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The skein relation for the $(\mathfrak{g}_2, V)$-link invariant

Anna-Barbara Berger and Ines Stassen

Abstract. Pulling back the weight system associated with the exceptional Lie algebra $\mathfrak{g}_2$ by a modification of the universal Vassiliev-Kontsevich invariant yields a link invariant; extending it to 3-nets, we derive a recursive algorithm for its evaluation.

Mathematics Subject Classification (1991). 57M25.

Keywords. Skein relations, monoidal category, universal Vassiliev–Kontsevich invariant, representation theory of Lie algebras.

0. Introduction

There is a well-known technique for the construction of Vassiliev link invariants: define a weight system (i.e. a linear form on the space of chord diagrams respecting certain relations) on the basis of some Lie algebraic data and pull it back by the universal Vassiliev-Kontsevich invariant. But unfortunately, the latter is not known explicitly enough to allow direct evaluation of these link invariants.

Efforts have been made to handle the universal Vassiliev-Kontsevich invariant by considering only "elementary" parts of links into which any link may be cut. This approach has been successful in so far as one may hope to find skein relations for the link invariants coming from Lie algebras—a skein relation being an equation implying a recursive algorithm for the computation of a link invariant, for example the one that determines the famous Jones polynomial up to normalization:

$t^2P($  $) - t^{-2}P($  $) = (t^{-1} - t)P($  $).$

It has been shown that the link invariants obtained from the classical simple Lie algebras $\mathfrak{sl}_n$, $\mathfrak{so}_n$, and $\mathfrak{sp}_n$ satisfy certain versions of the skein relation of the HOMFLY polynomial ($\mathfrak{sl}_n$; see [LM 1]) resp. the Kauffman polynomial ($\mathfrak{so}_n$, $\mathfrak{sp}_n$; see [LM 2]). But what about the exceptional simple Lie algebras?

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In this paper, we deal with the case of the exceptional Lie algebra $\mathfrak{g}_2$. By means of a generalization of the notion of links—since we have to introduce branchings, we call them 3-nets—we manage to establish a skein relation for the $(\mathfrak{g}_2, V)$-invariant, V being the 7-dimensional "standard" representation of $\mathfrak{g}_2$. As a by-product, we obtain an extension of the $(\mathfrak{g}_2, V)$-invariant to closed 3-nets. Kuperberg's skein relation for the quantum $\mathfrak{g}_2$-invariant (see [K]) turns out to be a special case of ours; it is not too surprising that there is a connection between these skein relations since the restrictions to knots of the two invariants coincide according to a result of Piunikhin's in [P].

We expect that our method can be adapted to the case of the other exceptional Lie algebras.

To the reader not familiar with Lie theory, we recommend [H] and [FH]. For an introduction to Vassiliev invariants and weight systems, see [BN 1]; a more general definition of weight systems is given in [V], section 6.

Overview over the categories and functors appearing in this paper:

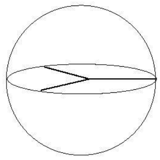
$$\begin{tikzcd} \mathcal{T}_{tpf} \arrow[r, "\hat{Z}_f"] \arrow[rrr, bend left, "I_{(\mathfrak{g}_2,V)}" , "\text{the } (\mathfrak{g}_2,V)\text{-invariant}"'] & \hat{\mathcal{D}}_3 \arrow[r, "\Psi"] & \mathcal{C}(\mathfrak{g}_2) \end{tikzcd}$$

$\mathcal{T}_{tpf}$	$\xrightarrow{\hat{Z}_f}$	$\hat{\mathcal{D}}_3$	$\xrightarrow{\Psi}$	$\mathcal{C}(\mathfrak{g}_2)$
the category of trivalent, parantesized, framed tangles; every 3-net corresponds to a morphism of this category	universal Vassiliev-Kontsevich invariant	the category of 3-diagrams (a generalization of chord diagrams)	$(\mathfrak{g}_2, V)$-weight system	a modification of a subcategory of the category of representations of $\mathfrak{g}_2$

1. 3-nets and 3-tangles

In this section, we will define 3-nets and 3-tangles. They are generalizations of links and tangles. We will also construct the category $\mathcal{T}_{tpf}$ of trivalent framed tangles, which will allow us to work in a category-theoretical setting.

A 3-net will be something like a "link with branchings". To describe the situation near a trivalent vertex (i.e. near a branching point), we will need the following notion:
Let B be the open unit ball in $\mathbf{R}^3$, i.e. $\{x \in \mathbf{R}^3; |x| < 1\}$, together with the distinguished subset $T := \{(t,0,0)|0 \le t < 1\} \cup \{(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)t|0 \le t < 1\} \cup \{(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)t|0 \le t \le 1\}$.

$B =$ 

Definition 1.1. *A* framed 3-net *is a subset N of $\mathbf{R}^3$ with a finite subset $\{t_1, \dots, t_n\} \subset N$ such that:*

(i) there exist disjoint open subsets $U_1, \dots U_n$ of $\mathbf{R}^3$ and diffeomorphisms $f_i : U_i \to B\,(i = 1 \dots n)$ such that U_i is a neighbourhood of t_i, $f(t_i) = (0,0,0)$, and $f_i(N \cap U_i) = T$,

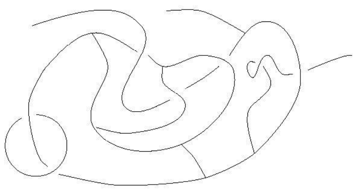
(ii) $\tilde{N} := N \backslash \left(\bigcup_{i=1}^{n} f_i^{-1}(\{x \in B; |x| < \frac{1}{2}\}) \right)$ is an embedded smooth closed compact 1-dimensional manifold,

together with:

(iii) a smooth vector field on N that is nowhere tangent to N (and in particullary nowhere zero)

The points $t_1, \dots, t_n$ are called trivalent vertices of N; boundary points x of $\tilde{N}$ with $x \notin U_i\,(\forall i)$ are called univalent vertices *of N.*

Observe that 3-nets without trivalent and univalent vertices are simply framed links. When we represent a framed 3-net N by a diagram, the framing of N is given by the blackboard framing[1].
The 3-net in the following figure is a 3-net with 7 trivalent and 3 univalent vertices.



As all 3-nets in this paper will be framed 3-nets, we will usually omit the word "framed".

Definition 1.2. *Two 3-nets N_1 and N_2 are* equivalent *if N_1 can be deformed into N_2 within the class of framed 3-nets by a smooth 1-parameter family of diffeomorphisms of $\mathbf{R}^3$.*
A closed *3-net is a 3-net without univalent vertices.*
A 3-net is planar *if it is equivalent to a 3-net M with $M \subset \mathbf{R}^2 \times \{0\}$ and the*

[1] i.e. the vector field assigned to N consists of vectors pointing upward

vector field assigned to M consists of vectors of the form $(0,0,1)$.

Remark 1.3. The overview in the introduction intimates that we will obtain an invariant of closed 3-nets that is composed of Vassiliev invariants (see also section 6). Vassiliev invariants are usually defined for oriented links, but we remind the reader that there is a definition for unoriented links (see e.g. [St]):
A link invariant[2] f is a *Vassiliev invariant of type m* if for any link L, any diagram $D(L)$ of L and any subset C of the set of crossings of $D(L)$ with cardinality greater than m the following equation holds:

$$\sum_{X\subset C}(-1)^{|X|}f([D(L)_X])=0,$$

where $|X|$ is the cardinality of X, $D(L)_X$ is the link diagram obtained form $D(L)$ by changing all the crossings in X, and $[D(L)_X]$ is a link with diagram $D(L)_X$ (such that the framing on the link is given by the blackboard framing of the diagram).
Of course, this definition can be extended to 3-nets.

Definition 1.4. *A* (framed) 3-tangle *is a framed 3-net N with $N\subset[0,1]\times\mathbf{R}^2$ such that the points of N lying in the planes $\{0\}\times\mathbf{R}^2$ and $\{1\}\times\mathbf{R}^2$ are exactly the univalent vertices of N and these lie on one of the lines $\{0\}\times\mathbf{R}\times\{0\}$ and $\{1\}\times\mathbf{R}\times\{0\}$. Additionally, we require that the normal plane of N in a univalent vertex v is parallel to the plane $\{0\}\times\mathbf{R}^2$, and the vector field assigned to N is $(0,0,1)$ in v.*

Definition 1.5. *Two 3-tangles T_1 and T_2 are equivalent if one can be deformed into the other within the class of 3-tangles by a smooth 1-parameter family of diffeomorphisms of $\mathbf{R}^3$.*
A 3-tangle is planar if it is equivalent to a 3-tangle M with $M\subset[0,1]\times\mathbf{R}\times\{0\}$ and the vector field assigned to M consists of vectors of the form $(0,0,1)$.

Now we will define the category of **t**rivalent, **p**arenthesized, **f**ramed tangles $\mathcal{T}_{tpf}$. It is an (unoriented) generalization of the category of non-associative tangles in [BN 2].

First, we will define non-associative words (which will be the objects of $\mathcal{T}_{tpf}$).

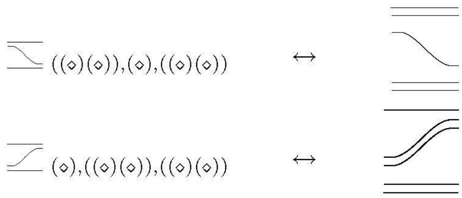
Definition 1.6. *A* non-associative word *is a word w in the alphabet $\{\diamond,),(\}$ such that w is equal to the empty word, $(\diamond)$, or (w_1w_2) where w_1,w_2 are non-associative words. For every word, we identify (w) with w.*
The length $l(w)$ *of a non-associative word w is the number of symbols $\diamond$ in w.*

[2] i.e. a function assigning to each link an element of an abelian group (usually $\mathbf{C}$) that is constant on the equivalence classes of framed links

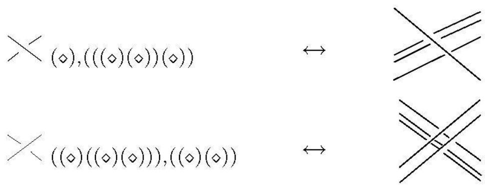
Example 1.7. $$l\,((\diamond)(((\diamond)((\diamond)((\diamond)(\diamond))))(\diamond))) = 6$$

Definition 1.8. *Let $\check{\mathcal{T}}_{tpf}$ be the monoidal* $\mathbf{C}$*-category whose objects are non-associative words (where the tensor product $w_1 \otimes w_2$ is (w_1w_2) and the unit object is the empty word) and whose morphisms are freely generated by the following morphisms:*

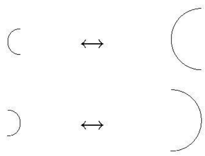
(G1) A morphism ${}_{v,w,x}$ and a morphism ${}_{v,w,x}$ for each triple (v,w,x) of non-empty non-associative coloured words: The sources of these morphisms are $((vw)x)$ and $(v(wx))$ and their targets are $(v(wx))$ and $((vw)x)$ respectively.
Graphically, we represent these morphisms as the following examples indicate (the parenthesation of source and target is encoded in the distances between the strands):



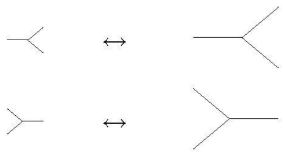
(G2) A morphism ${}_{v,w}$ and a morphism ${}_{v,w}$ for each pair (v,w) of non-empty non-associative words: The source of these morphisms is (v,w) and their target is (w,v). We will depict them as follows:



(G3) A morphism and a morphism with source $(\,)$ and $(\diamond\diamond)$ and target $(\diamond\diamond)$ and $(\,)$ respectively; graphically:



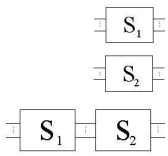
(G4) A morphism and a morphism with source $\diamond$ and $(\diamond\diamond)$ and target $(\diamond\diamond)$ and $\diamond$ respectively; graphically:



The graphical representation of the tensor product of two morphisms is obtained by putting the first above the second, the graphical representation of the composition $S_2 \circ S_1$ is obtained by glueing the graphical representation of S_2 to the one of S_1 from the right:

$S_1 \otimes S_2 \quad \leftrightarrow$

$S_2 \circ S_1 \quad \leftrightarrow$

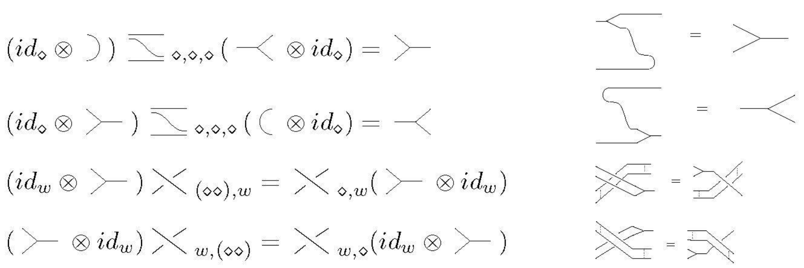


The graphical representations of the morphisms allow us to assign a 3-tangle to each morphism M (namely a 3-tangle T with the graphical representation of M as diagram, supplied with the blackboard framing).

Definition 1.9. *The monoidal* **C**-*category $\mathcal{T}_{tpf}$ is the category whose objects are the objects of $\check{\mathcal{T}}_{tpf}$ and whose morphisms are the equivalence classes of the morphisms of $\check{\mathcal{T}}_{tpf}$ under the following equivalence relation: Two morphisms from u to w are equivalent if they get assigned equivalent 3-tangles.*

Remark 1.10. The equivalence relation in the above definition can be described locally. For morphisms generated by (G1)-(G3) this is done in [BN 2]. If we take the generators in (G4) as well, we have to add the following relations for any word w:

Relation | Graphical representation



Remark 1.11. Observe that any 3-tangle, and in particular any closed 3-net, is equivalent to a 3-tangle assigned to a morphism of $\mathcal{T}_{tpf}$, and so the equivalence

classes of 3-tangles correspond exactly to a basis of the morphisms of $\mathcal{T}_{tpf}$. One might achieve this by taking much simpler generators (e.g. only generators without multiple strands), but then the local description of the equivalence relation given in remark 1.10 would be more complicated.

2. The universal Vassiliev-Kontsevich invariant extended to 3-tangles

Now we want to extend and adapt the universal Vassiliev-Kontsevich invariant to the 3-tangles in $\mathcal{T}_{tpf}$. Since this invariant operates by taking a diagram representing the given tangle as support and adding some chords, we have to generalize the notion of chord diagram and introduce trivalent vertices in the support, too.

Definition 2.1. *A* 3-diagram *is a finite trivalent graph K (by which we understand a graph with every vertex being either univalent or trivalent or else bivalent and adjacent to a loop) equipped with the following data:*

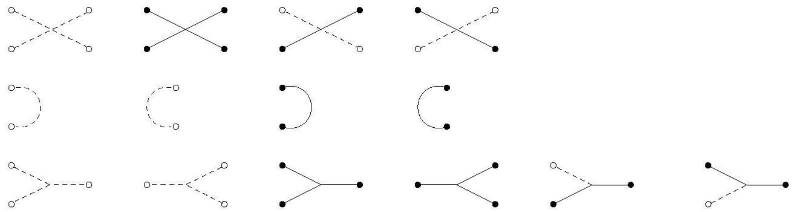
- *a colouring of the edges by - - - - - or —— such that there is not a vertex adjacent to two edges coloured by - - - - - and one coloured by ——*
- *a colouring of the univalent vertices by $\circ$ or $\bullet$ according to whether the edge arriving there is coloured by - - - - - or ——*
- *for every trivalent vertex x of K, a cyclic order of the edges arriving at x.*

The union of the edges coloured by —— is referred to as the support of the diagram; the edges coloured by - - - - - are called chords.
The degree of a 3-diagram is the number of trivalent vertices adjacent to at least one chord[3]*.*

Usually, we describe the 3-diagrams by graphical representations in the plane encoding the information about the cyclic order near the trivalent vertices by arranging the adjacent edges counterclockwise.

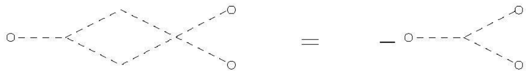
Definition 2.2. *The category $\mathcal{D}_3$ is a monoidal* **C***-category whose morphisms are linear combinations of certain graphical representations of 3-diagrams. It is given by the following data:*
objects: *$Obj(\mathcal{D}_3) := \bigcup_{n=0}^{\infty}\{\circ,\bullet\}^n$. The tensor product on $Obj(\mathcal{D}_3)$ is the juxtaposition.*
generators: *The morphism spaces are generated by:*

[3] Note that for a 3-diagram without univalent vertices adjacent to a chord, this is twice the classical degree.

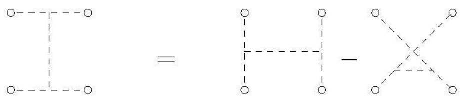


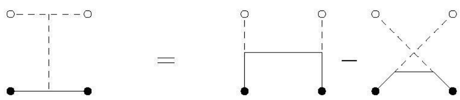
the source (resp. the target) being denoted on the left-(resp. right-)hand side from top to bottom[4]*.*
The tensor product of two morphisms is obtained by putting the first above the second, the composition by glueing together the corresponding entries of the target of the first and the source of the second.
relations: *Of course, different graphical representations of isomorphic 3-diagrams are to represent the same morphism; in addition, we impose the following relations*[5]*:*

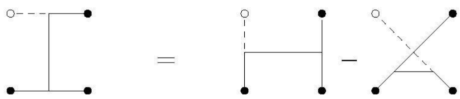
(AS1) 

(AS2) 

(AS3) 

(IHX1) 

(IHX2) 

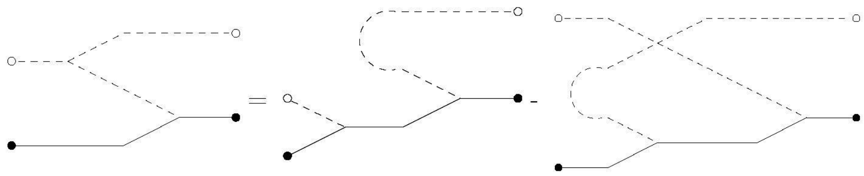
(IHX3) 

Obviously, the morphisms involved in the relations described above can be

[4] The apparent 4-valent vertices are no vertices at all - they are just crossings of two edges (there is no need to say that one of them passes over the other).

[5] The reader familiar with Bar-Natan's way of defining weight systems should pay attention to the sign in our relation *(AS3)*.

composed of the generators of $\mathcal{D}_3$; the relation *(IHX2)* for example can be written as follows:



But for obvious reasons, we refrain from doing so.

Remark 2.3. The relations *(AS1)*, *(AS2)*, and *(IHX1)* allow introducing trivalent vertices adjacent to three chords in a consistent way, which are convenient e.g. for the formulation of *(IHX2)*. The relations *(AS3)*, *(IHX2)*, and *(IHX3)* are required for the existence of the universal Vassiliev-Kontsevich invariant. See proof of 2.8 for *(AS3)* and proof of theorem 1 (1) in [BN 1] for *(IHX2)* (the 4T-relation); *(IHX3)* reflects a similar situation near a trivalent vertex.

Definition 2.4. *Let $\hat{\mathcal{D}}_3$ be the completion of the (graded) category $\mathcal{D}_3$.*

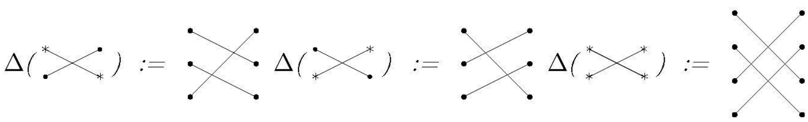
For convenience of notation, we define a functor Δ from a specialized version $\mathcal{D}_3^*$ of the category $\hat{\mathcal{D}}_3$ into $\hat{\mathcal{D}}_3$.

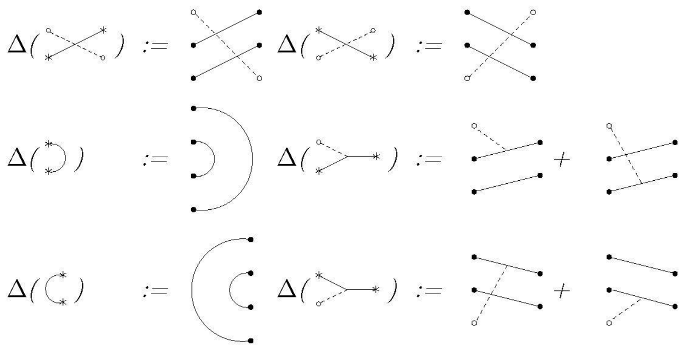
Definition 2.5. *Let $\mathcal{D}_3^*$ be the category whose objects and morphisms are those of $\hat{\mathcal{D}}_3$ together with some extra information: in the morphisms, some connected components of the support containing no trivalent vertices adjacent to three edges coloured by —— may be labelled by replacing the adjacent components of the source and/or the target by $*$.*
The composition of morphisms is to respect the labelling.
The relations of $\mathcal{D}_3^*$ are those of $\hat{\mathcal{D}}_3$ with any possible labelling.

If D is an arbitrary morphism in $\hat{\mathcal{D}}_3$, we denote by D_k the morphism of $\mathcal{D}_3^*$ that is D with the component departing from the k-th entry of the source labelled.

Definition 2.6. *Let Δ be the monoidal functor $\mathcal{D}_3^* \to \hat{\mathcal{D}}_3$ doubling the labelled components and taking the sum over all possible ways of lifting arriving chords to the new components of the support; thus, Δ is given by:*

$\Delta(*) \quad := \bullet\ \bullet \qquad \Delta(\bullet) \quad := \bullet \qquad \Delta(\circ) \quad := \circ$

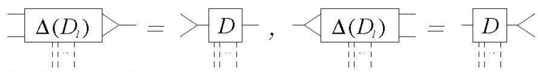




All the other generators (without any labelling) are not affected by Δ.

As an immediate consequence of the relation *(IHX3)*, we obtain:

Lemma 2.7.

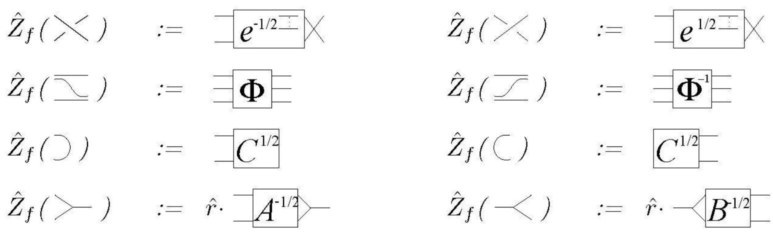


for any fitting 3-diagram D. □

Finally, we get to the definition of the universal Vassiliev-Kontsevich invariant:

Definition and Theorem 2.8. *The following assignments define a monoidal functor* $\hat{Z}_f : \mathcal{T}_{tpf} \to \hat{\mathcal{D}}_3$*, the (unoriented) universal Vassiliev-Kontsevich invariant:*

$\hat{Z}_f(u) := \bullet^n$*, where u is a non-associative word of length n.*

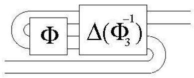


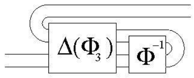
where $e^{\pm\frac{1}{2}\,\text{⌶}} := \sum_{n=0}^{\infty} (\pm\frac{1}{2})^n \frac{1}{n!}\ \text{⌶}^{\,n}$

Φ *is the Knizhnik-Zamolodchikov associator (for definition see [LM 3])*

$C := (\ \boxed{\Phi}\ )^{-1}$

$\hat{r} \in \mathbf{C} \setminus \{0\}$

$A :=$ 

$, B :=$ 

.

For generators with multiple strands, i.e. $_{u,v}$, $_{u,v}$, $_{u,v,w}$ *or* $_{u,v,w}$ *for some non-associative words u, v, and w, the image under* $\hat{Z}_f$ *is obtained by splitting the corresponding components of* $\hat{Z}_f(\ _{\diamond,\diamond})$, $\hat{Z}_f(\ _{\diamond,\diamond})$, $\hat{Z}_f(\ _{\diamond,\diamond,\diamond})$, *or* $\hat{Z}_f(\ _{\diamond,\diamond,\diamond})$ *respectively by repeated use of* Δ, *e.g.*

$$\hat{Z}_f(\ _{u,v}) = \underbrace{\Delta \ldots (\Delta}_{l(v)-1\,times} \ (\underbrace{\Delta \ldots (\Delta}_{l(u)-1\,times} \ (\hat{Z}_f(\ )_1)_{\ldots 1})_{l(u)+1})_{\ldots l(u)+1}).$$

Remark 2.9. As Φ, A, B, C are formal power series in certain 3-diagrams with degree 0-part 1, one may take their inverses and square roots by substituting x for their higher degree parts and expand the corresponding function of x in a Taylor series.

Remark 2.10. Since the number of trivalent vertices in a 3-tangle is invariant, $\hat{r}$ may be chosen freely.

Proof of 2.8. In section 1 of [MO], Murakami and Ohtsuki define the universal Vassiliev-Kontsevich invariant for *oriented* 3-tangles and prove that it is indeed an invariant. Their definition can be modified—without destroying the invariance of the functor—as follows: Omit the signs accounting for the orientation of the strands, and introduce the antisymmetry relation *(AS3)* instead of the corresponding (implicit) symmetry relation in the category of oriented 3-diagrams. In order to obtain as degree 1-part of $\hat{Z}_f(\ ) := \hat{Z}_f(\ ) - \hat{Z}_f(\ )$ the diagram in which the double point is replaced by a chord (arriving at the support on either side like this:), we have adjusted the sign in the exponent of the image of the crossings.

With this modified version, nothing can keep us from forgetting the orientations both of the 3-tangles and of the 3-diagrams. □

3. The $(\mathfrak{g}_2, V)$-weight system extended to 3-diagrams

Let $\mathcal{A}$ be the subspace of the morphisms of $\hat{\mathcal{D}}_3$ generated by diagrams with support S^1 and without univalent vertices. It has been known for some time that given a Lie algebra $\mathfrak{g}$, a representation V of $\mathfrak{g}$, and an ad-invariant symmetric non-degenerate bilinear form on $\mathfrak{g}$, one can construct a linear form—called weight system—on $\mathcal{A}$ if an orientation on the support is added (see e.g. [BN 1], section 2.4, or [V], section 6)[6]. In this section, we construct an unoriented and extended version of the weight

[6] Our approach is closer to the one described in [V].

system one gets from $\mathfrak{g}_2$ with its 7-dimensional standard representation[7] and the bilinear form $h\kappa$ (where h is any non-zero complex number and κ the Killing form on $\mathfrak{g}_2$). We do this by defining a functor Ψ from $\hat{\mathcal{D}}_3$ to $\mathcal{C}(\mathfrak{g}_2)$, a modification of a subcategory of the category of representations of $\mathfrak{g}_2$.

Denote by V the 7-dimensional irreducible representation and by L the adjoint representation of $\mathfrak{g}_2$. Let the highest weights of these representations be $(1,0)$ and $(0,1)$ respectively.

The following fact assures that V is selfdual (i.e. $V \cong V^*$) and that there exist $\mathfrak{g}_2$-linear embeddings $i_{\mathbf{C}}$ and i_V from $\mathbf{C}$ and V respectively into $V \otimes V$ unique up to scalars.

Fact 3.1.

$$V \otimes V \cong \mathbf{C} \oplus V \oplus L \oplus W$$

$$\textit{with} \quad Sym^2 V \cong \mathbf{C} \oplus W, \quad \wedge^2 V \cong V \oplus L$$

where W is the irreducible representation of $\mathfrak{g}_2$ with highest weight $(2,0)$.

For the construction of the functor Ψ, we will also need the following $\mathfrak{g}_2$-linear maps:

- $p_{\mathbf{C}} : V \otimes V \to \mathbf{C}$ and $p_V : V \otimes V \to V$, the projections belonging to the embeddings $i_{\mathbf{C}}$ and i_V (i.e. $p_{\mathbf{C}} \circ i_{\mathbf{C}} = id_{\mathbf{C}}$, $p_V \circ i_V = id_V$).
- $flip_{X \otimes Y} : X \otimes Y \to Y \otimes X$, the $\mathfrak{g}_2$-linear map taking $x \otimes y$ to $y \otimes x$ ($\forall x \in X, y \in Y$ for $X, Y \in \{V, L\}$).
- $cas : \mathbf{C} \to L \otimes L$ with $(h\kappa) \circ cas = 14 id_{\mathbf{C}}$. Observe that cas maps 1 to the Casimir belonging to $h\kappa$.

Now we define the category $\mathcal{C}(\mathfrak{g}_2)$ that will be the target of the functor Ψ. Let $\hat{h}$ be a formal parameter.

Definition 3.2. *The category $\mathcal{C}(\mathfrak{g}_2)$ is the monoidal $\mathbf{C}[[\hat{h}]]$-category with objects $Obj(\mathcal{C}(\mathfrak{g}_2)) := \{\mathbf{C}[[\hat{h}]] \otimes_{\mathbf{C}} U \,|\, U$ is a tensor product over $\mathbf{C}$ with factors V and $L\}$ and with the following morphism spaces:*

$$Mor_{\mathcal{C}(\mathfrak{g}_2)}(\mathbf{C}[[\hat{h}]] \otimes_{\mathbf{C}} U_1, \mathbf{C}[[\hat{h}]] \otimes_{\mathbf{C}} U_2) := \mathbf{C}[[\hat{h}]] \otimes_{\mathbf{C}} \mathrm{Hom}_{\mathfrak{g}_2}(U_1, U_2).$$

The definition of Ψ is contained in the following proposition whose proof will be omitted, because it just consists in checking straightforwardly that Ψ respects all relations required.

[7] "Standard" in the sense that every irreducible representation of $\mathfrak{g}_2$ occurs as a direct summand of some tensor power of V.

Proposition 3.3. *For any $\mathfrak{g}_2$-linear embeddings $i_{\mathbf{C}}$ and i_V of $\mathbf{C}$ and V respectively into $V \otimes V$, there exists $k \in \mathbf{C}$ for which we obtain a well defined $\mathbf{C}[[\hat{h}]]$-linear monoidal functor $\Psi : \hat{\mathcal{D}}_3 \to \mathcal{C}(\mathfrak{g}_2)$ by setting*

(i) $\Psi(\circ) := \mathbf{C}[[\hat{h}]] \otimes L$ $\quad \Psi(\bullet) := \mathbf{C}[[\hat{h}]] \otimes V$

(ii) $\Psi(\ ) := 1 \otimes flip_{L\otimes L}$ $\quad \Psi(\ ) := 1 \otimes flip_{V\otimes V}$

$\Psi(\ ) := 1 \otimes flip_{L\otimes V}$ $\quad \Psi(\ ) := 1 \otimes flip_{V\otimes L}$

(iii) $\Psi(\ ) := 1 \otimes h\kappa,$ $\quad \Psi(\ ) := 1 \otimes cas$

$\Psi(\ ) := 1 \otimes 7p_{\mathbf{C}}$ $\quad \Psi(\ ) := 1 \otimes i_{\mathbf{C}}$

(iv) $\Psi(\ ) := \hat{h} \otimes$ *Lie bracket on* $\mathfrak{g}_2$

$\Psi(\ ) := \hat{h} \otimes$ *dual*[8] *of the Lie bracket on* $\mathfrak{g}_2$

(v) $\Psi(\ ) := 1 \otimes kp_V$ $\quad \Psi(\ ) := 1 \otimes i_V$

(vi) $\Psi(\ ) := \hat{h} \otimes$ *representation*

$\Psi(\ ) := -\hat{h} \otimes$ *representation.*

Remark 3.4. The factor k in (v) depends on the embedding $i_V : V \to V \otimes V$. The value of k for a fixed embedding i_V can be found by solving the equation $(1 \otimes (p_{\mathbf{C}} \otimes id_V)) \circ (1 \otimes (id_V \otimes i_V)) = 1 \otimes kp_V$ representing the fact that $\Psi(\ ) = \Psi(\ )$ must hold.
The factor 7 in (iii) has been chosen to assure $\Psi(\ ) = \Psi(\ )$; it is independent of the embedding $i_{\mathbf{C}}$.
The formal parameter $\hat{h}$ has been introduced to assure the existence of $\Psi(D)$ for every morphism D of $\hat{\mathcal{D}}_3$: The powers of $\hat{h}$ induce a grading on the morphism spaces of $\mathcal{C}(\mathfrak{g}_2)$, and with respect to this grading Ψ is a grade preserving functor that is well defined in every degree and hence on the whole of $\hat{\mathcal{D}}_3$.

Remark 3.5. In the introduction to this section, we mentioned the construction of a weight system $\hat{\Psi}$ (out of Lie algebra information) given in [BN 1] and [V]. In these papers, the support of the diagrams is oriented, and so the reader familiar with them might ask if there is still a connection between Ψ and $\hat{\Psi}$. From the following observations, one can deduce that $\hat{\Psi}$ of an oriented diagram of degree m is exactly the degree m part of Ψ of the underlying unoriented one:

$$1 \otimes \hat{\Psi}(\ ) \quad = \quad \Psi(\ ) \qquad\qquad \hat{h}^{\sharp\text{chords}} \otimes \hat{\Psi}(\ ) \quad = \quad \Psi(\ )$$

---

8 Observe that $h\kappa$ induces an isomorphism from L to L^*.

$$1 \otimes \hat{\Psi}(\ ) \quad = \quad \Psi(\ ) \qquad\qquad \hat{h}^{\sharp\text{chords}} \otimes \hat{\Psi}(\ ) \quad = \quad \Psi(\ )$$

(where $\hat{\Psi}(D)$ is obviously regarded as map and not as tensor).

4. The skein relation for the $(\mathfrak{g}_2, V)$-invariant

What we have achieved so far, is the construction of an invariant for 3-tangles: $I_{(\mathfrak{g}_2,V)} := \Psi \circ \hat{Z}_f$. Unfortunately, we cannot evaluate it directly because the expression known for the associator Φ is not explicit enough to allow concrete computations. But we will derive a skein relation, i.e. recursive rules by which we can reduce the problem to finding the values for planar 3-tangles (with these, we will deal in the next section).

The idea is to cut out a small neighbourhood of a crossing and insert something else without changing the value of the invariant. The substitute for the crossing has to be a linear combination of small, simple 3-tangles with four univalent vertices. Obvious candidates for such are the inverse crossing, $\asymp$, and $)($; as their values will prove to be linearly independent in the space of endomorphisms of $V \otimes V$, these are not sufficient, and therefore, we include $\succ\!\prec$ into our considerations.

As $V \otimes V$ decomposes into four different irreducible representations (namely $\mathbf{C}$, V, L and W; see fact 3.1), each $\mathfrak{g}_2$-linear map $V \otimes V \to V \otimes V$ is determined by the four corresponding eigenvalues. To establish our skein relation, we have to ascertain the four eigenvalues of $I_{(\mathfrak{g}_2,V)}(\times)$, $I_{(\mathfrak{g}_2,V)}(\times)$, $I_{(\mathfrak{g}_2,V)}(\asymp)$, $I_{(\mathfrak{g}_2,V)}()()$, and $I_{(\mathfrak{g}_2,V)}(\succ\!\prec)$.

Eigenvalues of $I_{(\mathfrak{g}_2,V)}(\times)$ and $I_{(\mathfrak{g}_2,V)}(\times)$

The eigenvalues of $I_{(\mathfrak{g}_2,V)}(\times)$ and $I_{(\mathfrak{g}_2,V)}(\times)$ are the products of the corresponding eigenvalues of $\Psi(e^{\mp\frac{1}{2}\,\mathrm{I}})$ and $\Psi(\times)$.

Quite a bit of explicit calculation yields that the eigenvalues of $\Psi(\mathrm{I})$ are $\frac{\hat{h}^2}{2h}$ on $\mathbf{C}$, $\frac{\hat{h}^2}{4h}$ on V, 0 on L and $-\frac{\hat{h}^2}{12h}$ on W; accordingly, the eigenvalues of $\Psi(e^{\mp\frac{1}{2}\,\mathrm{I}})$ are $e^{\mp\frac{\hat{h}^2}{4h}}$, $e^{\mp\frac{\hat{h}^2}{8h}}$, 1, and $e^{\pm\frac{\hat{h}^2}{24h}}$, respectively.

The eigenvalue of $\Psi(\times)$ is 1 on $\mathrm{Sym}^2 V = \mathbf{C} \oplus W$ and -1 on $\wedge^2 V = V \oplus L$.

Eigenvalues of $I_{(\mathfrak{g}_2,V)}(\asymp)$

As $I_{(\mathfrak{g}_2,V)}(\asymp)$ is the identity on $V \otimes V$, its only eigenvalue is 1.

Eigenvalues of $I_{(\mathfrak{g}_2,V)}()()$

Let c be the scalar by which $\Psi(\boxed{C})$ operates on V. Then we have:

$$I_{(\mathfrak{g}_2,V)}()() = \Psi(\boxed{C^{1/2}}\ \boxed{C^{1/2}}) = c\Psi()() = 7c\, i_{\mathbf{C}} \circ p_{\mathbf{C}}.$$

Hence $I_{(\mathfrak{g}_2,V)}(\,)(\,)$ is 0 everywhere except on $\mathbf{C}$, where it has the eigenvalue $7c$.

Remark 4.1. Observe that:

$$I_{(\mathfrak{g}_2,V)}(\bigcirc) = \Psi \circ \hat{Z}_f(\bigcirc) = c\Psi(\bigcirc) = 7c.$$

As Piunikhin has shown in [P] that for framed knots the Reshetikhin-Turaev quantum invariants yield the same values as the invariants obtained by using the corresponding weight systems, we can use a result of Rosso and Jones in [RJ] to determine the value of c:

$$I_{(\mathfrak{g}_2,V)}(\bigcirc) = \prod_{\alpha\in\Delta_+} \frac{e^{\frac{\hat{h}^2 h}{2}(\lambda+\delta,\alpha)} - e^{-\frac{\hat{h}^2 h}{2}(\lambda+\delta,\alpha)}}{e^{\frac{\hat{h}^2 h}{2}(\delta,\alpha)} - e^{-\frac{\hat{h}^2 h}{2}(\delta,\alpha)}}$$

where Δ_+ is a possible choice for the set of positive roots of $\mathfrak{g}_2$
λ is the highest weight of V
$\delta := \sum_{R\in\Delta_+} R$
$(\ ,\ )$ is the bilinear form on the weight space of $\mathfrak{g}_2$ induced by the bilinear form $h\kappa$ on $\mathfrak{g}_2$.

Simplifying this expression and setting $q := e^{-\frac{\hat{h}^2}{24h}}$, we get:

$$7c = q^5 + q^4 + q + 1 + q^{-1} + q^{-4} + q^{-5}.$$

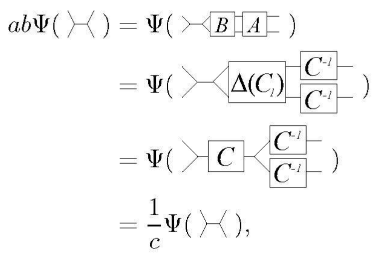
Eigenvalues of $I_{(\mathfrak{g}_2,V)}(\succ\!\prec)$

Let a (resp. b) be the eigenvalue of $\Psi(\boxed{A})$ (resp. $\Psi(\boxed{B})$) on V. Then we have:

$$I_{(\mathfrak{g}_2,V)}(\succ\!\prec) = \Psi(\boxed{A^{-1/2}}\succ\!\prec\boxed{B^{-1/2}}) = \frac{\hat{r}^2}{\sqrt{ab}}\Psi(\succ\!\prec).$$

Since $\Psi(\succ\!\prec)$ is 0 everywhere except on V, the parameter $\hat{r}$ occurs nowhere but in the eigenvalue of $I_{(\mathfrak{g}_2,V)}(\succ\!\prec)$ on V, which is $\frac{k\hat{r}^2}{\sqrt{ab}}$ (and $\neq 0$); therefore, we do not have to care about the factor $\frac{1}{\sqrt{ab}}$, but can simply shift the possibility of choice from $\hat{r}$ to $r := \frac{k\hat{r}^2}{\sqrt{ab}}$.

Remark 4.2. It is nonetheless possible to determine ab by using the following result of [LM 4] section 4: $AB = (C^{-1} \otimes C^{-1})\Delta(C_1)$, and lemma 2.7:



thus

$$ab = \frac{1}{c}.$$

Remark 4.3. The invariant $I_{(\mathfrak{g}_2,V)}$ takes actually values in $\mathbf{C}[\![\hat{h}]\!]$, but as long as we do not want to fix h and r, we can regard them as elements of $\mathbf{C}[\![\hat{h}, \frac{1}{h}, r]\!]$ and, accordingly, $\mathcal{C}(\mathfrak{g}_2)$ as $\mathbf{C}[\![\hat{h}, \frac{1}{h}, r]\!]$-category.

To summarize (recall that $q = e^{-\frac{\hat{h}^2}{24h}}$):

Eigen-value on	$I_{(\mathfrak{g}_2,V)}(\ )$	$I_{(\mathfrak{g}_2,V)}(\ )$	$I_{(\mathfrak{g}_2,V)}(\ )$	$I_{(\mathfrak{g}_2,V)}(\ )$	$I_{(\mathfrak{g}_2,V)}(\ )$
$\mathbf{C}$	q^{-6}	q^{6}	1	$7c$	0
V	$-q^{-3}$	$-q^{3}$	1	0	r
L	-1	-1	1	0	0
W	q	q^{-1}	1	0	0

The leftmost column can be expressed as a linear combination of the other columns; i.e. substituting this linear combination of the 3-tangles , , , and for a crossing in a 3-tangle does not change the value of $I_{(\mathfrak{g}_2,V)}$.

Theorem 4.4. *For the invariant $I_{(\mathfrak{g}_2,V)}$, the following skein relation holds:*

$$I_{(\mathfrak{g}_2,V)}(\ ) = \alpha I_{(\mathfrak{g}_2,V)}(\ ) + \beta I_{(\mathfrak{g}_2,V)}(\ ) + \gamma I_{(\mathfrak{g}_2,V)}(\ ) + \delta I_{(\mathfrak{g}_2,V)}(\ )$$

where
$$\begin{aligned} \alpha &:= q \\ \beta &:= q-1 \\ \gamma &:= \tfrac{1}{7c}(-q^7 + q^{-6} - q + 1) \\ \delta &:= \tfrac{1}{r}(q^4 - q^{-3} - q + 1). \end{aligned}$$

Since $I_{(\mathfrak{g}_2,V)}$ is a monoidal functor and invariant under ambient isotopy, we can deduce another skein relation as follows:

$$\begin{aligned} I_{(\mathfrak{g}_2,V)}(\;) &= I_{(\mathfrak{g}_2,V)}(\;) \\ &= \alpha I_{(\mathfrak{g}_2,V)}(\;) + \beta I_{(\mathfrak{g}_2,V)}(\;) + \gamma I_{(\mathfrak{g}_2,V)}(\;) + \delta I_{(\mathfrak{g}_2,V)}(\;) \\ &= \alpha I_{(\mathfrak{g}_2,V)}(\;) + \beta I_{(\mathfrak{g}_2,V)}()() + \gamma I_{(\mathfrak{g}_2,V)}(\;) + \delta I_{(\mathfrak{g}_2,V)}(\;). \end{aligned}$$

Combining these two versions of the skein relation, we obtain:

Corollary 4.5. *For the invariant $I_{(\mathfrak{g}_2,V)}$, the following skein relation holds*[9]*:*

$$I_{(\mathfrak{g}_2,V)}(\;) = \lambda I_{(\mathfrak{g}_2,V)}(\;) + \mu I_{(\mathfrak{g}_2,V)}()() + \rho I_{(\mathfrak{g}_2,V)}(\;) + \sigma I_{(\mathfrak{g}_2,V)}(\;)$$

where
$$\begin{aligned} \lambda &:= \tfrac{\alpha\gamma+\beta}{1-\alpha^2} \\ \mu &:= \tfrac{\alpha\beta+\gamma}{1-\alpha^2} \\ \rho &:= \tfrac{\delta}{1-\alpha^2} \\ \sigma &:= \tfrac{\alpha\delta}{1-\alpha^2}. \end{aligned}$$

It is clear that by means of this relation, every 3-tangle can be reduced to a linear combination of planar 3-tangles.

Remark 4.6. The invariant $I_{(\mathfrak{g}_2,V)} = \Psi \circ \hat{Z}_f$ of closed oriented 3-nets itself is not a Vassiliev invariant, but "consists of" Vassiliev invariants in the following sense: For each $m \in \mathbf{N}$, let $I^{(m)}_{(\mathfrak{g}_2,V)}$ be the function with values in $\mathbf{C}[\![\frac{1}{\hbar}, r]\!]$ such that $I_{(\mathfrak{g}_2,V)} = \sum_{m=0}^{\infty} I^{(m)}_{(\mathfrak{g}_2,V)} \hat{h}^{2m}$. Then $I^{(m)}_{(\mathfrak{g}_2,V)}$ is an invariant of type m.

5. Values of the $(\mathfrak{g}_2, V)$-invariant on closed planar 3-nets

In this section, we show how the value of $I_{(\mathfrak{g}_2,V)}$ on a closed planar 3-net can be calculated recursively.

The following lemma assures that it is sufficient to consider connected 3-nets.

Lemma 5.1. *If a closed 3-net N is equivalent to a 3-net consisting of two closed 3-nets N_1 and N_2 with $N_1 \subset \mathbf{R}^- \times \mathbf{R}^2, N_2 \subset \mathbf{R}^+ \times \mathbf{R}^2$, then*

$$I_{(\mathfrak{g}_2,V)}(N) = I_{(\mathfrak{g}_2,V)}(N_1) \cdot I_{(\mathfrak{g}_2,V)}(N_2). \qquad \square$$

[9] Setting $r = -(q^2 + q + 1 + q^{-2} + q^{-3} + q^{-4})$, we obtain the skein relation given in [K].

As, by definition, every planar 3-net is equivalent to a 3-net contained in $\mathbf{R}^2 \times \{0\}$ with only upward pointing vectors assigned, it is enough to calculate $I_{(\mathfrak{g}_2,V)}$ for these. Therefore, we assume that all planar 3-nets in this section are of this type.

Definition 5.2. *Let N be a planar 3-net. A* mesh *of N is the closure of a connected component of $(\mathbf{R}^2 \times \{0\}) \backslash N$. A n-*mesh *is a mesh with n trivalent vertices in the boundary.*

We will show how in any non-empty connected closed planar 3-net the number of meshes can be reduced without changing the value of the invariant. As we know that $I_{(\mathfrak{g}_2,V)}$(empty 3-net) $= 1$ (the empty 3-net is the unity in $\mathcal{T}_{tpf}$ and $I_{(\mathfrak{g}_2,V)}$ is a monoidal functor), this will allow us to calculate the invariant of a closed planar 3-net recursively.

Proposition 5.3. *Let N be a non-empty closed connected planar 3-net with m meshes. Then there exist closed planar 3-nets $N_1, \ldots, N_r$ (not necessarily connected) with fewer than m meshes and coefficients $\lambda_1, \ldots, \lambda_r \in \mathbf{C}$ such that $I_{(\mathfrak{g}_2,V)}(N) = \sum_{i=1}^{r} \lambda_i I_{(\mathfrak{g}_2,V)}(N_i)$.*

Proof. The idea is to cut out a mesh and replace it by a linear combination of pieces that lead to 3-nets with fewer meshes.

Thanks to the following lemma, the mesh we want to cut out can always be chosen to be a simply connected n-mesh with $n \leq 5$.

Lemma 5.4. *Let N be a planar non-empty closed connected 3-net. Then N has at least one simply connected n-mesh with $n \leq 5$.*

Proof of the lemma. If N is an embedded S^1, N has a bounded 0-mesh, and so in this case, the lemma holds.
Let $\sharp$ denote "number of" and let N be a 3-net without 0-mesh.
<u>Observation 1</u>: $\sharp$ vertices of $N - \sharp$ edges of $N + \sharp$ meshes of $N = 2$ (Euler characteristic of S^2).
<u>Observation 2</u>: If we assign to each mesh M the appropriate part of the contribu-

tions of its vertices, its edges, and its region, i.e.

$$\chi_M := \begin{cases} \;\;\frac{1}{3}\,(\sharp \text{ vertices of } M) \\ +\frac{1}{3}\,(\sharp \text{ vertices of } M \text{ for which all adjacent edges belong to } M) \\ +\frac{1}{3}\,(\sharp \text{ vertices of } M \text{ that belong only to } M) \\ -\frac{1}{2}\,(\sharp \text{ edges of } M) \\ -\frac{1}{2}\,(\sharp \text{ edges of } M \text{ that belong only to } M) \\ +1, \end{cases}$$

$$\text{then} \sum_{M \text{ mesh of } N} \chi_M = 2.$$

<u>Observation 3</u>: For a simply connected n-mesh M of N, we have $\chi_M = 1 - \frac{1}{6}n$.
<u>Observation 4</u>: If the unbounded mesh M' of N does not contain an edge belonging to M' only, then $\chi_{M'} \leq 1$.

If N does not contain an edge that belongs to one mesh only, the lemma is a consequence of observations 2, 3 and 4[10].

Now suppose that there are edges $e_1, \dots, e_k$ such that e_j belongs to only one mesh M_j. Note that forgetting such an edge e_j would split the 3-net N into two connected components N_{j1} and N_{j2}. As there are only finitely many edges e_j, there is an edge e_i for which at least for one $k \in \{1,2\}$ N_{ik} satisfies the following properties:
(i) N_{ik} does not contain an edge that belongs to only one mesh;
(ii) any mesh M of N that is also a mesh of N_{ik} is bounded.
The sum $\sum \chi_M$ over all meshes M of N mentioned in (ii) is greater than 0 (look at $\sum \chi_M$ with M considered as mesh of N_{ik}, use observation 4, and subtract $\frac{1}{6}$ for the influence of e_i), and thus observation 3 guarantees that at least one of these meshes has fewer than 6 vertices. □

To replace n-meshes for $n \leq 5$, we will use the following lemma:

Lemma 5.5. *The following equations hold:*

(o) $I_{(\mathfrak{g}_2,V)}(\;) = 7cI_{(\mathfrak{g}_2,V)}(\;) = 7c,$

(i) $I_{(\mathfrak{g}_2,V)}(\;) = 0,$

(ii) $I_{(\mathfrak{g}_2,V)}(\;) = rI_{(\mathfrak{g}_2,V)}(\;),$

(iii) $I_{(\mathfrak{g}_2,V)}(\;) = t \cdot I_{(\mathfrak{g}_2,V)}(\;)$ *with* $t := \frac{1}{\delta}(-q^3 + \alpha q^{-3} - \gamma),$

[10] Note that any bounded mesh M that does not contain an edge belonging only to M is simply connected.

$$(iv)\ I_{(\mathfrak{g}_2,V)}(\ ) = \frac{r^2q^5}{g(q^4+1)}(I_{(\mathfrak{g}_2,V)}(\ ) + I_{(\mathfrak{g}_2,V)}(\ ))$$
$$+\frac{rq^2(q^2+1)}{g}(I_{(\mathfrak{g}_2,V)}(\ ) + I_{(\mathfrak{g}_2,V)}(\ ))$$
$$\textit{with } g := q^6+q^5+q^4+q^2+q+1,$$

$$(v)\ I_{(\mathfrak{g}_2,V)}(\ ) = -d(I_{(\mathfrak{g}_2,V)}(\ ) + I_{(\mathfrak{g}_2,V)}(\ ) + I_{(\mathfrak{g}_2,V)}(\ )$$
$$+I_{(\mathfrak{g}_2,V)}(\ ) + I_{(\mathfrak{g}_2,V)}(\ ))$$
$$-d^2(I_{(\mathfrak{g}_2,V)}(\ ) + I_{(\mathfrak{g}_2,V)}(\ ) + I_{(\mathfrak{g}_2,V)}(\ )$$
$$+I_{(\mathfrak{g}_2,V)}(\ ) + I_{(\mathfrak{g}_2,V)}(\ ))$$
$$\textit{with } d := \frac{rq^3}{g}.$$

Proof of the Lemma. Equation (o) is proved in remark 4.1.

Equation (i) is true because $I_{(\mathfrak{g}_2,V)}(\ )$ is a $\mathfrak{g}_2$-linear map from $\mathbf{C}$ to V and must therefore be 0.

Equation (ii), we get from $I_{(\mathfrak{g}_2,V)}(\ ) = \Psi(\ A^{1/2}\ B^{1/2}\ ) \overset{\text{Sec. 4}}{=} r\Psi(\ )$.

To get equation (iii), we use the skein relation given in theorem 4.4 (rotated by 90°):

$$I_{(\mathfrak{g}_2,V)}(\ ) = \frac{1}{\delta}(I_{(\mathfrak{g}_2,V)}(\ )$$
$$-\alpha I_{(\mathfrak{g}_2,V)}(\ ) - \beta I_{(\mathfrak{g}_2,V)}(\ ) - \gamma I_{(\mathfrak{g}_2,V)}(\ )).$$

Note that $I_{(\mathfrak{g}_2,V)}(\ ) = I_{(\mathfrak{g}_2,V)}(\ ) \circ I_{(\mathfrak{g}_2,V)}(\ ) = -q^3 I_{(\mathfrak{g}_2,V)}(\ )$ (only the eigenvalue of $I_{(\mathfrak{g}_2,V)}(\ )$ on V matters).

Equation (iv) and (v) can be obtained in a similar way. For (v), it may help to use that $\ \sim\ $. □

As $I_{(\mathfrak{g}_2,V)}$ is a monoidal functor, equations (o)-(v) will still hold if the 3-nets depicted in the arguments of $I_{(\mathfrak{g}_2,V)}$ are parts of bigger 3-nets that are identical outside the depicted region for all arguments in the same equation. The observation that for all equations, the 3-nets appearing on the right-hand side have fewer meshes than the one on the left-hand side concludes the proof of the proposition. □

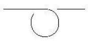
Now we are able to compute the value of the $I_{(\mathfrak{g}_2,V)}$-invariant for every oriented closed planar 3-net recursively. Planar 3-tangles that are not closed can often be reduced by the same technique, but because in this case, lemma 5.4 is no longer true, it may happen that we get stuck before reaching the empty 3-net (example: ).

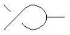
6. Some examples

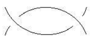
To do explicit calculations, the following lemmas may be helpful, the first comparing 3-nets to their mirror images, the second suggesting some short cuts.

Lemma 6.1. *As taking mirror images essentially comes to changing crossings, the value of $I_{(\mathfrak{g}_2,V)}$ on the mirror image of a 3-net N is obtained by substituting q^{-1} for q in $I_{(\mathfrak{g}_2,V)}(N)$.* $\square$

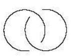
Lemma 6.2.

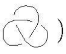
$I_{(\mathfrak{g}_2,V)}($  $) = q^6 I_{(\mathfrak{g}_2,V)}($  $)$

$I_{(\mathfrak{g}_2,V)}($  $) = -q^3 I_{(\mathfrak{g}_2,V)}($  $)$

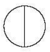
$I_{(\mathfrak{g}_2,V)}($  $) = \beta I_{(\mathfrak{g}_2,V)}($  $) + \alpha I_{(\mathfrak{g}_2,V)}($  $)$
$+\gamma q^{-6} I_{(\mathfrak{g}_2,V)}($  $) - \delta q^{-3} I_{(\mathfrak{g}_2,V)}($  $)$

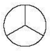
Example 6.3.

$I_{(\mathfrak{g}_2,V)}($  $) = 7c(q^7 + q^5 + q^2 + 1 + q^{-2} + q^{-5} + q^{-7})$

$I_{(\mathfrak{g}_2,V)}($  $) = 7c(q^8 + q^6 - q^5 + q^3 - q^2 + q - 1 + q^{-1} + q^{-4} - 2q^{-5} + 2q^{-6} - q^{-7} - q^{-9} - q^{-10} + q^{-11} - q^{-12} + q^{-13})$

$I_{(\mathfrak{g}_2,V)}($  $)$ $7c(q^{14} - q^{13} + 2q^{12} - 2q^{11} + q^9 - 2q^8 + 4q^7 - 4q^6 + 4q^5 - 2q^4 - q^3 + 3q^2 - 5q + 5 - 5q^{-1} + 3q^{-2} - q^{-3} - 2q^{-4} + 4q^{-5} - 4q^{-6} + 4q^{-7} - 2q^{-8} + q^{-9} - 2q^{-11} + 2q^{-12} - q^{-13} + q^{-14})$

$I_{(\mathfrak{g}_2,V)}($  $)$ $7cr$

$I_{(\mathfrak{g}_2,V)}($  $)$ $-7cq\frac{q^2-q+1}{q^4+1}r^2$

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