  $M$ , given by the form  $\omega$ . The positive components are defined as those with the internal normal vector field to  $N$  being positive. The union of all positive (negative) boundary components of  $N$  will be denoted by  $\partial_+ N$ , or  $\partial_- N$ , correspondingly.

Let  $p : N \rightarrow M$  denotes the natural projection. Then  $p^*\omega = dg$ , where  $g : N \rightarrow \mathbf{R}$  is a smooth function, determined up to a constant on each connected component of  $N$ . It is clear that  $g$  is constant on each connected component of  $\partial N$ . The points of  $\partial_+ N$  are points of local minimum of  $g$ ; the points of  $\partial_- N$  are points of local maximum of  $g$ . The map  $g$  sends the set  $S(g)$  of critical points of  $g$  diffeomorphically onto the set  $S(\omega)$ .

**2.2. Relative Lusternik - Schnirelman category.** We will use the well-known notion of relative Lusternik - Schnirelman category, cf. [Fa], [Fo], [S]. Let's recall it.

For any subset  $X \subset N$  containing  $\partial_+ N$  we will denote by  $\text{cat}_{(N, \partial_+ N)}(X)$  the minimal number  $k$  such that  $X$  can be covered by  $k + 1$  closed subsets

$$X \subset A_0 \cup A_1 \cup A_2 \cup \dots \cup A_k \subset N$$

with the following properties:

- (1)  $A_0$  contains  $\partial_+ N$  and the inclusion  $A_0 \rightarrow N$  is homotopic to a map  $A_0 \rightarrow \partial_+ N$  keeping the points of  $\partial_+ N \subset A$  fixed;
- (2) for  $j = 1, 2, \dots, k$ , each inclusion  $A_j \rightarrow N$  is null-homotopic.

We claim, that

$$\text{cat } S(\omega) = \text{cat } S(g) \geq \text{cat}_{(N, \partial_+ N)}(N). \quad (2-1)$$

This follows from known results, cf., for example, [Fo], Th. 4.2. We apply Theorem 4.2 of [Fo] to each of the connected components of  $N$  and to the restriction of function  $g$  on it; we use the additivity of the relative Lusternik - Schnirelman category with respect to disjoint union, cf. [Fo], Prop. 2.8.

Our next purpose will be to prove the inequality

$$\text{cat}_{(N, \partial_+ N)}(N) \geq \text{cl}(\xi) - 1. \quad (2-2)$$

Together with (2-1) this will complete the proof of the Theorem.

**2.3. The deformation complex.** The proof of (2-2) will consist of building a *polynomial deformation*, a finitely generated free cochain complex  $C^*$  over the ring  $P = \mathbf{Z}[\tau]$  of polynomials with integral coefficients, having properties (a), (b) described below. With the help of the deformation complex we will prove the Lifting Property, cf. Corollary 2.6, playing a crucial role in the proof.

In [F3] we show how the deformation complex leads to inequalities, which are stronger than the Novikov inequalities.

The construction of the deformation complex is similar to [F2]; the difference is that in the present paper we will work over the integers, and in [F2] over a field.

**Claim.** *Let  $E \rightarrow M$  be a flat vector bundle over  $M$ , admitting an integral lattice, and let  $\tilde{E}$  be a local system of free abelian groups over  $M$  such that  $\tilde{E} \otimes \mathbf{C} \simeq E$ . Denote by  $\tilde{E}_0 = p^*(\tilde{E})$ ; it is a local system over  $N$ . There exists a free finitely generated cochain complex  $C^*$  over the ring  $P = \mathbf{Z}[\tau]$  having the following properties:*

- (a) *for any nonzero complex number  $a \in \mathbf{C}^*$  there is a canonical isomorphism*

$$H^q(C^* \otimes_P \mathbf{C}_a) \xrightarrow{\simeq} H^q(M; a^{-\xi} \otimes E). \quad (2-3)$$

Here  $\mathbf{C}_a$  is  $\mathbf{C}$ , which is viewed as a  $P$ -module with the following structure:  $\tau x = ax$  for  $x \in \mathbf{C}$ .

(b) for  $a = 0$  there is a canonical evaluation isomorphism

$$H^q(C^* \otimes_P \mathbf{Z}_0) \rightarrow H^q(N, \partial_+ N; \tilde{E}_0), \tag{2-4}$$

where  $\mathbf{Z}_0$  is  $\mathbf{Z}$  with the following  $P$ -module structure:  $\tau x = 0$  for any  $x \in \mathbf{Z}$ .

To construct  $C^*$ , we shall assume that  $N$  is triangulated and  $\partial N$  is a subcomplex. Let  $i_{\pm} : V \rightarrow N$  be the inclusions, which identify  $V$  with  $\partial_{\pm} N$  correspondingly.  $\tilde{E}$  determines also an isomorphism of local systems  $\sigma : i_+^* \tilde{E}_0 \rightarrow i_-^* \tilde{E}_0$  over  $V$ .

Denote by  $C^q(N)$  and  $C^q(V)$  the free abelian groups of  $\tilde{E}_0$ -valued cochains;  $\delta_N : C^q(N) \rightarrow C^{q+1}(N)$  and  $\delta_V : C^q(V) \rightarrow C^{q+1}(V)$  will denote the corresponding coboundary homomorphisms.

Let  $C^q(N)[\tau]$  and  $C^{q-1}(V)[\tau]$  denote the free  $P$ -modules formed by polynomials with coefficients in the corresponding abelian groups; for example, an element  $c \in C^q(N)[\tau]$  is a formal sum  $c = \sum_{i \geq 0} c_i \tau^i$  with  $c_i \in C^q(N)$  and only finitely many  $c_i$ 's are nonzero. The  $P$ -module structure is given as follows:  $\tau \cdot c = \sum_{i \geq 0} c_i \tau^{i+1}$ . It is clear that  $C^q(N)[\tau]$  and  $C^{q-1}(V)[\tau]$  are free finitely generated  $P$ -modules.

The natural  $P$ -module extensions

$$\delta_N : C^q(N)[\tau] \rightarrow C^{q+1}(N)[\tau], \quad \text{and} \quad \delta_V : C^q(V)[\tau] \rightarrow C^{q+1}(V)[\tau]. \tag{2-5}$$

of the boundary homomorphisms act coefficientwise, so that  $\delta_N$  and  $\delta_V$  are  $P$ -homomorphisms. If  $\alpha = \sum_{i \geq 0} \alpha_i \tau^i \in C^q(N)[\tau]$ , then  $\delta_N(\alpha) = \sum_{i \geq 0} \delta_N(\alpha_i) \tau^i$ .

Define a finitely generated free cochain complex  $C^*$  over  $P = \mathbf{Z}[\tau]$  (the deformation complex) as follows:  $C^* = \oplus C^q$ , where

$$C^q = C^q(N)[\tau] \oplus C^{q-1}(V)[\tau].$$

Elements of chain complex  $C^q$  will be denoted as pairs  $(\alpha, \beta)$ , where  $\alpha \in C^q(N)[\tau]$  and  $\beta \in C^{q-1}(V)[\tau]$ . The differential  $\delta : C^q \rightarrow C^{q+1}$  is given by the following formula

$$\delta(\alpha, \beta) = (\delta_N(\alpha), (\sigma \otimes i_+^* - \tau i_-^*)(\alpha) - \delta_V(\beta)), \tag{2-6}$$

where  $\alpha \in C^q(N)[\tau]$  and  $\beta \in C^{q-1}(V)[\tau]$ . Obviously,  $C^*$  is the cylinder of the chain map  $\sigma \otimes i_+^* - \tau i_-^*$  with a shifted grading.

To show (a) we note that  $M$  is obtained from  $N$  by identifying all points  $i_+(v)$  with  $i_-(v)$ , where  $v \in V$ ; the flat bundle  $E$  over  $M$  is obtained from the flat bundle  $\tilde{E}$  over  $N$  by identifying the vectors  $e_+ \in \tilde{E}|_{\partial_+ N}$  and  $e_- \in \tilde{E}|_{\partial_- N}$  with  $\sigma i_+^*(e_+) = ai_-^*(e_-)$ . Hence  $H^q(M; a^{-\xi} \otimes E)$  can be identified with the cohomology of complex  $C^*(M; a^{-\xi} \otimes E)$ , consisting of cochains  $\alpha \in C^q(N)$  satisfying the boundary conditions

$$ai_-^*(\alpha) = \sigma \otimes i_+^*(\alpha) \in C^q(V).$$

The complex  $C^q \otimes_P \mathbf{C}_a = C^q(N) \oplus C^{q-1}(V)$  has the differential given by

$$\delta(\alpha, \beta) = (\delta_N(\alpha), (\sigma \otimes i_+^* - ai_-^*)(\alpha) - \delta_V(\beta)), \quad (2-7)$$

where  $\alpha \in C^q(N)$  and  $\beta \in C^{q-1}(V)$ . It is clear that there is a chain homomorphism  $C^*(M; a^{-\xi} \otimes E) \rightarrow C^* \otimes_P \mathbf{C}_a$  (acting by  $\alpha \mapsto (\alpha, 0)$ ). It is easy to see that it induces an isomorphism on the cohomology. Indeed, suppose that a cocycle  $\alpha \in C^q(M; a^{-\xi} \otimes E)$  bounds in the complex  $C^* \otimes_P \mathbf{C}_a$ . Then there are  $\alpha_1 \in C^{q-1}(N)$ ,  $\beta_1 \in C^{q-2}(V)$  such that  $\alpha = \delta_N(\alpha_1)$ ,  $\sigma \otimes i_+^*(\alpha_1) - ai_-^*(\alpha_1) - \delta_V(\beta_1) = 0$ . We may find a cochain  $\beta_2 \in C^{q-2}(N)$  such that  $\sigma i_+^*(\beta_2) = \beta_1$  and  $i_-^*(\beta_2) = 0$  (by extending  $\beta_1$  into a neighborhood of  $\partial_+ N$ ). Then setting  $\alpha_2 = \alpha_1 - \delta_N(\beta_2)$  we have

$$\alpha = \delta_N(\alpha_2), \quad \sigma i_+^*(\alpha_2) - ai_-^*(\alpha_2) = 0, \quad (2-8)$$

which means that  $\alpha$  also bounds in  $C^q(M; a^{-\xi} \otimes E)$ .

Similarly, suppose that  $(\alpha, \beta)$  is a cocycle of complex  $C^* \otimes_P \mathbf{C}_a$ . As above we may find a cochain  $\beta' \in C^{q-1}(N)$  with  $i_+^*(\beta') = \beta$  and  $i_-^*(\beta') = 0$ . Then  $(\alpha - \delta_N(\beta'), 0)$  is a cocycle of  $C^*(M; a^{-\xi} \otimes E)$  and it is cohomologous to the initial cocycle  $(\alpha, \beta)$ . This proves (a).

(b) follows similarly.  $\square$

**2.4. Relative deformation complex.** We will define now a relative version of the deformation complex  $C^*$ .

Let  $A \subset N$  be a simplicial subcomplex. We will assume that  $A$  is disjoint from  $\partial_+ N$ . Let  $C^q(N, A)$  denote the free abelian group of  $\tilde{E}_0$ -valued cochains on  $N$  which vanish on  $A$ . Let  $C^q(N, A)[\tau]$  be constructed similarly to  $C^q(N)[\tau]$ , cf. above. We define the complex  $C_A^*$  as follows:

$$C_A^q = C^q(N, A)[\tau] \oplus C^{q-1}(V)[\tau]. \quad (2-9)$$

The differential  $\delta : C_A^q \rightarrow C_A^{q+1}$  is defined by the following formula:

$$\delta(\alpha, \beta) = (\delta_{N,A}(\alpha), (\sigma i_+^* - \tau i_-^*)(\alpha) - \delta_V(\beta)), \quad (2-10)$$

where  $\alpha \in C^q(N, A)[\tau]$  and  $\beta \in C^{q-1}(V)[\tau]$ . Here  $\delta_{N,A} : C^q(N, A) \rightarrow C^{q+1}(N, A)$  and  $\delta_V : C^q(V) \rightarrow C^{q+1}(V)$  denote the coboundary homomorphisms and also their  $P$ -module extension.  $i_{\pm}^* : C^q(N, A) \rightarrow C^q(V)$  denote the restriction maps of cochains, and the same symbols denote also their polynomial extensions  $i_{\pm}^* : C^q(N, A)[\tau] \rightarrow C^q(V)[\tau]$ .

Similarly to (a) and (b) in 2.3 we have:

(a') for any  $a \in \mathbf{C}^*$  there is a natural isomorphism

$$H^i(C_A^* \otimes_P \mathbf{C}_a) \simeq H^i(M, p(A); a^{-\xi} \otimes E), \quad (2-11)$$

where  $p : N \rightarrow M$  is the identification map, cf. 2.1;

(b') also,

$$H^i(C_A^* \otimes_P \mathbf{Z}_0) \simeq H^i(N, A \cup \partial_+ N; \tilde{E}_0). \tag{2-12}$$

**2.5. Algebraic integers and lifting.** In this section it will become clear why our definition of the cup-length  $\text{cl}(\xi)$  involves the condition of not being a Dirichlet unit.

**Proposition 2.** *Suppose that  $A \subset N$  is a subcomplex, disjoint from  $\partial_+ N$ , such that the inclusion  $A \rightarrow N$  is homotopic to a map  $A \rightarrow \partial_+ N$ . Let  $a \in \mathbf{C}^*$  be a complex number, such that  $a^{-1}$  is not an algebraic integer. Then the homomorphism  $C_A^* \rightarrow \mathbf{C}^*$  induces an epimorphism on the cohomology*

$$H^i(C_A^* \otimes_P \mathbf{C}_a) \rightarrow H^i(\mathbf{C}^* \otimes_P \mathbf{C}_a), \quad i = 0, 1, 2, \dots \tag{2-13}$$

*Proof.* Let  $\mathbf{Z}_0$  denote the group  $\mathbf{Z}$  considered as a  $P$ -module with the trivial  $\tau$  action, i.e.  $\mathbf{Z}_0 = P/\tau P$ . We will show first that

$$H^i(C_A^* \otimes_P \mathbf{Z}_0) \rightarrow H^i(\mathbf{C}^* \otimes_P \mathbf{Z}_0) \tag{2-14}$$

is an epimorphism. We know from (2-4) and (2-12) that

$$H^i(C_A^* \otimes_P \mathbf{Z}_0) \simeq H^i(N, A \cup \partial_+ N; \tilde{E}_0) \quad \text{and} \quad H^i(\mathbf{C}^* \otimes_P \mathbf{Z}_0) \simeq H^i(N, \partial_+ N; \tilde{E}_0).$$

In the exact sequence

$$\dots \rightarrow H^i(N, A \cup \partial_+ N; \tilde{E}_0) \rightarrow H^i(N, \partial_+ N; \tilde{E}_0) \xrightarrow{j^*} H^i(A \cup \partial_+ N, \partial_+ N; \tilde{E}_0) \rightarrow \dots$$

$j^*$  acts trivially (since the inclusion  $(A \cup \partial_+ N, \partial_+ N) \rightarrow (N, \partial_+ N)$  is null-homotopic) and hence  $H^i(N, A \cup \partial_+ N; \tilde{E}_0) \rightarrow H^i(N, \partial_+ N; \tilde{E}_0)$  is an epimorphism. This proves that (2-14) is an epimorphism. Now, Proposition 2 follows from Proposition 3 below.  $\square$

**Proposition 3.** *Let  $C$  and  $D$  be chain complexes of free finitely generated  $P = \mathbf{Z}[\tau]$ -modules and let  $f : C \rightarrow D$  be a chain map. Suppose that for some  $q$  the induced map  $f_* : H_q(C \otimes_P \mathbf{Z}_0) \rightarrow H_q(D \otimes_P \mathbf{Z}_0)$  is an epimorphism; here  $\mathbf{Z}_0$  is  $\mathbf{Z}$  considered with the trivial  $P$ -action:  $\mathbf{Z}_0 = P/\tau P$ . Then for any complex number  $a \in \mathbf{C}^*$ , such that  $a^{-1}$  is not an algebraic integer, the homomorphism*

$$f_* : H_q(C \otimes_P \mathbf{C}_a) \rightarrow H_q(D \otimes_P \mathbf{C}_a) \tag{2-15}$$

*is an epimorphism; here  $\mathbf{C}_a$  denotes  $\mathbf{C}$  with  $\tau$  acting as the multiplication by  $a$ .*

*Proof.* Denote by  $Z_q(C), Z_q(D)$  the sets of cycles of  $C$  and  $D$  and by  $B_q(C)$  and  $B_q(D)$  the sets of their boundaries. Recall that the homological dimension of  $P$  is 2. We have the exact sequence

$$0 \rightarrow Z_q(C) \rightarrow C_q \rightarrow B_{q-1}(C) \rightarrow 0$$

and hence  $Z_q(C)$  is a free  $P$ -module (since  $B_{q-1}(C)$  is a submodule of a free module and so has a homological dimension  $\leq 1$ ). Similarly  $Z_q(D)$  is free.

Choose free bases for  $Z_q(C), Z_q(D)$  and  $D_{q+1}$ , and express in terms of these bases the map

$$f \oplus d : Z_q(C) \oplus D_{q+1} \rightarrow Z_q(D). \tag{2-16}$$

The resulting matrix  $\mathcal{G}$  is rectangular, with entries in  $P$ .

We claim: *there exist integers  $b_j \in \mathbf{Z}$  and minors  $A_j(\tau) \in P$  of the matrix  $\mathcal{G}$  of size  $\text{rk } Z_q(D) \times \text{rk } Z_q(D)$ , such that the polynomial with integer coefficients*

$$p(\tau) = \sum_j b_j A_j(\tau) \tag{2-17}$$

*satisfies*

$$p(0) = 1. \tag{2-18}$$

In fact, we will show that our claim is *equivalent* to the requirement that  $f_* : H_q(C \otimes_P \mathbf{Z}_0) \rightarrow H_q(D \otimes_P \mathbf{Z}_0)$  is an epimorphism. Namely, using the resolvent  $0 \rightarrow P \xrightarrow{\tau} P \rightarrow \mathbf{Z}_0 \rightarrow 0$  it is easy to see that  $\text{Tor}_1^P(B_{q-1}(C), \mathbf{Z}_0) = 0$  (since  $B_{q-1}(C)$  is a submodule of a free module). Hence we have the exact sequence

$$0 \rightarrow Z_q(C) \otimes_P \mathbf{Z}_0 \rightarrow C_q \otimes_P \mathbf{Z}_0 \rightarrow B_{q-1}(C) \otimes \mathbf{Z}_0 \rightarrow 0.$$

This means that  $Z_q(C) \otimes_P \mathbf{Z}_0 = Z_q(C \otimes_P \mathbf{Z}_0)$ , and  $B_{q-1}(C) \otimes_P \mathbf{Z}_0 = B_{q-1}(C \otimes_P \mathbf{Z}_0)$ . Hence, the hypothesis of the Proposition, the homomorphism

$$f \oplus d : (Z_q(C) \otimes_P \mathbf{Z}_0) \oplus (D_{q+1} \otimes_P \mathbf{Z}_0) \rightarrow Z_q(D) \otimes_P \mathbf{Z}_0$$

is an epimorphism. This epimorphism is described by the matrix  $\mathcal{G}(0)$ , where we substitute  $\tau = 0$  into  $\mathcal{G}$ . Therefore, there are minors  $A_j(\tau)$  of  $\mathcal{G}$  of size  $\text{rk } Z_q(D) \times \text{rk } Z_q(D)$ , so that the ideal in  $\mathbf{Z}$ , generated by the integers  $A_j(0)$  contains 1. This proves (2-18).

Since  $p(\tau)$  is an integral polynomial with  $p(0) = 1$  and  $a^{-1}$  is not an algebraic integer it follows that

$$p(a) \neq 0. \tag{2-19}$$

Let us show that (2-19) is equivalent to the statement that (2-15) is an epimorphism. We have the exact sequence

$$0 \rightarrow Z_q(C) \otimes_P \mathbf{C}_a \rightarrow C_q \otimes_P \mathbf{C}_a \rightarrow B_{q-1} \otimes \mathbf{C}_a \rightarrow 0$$

(here we may work over  $\mathbf{C}[\tau]$  which is a PID). Hence, similarly to the arguments above, we obtain that the map

$$f \oplus d : (Z_q(C) \otimes_P \mathbf{C}_a) \oplus (D_{q+1} \otimes_P \mathbf{C}_a) \rightarrow Z_q(D) \otimes_P \mathbf{C}_a \tag{2-20}$$

is described by the matrix  $\mathcal{G}$  with substitution  $\tau = a$ . We conclude that at least one of the  $\text{rk } Z_q(D) \times \text{rk } Z_q(D)$  minors  $A_j(a)$  is nonzero because of (2-19), and hence (2-20) and (2-15) are epimorphisms.  $\square$

**2.6. Corollary (Lifting Property).** *Let  $E \rightarrow M$  be a flat vector bundle admitting an integral lattice. Let  $a \in \mathbf{C}^*$  be a complex number, not an algebraic integer. Let  $A \subset M$  be a closed subset such that  $A = p(A')$ , where  $A' \subset N - \partial_+ N$  is a closed polyhedral subset such that the inclusion  $A' \rightarrow N$  is homotopic to a map with values in  $\partial_+ N$ . Then the restriction map*

$$H^q(M, A; a^\xi \otimes E) \rightarrow H^q(M; a^\xi \otimes E) \tag{2-21}$$

*is an epimorphism.*

*Proof.* We just combine the isomorphisms (2-3) and (2-11) and Proposition 2.  $\square$

**2.7. End of proof of Theorem 1.** We need to establish inequality (2-2). In other words, we want to prove the triviality of any cup-product

$$v_0 \cup v_1 \cup v_2 \cup \dots \cup v_{m+1} = 0, \quad \text{where } v_j \in H^{d_j}(M; E_j), \tag{2-22}$$

(where  $m$  denotes  $m = \text{cat}_{(N, \partial_+ N)}(N)$ ) assuming that  $d_j > 0$  for  $j = 0, 1, 2, \dots, m+1$ , and the bundles  $E_0$  and  $E_1$  are of the form  $a_i^\xi \otimes F_i$ , where  $i = 0, 1$ , with the numbers  $a_0, a_1 \in \mathbf{C}$  not Dirichlet units, and the bundles  $F_0$  and  $F_1$  admitting integral lattices.

Moreover, we will assume that one of the numbers  $a_0$  and  $a_1$  is not an algebraic integer. In the case when both  $a_0$  and  $a_1$  are algebraic integers, the inverse numbers  $a_0^{-1}$  and  $a_1^{-1}$  are not algebraic integers, and we shall apply the arguments following below to the form  $-\omega$  (representing the cohomology class  $-\xi$ ), which obviously has the same set of critical points.)

Since we may always rename the numbers  $a_0$  and  $a_1$ , we will assume below that  $a_0$  is not an algebraic integer.

Suppose that  $N$  can be covered by closed subsets  $A_0, A_1 \cup \dots \cup A_m = N$  so that  $A_0$  contains  $\partial_+ N$  and the inclusion  $A_0 \rightarrow N$  is homotopic to a map into  $\partial_+ N$  keeping the points of  $\partial_+ N$  fixed, (cf. 2.2), and for  $j = 1, 2, \dots, m$  the subset  $A_j$  is null-homotopic in  $N$ . Without loss of generality we may assume that all  $A_j$  are polyhedral.

Let  $U_\pm$  be a small cylindrical neighborhood of  $\partial_\pm N$  in  $N$ . We observe that for  $j = 2, 3, \dots, m+1$  we may lift the class  $v_j$  to a relative cohomology class lying in

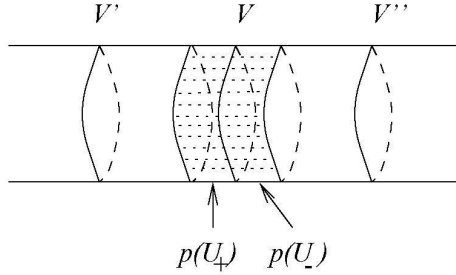


Figure 2.

$\tilde{v}_j \in H^{d_j}(M, B_j; E_j)$ , where  $B_j = p(A_{j-1} - U_+)$ , since  $B_j$  is null-homotopic in  $M$  and  $d_j > 0$ . Recall that  $p : N \rightarrow M$  denotes the natural identification map.

Applying Corollary 2.6, class  $v_0$  can be lifted to a class  $\tilde{v}_0 \in H^{d_0}(M, B_0; E_0)$ , where  $B_0 = p(A_0 - U_+)$ .

Let  $B_1$  be a closed cylindrical neighborhood of  $V$  in  $M$  containing  $\overline{p(U_-)} \cup \overline{p(U_+)}$ . We claim that we may lift the class  $v_1 \in H^{d_1}(M; E_1)$  to a class  $\tilde{v}_1 \in H^{d_1}(M, B_1; E_1)$ . We will use Corollary 2.6. First, find two shifts of  $V$  into  $M - B_1$ , one (denoted  $V'$ ) in the positive normal direction and the other (denoted  $V''$ ) in the negative normal direction (cf. Figure 2). If the number  $a_1$  is not an algebraic integer we may apply Corollary 2.6 to the cut  $V''$ . If the number  $a_1^{-1}$  is not an algebraic integer we may apply Corollary 2.6 to the cut  $V'$ .

Now, it is clear that the product  $v_0 \cup \dots \cup v_{m+1}$  must be trivial since it is obtained from the product  $\tilde{v}_0 \cup \dots \cup \tilde{v}_{m+1}$  (lying in  $H^d(M, \cup_{j=0}^{m+1} B_j; E)$ , where  $E = \otimes_{j=0}^{m+1} E_j$ ) by restricting onto  $M$ , and the group  $H^d(M, \cup_{j=0}^{m+1} B_j; E)$  vanishes, since  $M = \cup_{j=0}^{m+1} B_j$ .  $\square$

### §3. Proofs of Theorem 2 and Proposition 1

**3.1. Proof of Theorem 2.** Let  $\omega$  be a closed 1-form lying in a cohomology class  $\xi \in H^1(M; \mathbf{R})$  of rank  $= r > 1$ . Let  $S = S(\omega)$  denote the set of zeros of  $\omega$ . It is clear that  $\xi|_S = 0$ .

Let  $r$  be the rank of  $\xi$  and let  $\xi_1, \dots, \xi_r \in H^1(M; \mathbf{Z})$  be a basis of the free abelian group  $\text{Hom}(H_1(M)/\ker(\xi); \mathbf{Z})$ . We may write  $\xi = \sum_{i=1}^r \alpha_i \xi_i$ , and the coefficients are real  $\alpha_i \in \mathbf{R}$ .

Suppose that  $\xi_m$  is a sequence of rank 1 classes with  $\text{cl}(\xi_m) \geq \text{cl}(\xi)$ , which converges to  $\xi$  as  $m \rightarrow \infty$ , and each of the classes  $\xi_m$  vanishes on  $\ker(\xi)$ . Then we have  $\xi_m = \sum_i \alpha_{i,m} \xi_i$ , where  $\alpha_{i,m} = \lambda_m \cdot n_{i,m}$ ,  $\lambda_m \in \mathbf{R}$ , and  $n_{i,m} \in \mathbf{Z}$  for  $i = 1, 2, \dots, r$ . Each sequence  $\alpha_{i,m}$  converges to  $\alpha_i$  as  $m$  tends to  $\infty$ .

Choose a closed 1-form  $\omega_i$  in the class  $\xi_i$  for  $i = 1, \dots, r$ ; since  $\xi_i|_S = 0$  we may choose it so that it vanishes identically on a neighborhood of  $S$ . Define the

following sequence of closed 1-forms

$$\omega_m = \omega - \sum_{i=1}^r (\alpha_i - \alpha_{i,m}) \omega_i.$$

It is clear that  $\omega_m$  has rank 1 and for  $m$  large enough  $S(\omega_m) = S(\omega)$ . The cohomology class of  $\omega_m$  is  $\xi_m$ . By Theorem 1 we have  $\text{cat}(S(\omega)) \geq \text{cl}(\xi_m) - 1$ . Hence we obtain  $\text{cat}(S(\omega)) \geq \text{cl}(\xi) - 1$ .  $\square$

**3.2. Proof of Proposition 1.** It is clear that it is enough to prove (1-8) assuming that the classes  $\xi_1$  and  $\xi_2$  are integral  $\xi_\nu \in H^1(M_\nu; \mathbf{Z})$  for  $\nu = 1, 2$ . The general statement then follows automatically due to the nature of our definition of  $\text{cl}(\xi)$  for general  $\xi$ , cf. 1.8. One may use here an equivalent definition of the cup-length  $\text{cl}(\xi)$  for  $\text{rk}(\xi) > 1$ , which can be obtained from the definition given in 1.8 if in (1-5) we will additionally require that the approximating rank 1 classes  $\xi_m$  belong to  $H^1(M; \mathbf{Q})$ .

Position  $M_1$  and  $M_2$  so that their intersection is a small  $n$ -dimensional disk  $D^n$ , where  $n = \dim M_1 = \dim M_2$ , and then the connected sum  $M_1 \# M_2$  is obtained from the union  $M_1 \cup M_2$  by removing the interior of  $D^n$ . Let  $E$  be a flat bundle over the connected sum  $M_1 \# M_2$  and let  $E_\nu$  be a flat bundle over  $M_\nu$  so that

$$E|_{M_\nu - D^n} \simeq E_\nu|_{M_\nu - D^n}, \tag{3-1}$$

for  $\nu = 1, 2$ . Here we use the assumption that  $n > 2$  and so the sphere  $S^{n-1}$  is simply connected.

As follows from the Mayer - Vietoris sequence, there is a canonical isomorphism

$$\psi : H^q(M_1; E_1) \oplus H^q(M_2; E_2) \rightarrow H^q(M_1 \# M_2; E)$$

for  $0 < q < n$ . It will be clear from the rest of the proof that we do not need to worry about the case  $q = n$ .  $\psi$  is multiplicative in the following sense. Suppose that we have another flat bundle  $F$  over the connected sum  $M_1 \# M_2$  and let  $F_\nu$  be flat bundles over  $M_\nu$ ,  $\nu = 1, 2$ , satisfying condition (3-1). Then for any  $v \in H^i(M_1; E_1)$  and  $w \in H^j(M_1; F_1)$  with  $0 < i, 0 < j$ , and  $i + j < d$ , holds  $\psi(v \cup w, 0) = \psi(v, 0) \cup \psi(w, 0)$ . Similar property holds with respect to the other variable.

Suppose now that  $k = \text{cl}(\xi_1)$  and we have cohomology classes  $v_j \in H^{d_j}(M_1; E_j)$ , where  $j = 1, 2, \dots, k$ , satisfying all the properties of Definition 1.3; in particular, their product  $v_1 \cup \dots \cup v_k$  is non-trivial. Then  $\sum d_j = n$  (cf. 1.7). Extend each flat bundle  $E_j$  to a flat bundle  $\tilde{E}_j$  over  $M$ ; for  $j = 1, 2$  we will make this extension so, that  $\tilde{E}_1$  and  $\tilde{E}_2$  will still satisfy condition (1-2).

We will first assume that  $k > 2$ . Then the classes

$$u_j = \psi(v_j, 0) \in H^{d_j}(M; \tilde{E}_j), \quad j = 1, 2, \dots, k - 1,$$

have non-trivial cup product  $u_1 \cup \cdots \cup u_{k-1}$  and satisfy all the properties of Definition 1.3. Using the Poincaré duality (as in the proof of Corollary 1.4), we may find a non-trivial cup product  $u_1 \cup \cdots \cup u_{k-1} \cup u$ , where  $u \in H^{dk}(M; E^* \otimes \mathcal{L}_M)$ ,  $E = \otimes_{j=1}^{k-1} \tilde{E}_j$ , and  $\mathcal{L}_M$  is the orientation flat line bundle of  $M$ .

In case, when  $k = 2$  by the same reasons we will have a non-trivial cup-product  $u_1 \cup u$ , where  $u \in H^{d_2}(M; \tilde{E}_1^* \otimes \mathcal{L}_M)$  and the bundle  $\tilde{E}_1^* \otimes \mathcal{L}_M$  satisfies (1-2) assuming that  $E_1$  does.

This proves inequality  $\text{cl}(\xi) \geq \text{cl}(\xi_1)$ . Therefore  $\text{cl}(\xi) \geq \max\{\text{cl}(\xi_1), \text{cl}(\xi_2)\}$ .

The inverse inequality follows similarly, using the properties of the map  $\psi$  mentioned above.  $\square$

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