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Commentarii Mathematici Helvetici

Geometry for palindromic automorphism groups of free groups

Henry H. Glover and Craig A. Jensen

Dedicated to Peter Hilton:

Doc, note, I dissent. A fast never prevents a fatness. I diet on cod.

Abstract. We examine the palindromic automorphism group $\Pi A(F_n)$ of a free group F_n, a group first defined by Collins in [5] which is related to hyperelliptic involutions of mapping class groups, congruence subgroups of $SL_n(\mathbb{Z})$, and symmetric automorphism groups of free groups. Cohomological properties of the group are explored by looking at a contractible space on which $\Pi A(F_n)$ acts properly with finite quotient. Our results answer some conjectures of Collins and provide a few striking results about the cohomology of $\Pi A(F_n)$, such as that its rational cohomology is zero at the vcd.

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1. Introduction

Let $Aut(F_n)$ be the automorphism group of a free group F_n on n generators $a_1, a_2, \ldots, a_n$. A reduced word $x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_n^{\epsilon_n}$ is called a *palindrome* if it is equal to its reverse $x_n^{\epsilon_n} x_{n-1}^{\epsilon_{n-1}} \ldots x_1^{\epsilon_1}$. In [2] Collins defines the *palidromic automorphism group* $\Pi A(F_n)$ as the subgroup of $Aut(F_n)$ consisting of all automorphisms α for which $\alpha(a_i)$ is a palindrome for all i. He showed that the group was generated by three types of automorphisms:

- Maps $(a_i||a_j)$, $i \neq j$, which send $a_i \mapsto a_j a_i a_j$ and fix all other generators a_k.
- Maps σ_{a_i} which send $a_i \mapsto a_i^{-1}$ and fix all other generators a_k.
- Maps corresponding to elements of the symmetric group Σ_n which permute the $a_1, \ldots, a_n$ among themselves.

The portion of $\Pi A(F_n)$ generated by just the $(a_i||a_j)$ is called the *elementary palindromic automorphism group* of F_n and denoted $E\Pi A(F_n)$. Note that $\Pi A(F_n) = E\Pi A(F_n) \rtimes (\mathbb{Z}/2 \wr \Sigma_n)$. Collins showed that a set of defining relators for $E\Pi A(F_n)$ is given by relations of the form

(1) $(a_i||a_k)(a_j||a_k) = (a_j||a_k)(a_i||a_k)$

(2) $(a_i||a_k)(a_j||a_l) = (a_j||a_l)(a_i||a_k)$
(3) $(a_i||a_k)(a_j||a_k)(a_i||a_j) = (a_i||a_j)(a_j||a_k)(a_i||a_k)^{-1}$

He remarked how similar this was to the relations for the pure symmetric automorphism group $P\Sigma A(F_n)$ (see Gilbert's work in [10]):
(1) $(a_i|a_k)(a_j|a_k) = (a_j|a_k)(a_i|a_k)$
(2) $(a_i|a_k)(a_j|a_l) = (a_j|a_l)(a_i|a_k)$
(3) $(a_i|a_k)(a_j|a_k)(a_i|a_j) = (a_i|a_j)(a_j|a_k)(a_i|a_k)$
where $(a_i|a_j)$, $i \neq j$, sends $a_i \mapsto a_j^{-1}a_ia_j$ and fixes all other generators a_k.

On the basis of this, Collins conjectured that one could find the virtual cohomological dimension of $\Pi A(F_n)$ by employing the methods of [7], as he did for $\Sigma A(F_n)$ in [4]. He also speculated that $E\Pi A(F_n)$ is torsion free, just as $P\Sigma A(F_n)$ is. We are able to answer both of these questions in this paper, as well as obtaining several interesting facts about the cohomology of $\Pi A(F_n)$.

Theorem 1.1. *Let $\Pi A(F_n)$ be the palindromic automorphism group of the free group F_n on n letters and let $E\Pi A(F_n)$ be the subgroup of elementary palindromic automorphisms. Then*
a) *The virtual cohomological dimension of $\Pi A(F_n)$ is $n-1$.*
b) *(i) For the prime 2, the Krull dimension of $\hat{H}^*(\Pi A(F_n);\mathbb{Z}_{(2)})$ is n. For odd primes p, the Krull dimension of $\hat{H}^*(\Pi A(F_n);\mathbb{Z}_{(p)})$ is $\left[\frac{n}{p}\right]$.*
(ii) In the range where the Krull dimension of $\hat{H}^*(\Pi A(F_n);\mathbb{Z}_{(p)})$ is 1, the period is $2(p-1)$.
c) *The group $E\Pi A(F_n)$ is torsion free.*
d) *The cohomology group $H^{n-1}(\Pi A(F_n);\mathbb{Q}) = 0$.*
e) *If p is an odd prime and $n = p, p+1, p+2$, then the Farrell cohomology of the palindromic automorphism group is the same as that of the symmetric group on p elements:*

$$\hat{H}^*(\Pi A(F_n);\mathbb{Z}_{(p)}) \cong \hat{H}^*(\Sigma_p;\mathbb{Z}_{(p)}).$$

For analogous results concerning $Aut(F_n)$, see [3] and [11]. See [9] for the definition of the Farrell cohomology $\hat{H}^*(G;M)$ of a group G of finite vcd with coefficients in a G-module M and also see [2] for several useful properties of these cohomology groups.

The remainder of this paper is structured as follows. In section 2, we discuss $\Pi A(F_n)$ and note how it relates to some other groups, while in section 3 we introduce the space L_{σ_n} which $\Pi A(F_n)$ acts on and prove parts a) and d) of Theorem 1.1. Section 4 is concerned with a realization proposition which allows us to establish parts b) (i) and c) of the main theorem. Finally, section 5 looks in more detail at the cohomology of $\Pi A(F_n)$ at odd primes p and establishes parts b) (ii) and e) of the main theorem.

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paper.

2. Relationships with other groups

Let X_n be the spine of auter space (see [7], [13], [14]) and $Q_n = X_n/Aut(F_n)$. Let $\sigma_n \in Aut(F_n)$ be the automorphism which sends $a_i \mapsto a_i^{-1}$ for each i.

Define the θ-graph θ_m to be a graph with 2 vertices and $m+1$ edges, where each edge goes from one vertex to the other one. Choose one of the two vertices of θ_1 to be the basepoint $*$, and define the rose R_n to be the result of wedging together n copies of θ_1 at the basepoint.

The petals of the rose R_n can be identified with the generators a_i of F_n, so that $\pi_1(R_n,*) \cong F_n$. There is an action of $\langle\sigma_n\rangle = \mathbb{Z}/2$ on R_n given by inverting each petal of the rose. This action realizes the subgroup $\langle\sigma_n\rangle$ in the sense of [22] (also cf. [6].) An action of a group G on a graph Γ is *without inversions* if G does not send any edge e to its inverse $\bar{e}$, and an action is *reduced* if there are no G-invariant subforests in Γ. The action of σ_n on R_n is both without inversions and reduced. From now on, when we refer to a group action on a graph, it is assumed that the edges of the graph are subdivided as necessary to insure that the group acts without inversions.

Note that the palindromic automorphism group $\Pi A(F_n)$ is just $C_{Aut(F_n)}(\sigma_n)$. This follows because an easy argument shows that every element of $C_{Aut(F_n)}(\sigma_n)$ is palindromic, and because the generators of $\Pi A(F_n)$ are all in $C_{Aut(F_n)}(\sigma_n)$. For example, the generators $(a_i||a_j)$ are just products of σ_n-Nielsen transformations (see [20] where the G-Nielsen transformation $\langle e,f\rangle\Gamma$ of a G-graph Γ has the same vertex and edge set as Γ but where the terminal point of an edge eg, $g \in G$, in the new graph is the initial point of the edge fg in the original graph; this induces a map $\langle e,f\rangle$ from the fundamental groupoid of the first graph to that of the second where eg is sent to $(ef)g$ and all other edges are sent to themselves.) That is, if the petal a_i of the rose consists of the edges $\bar{e}_i f_i$, then $(a_i||a_j)$ is the composition $\langle e_i,\bar{f}_j\rangle \circ \langle e_i,e_j\rangle$.

As a note for the curious, it follows that $\Pi A(F_n)$ and $\Sigma A(F_n)$ are distinct groups for $n \geq 2$, since a direct argument shows that $\Sigma A(F_n)$ has no element of order 2 in its center. In addition, $E\Pi A(F_n)$ and $P\Sigma A(F_n)$ are also obviously distinct (for $n \geq 3$; for $n = 1,2$ they are the same group with the same presentation,) since the former abelianizes to an elementary abelian 2-group of rank $n(n-1)$ while the latter abelianizes to a free abelian group of rank $n(n-1)$.

In addition to its formal palindromic properties, the group $\Pi A(F_n)$ arises naturally from looking at hyperelliptic subgroups of mapping class groups (cf. Gries [12] for corresponding homological properties.) We have a commutative diagram

$$\begin{array}{ccccc} \Gamma_g^{2,pure} & \rightarrow & \Gamma_g^1 & \rightarrow & \Gamma_g \\ \downarrow & & \downarrow & & \downarrow \\ Aut(F_{2g}) & \rightarrow & Out(F_{2g}) & \rightarrow & GL_{2g}(\mathbb{Z}) \end{array} \tag{2.1}$$

where Γ_g is the mapping class group of an orientable surface of genus g, Γ_g^1 is the mapping class group of an orientable surface of genus g with 1 puncture, and $\Gamma_g^{2,pure}$ is the mapping class group of an orientable surface of genus g with two punctures, where each puncture is fixed pointwise. The map from $\Gamma_g^{2,pure}$ to $Aut(F_{2g})$ is obtained first by taking an intersection basis $a_1, b_1, \ldots, a_g, b_g$ for the fundamental group of the surface S with two punctures. One of the punctures should serve as the basepoint for fundamental group considerations. The other is treated as an actual puncture, so that the fundamental group of the surface minus this point is a free group F_{2g} on $2g$ generators. The map from $\Gamma_g^{2,pure}$ to $Aut(F_{2g})$ is now obtained by sending an element of $\Gamma_g^{2,pure}$ to the automorphism of F_{2g} that it induces. The map from Γ_g^1 to $Out(F_{2g})$ is obtained similarly.

Let $\psi \in \Gamma_g^{2,pure}$ be a hyperelliptic involution (see, for example, [8].) Then ψ has $2g+2$ fixed points, two of which are of course the punctures on the surface S. Choose loops $a_1, b_1, \ldots, a_g, b_g$, based at one of the punctures, which form an intersection basis for the surface S (say the ones described in [8] in the section on hyperelliptic Riemann surfaces.) By going along the top row of diagram 2.1 and then projecting downward, we see that the image of ψ in $GL_{2g}(\mathbb{Z})$ is $-I$. Let $\bar{\psi}$ be the image of ψ in $Aut(F_{2g})$. Our goal is to show that $\bar{\psi}$ is conjugate to σ_{2g} in $Aut(F_{2g})$.

Lemma 2.2. *If $\phi \in Aut(F_n)$ and the image of ϕ in $GL_n(\mathbb{Z})$ is $-I$, then the image of any conjugate $\alpha^{-1}\phi\alpha$ of ϕ, $\alpha \in Aut(F_n)$, is also $-I$.*

Proof. This follows directly since $-I$ is in the center of $GL_n(\mathbb{Z})$. □

Lemma 2.3. *If $\phi \in Aut(F_n)$ is an involution whose image in $GL_n(\mathbb{Z})$ is $-I$, then ϕ can be realized on a marked graph whose underlying graph is the rose.*

Proof. Realize ϕ on a reduced marked graph whose underlying graph is Λ. First, we show that if e is an edge of Λ which is not fixed by ϕ, then we can assume that one endpoint of e is the basepoint $*$. Let $f = \phi(e)$. Choose a shortest path γ from e to $*$. Since $stab(e) = \langle 1 \rangle$, $stab(e) \subseteq stab(h)$ for every h in the path γ. So we can apply a sequence of Nielsen transformations (see [20]) and slide e along γ to $*$. Note that since Λ is reduced, $\{e, f\}$ now forms either a rose R_2 based at $*$, or a θ_1. Proceeding in this manner, we can slide all of the edges of Λ not fixed by ϕ to the basepoint.

By way of contradiction, suppose that two of these θ_1-graphs (that have been moved so that one vertex of each θ_1 is the basepoint), say $\{e_1, f_1\}$ and $\{e_2, f_2\}$, share another common vertex in addition to the basepoint $*$. That is, suppose that there is a vertex $v \neq *$ and each of e_1, f_1, e_2, f_2 go from v to $*$. Say $g : R_n \to \Lambda$ is the marked graph, and recall that $\pi_1(R_n) = \langle a_1, \ldots, a_n \rangle = F_n$. By replacing ϕ by a conjugate if necessary (see Lemma 2.2) we can assume that g sends the petal a_1 of R_n to $e_1^{-1}f_1$, the petal a_2 to $e_2^{-1}f_2$, and the petal a_3 of R_n to $e_1^{-1}f_2$. So in

$\pi_1(\Lambda)$ we have

$$\phi \cdot a_1 = f_1^{-1}e_1 = g(a_1^{-1}),$$

$$\phi \cdot a_2 = f_2^{-1}e_2 = g(a_2^{-1}),$$

and

$$\phi \cdot a_3 = f_1^{-1}e_2 = g(a_1^{-1}a_3a_2^{-1}).$$

Hence the first column of $im(\phi)$ in $GL_n(\mathbb{Z})$ is $(-1,0,\dots,0)$, the second column is $(0,-1,0,\dots,0)$, and the third column is $(-1,-1,1,0,\dots,0)$. This contradicts the fact that $im(\phi) = -I$.

If the result of sliding to the basepoint $*$ all edges of Λ not fixed by ϕ yields a rose R_n, then we are done. Otherwise, suppose by way of contradiction that there exist edges e,f, and h of Λ such that

- Both e and f go from some vertex $v \neq *$ to $*$.
- $\phi(e) = f$.
- h goes from v to v.
- $\phi(h) = h$.

As before, say $g : R_n \to \Lambda$ is the marked graph. By replacing ϕ by a conjugate if necessary (see Lemma 2.2) we can assume that g sends the petal a_1 of R_n to $e^{-1}f$ and the petal a_2 of R_n to $e^{-1}hf$. So in $\pi_1(\Lambda)$ we have

$$\phi \cdot a_1 = f^{-1}e = g(a_1^{-1})$$

and

$$\phi \cdot a_2 = f^{-1}he = (e^{-1}f)^{-1}(e^{-1}hf)(e^{-1}f)^{-1} = g(a_1^{-1}a_2a_1^{-1}).$$

Hence the first column of $im(\phi)$ in $GL_n(\mathbb{Z})$ is $(-1,0,\dots,0)$ and the second column is $(-2,1,0,\dots,0)$. This contradicts the fact that $im(\phi) = -I$. □

Proposition 2.4. *If $\phi \in Aut(F_n)$ is an involution whose image in $GL_n(\mathbb{Z})$ is $-I$, then ϕ is conjugate in $Aut(F_n)$ to σ_n.*

Proof. Realize ϕ on a marked graph $g : R_n \to R_n$. By replacing ϕ by a conjugate if necessary, we can assume $g(a_i) = a_i$ for all i. The involution ϕ of the graph R_n must send the petal a_1 to some $a_j^{\pm 1}$, since it is a graph automorphism. But since $im(\phi) = -I \in GL_n(\mathbb{Z})$, we see that $a_j^{\pm 1}$ must be a_1^{-1}. Similarly, we can see that the graph automorphism ϕ sends, for each i, the petal a_i to the petal a_i^{-1}. This means our current ϕ is equal to σ_n (and thus our original ϕ, before we replaced it by a conjugate, was conjugate to σ_n.) □

The following corollary is immediate:

Corollary 2.5. *The image $\bar{\psi}$ of ψ in $Aut(F_{2g})$ is conjugate to σ_{2g}. Hence the hyperelliptic subgroup $C_{\Gamma_g^{2,pure}}(\psi)$ is conjugate to a subgroup of $C_{Aut(F_{2g})}(\sigma_{2g})$.*

As remarked in [5], the image of $\Pi A(F_n)$ in $GL_n(\mathbb{Z})$ is the subgroup of $GL_n(\mathbb{Z})$ consisting of invertible matrices where each column has exactly one odd entry (and the rest are even.) The subgroup is the semidirect product $\tilde{\Gamma}_2(\mathbb{Z}) \rtimes \Sigma_n$ where $\tilde{\Gamma}_2(\mathbb{Z})$ is the 2-congruence subgroup defined by the short exact sequence

$$\langle 1 \rangle \to \tilde{\Gamma}_2(\mathbb{Z}) \rightarrowtail GL_n(\mathbb{Z}) \twoheadrightarrow GL_n(\mathbb{Z}/2) \to \langle 1 \rangle$$

and Σ_n is standard inclusion of the symmetric group

$$\Sigma_n \subset GL_n(\mathbb{Z}).$$

3. A space for $\Pi A(F_n)$ to act on

We define a certain contractible space L_{σ_n}, related to auter space, which $\Pi A(F_n)$ acts on with finite stabilizers and finite quotient. This allows us to obtain some cohomological results.

A graph Γ is a *θ_1-tree of rank n* if there exists a pointed tree T such that Γ is obtained by "doubling" every edge of T into a θ_1-graph. That is, the vertex set of Γ is the same as the vertex set of T and for every edge e of T going from v to w, Γ has two edges e_1 and e_2, both of which go from v to w. There is a natural $\mathbb{Z}/2$-action on such a graph Γ, which is given by switching the two edges in each θ_1-graph. Note that the orbit space of Γ under this action is just the tree T.

Claim 3.1. *The reduced graphs Γ which realize the subgroup $\langle \sigma_n \rangle$ of $Aut(F_n)$ are exactly the θ_1-trees of rank n, where σ_n acts on the trees via their natural $\mathbb{Z}/2$-action.*

Proof. We have already mentioned that the rose R_n realizes σ_n. From Theorem 2 of [20], the other reduced graphs Γ which also realize σ are those that are Nielsen equivalent to R_n (up to an equivariant isomorphism.)

If e is an edge in one of the copies of θ_1 in R_n and f is an edge in a different θ_1 in R_n, and both e and f point toward the basepoint, then Nielsen transformation $\langle e, f \rangle$ has the result of pulling the θ_1-graph $\{e, \sigma_n e\}$ through the θ_1-graph $\{f, \sigma_n f\}$, so that now e terminates at the initial vertex of f, rather than at the basepoint $*$. In other words, the result of applying one Nielsen transformation to R_n is that of sliding one of the petals of R_n up though another petal.

A basic induction argument now yields that the result of applying a series of Nielsen transformations fo R_n will be some θ_1-tree Γ. □

Recall from [21] that an edge e of a G-graph Γ is *inessential* if it is in every maximal G-invariant subforest of Γ. A G-graph Γ is *inessential* if it has at least one inessential edge and is *essential* if it is not inessential. Let X_n^G be the fixed point subspace of X_n corresponding to some finite subgroup G of $Aut(F_n)$. From [17] (cf. part III of [18] and [21]), both the centralizer $C_{Aut(F_n)}(G)$ and the normalizer $N_{Aut(F_n)}(G)$ act on the contractible space X_n^G with finite stabilizers and finite quotient. Moreover, the space X_n^G G-equivariantly deformation retracts to the space L_G, where L_G is constructed from X_n^G by considering only essential marked graphs. Hence L_G is a good space to study if one wishes to calculate the cohomology of $C_{Aut(F_n)}(G)$ or $N_{Aut(F_n)}(G)$.

Further recall that a G-graph $\hat{\Gamma}$ is a G-equivariant *blowup* of a G-graph Γ if some G-invariant subforest F of $\hat{\Gamma}$ can be collapsed away to yield Γ. Let $\phi : R_n \to \Gamma$ be some reduced marked graph realizing $\langle \sigma_n \rangle$. From Claim 3.1, Γ is a θ_1-tree. Blow up Γ σ_n-equivariantly to some maximal essential blowup $\hat{\Gamma}$.

Claim 3.2. *The fixed points/cells of the action of σ_n on $\hat{\Gamma}$ are exactly the valence 2 vertices of $\hat{\Gamma}$.*

Proof. Note that vertices in $\hat{\Gamma}$ have valence 2 or 3 and that $\hat{\Gamma}$ is obtained from Γ by blowing up an oriented ideal forest (see [17], [21], [18].) Briefly, ideal edges (oriented ideal forests) correspond to subsets of edges (chains of subsets of edges) which are pulled away from existing vertices in order to create new graphs which collapse down to the original graph.)

No edge is in $Fix_{\sigma_n}(\hat{\Gamma})$ because no edge is in $Fix_{\sigma_n}(\Gamma)$ and so blowing up ideal edges will not create any new edges that are fixed under the action of σ_n (see [21] page 229.)

If a valence 3 vertex is in $Fix_{\sigma_n}(\hat{\Gamma})$, then at least one edge of $\hat{\Gamma}$ must be fixed by σ_n, which is a contradiction.

All of the valence 2 vertices of Γ are in $Fix_{\sigma_n}(\Gamma)$. New valence 2 vertices created as ideal edges are blown up correspond to either:

- Old vertices of Γ that used to be valence higher than 2 but have since had edges stripped (pulled away) from them. These are in $Fix_{\sigma_n}(\hat{\Gamma})$ because all vertices of Γ are in $Fix_{\sigma_n}(\Gamma)$.
- New valence 2 vertices inserted to insure that σ_n acts on $\hat{\Gamma}$ without inversions. These are also clearly in $Fix_{\sigma_n}(\hat{\Gamma})$. □

Note that $* \in Fix_{\sigma_n}(\Gamma)$. Cut $\hat{\Gamma}$ along each of its valence 2 vertices, yielding a graph $\hat{\Gamma}_{cut}$ with the same number of valence 3 vertices and edges as $\hat{\Gamma}$ had, no valence 2 vertices, and twice as many valence 1 vertices as $\hat{\Gamma}$ had valence 2 vertices.

Claim 3.3.

$$\hat{\Gamma}_{cut} = \hat{\Gamma}_1 \amalg \hat{\Gamma}_1,$$

the disjoint union of two trees $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$, where $\hat{\Gamma}_2 = \sigma_n \hat{\Gamma}_1$.

Proof. There is a covering map $p : \hat{\Gamma}_{cut} \to \hat{\Gamma}/\sigma_n$ obtained by mapping to the orbit space under the σ_n-action. The forest collapse that sends $\hat{\Gamma}$ to Γ is σ_n-equivariant, so it descends to a forest collapse of $\hat{\Gamma}/\sigma_n$ to Γ/σ_n. (That the quotient of the forest upstairs in $\hat{\Gamma}$ is also a forest in $\hat{\Gamma}/\sigma_n$ can be seen by an easy Euler characteristic argument.) But Γ is a θ_1-tree with a known σ_n-action on it, and Γ/σ_n is a tree (in fact, the underlying tree of the θ_1-tree.) Hence $\hat{\Gamma}/\sigma_n$ is a tree. Since $p : \hat{\Gamma}_{cut} \to \hat{\Gamma}/\sigma_n$ is a covering map with fiber two points, $\hat{\Gamma}_{cut}$ is as described. □

Let T be a pointed tree with $2n-1$ edges, all vertices valence either 1 or 3, where $*$ is one of the valence 1 vertices. (Then T has $n+1$ valence 1 vertices and $n-1$ valence 3 vertices.) Let T_1 and T_2 be two isomorphic copies of T, and let $f : T_1 \to T_2$ be an isomorphism. Define

$$\Gamma_T = \frac{T_1 \amalg T_2}{f(v) \sim v, \text{ for all valence 1 vertices } v \text{ of } T_1.}$$

Define a σ_n-action on Γ_T by

$$\sigma_n x = \begin{cases} f(x), & x \in T_1 \\ f^{-1}(x), & x \in T_2 \end{cases}$$

Proposition 3.4. *There is a bijective correspondence between trees T as above (with $2n-1$ edges, etc) and maximal, essential blowups $\hat{\Gamma}$ of reduced σ_n-graphs. The bijection is given by $T \mapsto \Gamma_T$.*

Proof. From claims 3.2 and 3.3, all blowups $\hat{\Gamma}$ have the required form. Finally, any Γ_T can easily be reduced to a θ_1-tree by collapsing edges, meaning that it is the blowup of such a graph. □

All maximal simplices in L_{σ_n} have the same dimension, from [21]. Maximal simplices in $L_{\sigma_n}/\Pi A(F_n)$ are constructed by taking chains of forest collapses from maximal blowups Γ_T. Alternatively, we can define a subforest of T to be a collection S of edges of T such that there is no path in S from one valence 1 vertex to another. (If there were such a path, then $S \cup \sigma_n S$ would be a cycle in Γ_T.) In this way, we can think of maximal simplices as coming from chains of subforests of various trees T.

Proof of part a) of Theorem 1.1: Since $\Pi A(F_n)$ acts on the contractible space L_{σ_n} with finite stabilizers and finite quotient, the vcd of $\Pi A(F_n)$ is at most the dimension of a maximal cell from $L_{\sigma_n}/\Pi A(F_n)$. Such a cell comes from a chain of forest collapses of a tree T with $2n-1$ edges, $n+1$ valence 1 vertices, and $n-1$ valence 3 vertices. Hence we can collapse at most $n-1$ of the valence 3 vertices into other vertices while doing forest collapses, resulting in maximal simplices of dimension $n-1$.

To show that the vcd of $\Pi A(F_n)$ is at least $n-1$, we note that the subgroup generated by $(a_i||a_n)$ for $i \in \{1, 2, \ldots, n-1\}$ is isomorphic to $\mathbb{Z}^{n-1} = \mathbb{Z} \times \ldots \times \mathbb{Z}$. $\square$

Lemma 3.5. *Let $\tilde{\Gamma}$ be the underlying graph of a particular marked graph in L_{σ_n}. Hence $\tilde{\Gamma}$ comes equipped with a σ_n-action. Let C be a simple closed curve in $\tilde{\Gamma}$. Then $Fix_{\sigma_n}(C)$ contains exactly two points, and $\sigma_n(C)$ is the curve $-C$, or C with the opposite of its original orientation.*

Proof. $\tilde{\Gamma}$ can be blown up (not necessarily uniquely) to some maximal essential graph $\Gamma = \Gamma_T$. Γ is the union of two isomorphic copies T_1 and T_2 of T, where T_1 and T_2 are attached along their corresponding valence 1 vertices.

As we collapse from Γ to $\tilde{\Gamma}$, the trees T_1 and T_2 collapse to trees $\tilde{T}_1$ and $\tilde{T}_2$. However, the attaching points for $\tilde{T}_1$ and $\tilde{T}_2$ are no longer necessarily just the valence 1 vertices, and could be other vertices as well.

Let α_1 be a taut path in $\tilde{T}_1$ from one attaching point v_1 to some other attaching point v_2, where furthermore there are no attaching points in the interior of α_1. Let $\alpha_2 = \sigma_n(\alpha_1)$ be the corresponding path in $\tilde{T}_2$. Then $\alpha_1\bar{\alpha}_2$ is a simple closed curve and $\sigma_n(\alpha_1\bar{\alpha}_2) = \alpha_2\bar{\alpha}_1$, or the original curve oriented in the other direction. Hence the curve $\alpha_1\bar{\alpha}_2$ satisfies the conclusions of the lemma. Our goal is to show that any simple closed curve C takes this form.

Let C_1 be the portion of C that is in $\tilde{T}_1$ and let C_2 be the portion of C that is in $\tilde{T}_2$. Since C is a cycle and yet both $\tilde{T}_1$ and $\tilde{T}_2$ are trees, both C_1 and C_2 are nonempty. In fact, there must be a path α_1 in C_1 from one attaching point v_1 to some other attaching point v_2, where there are no other attaching points in the interior of the path. Since C is a simple closed curve, α_1 must be a taut path. Let $\alpha_2 = C - \alpha_1$, some other taut path in $\tilde{\Gamma}$ from v_1 to v_2.

Let $p : \tilde{\Gamma} \to \tilde{T}_1$ be the map given by taking the quotient space under the action of σ_n. Note that $p(\alpha_2)$ is a path from v_1 to v_2. Since $\tilde{T}_1$ is a tree and α_1 is the unique taut path in $\tilde{T}_1$ from v_1 to v_2, this gives us that edges$(\alpha_1) \subseteq$ edges$(p(\alpha_2))$. Hence if e is an edge in α_1, then $p^{-1}(e)$ is two edges, $e \in C_1$ and $\sigma_n(e) \in C_2$. It follows that all of the oriented edges of the simple closed curve $\alpha_1\sigma_n(\bar{\alpha}_1)$ are in the simple closed curve C. Hence $C = \alpha_1\sigma_n(\bar{\alpha}_1)$. $\square$

Lemma 3.6. *Let $\tilde{\Gamma}$ be the underlying graph of a particular marked graph in L_{σ_n}. Hence $\tilde{\Gamma}$ comes equipped with a σ_n-action. Choose an edge e in $\tilde{\Gamma}$. Among all simple closed curves D which pass through e, choose one curve C for which the distance from the curve to the basepoint $*$ is minimal. Then $Fix_{\sigma_n}(C)$ is two points v_1 and v_2. One of these two points in the closest point in C to the basepoint $*$ and one of them is the farthest point in C to the basepoint $*$.*

Proof. Using the notation of the proof of Lemma 3.5, C is the result of following some path α_1 in $\tilde{T}_1$ and then $\bar{\alpha}_2 = \sigma_n(\bar{\alpha}_1)$ in $\tilde{T}_2$, where α_1 goes from the attaching point v_1 to the attaching point v_2.

Since α_1 is a path in a pointed tree, there is a unique vertex w in α_1 which is closest to $*$. By way of contradiction, suppose that $w \notin \{v_1, v_2\}$. Then w is not an attaching point. Let β_1 be the unique taut path in $\tilde{T}_1$ from $*$ to w. Let γ_1 be the unique subpath of β_1 which contains w and exactly one attaching point y. Now let δ_1 be the path in $\tilde{T}_1$ which starts at y, follows γ_1 along to w, and then either follows C from w to v_2 or $-C$ from w to v_1 (where we choose whichever possibility insures that $\pm e \in \delta_1$.) Then the simple closed curve $\delta_1\sigma_n(\bar{\delta}_1)$ is closer to $*$ than C is, which is a contradiction. Hence w is v_1 or v_2, and the lemma follows. □

Proposition 3.7. *Let $\tilde{\Gamma}$ be a graph which occurs as an underlying graph of marked graphs in L_{σ_n}. Then there is only one possible σ_n-action on $\tilde{\Gamma}$.*

Proof. We see that our task is to show that a unique σ_n-action is determined by the properties about simple closed curves listed in Lemmas 3.5 and 3.6.

Define an action η on $\tilde{\Gamma}$ as follows. Let e be an oriented edge of $\tilde{\Gamma}$. Among all simple closed curves D which pass through e, choose one path C for which the distance from the curve to the basepoint $*$ is minimal. Let v_1 be a point on C which is closest to the basepoint. Let n be the edge-path distance in C from v_1 to e. Then there is an orientation $\epsilon \in \{-1, 1\}$ such that if you traverse ϵC starting at v_1 and go n edges, you get to e. Define $\eta(e)$ to be the result of traversing $-\epsilon C$ starting at v_1 and then going n more edges.

By Lemma 3.5 and 3.6, the action η is well defined and if any σ_n acts on $\tilde{\Gamma}$, then the σ_n-action and the η-action coincide. □

Denote by Q_{σ_n} the quotient space $L_{\sigma_n}/\Pi A(F_n)$.

Corollary 3.8. *If two marked graphs in L_{σ_n} have the same underlying graph, then they correspond to the same vertex in Q_{σ_n}. That is, the moduli space Q_{σ_n} can be formed by looking only at the poset structure of the underlying graphs of marked graphs in L_{σ_n}.*

Proof. From Proposition 3.7, any underlying graph of a marked graph in L_{σ_n} has only one possible σ_n-action. But from Corollary 10.4 of [21], $\Pi A(F_n)$ acts transitively on the set of marked σ_n-graphs based on the same σ_n-graph. The result follows. □

The simplices of L_{σ_n} group themselves into cubes, as described in §3 of [15]. In [13], Hatcher and Vogtmann show that the quotients in Q_n of cubes in X_n have the rational homology of balls. They use this to create a cubical chain complex which has the same rational homology as Q_n. Our goal here is to establish a similar result for the cubes of maximal dimension in Q_{σ_n}, where this time we want the quotients of cubes to have the $\mathbb{Z}_{(p)}$-cohomology of balls, where p is any odd prime.

Following [14], we consider a maximal cube in L_{σ_n}. It is given by considering a maximal essential marked graph $\phi : R_n \to \Gamma_T$ and considering some maximal subforest S of T. Recall that by a subforest of T we mean a subset S of the edges of T where there is no path in S from one valence 1 vertex of T (or equivalently, one terminal edge of T) to another. From part a) of Theorem 1.1, S has $n-1$ edges in it. The cube corresponding to the pair (T, S) can thus be thought of as imbedded in $\mathbb{R}^{n-1}$, where each coordinate vector is an edge of the cube, the graph obtained by collapsing each edge of $S \cup \sigma_n(S)$ is at the origin, and Γ_T is at $(1, 1, \dots, 1)$. Let $Aut(T, S)$ be the group of all (pointed) automorphisms of the tree T which take S to S. The group $stab_{\Pi A(F_n)}(T, S) = \langle\sigma_n\rangle \times Aut(T, S)$ acts linearly on the cube by permuting the coordinates of $\mathbb{R}^{n-1}$, and fixes the diagonal from $(0, 0, \dots, 0)$ to $(1, 1, \dots, 1)$. (The involution σ_n acts trivially on the cube, of course, since all of these cubes are coming from $L_{\sigma_n} \subset X_n^{\sigma_n}$.) Hence, just as in [15], the quotient of the cube in Q_{σ_n} is a cone with base $S^{n-2}/Aut(T, S)$, where S^{n-2} is the boundary of the cube.

Lemma 3.9. *The finite group $Aut(T)$ (and hence its subgroup $Aut(T, S)$) is all 2-torsion.*

Proof. Let $\xi \in Aut(T)$. Now ξ must take the basepoint to the basepoint, and so it must take the unique edge attached to the basepoint to itself. For each n, let E_n be the edges in T which are at most distance n from $*$. So E_0 is just one edge, and ξ fixes it as already mentioned. Since all nonterminal vertices of T have valence 3, an inductive argument yields that ξ^{2^n} fixes E_n pointwise. □

The following is an analog of Proposition 3.1 of [15]:

Proposition 3.10. *$S^{n-2}/Aut(T, S)$ has the $\mathbb{Z}_{(p)}$-cohomology of an $(n-2)$-sphere or a ball. The latter possibility happens when there is an element of $Aut(T, S)$ which induces an odd permutation of the edges of S.*

Proof. The finite group $Aut(T, S)$, which is all 2-torsion, acts cellularly on S^{n-2}, where the stabilizer of a cell fixes it pointwise. We use the spectral sequence for equivariant cohomology (cf. [2] VII §7):

$$E_1^{r,s} = \prod_{[\delta]\in\Delta_n^r} H^s(stab(\delta); \mathbb{Z}_{(p)}) \Rightarrow H^{r+s}_{Aut(T,S)}(S^{n-2}; \mathbb{Z}_{(p)}) \tag{3.11}$$

where $[\delta]$ ranges over the set Δ^r of orbits of r-simplices δ in S^{n-2}. Since $Aut(T, S)$ is all 2-torsion and finite, so are all of the $stab(\delta)$. Hence if $s > 0$, $H^s(stab(\delta); \mathbb{Z}_{(p)}) = 0$. So the above spectral sequence converges to $H^r(S^{n-2}/Aut(T, S); \mathbb{Z}_{(p)})$.

But another filtration yields a spectral sequence with

$$E_2^{r,s} = H^r(Aut(T, S); H^s(S^{n-2}; \mathbb{Z}_{(p)})) \Rightarrow H^{r+s}_{Aut(T,S)}(S^{n-2}; \mathbb{Z}_{(p)}) \tag{3.12}$$

It follows that $E_2^{r,s} = 0$ unless (r,s) is $(0,0)$ or $(0,n-2)$. Hence $E_2^{0,0} = \mathbb{Z}_{(p)}$ and $E_2^{0,n-2} = H^{n-2}(S^{n-2};\mathbb{Z}_{(p)})^{Aut(T,S)}$. The latter group of invariants is $\mathbb{Z}_{(p)}$ if the action of $Aut(T,S)$ on S^{n-2} preserves orientation and 0 otherwise. The last assertion in the proposition follows from Corollary 3.2 of [15]. □

Theorem 3.13. *The top dimensional cohomology group of Q_{σ_n} vanishes. That is, $H^{n-1}(Q_{\sigma_n};\mathbb{Z}_{(p)}) = 0$.*

Proof. We show that the quotient of every maximal cube (T,S) has a free face, so that the interior of the quotient of the cube can be collapsed away. If we can do this, then Q_{σ_n} will have the same $\mathbb{Z}_{(p)}$-cohomology as an $(n-2)$-dimensional complex, and we will be done.

In the degenerate case where there is an element of $Aut(T,S)$ which induces an odd permutation of the edges of S, then the quotient of the cube (T,S) is not itself a cube. In this case, the diagonal from $(0,\dots,0)$ to $(1,\dots,1)$ is exposed in the quotient, and any $(n-2)$-dimensional simplex in the quotient which lies next to the diagonal is a free face.

In the nondegenerate case, the quotient of the cube (T,S) is itself a cube, although its boundary might be self indentified in various ways. Since the subforest S of T is maximal, S must contain at least one terminal edge e. That is, one of the two vertices of e is a valence 1 vertex or attaching point. Let $\tilde{\Gamma}$ be the graph obtained from $\Gamma = \Gamma_T$ by collapsing the subforest $\{e, \sigma_n(e)\}$. The graph $\tilde{\Gamma}$ has a maximal subforest corresponding to collapsing the edges e and $\sigma_n(e)$ from the forest $S \cup \sigma_n(S)$ of Γ. Hence we see that collapsing e gives us a face, which we will denote by $(T/e, S/e)$, of the cube (T,S). It can be shown that this face corresponds to a (nondegenerate, cubical) face of the quotient of the cube (T,S) because

Claim 3.14. *There is an natural injection of $Aut(\tilde{\Gamma}) = \langle\sigma_n\rangle \times Aut(T/e)$ into $Aut(\Gamma) = \langle\sigma_n\rangle \times Aut(T)$. Define the lift $\hat{\phi}$ of an automorphism $\phi \in Aut(T/e)$ by sending an edge f to $\phi(f)$ if $f \neq e$ and letting $\hat{\phi}(e) = e$.*

Proof. Denote by v the valence 1 vertex of $e \in T$ (the attaching point) and let w be the other vertex of e. In T/e, $w = v$. We must show that ϕ sends w to w. This follows automatically, however, as $w = v = \sigma_n(w) = \sigma_n(v)$ is the only valence 4 vertex of $\tilde{\Gamma}$ and so any automorphism of the graph must fix it. Let f and g be the two other edges in T which share the vertex w. Now if $v = *$ then ϕ could possibly exchange f and g, but this is fine as the lift $\hat{\phi}$ also can. If $v \neq *$, then one of f or g must be closer to the basepoint, and so ϕ must fix both f and g. Regardless, $\hat{\phi}$ can be defined as in the statement of the claim. □

Warning: Note that if e is not a terminal edge, the above claim is false. Collapsing an interior edge somtimes allows you to construct automorphisms with 3-torsion, which obviously cannot be lifted to $Aut(T)$.

No automorphism ϕ of $(T/e, S/e)$ can induce an odd permutation of the edges in S/e, else the lift $\hat{\phi}$ of ϕ to T would induce an odd permutation of the edge of S. Since $Aut(T)$ is all 2-torsion, it follows from the above claim that $Aut(T/e)$ is also all 2-torsion. Hence the same spectral sequence argument used in Proposition 3.10 yields that the quotient of the cube corresponding to $(T/e, S/e)$ actually is a $\mathbb{Z}_{(p)}$-cohomology cube.

It remains to be shown that the cubical face corresponding to $(T/e, S/e)$ is free. First, if another subforest S' with an edge e' of T gives a cube with a face isomorphic to $(S/e, S/e)$, then e' must also be a terminal edge of S'. Hence the isomorphism $(T/e, S/e) \to (T/e', S'/e')$ maps the vertex that e collapsed into to the the vertex that e' collapsed into, and so we can lift the isomorphism to one from $(T, S) \to (T, S')$.

Second, we must show that blowing up the vertex w in $\tilde{\Gamma}$ only yields graphs isomorphic to Γ. This follows by considering the ways that the vertex w in $\tilde{\Gamma}$ can be blown up. Say that the edges f, g, $\sigma_n(f)$, and $\sigma_n(g)$ are the ones incident to w. If the ideal edge orbit $\sigma_n\{f, g\}$ is blown up, we get back Γ exactly, and if $\sigma_n\{f, \sigma_n(g)\}$ is blown up, we get a graph isomorphic to Γ. As these are the only ways to blow up the graph σ_n-equivariantly into another essential graph, we are done. $\square$

Corollary 3.15. $H^{n-1}(Q_{\sigma_n}; \mathbb{Q}) = H^{n-1}(\Pi A(F_n); \mathbb{Q}) = 0$.

Proof. That $H^{n-1}(Q_{\sigma_n}; \mathbb{Q}) = 0$ follows immediately from Theorem 3.13. Recall that $\Pi A(F_n)$ acts with finite stabilizers and finite quotient Q_{σ_n} on the contractible space L_{σ_n}. Since the stabilizers are finite, their rational cohomology vanishes, and the standard equivariant spectral sequence yields that $H^*(Q_{\sigma_n}; \mathbb{Q}) = H^*(\Pi A(F_n); \mathbb{Q})$. $\square$

Note that part d) of Theorem 1.1 follows from the above Corollary.

As a final remark for this section, we show that L_{σ_n} is an $\underline{E}\Pi A(F_n)$ (cf. [19]); that is, for finite subgroups G of $\Pi A(F_n)$, the fixed point subcomplex $L_{\sigma_n}^G$ is contractible. This follows directly from the corresponding property of $Aut(F_n)$. The following proposition is unneccesary in the specific case of L_{σ_n}, since (proof omitted) L_{σ_n} actually equals $X_n^{\sigma_n}$. This does not normally happen (for example, the spaces $L_{P_n \times \sigma_n}$ mentioned later in Fact 5.4 are not equal to the corresponding fixed point space of X_n), however, and thus it seems worth noting the more general fact.

Proposition 3.16. *Let S be a finite subgroup of $Aut(F_n)$ and let $\mathcal{S}$ be either $C_{Aut(F_n)}(S)$ or $N_{Aut(F_n)}(S)$. Let L_S be the retract, defined by Krstic and Vogtmann and consisting of essential marked graphs, of the fixed point subcomplex X_n^S of the spine of auter space X_n. Then L_S is an $\underline{E}\mathcal{S}$ space.*

Sketch of Proof. Let H be a finite subgroup of $\mathcal{S}$ and let G be the (finite, because $HSH^{-1} = S$) subgroup generated by H and S. Then $X_n^G = (X_n^S)^H = (X_n^H)^S$, and X_n^G is contractible from [17]. It remains to be shown that $X_n^G = (X_n^S)^H$ deformation retracts to $(L_S)^H$.

Given a marked graph Γ representing a vertex of $(X_n^S)^H$, we must show (see Proposition 3.3 of [21]) that for every edge e in Γ and every $h \in H$, e is S-inessential if and only if he is S-inessential. This follows automatically from Corollary 4.5 of [21], which characterizes essential edges by looking at the stabilizers (in S) of paths in Γ. Since $HSH^{-1} = H$, the stabilizers in h-translates of such paths are still in S and are isomorphic (conjugate by h) to those of the original path. □

4. A realization proposition

Let $\hat{A}$ be a finite subgroup of $\Pi A(F_n)$ and let A be the (finite) subgroup generated by $\hat{A}$ and σ_n. By Zimmerman's [22] realization theorem, we can realize A by an action on an A-reduced graph Γ. From the proposition below, Γ is also $\langle\sigma_n\rangle$-reduced; that is, Γ is a θ_1-tree.

Note that the corresponding statement is not true in $Out(F_n)$ (have $\mathbb{Z}/p \times \langle\sigma_{p-1}\rangle$ act on a θ-graph θ_{p-1}) and certainly would not be true in $Aut(F_n)$ if the σ_n-action were replaced by some other $\mathbb{Z}/2$-action.

Proposition 4.1. *Let $A \subseteq \Pi A(F_n)$ be a finite subgroup of the palindromic automorphism group with $\sigma_n \in A$. Realize A by an action on an A-reduced marked graph $\phi : R_n \to \Gamma$. Then $\phi : R_n \to \Gamma$ is also a $\langle\sigma_n\rangle$-reduced marked graph.*

Proof. As before, let $F_n = \langle a_1, \dots, a_n\rangle$ and identify the petals of the rose R_n with the generators a_i. Note that Γ has no separating edges, else it would not be A-reduced. In this proof, when we refer to concepts such as the number of times an edge e of Γ occurs in some $\phi(a_i)$, we mean that we should take the unique taut path in Γ, starting and ending at $*$, which is homotopic to the path $\phi(a_i)$ in Γ, and then count the number of times e occurs in this taut path. By way of contradiction, suppose Γ is not $\langle\sigma_n\rangle$-reduced. Let $e_1 \in \Gamma$ be an edge of minimal distance to the basepoint $*$ such that $\{e_1, \sigma_n e_1\}$ is a forest.

CASE 1: $e_1 = \sigma_n e_1$. Since e_1 is not a separating edge of Γ, we can choose a nontrivial cycle μ, starting and ending at $*$, which has just one occurence of e_1 and none of e_1^{-1}. If for all $i = 1, \dots, n$, the cycles $\phi(a_i)$ have an even number of occurences of $e_1^{\pm 1}$, then we could not write μ as a product of them and their inverses. So some $\phi(a_j)$ has an odd number of occurences of $e_i^{\pm 1}$ in it. Say that the exponent sum of e_1 in $\phi(a_j)$ is k, k odd. Then the exponent sum of e_1 in $\sigma_n\phi(a_j)$ is still k, but the exponent sum of $\phi(a_j^{-1})$ is $-k$. This contradicts the fact that $\sigma_n a_j = a_j^{-1}$.

CASE 2: $e_1 \neq \sigma_n e_1$. Let α be a shortest length path from $*$ to e_1. Say without loss of generality that e_1 is the oriented edge from v to w and that α goes from $*$ to v. Let $f_1 = \sigma_n e_1$. Then $\sigma_n \alpha$ is a shortest length path from $*$ to f_1. Now $v = \sigma_n v$, else we could write $\alpha = \beta b$ and get $\{b, \sigma_n b\}$ as a σ_n-invariant forest closer to $*$ than $\{e_1, f_1\}$ is. (If $|\alpha| = 0$, then $v = *$ and so $\sigma_n(v) = v$ necessarily.) So we have both α and $\sigma_n \alpha$ are paths from $*$ to v Moreover, $w \neq \sigma_n w$ (else $\{e_1, \sigma_n e_1\}$ is not a forest.) Now $Ae_1 = A\{e_1, f_1\}$ is not a forest, since Γ is A-reduced. Hence we can choose some simple closed curve μ in $Ae_1 \subseteq \Gamma$ that contains e_1. There must exist some $ae_1^{\pm 1} \in \mu$, $ae_1^{\pm 1} \notin \{e_1, e_1^{-1}\}$, such that $aw = w$. Why? Otherwise we could deformation retract μ to the set of vertices $\{\hat{a}v : \hat{a}e_1^{\pm 1} \in \mu\}$, which contradicts the fact that μ is a simple closed curve. Now $ae_1 \neq f_1$, as $\sigma_n w \neq w$. Hence $\alpha e_1 (ae_1)^{-1} (a\alpha)^{-1}$ is a nontrivial cycle starting and ending at $*$ which contains exactly one occurence of e_1 and none of f_1. So there must be a $\phi(a_j^\epsilon)$, $\epsilon \in \{-1, 1\}$, which contains an odd number of occurences of $e_1^{\pm 1}$ and an even number of occurences of $f_1^{\pm 1}$. (If we had some even/odd $\phi(a_j)$, then we could act by σ_n to get odd/even, and this would be a $\phi(a_j^{-1})$. Otherwise, all $\phi(a_i)$ are all even/even or odd/odd, and so combine together just to get more even/even or odd/odd loops.) This is a contradiction, however, because $\phi(a_j^{-\epsilon})$ still has an odd number of occurences of $e_1^{\pm 1}$ while $\phi(\sigma_n a_j^\epsilon)$ has an even number of occurences of $e_1^{\pm 1}$. □

Proof of part b) (i) of Theorem 1.1. From the action of $(\mathbb{Z}/2)^n$ on the rose R_n such that the ith generator inverts the ith petal and leaves all others fixed, we know that the Krull dimension at the prime 2 is at least n. Similarly, there is an action of $(\mathbb{Z}/p)^{\left[\frac{n}{p}\right]}$ on R_n where the first $\mathbb{Z}/p$ rotates the first p petals, the second $\mathbb{Z}/p$ rotates the next p petals, etc. Hence the Krull dimension at the prime p is at least $\left[\frac{n}{p}\right]$.

Let A be a maximal rank elementary abelian subgroup of $\Pi A(F_n)$. From Proposition 4.1, we can realize A by an action of A on a σ_n-graph Γ which is both A-reduced and σ_n-reduced. That is, we have an action of A on a pointed Θ_1-tree Γ. Since elements of A must preserve basepoints, the action of A on the tree Γ/σ_n does not invert edges. Hence we have inclusions

$$A \rightarrowtail (\mathbb{Z}/2)^n \rtimes Aut_*(\Gamma/\sigma_n) \rightarrowtail (\mathbb{Z}/2)^n \rtimes \Sigma_n = \mathbb{Z}/2 \wr \Sigma_n.$$

The result (for 2 or odd primes p) now follows from standard facts about Σ_n (cf. Theorem 1.3 in Chapter VI of [1].) □

Proof of part c) of Theorem 1.1. We sketch the proof, which uses standard methods. Suppose that some $A = \mathbb{Z}/p$ lies in $E\Pi A(F_n)$. From Proposition 4.1, we can realize A by an action of A on a σ_n-graph Γ which is both A-reduced and σ_n-reduced. Let $\phi : R_n \to \Gamma$ be the corresponding marked graph. Let $T = \Gamma/\sigma_n$, a pointed tree with an A-action on it. First, suppose that the A-action on T is nontrivial. (This will always be the case if p is odd.) Then there are two edges e_1 and e_2, both oriented so that their terminal vertices are closer to the basepoint than their initial vertices, of Γ such that a generator of A rotates the edge $[e_1]$ into the edge $[e_2]$ in T. Some generator a_{i_1} of F_n must be such that $\phi(a_{i_1})$ contains an odd number of occurences of e_1 in it. Choose a_{i_2} similarly. Then $\phi(a_{i_j})$ is a palindromic word in the edges of Γ with either $\bar{e}_{i_j}\sigma_n(e_{i_j})$ or $\sigma_n(\bar{e}_{i_j})e_{i_j}$ in the middle of the palindrome. The generator of A (thought of as an element of $E\Pi A(F_n)$) must send a_{i_1} to a palindrome with either a_{i_2} or $a_{i_2}^{-1}$ in the center of it. This contradicts the fact that all elements of $E\Pi A(F_n)$ send generators a_i to palidromes with a_i in the center of them.

The only remaining case is where $p = 2$ and A acts trivally on T. So A is a subgroup of the group $(\mathbb{Z}/2)^n$ of graph automorphisms of Γ which act by inverting the Θ_1's in the Θ_1-tree Γ. Hence the generator of A corresponds to an element of $\Pi A(F_n)$ which, for at least one i, sends a_i to a palindrome with a_i^{-1} in its center. As none of these automorphisms are in $E\Pi A(F_n)$, we again have a contradiction. $\square$

5. Cohomology of $\Pi A(F_n)$ at odd primes p

Let p be an odd prime (as will always be the case from now on in this paper.) We wish to calculate the Farrell cohomology of $\Pi A(F_n)$ using Ken Brown's [2] normalizer spectral sequence, which states that

$$E_1^{r,s} = \prod_{(P_0\subset\cdots\subset P_r)\in|\mathcal{B}|_r} \hat{H}^s\Big(\bigcap_{i=0}^{r} N_G(P_i);\mathbb{Z}_{(p)}\Big) \Rightarrow \hat{H}^{r+s}(G;\mathbb{Z}_{(p)}) \tag{5.1}$$

where G is a group with finite virtual cohomological dimension, $\mathcal{A}$ is the poset of nontrivial elementary abelian p-subgroups of G, $\mathcal{B}$ is the poset of conjugacy classes of nontrivial elementary abelian p-subgroups of G, and $|\mathcal{B}|_r$ is the set of r-simplices in the realization $|\mathcal{B}|$.

A first step toward performing such a calculation is calculating $|\mathcal{B}|$. In other words, we wish to calculate conjugacy classes of elementary abelian subgroups $P \subset \Pi A(F_n)$. By Proposition 4.1, we can realize such finite groups P by reduced actions on θ_1-trees.

If $n \geq p$, define a particular subgroup $P_n \cong \mathbb{Z}/p$ of $\Pi A(F_n)$ by letting P_n act on the rose R_n by rotating its first p leaves and leaving the last $n-p$ leaves fixed. That is, P_n corresponds to automorphisms which rotate the first p generators $a_1, \ldots, a_p$ and leave the remaining generators fixed.

Corollary 5.2. *If $p \leq n \leq 2p-1$, then*

$$\hat{H}^*(\Pi A(F_n);\mathbb{Z}_{(p)}) \cong \hat{H}^*(N_{Aut(F_n)}(P_n \times \langle\sigma_n\rangle);\mathbb{Z}_{(p)}).$$

Proof. We show that P_n is the only conjugacy class of nontrivial elementary abelian p-subgroups that is in $\Pi A(F_n)$. By Proposition 4.1, we see that an arbitrary nontrivial elementary abelian p-subgroup A comes from some action on a θ_1-tree with p-symmetry. Since $p \leq n \leq 2p-1$, the only possibility is that A acts on a θ_1-tree Γ by rotating p of the θ_1-leaves and leaving the other $n-p$ θ_1-edges in the tree fixed. But it is clear that a product of $(P_n \times \langle\sigma_n\rangle)$-Nielsen transformations takes the rose R_n to the graph Γ, and hence we see that A and P_n are conjugate to each other in $\Pi A(F_n)$.

By the normalizer spectral sequence 5.1, this yields that

$$\hat{H}^*(\Pi A(F_n);\mathbb{Z}_{(p)}) \cong \hat{H}^*(N_{\Pi A(F_n)}(P_n);\mathbb{Z}_{(p)}).$$

But since p is an odd prime, it is easy to see that

$$N_{\Pi A(F_n)}(P_n) = N_{Aut(F_n)}(P_n \times \langle\sigma_n\rangle).$$

□

Proposition 5.3.

$$N_{Aut(F_n)}(P_n \times \langle\sigma_n\rangle) \cong N_{\Sigma_p}(P_n) \times (F_m \rtimes (\langle\sigma_p\rangle \times \Pi A(F_m)))$$

where $m = n-p$, $\Pi A(F_m)$ acts on the F_m in the semidirect product in the natural way, and σ_p acts on F_m as σ_m does.

Proof. The $N_{\Sigma_p}(P_n)$ in the above decomposition comes from automorphisms of F_n which permute the first p generators and leave the remaining m fixed. The F_m being acted upon in the semidirect product structure above has ith generator $(a_1||a_{p+i})(a_2||a_{p+i})\dots(a_p||a_{p+i})$. The σ_p is the involution which inverts the first p generators of F_n and leaves the remaining m fixed. Finally, the $\Pi A(F_m)$ comes from automorphisms which fix the first p generators of F_n and act on the last m generators by identifying the subgroup $\langle a_{p+1}, a_{p+2}, \dots, a_n\rangle$ with F_m.

Consider the action of $P_n \times \langle\sigma_n\rangle$ on the rose R_n. P_n rotates the first p petals. Label the first p petals of the rose as $a_1, \dots, a_p$ as before, but label the last m petals as $b_1, \dots, b_m$.

Since $|P_n| = p$ is an odd prime, $N_{Aut(F_n)}(P_n \times \langle\sigma_n\rangle) \subseteq N_{Aut(F_n)}(P_n)$ and in Lemma 5.1 of [16], we calculated

$$N_{Aut(F_n)}(P_n) \cong N_{\Sigma_p}(P_n) \times ((F_m \times F_m) \rtimes (\langle\sigma_p\rangle \times Aut(F_m))),$$

where the first F_m in $F_m \times F_m$ is the free group on the P_n-Nielsen transformations $\langle a_1, b_i^{-1}\rangle$ for $i \in \{1, \dots, m\}$ and the latter F_m is the free group on the P_n-Nielsen transformations $\langle a_1^{-1}, b_i^{-1}\rangle$, $i \in \{1, \dots, m\}$. Note that $\langle\sigma_p\rangle$ acts on $F_m \times F_m$ via $\sigma_p\langle a_1, b_i^{-1}\rangle\sigma_p = \langle a_1^{-1}, b_i^{-1}\rangle$ and $\sigma_p\langle a_1^{-1}, b_i^{-1}\rangle\sigma_p = \langle a_1, b_i^{-1}\rangle$. In other words, if $(b,c) \in F_m \times F_m$ then $\sigma_p(b,c)\sigma_p = (c,b)$.

Let G be the subgroup

$$N_{\Sigma_p}(P_n) \times (F_m \rtimes (\langle \sigma_p \rangle \times C_{Aut(F_m)}(\sigma_m)))$$

of $N_{Aut(F_n)}(P_n)$, where F_m is the free group on the generators $\langle a_1, b_i \rangle \circ \langle a_1^{-1}, b_i^{-1} \rangle$ for $i \in \{1, \ldots, m\}$, and $C_{Aut(F_m)}(\sigma_m)$ is included in $Aut(F_m)$ in the obvious way. It follows directly that $G \subseteq N_{Aut(F_n)}(P_n \times \langle \sigma_n \rangle)$. To prove the proposition, we must show that they are equal.

Take an arbitrary

$$x \in N_{Aut(F_n)}(P_n \times \langle \sigma_n \rangle) \subseteq N_{\Sigma_p}(P_n) \times ((F_m \times F_m) \rtimes (\langle \sigma_p \rangle \times Aut(F_m))).$$

Say $x = abcde$, where $a \in N_{\Sigma_p}(P_n)$, $(b, c) \in F_m \times F_m$, $d \in \langle \sigma_p \rangle$, and $e \in Aut(F_m)$. Since $a, d \in N_{Aut(F_n)}(P_n \times \langle \sigma_n \rangle)$, $a^{-1}xd^{-1} = bce \in N_{Aut(F_n)}(P_n \times \langle \sigma_n \rangle)$ also. So $bce \in \Pi A(F_n)$ and $(bce)\sigma_n(bce)^{-1} = \sigma_n$. This means that the map $(bce)\sigma_n(bce)^{-1}$ sends a_i to a_i^{-1} for $i \in \{1, \ldots, p\}$ and b_i to b_i^{-1} for $i \in \{1, \ldots, m\}$. Now both σ_n and e restrict to maps in $Aut(\langle b_1, \ldots, b_m \rangle)$ and moreover b and c both restrict to the identity map in $Aut(\langle b_1, \ldots, b_m \rangle)$. Hence for $i \in \{1, \ldots, m\}$, we have

$$b_i^{-1} = (bce)\sigma_n(bce)^{-1}(b_i) = e\sigma_n e^{-1}(b_i),$$

and we see that $e\sigma_n e^{-1}$ restricts to σ_m in $Aut(F_m)$. As $e \in Aut(F_m)$, this means $e \in C_{Aut(F_m)}(\sigma_m)$. Hence $e \in \Pi A(F_n)$ also. Since $bce \in \Pi A(F_n)$, this gives $bc \in \Pi A(F_n)$. In other words, we have

$$(b, c) \in (F_m \times F_m) \subseteq N_{\Sigma_p}(P_n) \times ((F_m \times F_m) \rtimes (\langle \sigma_p \rangle \times Aut(F_m)))$$

and

$$(b, c) \in \Pi A(F_n).$$

It follows that

$$\begin{aligned} (b, c) &= \sigma_n(b, c)\sigma_n \\ &= \sigma_m\sigma_p(b, c)\sigma_p\sigma_m \\ &= \sigma_m(c, b)\sigma_m \\ &= (\sigma_m(c), \sigma_m(b)). \end{aligned}$$

So $b = \sigma_m(c)$ and $c = \sigma_m(b)$. In summary, we have shown that an arbitrary element $x = abcde \in N_{Aut(F_n)}(P_n \times \langle \sigma_n \rangle)$ has $c = \sigma_m(b)$ and $e \in C_{Aut(F_m)}(\sigma_m)$. Thus $x \in G$, as desired. □

The group $N_{Aut(F_n)}(P_n \times \langle \sigma_n \rangle)$ acts on the contractible space $L_{P_n \times \langle \sigma_n \rangle}$ with finite stabilizers and finite quotient $Q_{P_n \times \langle \sigma_n \rangle} = L_{P_n \times \langle \sigma_n \rangle} / N_{Aut(F_n)}(P_n \times \langle \sigma_n \rangle)$.

Define a *p-admissible tree* T to be a triple $(T, \circ, A)$ where T is a pointed tree, $\circ$ is a vertex of T (which may be the basepoint $*$), A is a subset of the vertices of T called the *set of attaching points*, $* \in A$, and all valence 1 vertices of T are in A. For a p-admissible tree T, define the *corresponding graph* Γ_T as follows: Take two isomorphic copies T_1 and T_2 of the tree T, and let $f : T_1 \to T_2$ be an isomorphism. Then let Γ_T^{pre} be the graph

$$\Gamma_T^{pre} = \frac{T_1 \amalg T_2}{f(v) \sim v, \text{ for all attaching points } v \text{ in } A.}$$

Let θ_{p-1} be a θ-graph with p edges and two vertices v_1 and v_2. Let $\circ_1$ be the $\circ$-vertex in T_1 and let $\circ_2 = f(\circ_1)$ be the $\circ$-vertex in T_2. Finally, let

$$\Gamma_T = \frac{\Gamma_T^{pre} \amalg \theta_{p-1}}{\circ_1 \sim v_1, \circ_2 \sim v_2}$$

If $\pi_1(\Gamma_T) \cong F_n$, then say T is a p-admissible tree of *rank* n.

If T is a p-admissible tree of rank n, define a $\langle\sigma_n\rangle$-action on the edges of Γ_T by

$$\sigma_n x = \begin{cases} f(x), & x \in T_1 \\ f^{-1}(x), & x \in T_2 \\ x^{-1}, & x \in \theta_{p-1} \end{cases}$$

Since this action inverts the edges of the θ-graph in Γ_T, we then need to subdivide these edges so that the group acts without inversions. Next, define a P_n-action on Γ_T by having P_n fix Γ_T^{pre} and rotate the edges of θ_{p-1} cyclically. In this way, Γ_T is a $(P_n \times \langle\sigma_n\rangle)$-graph.

A p-admissible tree is T *reduced* if the corresponding $(P_n \times \langle\sigma_n\rangle)$-graph Γ_T is reduced; that is, if all vertices of T are attaching points. Similarly, a p-admissible tree T is a *maximal* if the attaching points of T are exactly its valence 1 vertices, the valence 2 vertices of T consist of just the point $\circ$, and T has no vertices with valence 4 or more. As before, a *subforest* of T is a collection of edges S of T such that there is no path in S from one attaching point to another. Lastly, isomorphisms of p-admissible trees must be graph isomorphisms which take $*$ to $*$, $\circ$ to $\circ$, and A to A.

The following facts about $(P_n \times \langle\sigma_n\rangle)$-graphs are all proven in similar ways to the analogous facts about σ_n-graphs.

Fact 5.4.

(1) *There is a bijective correspondence between reduced* p*- admissible trees of rank* n *and the underlying graphs of* $(P_n \times \langle\sigma_n\rangle)$*- reduced marked graphs, given by* $T \to \Gamma_T$.

(2) *There is a bijective correspondence between maximal* p*-admissible trees of rank* n *and the underlying graphs of maximal essential marked* $(P_n \times \langle\sigma_n\rangle)$*-graphs, given by* $T \to \Gamma_T$.

(3) *The virtual cohomological dimension of* $N_{Aut(F_n)}(P_n \times \langle\sigma_n\rangle)$ *is* $m = n - p$.

(4) *Let* Γ *be a graph which occurs as the underlying graph of a marked graph in* $L_{P_n\times\langle\sigma_n\rangle}$. *Then there is only one possible* σ_n*-action on* Γ.

(5) *If two marked graphs in* $L_{P_n\times\langle\sigma_n\rangle}$ *have underlying graphs which correspond to the same* p*-admissible tree, then they correspond to the same vertex in* $Q_{P_n\times\langle\sigma_n\rangle}$. *That is, we can form the moduli space* $Q_{P_n\times\langle\sigma_n\rangle}$ *by looking only at the poset structure of the* p*-admissible trees corresponding to marked graphs in* $L_{P_n\times\langle\sigma_n\rangle}$.

(6) *The top dimensional cohomology class of* $Q_{P_n\times\langle\sigma_n\rangle}$, *with coefficients in* $\mathbb{Z}_{(p)}$, *vanishes. That is,* $H^{n-p}(Q_{P_n\times\langle\sigma_n\rangle};\mathbb{Z}_{(p)}) = 0$.

(7) $H^{n-p}(Q_{P_n\times\langle\sigma_n\rangle};\mathbb{Q}) = H^{n-p}(N_{Aut(F_n)}(P_n \times \langle\sigma_n\rangle);\mathbb{Q}) = 0$.

Note that (4) and (5) above are a little bit different from their analogs Proposition 3.7 and Corollary 3.8. Basically, the underlying graphs Γ always have just one possible σ_n-action, as before, but it is conceivable (for example, if the graph contains two or more copies of θ_{p-1} inside it and we must decide which one P_n rotates) that it might have several possible P_n-actions. That is why we talk about p-admissible trees instead in (5), since the vertex $\circ$ in the tree determines where the p edges that P_n rotates are located.

Fact 5.4 allows us to show

Proposition 5.5. *If* $p \le n \le 2p-1$, *then*

$$\hat{H}^t(N_{Aut(F_n)}(P_n \times \langle\sigma_n\rangle);\mathbb{Z}_{(p)}) \cong \begin{cases} \mathbb{Z}/p & t \equiv 0 \ (mod\ 2(p-1)) \\ H^r(Q_{P_n\times\langle\sigma_n\rangle};\mathbb{Z}/p) & t \equiv r \ (mod\ 2(p-1)), \\ & 1 \le r \le n-p-1 \\ 0 & t \equiv r \ (mod\ 2(p-1)), \\ & n-p \le r \le 2p-3 \end{cases}$$

Proof. We use the equivariant cohomology spectral sequence for $N_{Aut(F_n)}(P_n \times \langle\sigma_n\rangle)$ acting on the contractible space $L_{P_n\times\langle\sigma_n\rangle}$ with finite stabilizers and finite quotient $Q_{P_n\times\langle\sigma_n\rangle}$. The equivariant cohomology spectral sequence for this action is

$$\begin{aligned} E_1^{r,s} = \textstyle\prod_{[\delta]\in\Delta_n^r} & \hat{H}^s(stab_{N_{Aut(F_n)}(P_n\times\langle\sigma_n\rangle)}(\delta);\mathbb{Z}_{(p)}) \\ & \Rightarrow \hat{H}^{r+s}(N_{Aut(F_n)}(P_n \times \langle\sigma_n\rangle));\mathbb{Z}_{(p)}) \end{aligned}$$

where $[\delta]$ ranges over the set Δ_n^r of orbits of r-simplices δ in $L_{P_n\times\langle\sigma_n\rangle}$.

From the decomposition

$$N_{Aut(F_n)}(P_n \times \langle\sigma_n\rangle) \cong N_{\Sigma_p}(P_n) \times (F_m \rtimes (\langle\sigma_p\rangle \times C_{Aut(F_m)}(\sigma_m)))$$

we see that $(F_m \rtimes (\langle\sigma_p\rangle \times C_{Aut(F_m)}(\sigma_m)))$ has p-rank 0. Since $N_{\Sigma_p}(P_n)$ acts trivially on marked graphs in $L_{P_n\times\langle\sigma_n\rangle}$ by permuting the edges of the θ-graph attached at $\circ$, it follows that for every simplex δ we have

$$\hat{H}^*(stab_{N_{Aut(F_n)}(P_n\times\langle\sigma_n\rangle)}(\delta);\mathbb{Z}_{(p)}) \cong \hat{H}^*(N_{\Sigma_p}(P_n);\mathbb{Z}_{(p)}) \cong \hat{H}^*(\Sigma_p;\mathbb{Z}_{(p)}).$$

The $E_1^{r,s}$-page of the spectral sequence is 0 in the rows where $s \neq k\cdot 2(p-1)$ and a copy of the cellular cochain complex with $\mathbb{Z}/p$-coefficients of the $(n-p)$-dimensional complex $Q_{P_n\times\langle\sigma_n\rangle}$ in rows $k\cdot 2(p-1)$. It follows that the E_2-page has the form:

$$E_2^{r,s} = \begin{cases} \mathbb{Z}/p & r=0 \text{ and } s=k\cdot 2(p-1) \\ H^r(Q_{P_n\times\langle\sigma_n\rangle};\mathbb{Z}/p) & 1\le r\le n-p \text{ and } s=k\cdot 2(p-1) \\ 0 & \text{otherwise} \end{cases}$$

Hence we see that the spectral sequence converges at the E_2-page.

That $H^{n-p}(Q_{P_n\times\langle\sigma_n\rangle};\mathbb{Z}/p)=0$ follows from part 6 of Fact 5.4 and universal coefficients. □

Note that the above proposition immediately proves part b) (ii) of Theorem 1.1.

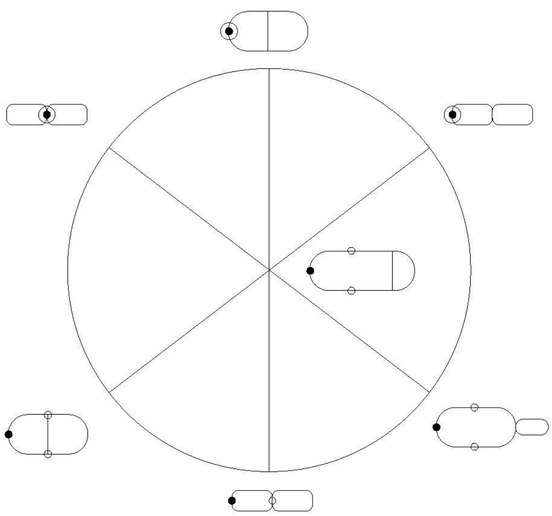

Figure 1. Simplices from the first maximal graph

By examining the space $Q_{P_n\times\langle\sigma_n\rangle}$ in low dimensions where $m\in\{0,1,2\}$ and showing that it is contractible, we have the following corollary, which will give us part e) of Theorem 1.1:

Corollary 5.6. *If* $m = n - p \in \{0, 1, 2\}$, *then*

$$\hat{H}^*(\Pi A(F_n); \mathbb{Z}_{(p)}) \cong \hat{H}^*(N_{Aut(F_n)}(P_n \times \langle \sigma_n \rangle); \mathbb{Z}_{(p)}) \cong \hat{H}^*(\Sigma_p; \mathbb{Z}_{(p)}).$$

Proof. *CASE 1:* $m = 0$. Then $Q_{P_n \times \langle \sigma_n \rangle}$ is a point.
CASE 2: $m = 1$. Then $Q_{P_n \times \langle \sigma_n \rangle}$ is a contractible 1-dimensional complex with 3 vertices and two edges. Define the maximal p-admissible tree T of rank n to be the tree with three vertices $*$, $\circ$, v and two edges e_1, e_2 where e_1 goes from $*$ to $\circ$ and e_2 goes from $\circ$ to v. The middle vertex of the 1-dimensional complex $Q_{P_n \times \langle \sigma_n \rangle}$ corresponds to the graph Γ_T. The other two vertices and two edges $Q_{P_n \times \langle \sigma_n \rangle}$ correspond to the two possible ways that Γ_T can be collapsed equivariantly.

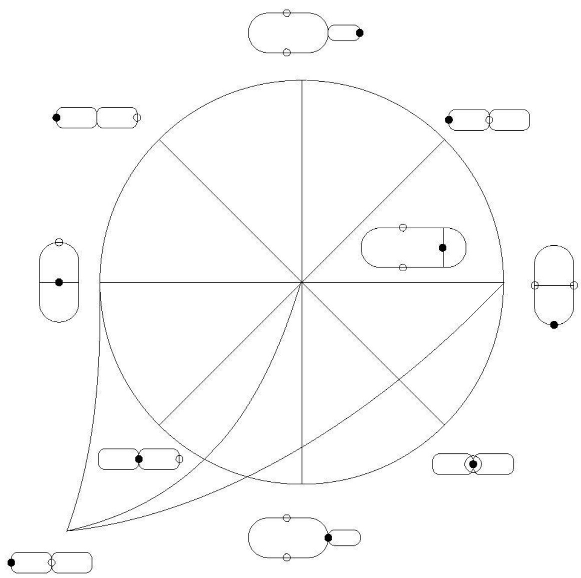

Figure 2. Simplices from the second maximal graph

CASE 3: m=2. Then $Q_{P_n \times \langle \sigma_n \rangle}$ is a 2-dimensional complex with 13 vertices, 28 edges, and 16 two-simplices. There are two maximal graphs in $Q_{P_n \times \langle \sigma_n \rangle}$. Simplices coming from the first graph are listed in figure 1 and simplices from the second graph are listed in figure 2. In figures 1 and 2, the maximal graphs are listed in the center. These maximal graphs can be collapsed in various ways, and these are listed around the periphery of the figures. In the graphs, a solid dot indicates the basepoint $*$ and the hollow dots represent attaching points $\circ$ for the θ-graph θ_{p-1}. If there is only one hollow dot in a graph, both ends of the θ-graph should be attached to that one vertex. Upon identifying the boundaries of the simplices listed in figures 1 and 2, we obtain the complex $Q_{P_n \times \langle \sigma_n \rangle}$ pictured in figure 3. The complex is homeomorphic to the fletching of a dart, three half disks, all identified along a common line in their boundary. This complex is clearly contractible. $\square$

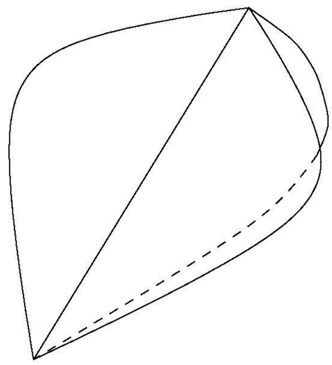

Figure 3. The complete complex $Q_{P_n \times \langle \sigma_n \rangle}$

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