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Commentarii Mathematici Helvetici

# Periodic ends, growth rates, Hölder dynamics for automorphisms of free groups

Gilbert Levitt and Martin Lustig

**Abstract.** Let $F_n$ be the free group of rank $n$, and $\partial F_n$ its boundary (or space of ends).

For any $\alpha \in \operatorname{Aut} F_n$, the homeomorphism $\partial\alpha$ induced by $\alpha$ on $\partial F_n$ has at least two periodic points of period $\leq 2n$. Periods of periodic points of $\partial\alpha$ are bounded above by a number $M_n$ depending only on $n$, with $\log M_n \sim \sqrt{n \log n}$ as $n \to +\infty$.

Using the canonical Hölder structure on $\partial F_n$, we associate an algebraic number $\lambda \geq 1$ to any attracting fixed point $X$ of $\partial\alpha$; if $\lambda > 1$, then for any $Y$ close to $X$ the sequence $\partial\alpha^p(Y)$ approaches $X$ at about the same speed as $e^{-\lambda p}$. This leads to a set of Hölder exponents $\Lambda_h(\Phi) \subset (1, +\infty)$ associated to any $\Phi \in \operatorname{Out} F_n$. This set coincides with the set of nontrivial exponential growth rates of conjugacy classes of $F_n$ under iteration of $\Phi$.

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**Keywords.** Free group, automorphism, boundary, periodic point, growth rate, Hölder.

## Introduction and statement of results

Let $\varphi$ be a homeomorphism of a closed surface $\Sigma$, with $\chi(\Sigma) < 0$. In [14], Nielsen studied $\varphi$ by lifting it to the universal covering $D$ of $\Sigma$ and considering the induced homeomorphism $f$ on the circle at infinity $S$. In more algebraic terms, the mapping class of $\varphi$ corresponds to an outer automorphism $\Phi$ of $\pi_1\Sigma$, various lifts of $\varphi$ to $D$ correspond to various automorphisms $\alpha$ of $\pi_1\Sigma$ representing $\Phi$, and $f : S \to S$ corresponds to the homeomorphism $\partial\alpha$ induced by $\alpha$ on the boundary of the group $\pi_1\Sigma$.

Let $F_n$ be the free group of rank $n$. We will study automorphisms $\alpha$ of $F_n$, and outer automorphisms $\Phi \in \operatorname{Out} F_n$, through the homeomorphisms $\partial\alpha$ induced on the boundary $\partial F_n$. The space $\partial F_n$, homeomorphic to a Cantor set if $n \geq 2$, may be viewed as the (Gromov) boundary of $F_n$, or its space of ends, or the set of right-infinite reduced words in the generators and their inverses.

In the case of a surface group, Nielsen proved among many other things that $f = \partial\alpha : S \to S$ always has at least two periodic points. Furthermore, the period of these points may be bounded in terms of $|\chi(\Sigma)|$.

Our first main result has a similar flavor.

**Theorem 1.** *Let $\alpha \in \operatorname{Aut} F_n$.*

*(1) The homeomorphism $\partial\alpha : \partial F_n \to \partial F_n$ has at least two periodic points of period $\leq 2n$. If it has only one orbit of periodic points, then this orbit has order two.*

*(2) Suppose $X \in \partial F_n$ is periodic of period $q$ under $\partial\alpha$. Then $q \leq M_n$, where $M_n$ depends only on $n$ and $\log M_n \sim \sqrt{n \log n}$ as $n \to \infty$.*

The bound $2n$ and the bound on $q$ are sharp. The quantity $\sqrt{n \log n}$ is asymptotic to the logarithm of the maximum order of torsion elements in $\operatorname{Aut} F_n$, see [11]. As a special case of assertion 2, there is a bound depending only on $n$ for periods of elements $g \in F_n$ under the action of $\alpha$. One may also establish a uniform bound for periods of conjugacy classes under the action of $\Phi \in \operatorname{Out} F_n$. It is proved in [9] that, for "most" $\alpha \in \operatorname{Aut} F_n$, the homeomorphism $\partial\alpha$ has exactly two fixed points, and no other periodic point.

Like many results of the present paper, the proof of Theorem 1 uses $\mathbf{R}$-trees and techniques introduced in [5]. The proof of assertion 2 uses the main result of [5], and Bestvina-Handel's bound [1] for the rank of the fixed subgroup (the "Scott conjecture").

Let us now consider local properties of fixed points of $\partial\alpha$, using the canonical Hölder structure on $\partial F_n$ (see [3, 7]). Let $X$ be a fixed point of $\partial\alpha$ not belonging to the limit set of the fixed subgroup $\operatorname{Fix}\alpha \subset F_n$. It is either attracting or repelling [5]. In the attracting case, we show that, for $Y \in \partial F_n$ close enough to $X$, the sequence $\partial\alpha^p(Y)$ converges to $X$ *super-exponentially* in the sense that

$$\lim_{p\to+\infty} \frac{1}{p} \log d(\partial\alpha^p(Y), X) = -\infty,$$

where $d$ is any distance on $\partial F_n$ defining the Hölder structure. We say that $X$ is *superattracting* (see the beginning of Section 4 for a detailed discussion).

**Theorem 2.** *Let $\alpha \in \operatorname{Aut} F_n$. Let $X \in \partial F_n$ be a superattracting fixed point of $\partial\alpha$. There exists an algebraic number $\lambda = \lambda(\alpha, X) \geq 1$ such that*

$$\lim_{p\to+\infty} \frac{1}{p} \log\Big(-\log d(\partial\alpha^p(Y), X)\Big) = \log\lambda$$

*for $Y \in \partial F_n$ close to $X$ (and $d$ a distance on $\partial F_n$ as above).*

Thus, when $\lambda > 1$, the sequence $\partial\alpha^p(Y)$ converges to $X$ at about the same speed as $f^p(x)$ approaches 0, where $f$ is the map $x \mapsto x^\lambda : [0,1) \to [0,1)$.

**Example.** Consider $\alpha : F_2 \to F_2$ given by $\alpha(a) = aba$, $\alpha(b) = ab$. The number associated to $X = \lim_{p\to+\infty} \alpha^p(a) = ababaaba\ldots$ is the Perron-Frobenius eigenvalue of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. On the other hand, for the polynomially growing

$\alpha : F_3 \to F_3$ given by $\alpha(a) = a$, $\alpha(b) = ba$, $\alpha(c) = cba$, the number associated to the superattracting point $X = \lim_{p\to+\infty} \alpha^p(c) = cbaba^2ba^3\ldots$ equals 1.

We now associate a canonical set of *Hölder exponents* $\Lambda_h(\Phi) \subset (1, +\infty)$ to any $\Phi \in \operatorname{Out} F_n$. View $\Phi$ as a collection of automorphisms $\alpha \in \operatorname{Aut} F_n$. We say that $\mu > 1$ belongs to $\Lambda_h(\Phi)$ if there exist $\beta \in \Phi^q$, with $q \geq 1$, and a superattracting fixed point $X$ of $\partial\beta$ with $\lambda(\beta, X) = \mu^q$. The set $\Lambda_h(\Phi)$ is a conjugacy invariant of $\Phi$.

**Example.** If $\Phi$ is induced by a homeomorphism $\varphi$ of a compact surface $\Sigma$ with $\pi_1\Sigma \simeq F_n$, then $\Lambda_h(\Phi)$ consists of (roots of) the expansion factors of the pseudo-Anosov pieces of $\varphi$. They are algebraic units.

If $\alpha \in \operatorname{Aut} F_3$ is given by $\alpha(a) = ab^{-1}$, $\alpha(b) = bac^{-1}$, $\alpha(c) = ca^{-3}$ (see [6, Example II.7]), then $\Lambda_h(\Phi)$ consists of the real root $\lambda$ of $x^3 - 3x^2 + 2x - 3$. Note that $\lambda$ is not an algebraic unit, and therefore cannot be read off the graph of groups constructed by Sela in Theorem 4.1 of [15].

**Theorem 3.** *Given $\Phi \in \operatorname{Out} F_n$, the set of Hölder exponents $\Lambda_h(\Phi)$ equals the set $\Lambda(\Phi)$ of nontrivial exponential growth rates of conjugacy classes of $F_n$ under iteration of $\Phi$.*

The *(exponential) growth rate* of a conjugacy class $\gamma$ is $\lambda(\gamma) = \lim_{p\to+\infty} |\Phi^p(\gamma)|^{1/p}$ (see Proposition 3.3). It is nontrivial if $\lambda(\gamma) > 1$. It will be shown in [10] that $\Lambda(\Phi)$ has at most $[\frac{3n-2}{4}]$ elements and consists of certain Perron-Frobenius eigenvalues of the transition matrix associated to a relative train track representative of $\Phi$.

This paper is organized as follows. In Section 1 we prove the existence of periodic points for $\partial\alpha$. The proof of Theorem 1 is completed in Section 2 (Theorems 2.1 and 2.3). In Section 3 we briefly discuss growth rates. We start Section 4 by a general discussion of superattractivity, valid for an arbitrary hyperbolic group. We then prove Theorem 2.

## 1. Existence of periodic points

Let $F_n$ be a free group. We consider its boundary $\partial F_n$, equipped with the natural action of $F_n$ by left-translations. It is a Cantor set if $n \geq 2$ (it consists of two points if $n = 1$). In section 4, we will view $\partial F_n$ as a set of right-infinite reduced words. A finitely generated subgroup $J \subset F_n$ is quasiconvex [16]. In particular, we can identify the boundary (or limit set) $\partial J$ with a subset of $\partial F_n$.

An automorphism $\alpha \in \operatorname{Aut} F_n$ is a quasi-isometry of $F_n$. It induces a homeomorphism $\partial\alpha : \partial F_n \to \partial F_n$, and a homeomorphism $\overline{\alpha} = \alpha \cup \partial\alpha$ on the compact space $\overline{F}_n = F_n \cup \partial F_n$.

The fixed subgroup $\operatorname{Fix}\alpha = \{g \in F_n \mid \alpha(g) = g\}$ has finite rank (Gersten, see

e.g. [2]). Its boundary $\partial(\operatorname{Fix}\alpha)$ is a subspace of $\partial F_n$ upon which $\partial\alpha$ acts as the identity. Note that for any integer $q$ the subgroup $\operatorname{Fix}\alpha^q$ is $\alpha$-invariant (i.e. it is mapped to itself by $\alpha$).

Following Nielsen [14], we say that a fixed point $X$ of $\partial\alpha$ is *singular* if $X \in \partial(\operatorname{Fix}\alpha)$, *regular* otherwise.

It is shown in [5, Proposition 1.1] that a regular fixed point $X$ of $\partial\alpha$ is either *attracting* or *repelling*. Attracting means that $\overline{\alpha}^p(Y)$ converges to $X$ for every $Y$ in a neighborhood of $X$ in $F_n \cup \partial F_n$ (as $p \to +\infty$), repelling means attracting for $\alpha^{-1}$ (see a detailed discussion in Section 4).

We say that $X \in \partial F_n$ is *periodic* if there exists $q \geq 1$ with $\partial\alpha^q(X) = X$. The smallest such $q$ is the *period* of $X$ and the set $\{X, \partial\alpha(X), \ldots, \partial\alpha^{q-1}(X)\}$ is a *periodic orbit of order* $q$. We define $X$ to be regular, attracting... if it is as a fixed point of $\partial\alpha^q$. We give a similar definition for a periodic orbit, noting that all its elements have the same type.

**Theorem 1.1.** *Let $\alpha \in \operatorname{Aut} F_n$. The homeomorphism $\partial\alpha : \partial F_n \to \partial F_n$ has at least two periodic points. More precisely, either $\partial\alpha$ has at least two periodic orbits, or the unique periodic orbit has order 2 and is the boundary of an $\alpha$-invariant infinite cyclic subgroup.*

**Example 1.2.** We construct a $\partial\alpha$ with only one periodic orbit. First define $\beta : F_2 \to F_2$ by $a \mapsto a$, $b \mapsto aba$. Then $\partial\beta$ has two singular fixed points $a^{\pm\infty} = \lim_{p\to+\infty} a^{\pm p}$. It is easily checked that these are the only periodic points of $\partial\beta$. The automorphism $\beta$ is the square of $\alpha : a \mapsto a^{-1}, b \mapsto a^{-1}b^{-1}$. The map $\partial\alpha$ permutes $a^\infty$ and $a^{-\infty}$.

The proof of Theorem 1.1 (to be found below) uses an $\alpha$-invariant $\mathbf{R}$-tree $T$. The main properties of $T$ are summarized as follows.

**Theorem 1.3.** **([5])** *For every automorphism $\alpha$ of $F_n$ there exists an $\mathbf{R}$-tree $T$ and a number $\lambda \geq 1$ such that:*

*(1) $F_n$ acts on $T$ non-trivially, minimally, with trivial arc stabilizers.*

*(2) There exists a homothety $H\colon T \to T$ with stretching factor $\lambda$ such that*

$$\alpha(g)H = Hg$$

*for all $g \in F_n$ (viewing elements of $F_n$ as isometries of $T$). If $\lambda = 1$, then $T$ is simplicial.*

*(3) There exists an $F_n$-equivariant injection $j : \partial T \to \partial F_n$ satisfying $\partial\alpha \circ j = j \circ H$.* □

Furthermore:

**Theorem 1.4.** **([6])** *Given $Q \in T$, its stabilizer $\operatorname{Stab} Q$ has rank $\leq n-1$, and the*

*action of* $\operatorname{Stab} Q$ *on* $\pi_0(T \setminus \{Q\})$ *has at most* $2n$ *orbits. The number of* $F_n$*-orbits of branch points is at most* $2n-2$. □

A *homothety* is a map $H$ such that $d(Hx, Hy) = \lambda d(x, y)$ for some $\lambda > 0$ (the *stretching factor*). We denote $\partial T$ the set of equivalence classes of infinite rays $\rho : (0, +\infty) \to T$, and again $H : \partial T \to \partial T$ the induced map. See [5, Sections 2 and 3] for other definitions and a proof of Theorem 1.3. Theorem 1.4 follows from Theorem III.2 of [6]. Given $\alpha$ and $T$, the number $\lambda$ and the homothety $H$ satisfying $\alpha(g)H = Hg$ are unique.

A homothety $H$ with $\lambda > 1$ has a unique fixed point $Q$, which may be in $T$ or only in its metric completion $\overline{T}$. We define an *eigenray* of $H$ as in [5], as an isometric map $\rho : (0, \infty) \to T$ such that $\rho(\lambda x) = H\rho(x)$. We note:

**Proposition 1.5.** *If* $HR = R$*, the stabilizer* $\operatorname{Stab} R$ *is* $\alpha$*-invariant. If* $\rho$ *is an eigenray, then* $j(\rho)$ *is a fixed point of* $\partial\alpha$*. Now suppose* $\lambda > 1$*, and let* $Q$ *be the fixed point of* $H$*. If* $Q \in \overline{T} \setminus T$*, then there exists a unique eigenray. If* $Q \in T$*, then any component of* $T \setminus \{Q\}$ *that is fixed by* $H$ *contains a unique eigenray.* □

*Proof of Theorem 1.1.* First assume that the fixed subgroup $\operatorname{Fix} \alpha^q$ is nontrivial for some $q \geq 1$. If it is cyclic, its two boundary points are either fixed points of $\partial\alpha$ or a periodic orbit of order 2. If $\operatorname{Fix} \alpha^q$ has rank $\geq 2$, we get uncountably many periodic orbits. From now on we assume that $\operatorname{Fix} \alpha^q$ is trivial for every $q$, and we construct an attracting periodic orbit of $\partial\alpha$. The same argument, applied to $\alpha^{-1}$, will yield a second orbit.

Let $T$ be as in Theorem 1.3. If $H$ fixes some $Q \in T$ with $\operatorname{Stab} Q$ nontrivial, recall that $\operatorname{Stab} Q$ is $\alpha$-invariant. Since it has rank less that $n$ and $\partial\operatorname{Stab} Q$ embeds into $\partial F_n$, we will be able to use induction on $n$ (of course $n = 1$ is trivial). Also note that, if $\rho$ is an eigenray of $H$ (with $\lambda > 1$), then the fixed point $j(\rho)$ of $\partial\alpha$ is attracting (see the proof of Assertion 2 of Proposition 4.4 in [5]).

Recall that we want to find an attracting periodic orbit of $\partial\alpha$. First assume $\lambda > 1$. Let $Q \in \overline{T}$ be the fixed point of $H$. If $Q \in \overline{T} \setminus T$, there is an eigenray $\rho$ and $j(\rho)$ is an attracting fixed point of $\partial\alpha$. Suppose $Q \in T$. If $\operatorname{Stab} Q$ is nontrivial, we use induction on $n$. Otherwise $T \setminus \{Q\}$ has at most $2n$ components by Theorem 1.4, and some power of $H$ has an eigenray. This gives an attracting periodic orbit as before.

Now we assume $\lambda = 1$. In this case $T$ is simplicial and $H$ is an isometry.

First suppose $H$ fixes some $Q$. We may assume $\operatorname{Stab} Q$ is trivial (otherwise, we do induction). Then some $H^k$ fixes an edge $e$. Replacing $\alpha$ by $\alpha^k$, we assume $k = 1$. Collapse to a point every edge not in the orbit of $e$ (under the action of $F_n$). We get a new tree $T'$ with an isometry $H'$ satisfying the conclusions of Theorem 1.3. The map $H'$ fixes some point with nontrivial stabilizer (since all vertices now have nontrivial stabilizer) and we use induction.

The last possibility if that $H$ is a hyperbolic isometry of $T$. In this case $H$ has

a translation axis $A$ and fixes two ends of $T$. Orienting $A$ by the action of $H$, we consider the positive end $A^+$ of $A$ and the associated fixed point $X^+ = j(A^+)$ of $\partial\alpha$. We complete the proof by showing that $X^+$ is not repelling (and therefore is attracting since we assume $\operatorname{Fix}\alpha^q$ trivial for all $q$). Choose any point $Q \in A$, and $g \in F_n$ acting on $T$ as a hyperbolic isometry whose axis has compact intersection with $A$. Writing $\alpha^p(g)Q = H^p g H^{-p} Q$ we see that the projection of $\alpha^p(g)Q$ onto $A$ goes to $A^+$ as $p \to \infty$. By Section 3 of [5] this implies $\lim_{p\to\infty} \alpha^p(g) = X^+$. Thus $X^+$ cannot be repelling. □

## 2. Bounding periods

**Theorem 2.1.** *Let $\alpha \in \operatorname{Aut} F_n$. Suppose $X \in \partial F_n$ is periodic of period $q$ under $\partial\alpha$. Then $q \le M_n$, where $M_n$ depends only on $n$ and $\log M_n \sim \sqrt{n \log n}$ as $n \to \infty$.*

The quantity $\sqrt{n \log n}$ is Landau's asymptotic estimate for $\log g(n)$, where $g(n)$ is the maximum order of elements in the symmetric group $S_n$ [8]. It is shown in [11] that the same estimate holds for the maximum order of torsion elements in $GL(n, \mathbf{Z})$ and $\operatorname{Aut} F_n$.

We first prove the following special case of Theorem 2.1:

**Lemma 2.2.** *If $g \in F_n$ is periodic of period $q$ under $\alpha \in \operatorname{Aut} F_n$, then $q \le A_n$, where $A_n$ is the maximum order of torsion elements in $\operatorname{Aut} F_n$.*

*Proof.* The subgroup $\operatorname{Fix}\alpha^q$ is $\alpha$-invariant, and the restriction of $\alpha$ has order exactly $q$ in $\operatorname{Aut}(\operatorname{Fix}\alpha^q)$. Since the rank of $\operatorname{Fix}\alpha^q$ is $\le n$ by [1], and $\operatorname{Aut} F_k$ naturally embeds into $\operatorname{Aut} F_\ell$ for $k < \ell$, the group $\operatorname{Aut} F_n$ contains an element of order $q$. □

**Remark.** Before the Scott conjecture was proved, Stallings showed [17] that, for a given $\alpha$, there is a bound for periods of elements $g \in F_n$.

*Proof of Theorem 2.1.* Lemma 2.2 shows that singular periodic points of $\partial\alpha$ have period $\le A_n$. Now suppose $X$ is regular, say attracting.

The points $X, \partial\alpha(X), \ldots, \partial\alpha^{q-1}(X)$ are attracting fixed points of $\partial\alpha^q$. By Theorem 1 of [5], the action of $\operatorname{Fix}\alpha^q$ on the set of attracting fixed points of $\partial\alpha^q$ has at most $2n$ orbits. Thus there exist $r \le 2n$ and $u \in \operatorname{Fix}\alpha^q$ such that

$$\partial\alpha^r(X) = uX.$$

By Lemma 2.2 we have

$$\alpha^s(u) = u$$

for some $s \le A_n$.

The above equations yield $\partial\alpha^{rs}(X) = aX$ with

$$a = \alpha^{(s-1)r}(u)\dots\alpha^{r}(u)u.$$

If $a = 1$ we get $q \le rs \le 2nA_n$. Otherwise we note that $a \in \operatorname{Fix}\alpha^s$, and from $X = \partial\alpha^{qrs}(X) = a^q X$ we conclude that $X$ is singular, a contradiction.

We have thus shown $q \le M_n = 2nA_n$. Since $\log A_n \sim \sqrt{n\log n}$ by [11], we have $\log M_n \sim \sqrt{n\log n}$. □

**Remark.** The bound $q \le 2nA_n$ is not quite sharp. But if $\alpha \in \operatorname{Aut} F_n$ has order $A_n$ then generic points of $\partial F_n$ have period $A_n$ under $\partial\alpha$. Therefore the estimate $\log M_n \sim \sqrt{n\log n}$ cannot be improved.

**Theorem 2.3.** *For any $\alpha \in \operatorname{Aut} F_n$, the map $\partial\alpha : \partial F_n \to \partial F_n$ has at least two periodic points of period $\le 2n$.*

For the automorphism defined by $a_i \mapsto a_{i+1}$ $(1 \le i \le n-1)$, $a_n \mapsto a_1^{-1}$, every point of $\partial F_n$ has period $2n$.

*Proof.* There are two cases. If $\alpha$ has no periodic element $g \ne 1$, then $\partial\alpha$ has at most $2n$ periodic points of a given type (attracting or repelling) by Theorem 1 of [5]. The other case is taken care of by the following result. □

**Proposition 2.4.** *Let $\alpha \in \operatorname{Aut} F_n$. If there is a nontrivial $\alpha$-periodic element $g \in F_n$, then there is one of period $\le 2n$.*

*Proof.* Let $q$ be the smallest period of nontrivial periodic elements. Arguing as in the proof of Lemma 2.2, we may assume that $\alpha$ has order $q$. Such an $\alpha$ may be realized as an automorphism of a graph ([4], [18]): there exist a finite graph $\Lambda$, an automorphism $f$ of $\Lambda$ fixing a vertex $v$, and an isomorphism $F_n \to \pi_1(\Lambda, v)$ such that the following diagram commutes:

$$\begin{array}{ccc} F_n & \xrightarrow{\alpha} & F_n \\ \downarrow & & \downarrow \\ \pi_1(\Lambda, v) & \xrightarrow{f_*} & \pi_1(\Lambda, v). \end{array}$$

We choose $\Lambda$ with minimal number of vertices. We claim that the action of $\mathbf{Z}/q\mathbf{Z} = \langle f \rangle$ on the set of germs of edges at $v$ is free. This will show $q \le 2n$ since $v$ has valence at most $2n$.

Assume the action is not free. Then some $f^r$ $(1 \le r \le q-1)$ fixes an edge containing $v$. Let $\Lambda_0$ be the component of the fixed point set of $f^r$ containing $v$. It is a tree since otherwise $\alpha$ would have a nontrivial periodic element of period $\le r$. We may therefore collapse $\Lambda_0$ to a point, contradicting the choice of $\Lambda$. □

## 3. Growth rates

In this section we fix $\Phi \in \operatorname{Out} F_n$, and sometimes also an automorphism $\alpha \in \Phi$. We write $|g|$ for the word length of $g \in F_n$, and $|\gamma|$ for the length of a conjugacy class $\gamma$ (equal to the length of a cyclically reduced word representing $\gamma$).

Let $M$ be the transition matrix of a relative train track map representing $\Phi$ (see [1]). The largest positive eigenvalue (spectral radius) of the matrix $M$ is denoted $\lambda(\Phi)$, or $\lambda(\alpha)$. It is an algebraic integer of degree bounded by $3n-3$.

For $g \in F_n$, the length of $\alpha^p(g)$ is bounded from above by a constant times $\|M\|^p\,|g|$. If $\lambda(\Phi)=1$, the growth of $\alpha^p(g)$ is polynomial and $\Phi$ is called *polynomially growing*. For future reference we note:

**Remark 3.1.** Given $\nu > \lambda(\alpha)$, there exists $C>0$ such that $|\alpha^p(g)| \le C\nu^p|g|$ for all $g \in F_n$ and $p \ge 1$.

Now let $\ell : F_n \to \mathbf{R}^+$ be the length function of an action of $F_n$ on an $\mathbf{R}$-tree $T$. It is bounded from above by a constant times word length. In particular, if $T$ is an $\alpha$-invariant $\mathbf{R}$-tree as in Theorem 1.3, we have (up to multiplicative constants) $\lambda^p\ell(g) = \ell(\alpha^p(g)) \le |\alpha^p(g)| \le \|M\|^p|g|$ and therefore $\lambda \le \lambda(\alpha)$. Conversely:

**Proposition 3.2.** *There exists an $\alpha$-invariant $\mathbf{R}$-tree $T$ as in Theorem 1.3 with $\lambda = \lambda(\alpha)$.*

*Proof.* This is proved by the same arguments as in [5, section 2], but instead of using only the top stratum of the train track (which may lead to $\lambda < \lambda(\alpha)$) we use the whole relative train track and an eigenvector $v$ of $M$ associated to $\lambda(\alpha)$. One shows that the resulting action on an $\mathbf{R}$-tree $T$ is nontrivial and has trivial arc stabilizers as in [5]. Minimality of the action may be achieved by restricting to the minimal invariant subtree. It is often more convenient, though, to work with the metric completion $\overline{T}$ of $T$ so as to ensure that $H$ has a fixed point $Q$ when $\lambda(\alpha) > 1$. □

Now let $J$ be a finitely generated malnormal subgroup of $F_n$ (recall that $J$ is *malnormal* if $gJg^{-1} \cap J \ne \{1\} \implies g \in J$). We say that $J$ is *$\Phi$-periodic* if there exist $q \ge 1$ and $\beta \in \Phi^q$ with $\beta(J) = J$. Note that, by malnormality, the class of $\beta$ in $\operatorname{Out} J$ is uniquely determined.

For example, suppose that $T$ is an $\mathbf{R}$-tree as in Theorem 1.3 and $J = \operatorname{Stab} Q$ for some branch point $Q$. Then $J$ is malnormal (because arc stabilizers are trivial). By Theorem 1.4, it has rank $< n$. We claim that it is $\Phi$-periodic. Indeed, by Theorem 1.4 there exist $m \in F_n$ and $q \ge 1$ such that $mH^q$ fixes $Q$. Denoting $i_m(g) = mgm^{-1}$, the automorphism $\beta = i_m \circ \alpha^q \in \Phi^q$ maps $J$ to itself.

If $J$ is finitely generated, malnormal, $\Phi$-periodic, we define $\lambda_J = \lambda(\beta_{|J})^{\frac{1}{q}}$.

**Proposition 3.3.** *Let $\Phi \in \operatorname{Out} F_n$.*

*(1) Each conjugacy class $\gamma$ in $F_n$ has a growth rate $\lambda(\gamma) = \lim_{p\to+\infty} |\Phi^p(\gamma)|^{1/p}$.*

*(2) Given $\lambda \geq 1$, the following are equivalent:*

- *$\lambda = \lambda(\gamma)$ for some conjugacy class $\gamma$.*
- *$\lambda = \lambda_J$ for some malnormal $\Phi$-periodic subgroup $J \subset F_n$ of rank $\leq n$.*

The existence of the limit in assertion 1 is folklore (compare [1, Remark 1.8]). Simple examples show that one cannot restrict to free factors in assertion 2.

*Proof.* The proof is by induction on $n$. Let $T$ be an $\alpha$-invariant $\mathbf{R}$-tree with $\lambda = \lambda(\Phi)$ (see Proposition 3.2). We distinguish two cases, by evaluating the length function on $\gamma$.

If $\ell(\gamma) > 0$, we write $|\Phi^p(\gamma)| \geq \ell(\Phi^p(\gamma)) = \lambda^p \ell(\gamma)$ (up to a constant) and we conclude that $\gamma$ has growth rate $\lambda(\gamma) = \lambda = \lambda(\Phi)$ (recall that the exponential growth of $\Phi^p(\gamma)$ is bounded from above by $\lambda(\Phi)$). Note that there exist classes with $\ell(\gamma) > 0$, hence there exist classes with growth rate $\lambda(\Phi)$.

If $\ell(\gamma) = 0$, an element $g \in F_n$ representing $\gamma$ fixes some branch point $Q \in T$, and we argue by induction by considering $\gamma$ as a conjugacy class in $J = \operatorname{Stab} Q$. We have pointed out earlier that $J$ is malnormal, $\Phi$-periodic, of rank $< n$. If $\beta = i_m \circ \alpha^q$ leaves $J$ invariant, note that, by quasiconvexity of $J$, the growth rate of $\gamma$ under $\beta_{|J}$ is the same as the growth rate of $\gamma$, viewed as a conjugacy class in $F_n$, under $\Phi^q$.

These arguments show that every $\gamma$ has a growth rate, which is of the form $\lambda_J$ with $J$ as in the proposition. Conversely, given $J$, let $\ell_J$ be the length function of a $\beta_{|J}$-invariant tree with $\lambda = \lambda(\beta_{|J})$. Conjugacy classes with $\ell_J(\gamma) > 0$ have growth rate $\lambda_J$ under $\Phi$. □

**Definition.** We call $\lambda(\Phi)$ the *top growth rate* of $\Phi$. The *set of growth rates* $\Lambda(\Phi) \subset (1, \infty)$ consists of the growth rates $\lambda(\gamma)$ which are bigger than 1.

Note that $\Lambda(\Phi)$ consists of algebraic integers of degree $\leq 3n-3$, and that $\lambda(\Phi)$ is the largest element of $\Lambda(\Phi) \cup \{1\}$. See [10] for more results about $\Lambda(\Phi)$.

## 4. Hölder dynamics

### Superattractivity

The discussion in this subsection (including Proposition 4.1) is valid for automorphisms of arbitrary (word) hyperbolic groups, but for simplicity we restrict to the case of $F_n$ (the generalization is almost immediate using [13]).

Fixing a free basis of $F_n$, we may view $\partial F_n$ as the set of right-infinite reduced words. Let $X \in \partial F_n$ be a fixed point of the homeomorphism $\partial\alpha$ induced by

$\alpha \in \operatorname{Aut} F_n$ on $\partial F_n$. We say that $X$ is *singular* if it belongs to the limit set of the fixed subgroup $\operatorname{Fix}\alpha$, *regular* otherwise (recall that $\operatorname{Fix}\alpha$ has finite rank).

As explained in [5], there is a basic trichotomy: *either $X$ is singular, or $X$ is attracting, or $X$ is repelling (i.e. attracting for $\alpha^{-1}$).* Attractivity has a strong meaning here (see section 1 of [5]): given $A$, there exists $m$ such that for $Y \in F_n \cup \partial F_n$

$$c_X Y \geq m \implies c_X(\partial\alpha(Y)) - c_X Y > A, \tag{1}$$

where $c_X Y$ is the length of the maximal common initial segment between the reduced words $X$ and $Y$ (i.e. the Gromov scalar product $< X, Y >$ with basepoint at the identity in the Cayley graph).

In particular, we have $\lim_{p\to+\infty} \overline{\alpha}^p(Y) = X$ uniformly on a neighborhood of $X$ in $F_n \cup \partial F_n$ if $X$ is attracting (whereas if $X$ is singular there are fixed points of $\alpha$ in $F_n$ arbitrarily close to $X$). For the automorphism $\beta$ studied in Example 1.2, the (singular) fixed points $a^{\pm\infty}$ of $\partial\beta$ are partly repelling and partly attracting: for any $k \in \mathbf{Z}$ we have $\lim_{p\to+\infty} \partial\beta^p(a^k bY) = a^\infty$ if $Y$ is a right-infinite reduced word not starting with $b^{-1}$, but $\lim_{p\to+\infty} \partial\beta^p(a^k b^{-1}Y) = a^{-\infty}$ if $Y$ does not start with $b$.

Also note that an isolated fixed point of $\partial\alpha$ is singular if and only if it belongs to the limit set of an $\alpha$-invariant cyclic subgroup (for the "only if" direction, simply observe that $\alpha$ leaves invariant the stabilizer of $X$ for the action of $F_n$ on $\partial F_n$). In particular, the natural action of $\operatorname{Fix}\alpha$ on the set of regular fixed points of $\partial\alpha$ is free. This action has finitely many orbits [2], indeed it follows from [5] that the number of orbits is at most $4n$. It is not clear to us whether there is a bound depending only on $G$ when $G$ is an arbitrary hyperbolic group.

Now recall that the boundary of $F_n$ (of any hyperbolic group, in fact) has a canonical *Hölder structure* (see [3], [7]). It may be viewed as a collection $\mathcal{D}$ of distance functions on $\partial F_n$ that are pairwise bi-Hölder equivalent: Given $d, d' \in \mathcal{D}$, there exist $C > 0$ and $\beta \in (0, 1]$ such that $\frac{1}{C} d^{\frac{1}{\beta}} \leq d' \leq C d^\beta$. This Hölder structure is preserved by $\partial\alpha$ for every $\alpha \in \operatorname{Aut} F_n$. If $J \subset F_n$ has finite rank, the inclusion $\partial J \hookrightarrow \partial F_n$ is bi-Hölder.

We represent the Hölder structure by the visual metrics $d_\varepsilon(X, Y) = \exp(-\varepsilon c_X Y)$.

Let $X \in \partial F_n$ be a fixed point of $\partial\alpha$, and $d = d_\varepsilon$ a visual metric. If $X$ is regular, attracting, it follows from (1) that

$$\lim_{Y\to X} \frac{d(\partial\alpha(Y), X)}{d(Y, X)} = 0. \tag{2}$$

If $X$ is repelling or singular, however, the above quotient is bounded away from 0 on a neighborhood of $X$ (if $X$ is singular, $c_X(\partial\alpha(Y)) - c_X Y$ is bounded near $X$ because $\operatorname{Fix}\alpha$ is quasiconvex and $\alpha$ is a quasi-isometry).

Thus (2) is a metric characterization of attracting regular fixed points, similar to the characterization of a superattracting fixed point $c$ of a holomorphic map

$f : \mathbf{C} \to \mathbf{C}$ by $f'(c) = 0$. For this reason, we call an attracting regular point *superattracting* (and a repelling regular point *superrepelling*).

Of course the map $\partial\alpha$ is a homeomorphism, and superattracting fixed points may exist only because $\partial\alpha$ is bi-Hölder but in general not bi-Lipschitz. For instance, if $t$ is any lift to the Poincaré disc of a pseudo-Anosov diffeomorphism of a closed hyperbolic surface, then the homeomorphism induced by $t$ on the circle at infinity is never bi-Lipschitz (see Remark (22.14) in [12]).

Characterization (2) above does not depend on the chosen visual metric $d$, but it is not valid for arbitrary metrics in $\mathcal{D}$. The following characterization will apply to every $d \in \mathcal{D}$.

**Proposition 4.1.** *Let $\alpha \in \operatorname{Aut} F_n$. A fixed point $X$ of $\partial\alpha$ is superattracting if and only if*

$$\lim_{p\to+\infty} \frac{1}{p} \log d(\partial\alpha^p(Y), X) = -\infty$$

*for $Y \in \partial F_n$ close to $X$, where $d$ is any metric on $\partial F_n$ defining the Hölder structure.*

This equation means that $\partial\alpha^p(Y)$ converges to $X$ super-exponentially as $p \to \infty$. Unlike (2), it is true for every metric in $\mathcal{D}$ if it is true for one.

*Proof.* We may assume that $d$ is a visual metric. Suppose $X$ is superattracting. We have to prove $\lim_{p\to\infty} \frac{1}{p} c_X(\partial\alpha^p(Y)) = +\infty$ for $Y$ close to $X$. Given $A > 0$, let $m$ be as in (1). If $\lim_{p\to\infty} \partial\alpha^p(Y) = X$, there exists $n_0$ such that $c_X(\partial\alpha^p(Y)) \geq m$ for $p \geq n_0$. For $p$ large, we then have

$$c_X(\partial\alpha^p(Y)) \geq A(p - n_0) + m,$$

and the result follows.

Conversely, if $X$ is singular, then $c_X(\partial\alpha(Z)) - c_X Z$ is bounded in a neighborhood of $X$, and therefore $\frac{1}{p} \log d(\partial\alpha^p(Y), X)$ is bounded from below as $p \to \infty$. □

## Speed of convergence

We consider $\alpha \in \operatorname{Aut} F_n$, and the associated $\Phi \in \operatorname{Out} F_n$. Recall that $\Lambda(\Phi) \subset (1, \infty)$ is the set of nontrivial growth rates. It may also be viewed as a set of $\lambda_J$ (see Proposition 3.3).

**Theorem 4.2.** *Let $\alpha \in \operatorname{Aut} F_n$. Let $X \in \partial F_n$ be a superattracting fixed point of $\partial\alpha$. There exists $\lambda = \lambda(\alpha, X) \in \Lambda(\Phi) \cup \{1\}$ such that*

$$\lim_{p\to+\infty} \frac{1}{p} \log\Big( -\log d(\partial\alpha^p(Y), X)\Big) = \log\lambda \tag{3}$$

*for $Y \in \partial F_n$ close to $X$ (and any metric $d$ on $\partial F_n$ defining the Hölder structure).*

*Conversely, given $\mu \in \Lambda(\Phi)$, there exist $q \geq 1$, an automorphism $\beta \in \Phi^q$, and a superattracting fixed point $X$ of $\partial\beta$ with $\lambda(\beta, X) = \mu^q$.*

It follows that the set $\Lambda_h(\Phi)$ of Hölder exponents defined in the introduction equals $\Lambda(\Phi)$. Note that replacing $d$ by a metric bi-Hölder equivalent to $d$ does not affect the validity of (3).

*Proof of Theorem 4.2.* We fix a basis of $F_n$ and we consider the corresponding Cayley tree $\Gamma$.

Let $X$ be a superattracting fixed point of $\partial\alpha$. We need to prove

$$\lim_{p\to+\infty} \frac{1}{p} \log c_X(\partial\alpha^p(Y)) = \log\lambda.$$

We will bound the left-hand side, first from above and then from below.

**Lemma 4.3.** *Suppose $X \in \partial J$, with $J \subset F_n$ a finitely generated $\alpha$-invariant malnormal subgroup. Then*

$$\limsup_{p\to+\infty} \frac{1}{p} \log c_X(\partial\alpha^p(Y)) \leq \log\lambda_J$$

*for all $Y \in \partial F_n$.*

Recall that $\lambda_J$ is the top growth rate of $\alpha_{|J}$.

*Proof.* Let $x_{t_p}$ be the projection of $\partial\alpha^p(Y)$ onto the geodesic from $1 \in F_n$ to $X$ in $\Gamma$. By quasiconvexity of $J$, we can find $j_p \in J$ within a fixed distance of $x_{t_p}$. We need to prove $\limsup_{p\to+\infty} \frac{1}{p} \log|j_p| \leq \log\lambda_J$. We will work with word length $|j_p|_J$ in $J$, which is comparable to $|j_p|$.

Define $w_p \in J$ by $j_p = \alpha(j_{p-1})w_p$. Since $\alpha$ is a quasi-isometry, there is a uniform bound for $|w_p|$, hence also for $|w_p|_J$ because $J$ is quasiconvex. Now write

$$j_p = \alpha^p(j_0)\alpha^{p-1}(w_1)\cdots\alpha(w_{p-1})w_p.$$

For $\nu > \lambda_J$ we have

$$|j_p|_J \leq C\nu^p|j_0|_J + C\nu^{p-1}|w_1|_J + \cdots + C\nu|w_{p-1}|_J + |w_p|_J,$$

with $C$ given by Remark 3.1. Thus $|j_p|_J = O(\nu^p)$ for all $\nu > \lambda_J$, and the lemma is proved. □

**Corollary 4.4.** *Theorem 4.2 holds if $\alpha$ is polynomially growing (i.e. $\lambda(\alpha) = 1$), with $\lambda(\alpha, X) = 1$.* □

Fix a subgroup $J$ as in Lemma 4.3, and consider an $\mathbf{R}$-tree $T$ with an action of $J$ satisfying the conditions of Theorem 1.3 with respect to $\alpha_{|J}$. Using Proposition 3.2, we assume that the stretching factor of the homothety $H$ equals $\lambda_J$. Suppose furthermore $\lambda_J > 1$.

**Lemma 4.5.** *Suppose $X = j(\rho)$, where $\rho$ is an eigenray of $H : T \to T$ (in particular, $X \in \partial J$). Then*

$$\liminf_{p\to+\infty} \frac{1}{p} \log c_X(\partial\alpha^p(Y)) \geq \log \lambda_J$$

*for $Y \in \partial F_n$ close enough to $X$.*

*Proof.* With the notations of Section 1, let $Q \in \overline{T}$ be the fixed point of $H$ (i.e. the origin of $\rho$). Choose $j_p$ as in the proof of Lemma 4.3 and define $d_p$ as $\overline{d}(Q, j_pQ)$ (where $\overline{d}$ denotes the distance in $\overline{T}$). Note that

$$\overline{d}(Q, \alpha(j_p)Q) = \overline{d}(Q, \alpha(j_p)HQ) = \overline{d}(Q, Hj_pQ) = \lambda_J\overline{d}(Q, j_pQ).$$

On the other hand, recall that the distance in $J$ from $\alpha(j_p)$ to $j_{p+1}$ is bounded independently of $p$ (and of $Y$). Thus we obtain an inequality of the form

$$d_{p+1} \geq \lambda_J d_p - A,$$

with $A$ independent of $p$ and $Y$.

If $Y$ is close enough to $X$ in $\partial F_n$, then $j_0$ is close to $X$ in $J \cup \partial J$, and therefore $d_0$ is large (by bounded backtracking, see section 3 of [5]). This implies

$$\liminf_{p\to+\infty} \frac{1}{p} \log d_p \geq \log \lambda_J.$$

Finally, we observe that $d_p = \overline{d}(Q, j_pQ)$ is bounded above by a constant times $|j_p|_J$, hence by a constant times $|j_p|$. □

Now we complete the proof of Theorem 4.2. If $\lambda(\alpha) = 1$, then we are done by Corollary 4.4. Assume $\lambda(\alpha) > 1$, and consider a tree $T$ as in Proposition 3.2, with stretching factor $\lambda(\alpha)$. If $X = j(\rho)$ as in Lemma 4.5, we are done, with $\lambda = \lambda(\alpha)$. If not, then by Proposition 4.3 of [5] we have $X \in \partial\mathrm{Stab}\, Q$, where $Q \in T$ is the fixed point of $H$ (recall that points of $\overline{T} \setminus T$ have trivial stabilizer).

The subgroup $\mathrm{Stab}\, Q$ is $\alpha$-invariant, malnormal, and has rank $< n$ (see section 3). Repeat the argument, working with $\alpha_{|\mathrm{Stab}\, Q}$. After a finite number of steps we find that $X \in \partial J$ (with $J$ invariant, malnormal, of rank $< n$), and either $\lambda_J = 1$ or $X = j(\rho)$. It follows from Lemmas 4.3 and 4.5 that Theorem 4.2 holds, with $\lambda(\alpha, X) = \lambda_J$.

Conversely, consider $\mu \in \Lambda(\Phi)$. First suppose $\mu = \lambda(\alpha)$. Consider an $\mathbf{R}$-tree $T$ as in Theorem 1.3, with $\lambda = \lambda(\alpha)$. By Theorem 1.4 and Proposition 1.5, there exist $m \in F_n$ and $q \geq 1$ such that $mH^q$ has an eigenray $\rho$. Let $\beta = i_m \circ \alpha^q$, with $i_m(g) = mgm^{-1}$. Then $X = j(\rho)$ is a fixed point of $\partial\beta$, and $\lambda(\beta, X) = \lambda(\beta) = \mu^q$.

For arbitrary $\mu = \lambda_J \in \Lambda(\Phi)$, let $\alpha' \in \Phi^r$ leave $J$ invariant. The previous argument yields $\beta \in \Phi^{rq}$ and a fixed point $X$ of $\partial\beta$ in $\partial J$ such that $\lambda(\beta_{|J}, X) = \mu^{rq}$. Since the inclusion $\partial J \hookrightarrow \partial F_n$ is bi-Hölder, $\lambda(\beta, X) = \lambda(\beta_{|J}, X)$ has the required form. □

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## References

[1] M. Bestvina, M. Handel, Train tracks for automorphisms of the free group *Ann. Math.* **135** (1992), 1–51.

[2] D. Cooper, Automorphisms of free groups have finitely generated fixed point sets, *Jour. Alg.* **111** (1987), 453–456.

[3] M. Coornaert, T. Delzant, A. Papadopoulos, *Géométrie et théorie des groupes*, Lecture Notes **1441** Springer-Verlag, 1990.

[4] M. Culler, Finite groups of outer automorphisms of a free group, *Contemp. Math.* **33** (1984), 197–207.

[5] D. Gaboriau, A. Jaeger, G. Levitt, M. Lustig, An index for counting fixed points of automorphisms of free groups, *Duke Math. Jour.* **93** (1998), 425–452.

[6] D. Gaboriau, G. Levitt, The rank of actions on $\mathbf{R}$-trees, *Ann. Sc. ENS* **28** (1995), 549–570.

[7] E. Ghys, P. de la Harpe (eds.), Sur les groupes hyperboliques d'après Mikhael Gromov. In: *Progress in Mathematics 83*, Birkhäuser, 1990.

[8] E. Landau, Über die Maximalordnung der Permutation gegebenen Grades, *Archiv der Math. und Phys.*, Ser. 3, **5** (1903), 92–103.

[9] G. Levitt, M. Lustig, Most automorphisms of a hyperbolic group have very simple dynamics, *Ann. Sc. ENS*, to appear (available from http://picard.ups-tlse.fr/~levitt/).

[10] G. Levitt, M. Lustig, Growth rates of automorphisms of free groups, in preparation.

11] G. Levitt, J.-L. Nicolas, On the maximum order of torsion elements in $GL(n, \mathbf{Z})$ and $\mathrm{Aut}\,(F_n)$, *Jour. Alg.* **208** (1998), 630–642.

[12] G. D. Mostow, Strong rigidity of locally symmetric spaces, *Ann. Math. Studies* **78**, Princeton Univ. Press.

[13] W. D. Neumann, The fixed subgroup of an automorphism of a word hyperbolic group is rational, *Inv. Math.* **110** (1992), 147–150.

[14] J. Nielsen, Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen, *Acta Math.* **50** (1927), 189–358; II, *Acta Math.* **53** (1929), 1–76; III; *Acta Math.* **58** (1932), 87–167; English transl. in *Collected mathematical papers* Birkhäuser, 1986.

[15] Z. Sela, The Nielsen-Thurston classification and automorphisms of a free group I, *Duke Math. Jour.* **84** (1996), 379–397.

[16] H. Short, Quasiconvexity and a theorem of Howson's, pp. 168–176 in *Group theory from a geometrical viewpoint* (E. Ghys, A. Haefliger, A. Verjovsky, eds.), World Scientific, 1991.

[17] J. R. Stallings, Finiteness properties of matrix representations, *Ann. Math.* **124** (1986), 337–346.

[18] B. Zimmermann, Über Homöomorphismen $n$-dimensionaler Henkelkörper und endliche Erweiterungen von Schottky-Gruppen, *Comm. Math. Helv.* **56** (1981), 474–486.

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