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## Periodic ends, growth rates, Hölder dynamics for automorphisms of free groups

Gilbert Levitt and Martin Lustig

**Abstract.** Let  $F_n$  be the free group of rank  $n$ , and  $\partial F_n$  its boundary (or space of ends).

For any  $\alpha \in \text{Aut } F_n$ , the homeomorphism  $\partial\alpha$  induced by  $\alpha$  on  $\partial F_n$  has at least two periodic points of period  $\leq 2n$ . Periods of periodic points of  $\partial\alpha$  are bounded above by a number  $M_n$  depending only on  $n$ , with  $\log M_n \sim \sqrt{n \log n}$  as  $n \rightarrow +\infty$ .

Using the canonical Hölder structure on  $\partial F_n$ , we associate an algebraic number  $\lambda \geq 1$  to any attracting fixed point  $X$  of  $\partial\alpha$ ; if  $\lambda > 1$ , then for any  $Y$  close to  $X$  the sequence  $\partial\alpha^p(Y)$  approaches  $X$  at about the same speed as  $e^{-\lambda^p}$ . This leads to a set of Hölder exponents  $\Lambda_h(\Phi) \subset (1, +\infty)$  associated to any  $\Phi \in \text{Out } F_n$ . This set coincides with the set of nontrivial exponential growth rates of conjugacy classes of  $F_n$  under iteration of  $\Phi$ .

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## Introduction and statement of results

Let  $\varphi$  be a homeomorphism of a closed surface  $\Sigma$ , with  $\chi(\Sigma) < 0$ . In [14], Nielsen studied  $\varphi$  by lifting it to the universal covering  $D$  of  $\Sigma$  and considering the induced homeomorphism  $f$  on the circle at infinity  $S$ . In more algebraic terms, the mapping class of  $\varphi$  corresponds to an outer automorphism  $\Phi$  of  $\pi_1 \Sigma$ , various lifts of  $\varphi$  to  $D$  correspond to various automorphisms  $\alpha$  of  $\pi_1 \Sigma$  representing  $\Phi$ , and  $f : S \rightarrow S$  corresponds to the homeomorphism  $\partial\alpha$  induced by  $\alpha$  on the boundary of the group  $\pi_1 \Sigma$ .

Let  $F_n$  be the free group of rank  $n$ . We will study automorphisms  $\alpha$  of  $F_n$ , and outer automorphisms  $\Phi \in \text{Out } F_n$ , through the homeomorphisms  $\partial\alpha$  induced on the boundary  $\partial F_n$ . The space  $\partial F_n$ , homeomorphic to a Cantor set if  $n \geq 2$ , may be viewed as the (Gromov) boundary of  $F_n$ , or its space of ends, or the set of right-infinite reduced words in the generators and their inverses.

In the case of a surface group, Nielsen proved among many other things that  $f = \partial\alpha : S \rightarrow S$  always has at least two periodic points. Furthermore, the period of these points may be bounded in terms of  $|\chi(\Sigma)|$ .

Our first main result has a similar flavor.

**Theorem 1.** *Let  $\alpha \in \text{Aut } F_n$ .*

- (1) *The homeomorphism  $\partial\alpha : \partial F_n \rightarrow \partial F_n$  has at least two periodic points of period  $\leq 2n$ . If it has only one orbit of periodic points, then this orbit has order two.*
- (2) *Suppose  $X \in \partial F_n$  is periodic of period  $q$  under  $\partial\alpha$ . Then  $q \leq M_n$ , where  $M_n$  depends only on  $n$  and  $\log M_n \sim \sqrt{n \log n}$  as  $n \rightarrow \infty$ .*

The bound  $2n$  and the bound on  $q$  are sharp. The quantity  $\sqrt{n \log n}$  is asymptotic to the logarithm of the maximum order of torsion elements in  $\text{Aut } F_n$ , see [11]. As a special case of assertion 2, there is a bound depending only on  $n$  for periods of elements  $g \in F_n$  under the action of  $\alpha$ . One may also establish a uniform bound for periods of conjugacy classes under the action of  $\Phi \in \text{Out } F_n$ . It is proved in [9] that, for “most”  $\alpha \in \text{Aut } F_n$ , the homeomorphism  $\partial\alpha$  has exactly two fixed points, and no other periodic point.

Like many results of the present paper, the proof of Theorem 1 uses  $\mathbf{R}$ -trees and techniques introduced in [5]. The proof of assertion 2 uses the main result of [5], and Bestvina-Handel’s bound [1] for the rank of the fixed subgroup (the “Scott conjecture”).

Let us now consider local properties of fixed points of  $\partial\alpha$ , using the canonical Hölder structure on  $\partial F_n$  (see [3, 7]). Let  $X$  be a fixed point of  $\partial\alpha$  not belonging to the limit set of the fixed subgroup  $\text{Fix } \alpha \subset F_n$ . It is either attracting or repelling [5]. In the attracting case, we show that, for  $Y \in \partial F_n$  close enough to  $X$ , the sequence  $\partial\alpha^p(Y)$  converges to  $X$  *super-exponentially* in the sense that

$$\lim_{p \rightarrow +\infty} \frac{1}{p} \log d(\partial\alpha^p(Y), X) = -\infty,$$

where  $d$  is any distance on  $\partial F_n$  defining the Hölder structure. We say that  $X$  is *superattracting* (see the beginning of Section 4 for a detailed discussion).

**Theorem 2.** *Let  $\alpha \in \text{Aut } F_n$ . Let  $X \in \partial F_n$  be a superattracting fixed point of  $\partial\alpha$ . There exists an algebraic number  $\lambda = \lambda(\alpha, X) \geq 1$  such that*

$$\lim_{p \rightarrow +\infty} \frac{1}{p} \log \left( -\log d(\partial\alpha^p(Y), X) \right) = \log \lambda$$

*for  $Y \in \partial F_n$  close to  $X$  (and  $d$  a distance on  $\partial F_n$  as above).*

Thus, when  $\lambda > 1$ , the sequence  $\partial\alpha^p(Y)$  converges to  $X$  at about the same speed as  $f^p(x)$  approaches 0, where  $f$  is the map  $x \mapsto x^\lambda : [0, 1) \rightarrow [0, 1)$ .

**Example.** Consider  $\alpha : F_2 \rightarrow F_2$  given by  $\alpha(a) = aba$ ,  $\alpha(b) = ab$ . The number associated to  $X = \lim_{p \rightarrow +\infty} \alpha^p(a) = ababaaba \dots$  is the Perron-Frobenius eigenvalue of the matrix  $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ . On the other hand, for the polynomially growing

$\alpha : F_3 \rightarrow F_3$  given by  $\alpha(a) = a$ ,  $\alpha(b) = ba$ ,  $\alpha(c) = cba$ , the number associated to the superattracting point  $X = \lim_{p \rightarrow +\infty} \alpha^p(c) = cbaba^2ba^3 \dots$  equals 1.

We now associate a canonical set of *Hölder exponents*  $\Lambda_h(\Phi) \subset (1, +\infty)$  to any  $\Phi \in \text{Out } F_n$ . View  $\Phi$  as a collection of automorphisms  $\alpha \in \text{Aut } F_n$ . We say that  $\mu > 1$  belongs to  $\Lambda_h(\Phi)$  if there exist  $\beta \in \Phi^q$ , with  $q \geq 1$ , and a superattracting fixed point  $X$  of  $\partial\beta$  with  $\lambda(\beta, X) = \mu^q$ . The set  $\Lambda_h(\Phi)$  is a conjugacy invariant of  $\Phi$ .

**Example.** If  $\Phi$  is induced by a homeomorphism  $\varphi$  of a compact surface  $\Sigma$  with  $\pi_1 \Sigma \simeq F_n$ , then  $\Lambda_h(\Phi)$  consists of (roots of) the expansion factors of the pseudo-Anosov pieces of  $\varphi$ . They are algebraic units.

If  $\alpha \in \text{Aut } F_3$  is given by  $\alpha(a) = ab^{-1}$ ,  $\alpha(b) = bac^{-1}$ ,  $\alpha(c) = ca^{-3}$  (see [6, Example II.7]), then  $\Lambda_h(\Phi)$  consists of the real root  $\lambda$  of  $x^3 - 3x^2 + 2x - 3$ . Note that  $\lambda$  is not an algebraic unit, and therefore cannot be read off the graph of groups constructed by Sela in Theorem 4.1 of [15].

**Theorem 3.** *Given  $\Phi \in \text{Out } F_n$ , the set of Hölder exponents  $\Lambda_h(\Phi)$  equals the set  $\Lambda(\Phi)$  of nontrivial exponential growth rates of conjugacy classes of  $F_n$  under iteration of  $\Phi$ .*

The (exponential) growth rate of a conjugacy class  $\gamma$  is  $\lambda(\gamma) = \lim_{p \rightarrow +\infty} |\Phi^p(\gamma)|^{1/p}$  (see Proposition 3.3). It is nontrivial if  $\lambda(\gamma) > 1$ . It will be shown in [10] that  $\Lambda(\Phi)$  has at most  $\lfloor \frac{3n-2}{4} \rfloor$  elements and consists of certain Perron-Frobenius eigenvalues of the transition matrix associated to a relative train track representative of  $\Phi$ .

This paper is organized as follows. In Section 1 we prove the existence of periodic points for  $\partial\alpha$ . The proof of Theorem 1 is completed in Section 2 (Theorems 2.1 and 2.3). In Section 3 we briefly discuss growth rates. We start Section 4 by a general discussion of superattractivity, valid for an arbitrary hyperbolic group. We then prove Theorem 2.

## 1. Existence of periodic points

Let  $F_n$  be a free group. We consider its boundary  $\partial F_n$ , equipped with the natural action of  $F_n$  by left-translations. It is a Cantor set if  $n \geq 2$  (it consists of two points if  $n = 1$ ). In section 4, we will view  $\partial F_n$  as a set of right-infinite reduced words. A finitely generated subgroup  $J \subset F_n$  is quasiconvex [16]. In particular, we can identify the boundary (or limit set)  $\partial J$  with a subset of  $\partial F_n$ .

An automorphism  $\alpha \in \text{Aut } F_n$  is a quasi-isometry of  $F_n$ . It induces a homeomorphism  $\partial\alpha : \partial F_n \rightarrow \partial F_n$ , and a homeomorphism  $\bar{\alpha} = \alpha \cup \partial\alpha$  on the compact space  $\bar{F}_n = F_n \cup \partial F_n$ .

The fixed subgroup  $\text{Fix } \alpha = \{g \in F_n \mid \alpha(g) = g\}$  has finite rank (Gersten, see

e.g. [2]). Its boundary  $\partial(\text{Fix } \alpha)$  is a subspace of  $\partial F_n$  upon which  $\partial\alpha$  acts as the identity. Note that for any integer  $q$  the subgroup  $\text{Fix } \alpha^q$  is  $\alpha$ -invariant (i.e. it is mapped to itself by  $\alpha$ ).

Following Nielsen [14], we say that a fixed point  $X$  of  $\partial\alpha$  is *singular* if  $X \in \partial(\text{Fix } \alpha)$ , *regular* otherwise.

It is shown in [5, Proposition 1.1] that a regular fixed point  $X$  of  $\partial\alpha$  is either *attracting* or *repelling*. Attracting means that  $\alpha^p(Y)$  converges to  $X$  for every  $Y$  in a neighborhood of  $X$  in  $F_n \cup \partial F_n$  (as  $p \rightarrow +\infty$ ), repelling means attracting for  $\alpha^{-1}$  (see a detailed discussion in Section 4).

We say that  $X \in \partial F_n$  is *periodic* if there exists  $q \geq 1$  with  $\partial\alpha^q(X) = X$ . The smallest such  $q$  is the *period* of  $X$  and the set  $\{X, \partial\alpha(X), \dots, \partial\alpha^{q-1}(X)\}$  is a *periodic orbit of order  $q$* . We define  $X$  to be regular, attracting... if it is as a fixed point of  $\partial\alpha^q$ . We give a similar definition for a periodic orbit, noting that all its elements have the same type.

**Theorem 1.1.** *Let  $\alpha \in \text{Aut } F_n$ . The homeomorphism  $\partial\alpha : \partial F_n \rightarrow \partial F_n$  has at least two periodic points. More precisely, either  $\partial\alpha$  has at least two periodic orbits, or the unique periodic orbit has order 2 and is the boundary of an  $\alpha$ -invariant infinite cyclic subgroup.*

**Example 1.2.** We construct a  $\partial\alpha$  with only one periodic orbit. First define  $\beta : F_2 \rightarrow F_2$  by  $a \mapsto a, b \mapsto aba$ . Then  $\partial\beta$  has two singular fixed points  $a^{\pm\infty} = \lim_{p \rightarrow +\infty} a^{\pm p}$ . It is easily checked that these are the only periodic points of  $\partial\beta$ . The automorphism  $\beta$  is the square of  $\alpha : a \mapsto a^{-1}, b \mapsto a^{-1}b^{-1}$ . The map  $\partial\alpha$  permutes  $a^\infty$  and  $a^{-\infty}$ .

The proof of Theorem 1.1 (to be found below) uses an  $\alpha$ -invariant  $\mathbf{R}$ -tree  $T$ . The main properties of  $T$  are summarized as follows.

**Theorem 1.3.** ([5]) *For every automorphism  $\alpha$  of  $F_n$  there exists an  $\mathbf{R}$ -tree  $T$  and a number  $\lambda \geq 1$  such that:*

- (1)  $F_n$  acts on  $T$  non-trivially, minimally, with trivial arc stabilizers.
- (2) There exists a homothety  $H : T \rightarrow T$  with stretching factor  $\lambda$  such that

$$\alpha(g)H = Hg$$

for all  $g \in F_n$  (viewing elements of  $F_n$  as isometries of  $T$ ). If  $\lambda = 1$ , then  $T$  is simplicial.

- (3) There exists an  $F_n$ -equivariant injection  $j : \partial T \rightarrow \partial F_n$  satisfying  $\partial\alpha \circ j = j \circ H$ .  $\square$

Furthermore:

**Theorem 1.4.** ([6]) *Given  $Q \in T$ , its stabilizer  $\text{Stab } Q$  has rank  $\leq n - 1$ , and the*

action of  $\text{Stab } Q$  on  $\pi_0(T \setminus \{Q\})$  has at most  $2n$  orbits. The number of  $F_n$ -orbits of branch points is at most  $2n - 2$ .  $\square$

A *homothety* is a map  $H$  such that  $d(Hx, Hy) = \lambda d(x, y)$  for some  $\lambda > 0$  (the *stretching factor*). We denote  $\partial T$  the set of equivalence classes of infinite rays  $\rho : (0, +\infty) \rightarrow T$ , and again  $H : \partial T \rightarrow \partial T$  the induced map. See [5, Sections 2 and 3] for other definitions and a proof of Theorem 1.3. Theorem 1.4 follows from Theorem III.2 of [6]. Given  $\alpha$  and  $T$ , the number  $\lambda$  and the homothety  $H$  satisfying  $\alpha(g)H = Hg$  are unique.

A homothety  $H$  with  $\lambda > 1$  has a unique fixed point  $Q$ , which may be in  $T$  or only in its metric completion  $\overline{T}$ . We define an *eigenray* of  $H$  as in [5], as an isometric map  $\rho : (0, \infty) \rightarrow T$  such that  $\rho(\lambda x) = H\rho(x)$ . We note:

**Proposition 1.5.** *If  $HR = R$ , the stabilizer  $\text{Stab } R$  is  $\alpha$ -invariant. If  $\rho$  is an eigenray, then  $j(\rho)$  is a fixed point of  $\partial\alpha$ . Now suppose  $\lambda > 1$ , and let  $Q$  be the fixed point of  $H$ . If  $Q \in \overline{T} \setminus T$ , then there exists a unique eigenray. If  $Q \in T$ , then any component of  $T \setminus \{Q\}$  that is fixed by  $H$  contains a unique eigenray.*  $\square$

*Proof of Theorem 1.1.* First assume that the fixed subgroup  $\text{Fix } \alpha^q$  is nontrivial for some  $q \geq 1$ . If it is cyclic, its two boundary points are either fixed points of  $\partial\alpha$  or a periodic orbit of order 2. If  $\text{Fix } \alpha^q$  has rank  $\geq 2$ , we get uncountably many periodic orbits. From now on we assume that  $\text{Fix } \alpha^q$  is trivial for every  $q$ , and we construct an attracting periodic orbit of  $\partial\alpha$ . The same argument, applied to  $\alpha^{-1}$ , will yield a second orbit.

Let  $T$  be as in Theorem 1.3. If  $H$  fixes some  $Q \in T$  with  $\text{Stab } Q$  nontrivial, recall that  $\text{Stab } Q$  is  $\alpha$ -invariant. Since it has rank less than  $n$  and  $\partial\text{Stab } Q$  embeds into  $\partial F_n$ , we will be able to use induction on  $n$  (of course  $n = 1$  is trivial). Also note that, if  $\rho$  is an eigenray of  $H$  (with  $\lambda > 1$ ), then the fixed point  $j(\rho)$  of  $\partial\alpha$  is attracting (see the proof of Assertion 2 of Proposition 4.4 in [5]).

Recall that we want to find an attracting periodic orbit of  $\partial\alpha$ . First assume  $\lambda > 1$ . Let  $Q \in \overline{T}$  be the fixed point of  $H$ . If  $Q \in \overline{T} \setminus T$ , there is an eigenray  $\rho$  and  $j(\rho)$  is an attracting fixed point of  $\partial\alpha$ . Suppose  $Q \in T$ . If  $\text{Stab } Q$  is nontrivial, we use induction on  $n$ . Otherwise  $T \setminus \{Q\}$  has at most  $2n$  components by Theorem 1.4, and some power of  $H$  has an eigenray. This gives an attracting periodic orbit as before.

Now we assume  $\lambda = 1$ . In this case  $T$  is simplicial and  $H$  is an isometry.

First suppose  $H$  fixes some  $Q$ . We may assume  $\text{Stab } Q$  is trivial (otherwise, we do induction). Then some  $H^k$  fixes an edge  $e$ . Replacing  $\alpha$  by  $\alpha^k$ , we assume  $k = 1$ . Collapse to a point every edge not in the orbit of  $e$  (under the action of  $F_n$ ). We get a new tree  $T'$  with an isometry  $H'$  satisfying the conclusions of Theorem 1.3. The map  $H'$  fixes some point with nontrivial stabilizer (since all vertices now have nontrivial stabilizer) and we use induction.

The last possibility is that  $H$  is a hyperbolic isometry of  $T$ . In this case  $H$  has

a translation axis  $A$  and fixes two ends of  $T$ . Orienting  $A$  by the action of  $H$ , we consider the positive end  $A^+$  of  $A$  and the associated fixed point  $X^+ = j(A^+)$  of  $\partial\alpha$ . We complete the proof by showing that  $X^+$  is not repelling (and therefore is attracting since we assume  $\text{Fix } \alpha^q$  trivial for all  $q$ ). Choose any point  $Q \in A$ , and  $g \in F_n$  acting on  $T$  as a hyperbolic isometry whose axis has compact intersection with  $A$ . Writing  $\alpha^p(g)Q = H^p g H^{-p}Q$  we see that the projection of  $\alpha^p(g)Q$  onto  $A$  goes to  $A^+$  as  $p \rightarrow \infty$ . By Section 3 of [5] this implies  $\lim_{p \rightarrow \infty} \alpha^p(g) = X^+$ . Thus  $X^+$  cannot be repelling.  $\square$

## 2. Bounding periods

**Theorem 2.1.** *Let  $\alpha \in \text{Aut } F_n$ . Suppose  $X \in \partial F_n$  is periodic of period  $q$  under  $\partial\alpha$ . Then  $q \leq M_n$ , where  $M_n$  depends only on  $n$  and  $\log M_n \sim \sqrt{n \log n}$  as  $n \rightarrow \infty$ .*

The quantity  $\sqrt{n \log n}$  is Landau's asymptotic estimate for  $\log g(n)$ , where  $g(n)$  is the maximum order of elements in the symmetric group  $S_n$  [8]. It is shown in [11] that the same estimate holds for the maximum order of torsion elements in  $GL(n, \mathbf{Z})$  and  $\text{Aut } F_n$ .

We first prove the following special case of Theorem 2.1:

**Lemma 2.2.** *If  $g \in F_n$  is periodic of period  $q$  under  $\alpha \in \text{Aut } F_n$ , then  $q \leq A_n$ , where  $A_n$  is the maximum order of torsion elements in  $\text{Aut } F_n$ .*

*Proof.* The subgroup  $\text{Fix } \alpha^q$  is  $\alpha$ -invariant, and the restriction of  $\alpha$  has order exactly  $q$  in  $\text{Aut}(\text{Fix } \alpha^q)$ . Since the rank of  $\text{Fix } \alpha^q$  is  $\leq n$  by [1], and  $\text{Aut } F_k$  naturally embeds into  $\text{Aut } F_\ell$  for  $k < \ell$ , the group  $\text{Aut } F_n$  contains an element of order  $q$ .  $\square$

**Remark.** Before the Scott conjecture was proved, Stallings showed [17] that, for a given  $\alpha$ , there is a bound for periods of elements  $g \in F_n$ .

*Proof of Theorem 2.1.* Lemma 2.2 shows that singular periodic points of  $\partial\alpha$  have period  $\leq A_n$ . Now suppose  $X$  is regular, say attracting.

The points  $X, \partial\alpha(X), \dots, \partial\alpha^{q-1}(X)$  are attracting fixed points of  $\partial\alpha^q$ . By Theorem 1 of [5], the action of  $\text{Fix } \alpha^q$  on the set of attracting fixed points of  $\partial\alpha^q$  has at most  $2n$  orbits. Thus there exist  $r \leq 2n$  and  $u \in \text{Fix } \alpha^q$  such that

$$\partial\alpha^r(X) = uX.$$

By Lemma 2.2 we have

$$\alpha^s(u) = u$$

for some  $s \leq A_n$ .

The above equations yield  $\partial\alpha^{rs}(X) = aX$  with

$$a = \alpha^{(s-1)r}(u) \dots \alpha^r(u)u.$$

If  $a = 1$  we get  $q \leq rs \leq 2nA_n$ . Otherwise we note that  $a \in \text{Fix } \alpha^s$ , and from  $X = \partial\alpha^{qrs}(X) = a^q X$  we conclude that  $X$  is singular, a contradiction.

We have thus shown  $q \leq M_n = 2nA_n$ . Since  $\log A_n \sim \sqrt{n \log n}$  by [11], we have  $\log M_n \sim \sqrt{n \log n}$ .  $\square$

**Remark.** The bound  $q \leq 2nA_n$  is not quite sharp. But if  $\alpha \in \text{Aut } F_n$  has order  $A_n$  then generic points of  $\partial F_n$  have period  $A_n$  under  $\partial\alpha$ . Therefore the estimate  $\log M_n \sim \sqrt{n \log n}$  cannot be improved.

**Theorem 2.3.** *For any  $\alpha \in \text{Aut } F_n$ , the map  $\partial\alpha : \partial F_n \rightarrow \partial F_n$  has at least two periodic points of period  $\leq 2n$ .*

For the automorphism defined by  $a_i \mapsto a_{i+1}$  ( $1 \leq i \leq n-1$ ),  $a_n \mapsto a_1^{-1}$ , every point of  $\partial F_n$  has period  $2n$ .

*Proof.* There are two cases. If  $\alpha$  has no periodic element  $g \neq 1$ , then  $\partial\alpha$  has at most  $2n$  periodic points of a given type (attracting or repelling) by Theorem 1 of [5]. The other case is taken care of by the following result.  $\square$

**Proposition 2.4.** *Let  $\alpha \in \text{Aut } F_n$ . If there is a nontrivial  $\alpha$ -periodic element  $g \in F_n$ , then there is one of period  $\leq 2n$ .*

*Proof.* Let  $q$  be the smallest period of nontrivial periodic elements. Arguing as in the proof of Lemma 2.2, we may assume that  $\alpha$  has order  $q$ . Such an  $\alpha$  may be realized as an automorphism of a graph ([4], [18]): there exist a finite graph  $\Lambda$ , an automorphism  $f$  of  $\Lambda$  fixing a vertex  $v$ , and an isomorphism  $F_n \rightarrow \pi_1(\Lambda, v)$  such that the following diagram commutes:

$$\begin{array}{ccc} F_n & \xrightarrow{\alpha} & F_n \\ \downarrow & & \downarrow \\ \pi_1(\Lambda, v) & \xrightarrow{f_*} & \pi_1(\Lambda, v). \end{array}$$

We choose  $\Lambda$  with minimal number of vertices. We claim that the action of  $\mathbf{Z}/q\mathbf{Z} = \langle f \rangle$  on the set of germs of edges at  $v$  is free. This will show  $q \leq 2n$  since  $v$  has valence at most  $2n$ .

Assume the action is not free. Then some  $f^r$  ( $1 \leq r \leq q-1$ ) fixes an edge containing  $v$ . Let  $\Lambda_0$  be the component of the fixed point set of  $f^r$  containing  $v$ . It is a tree since otherwise  $\alpha$  would have a nontrivial periodic element of period  $\leq r$ . We may therefore collapse  $\Lambda_0$  to a point, contradicting the choice of  $\Lambda$ .  $\square$



### 3. Growth rates

In this section we fix  $\Phi \in \text{Out } F_n$ , and sometimes also an automorphism  $\alpha \in \Phi$ . We write  $|g|$  for the word length of  $g \in F_n$ , and  $|\gamma|$  for the length of a conjugacy class  $\gamma$  (equal to the length of a cyclically reduced word representing  $\gamma$ ).

Let  $M$  be the transition matrix of a relative train track map representing  $\Phi$  (see [1]). The largest positive eigenvalue (spectral radius) of the matrix  $M$  is denoted  $\lambda(\Phi)$ , or  $\lambda(\alpha)$ . It is an algebraic integer of degree bounded by  $3n - 3$ .

For  $g \in F_n$ , the length of  $\alpha^p(g)$  is bounded from above by a constant times  $\|M\|^p |g|$ . If  $\lambda(\Phi) = 1$ , the growth of  $\alpha^p(g)$  is polynomial and  $\Phi$  is called *polynomially growing*. For future reference we note:

**Remark 3.1.** Given  $\nu > \lambda(\alpha)$ , there exists  $C > 0$  such that  $|\alpha^p(g)| \leq C\nu^p |g|$  for all  $g \in F_n$  and  $p \geq 1$ .

Now let  $\ell : F_n \rightarrow \mathbf{R}^+$  be the length function of an action of  $F_n$  on an  $\mathbf{R}$ -tree  $T$ . It is bounded from above by a constant times word length. In particular, if  $T$  is an  $\alpha$ -invariant  $\mathbf{R}$ -tree as in Theorem 1.3, we have (up to multiplicative constants)  $\lambda^p \ell(g) = \ell(\alpha^p(g)) \leq |\alpha^p(g)| \leq \|M\|^p |g|$  and therefore  $\lambda \leq \lambda(\alpha)$ . Conversely:

**Proposition 3.2.** *There exists an  $\alpha$ -invariant  $\mathbf{R}$ -tree  $T$  as in Theorem 1.3 with  $\lambda = \lambda(\alpha)$ .*

*Proof.* This is proved by the same arguments as in [5, section 2], but instead of using only the top stratum of the train track (which may lead to  $\lambda < \lambda(\alpha)$ ) we use the whole relative train track and an eigenvector  $v$  of  $M$  associated to  $\lambda(\alpha)$ . One shows that the resulting action on an  $\mathbf{R}$ -tree  $T$  is nontrivial and has trivial arc stabilizers as in [5]. Minimality of the action may be achieved by restricting to the minimal invariant subtree. It is often more convenient, though, to work with the metric completion  $\bar{T}$  of  $T$  so as to ensure that  $H$  has a fixed point  $Q$  when  $\lambda(\alpha) > 1$ .  $\square$

Now let  $J$  be a finitely generated malnormal subgroup of  $F_n$  (recall that  $J$  is *malnormal* if  $gJg^{-1} \cap J \neq \{1\} \implies g \in J$ ). We say that  $J$  is  $\Phi$ -periodic if there exist  $q \geq 1$  and  $\beta \in \Phi^q$  with  $\beta(J) = J$ . Note that, by malnormality, the class of  $\beta$  in  $\text{Out } J$  is uniquely determined.

For example, suppose that  $T$  is an  $\mathbf{R}$ -tree as in Theorem 1.3 and  $J = \text{Stab } Q$  for some branch point  $Q$ . Then  $J$  is malnormal (because arc stabilizers are trivial). By Theorem 1.4, it has rank  $< n$ . We claim that it is  $\Phi$ -periodic. Indeed, by Theorem 1.4 there exist  $m \in F_n$  and  $q \geq 1$  such that  $mH^q$  fixes  $Q$ . Denoting  $i_m(g) = mgm^{-1}$ , the automorphism  $\beta = i_m \circ \alpha^q \in \Phi^q$  maps  $J$  to itself.

If  $J$  is finitely generated, malnormal,  $\Phi$ -periodic, we define  $\lambda_J = \lambda(\beta|_J)^{\frac{1}{q}}$ .

**Proposition 3.3.** *Let  $\Phi \in \text{Out } F_n$ .*

- (1) *Each conjugacy class  $\gamma$  in  $F_n$  has a growth rate  $\lambda(\gamma) = \lim_{p \rightarrow +\infty} |\Phi^p(\gamma)|^{1/p}$ .*
- (2) *Given  $\lambda \geq 1$ , the following are equivalent:*
  - $\lambda = \lambda(\gamma)$  for some conjugacy class  $\gamma$ .
  - $\lambda = \lambda_J$  for some malnormal  $\Phi$ -periodic subgroup  $J \subset F_n$  of rank  $\leq n$ .

The existence of the limit in assertion 1 is folklore (compare [1, Remark 1.8]). Simple examples show that one cannot restrict to free factors in assertion 2.

*Proof.* The proof is by induction on  $n$ . Let  $T$  be an  $\alpha$ -invariant  $\mathbf{R}$ -tree with  $\lambda = \lambda(\Phi)$  (see Proposition 3.2). We distinguish two cases, by evaluating the length function on  $\gamma$ .

If  $\ell(\gamma) > 0$ , we write  $|\Phi^p(\gamma)| \geq \ell(\Phi^p(\gamma)) = \lambda^p \ell(\gamma)$  (up to a constant) and we conclude that  $\gamma$  has growth rate  $\lambda(\gamma) = \lambda = \lambda(\Phi)$  (recall that the exponential growth of  $\Phi^p(\gamma)$  is bounded from above by  $\lambda(\Phi)$ ). Note that there exist classes with  $\ell(\gamma) > 0$ , hence there exist classes with growth rate  $\lambda(\Phi)$ .

If  $\ell(\gamma) = 0$ , an element  $g \in F_n$  representing  $\gamma$  fixes some branch point  $Q \in T$ , and we argue by induction by considering  $\gamma$  as a conjugacy class in  $J = \text{Stab } Q$ . We have pointed out earlier that  $J$  is malnormal,  $\Phi$ -periodic, of rank  $< n$ . If  $\beta = i_m \circ \alpha^q$  leaves  $J$  invariant, note that, by quasiconvexity of  $J$ , the growth rate of  $\gamma$  under  $\beta|_J$  is the same as the growth rate of  $\gamma$ , viewed as a conjugacy class in  $F_n$ , under  $\Phi^q$ .

These arguments show that every  $\gamma$  has a growth rate, which is of the form  $\lambda_J$  with  $J$  as in the proposition. Conversely, given  $J$ , let  $\ell_J$  be the length function of a  $\beta_J$ -invariant tree with  $\lambda = \lambda(\beta_J)$ . Conjugacy classes with  $\ell_J(\gamma) > 0$  have growth rate  $\lambda_J$  under  $\Phi$ .  $\square$

**Definition.** We call  $\lambda(\Phi)$  the *top growth rate* of  $\Phi$ . The *set of growth rates*  $\Lambda(\Phi) \subset (1, \infty)$  consists of the growth rates  $\lambda(\gamma)$  which are bigger than 1.

Note that  $\Lambda(\Phi)$  consists of algebraic integers of degree  $\leq 3n - 3$ , and that  $\lambda(\Phi)$  is the largest element of  $\Lambda(\Phi) \cup \{1\}$ . See [10] for more results about  $\Lambda(\Phi)$ .

## 4. Hölder dynamics

### Superattractivity

The discussion in this subsection (including Proposition 4.1) is valid for automorphisms of arbitrary (word) hyperbolic groups, but for simplicity we restrict to the case of  $F_n$  (the generalization is almost immediate using [13]).

Fixing a free basis of  $F_n$ , we may view  $\partial F_n$  as the set of right-infinite reduced words. Let  $X \in \partial F_n$  be a fixed point of the homeomorphism  $\partial\alpha$  induced by

$\alpha \in \text{Aut } F_n$  on  $\partial F_n$ . We say that  $X$  is *singular* if it belongs to the limit set of the fixed subgroup  $\text{Fix } \alpha$ , *regular* otherwise (recall that  $\text{Fix } \alpha$  has finite rank).

As explained in [5], there is a basic trichotomy: *either  $X$  is singular, or  $X$  is attracting, or  $X$  is repelling (i.e. attracting for  $\alpha^{-1}$ )*. Attractivity has a strong meaning here (see section 1 of [5]): given  $A$ , there exists  $m$  such that for  $Y \in F_n \cup \partial F_n$

$$c_X Y \geq m \implies c_X(\partial\alpha(Y)) - c_X Y > A, \quad (1)$$

where  $c_X Y$  is the length of the maximal common initial segment between the reduced words  $X$  and  $Y$  (i.e. the Gromov scalar product  $\langle X, Y \rangle$  with basepoint at the identity in the Cayley graph).

In particular, we have  $\lim_{p \rightarrow +\infty} \bar{\alpha}^p(Y) = X$  uniformly on a neighborhood of  $X$  in  $F_n \cup \partial F_n$  if  $X$  is attracting (whereas if  $X$  is singular there are fixed points of  $\alpha$  in  $F_n$  arbitrarily close to  $X$ ). For the automorphism  $\beta$  studied in Example 1.2, the (singular) fixed points  $a^{\pm\infty}$  of  $\partial\beta$  are partly repelling and partly attracting: for any  $k \in \mathbf{Z}$  we have  $\lim_{p \rightarrow +\infty} \partial\beta^p(a^k b Y) = a^\infty$  if  $Y$  is a right-infinite reduced word not starting with  $b^{-1}$ , but  $\lim_{p \rightarrow +\infty} \partial\beta^p(a^k b^{-1} Y) = a^{-\infty}$  if  $Y$  does not start with  $b$ .

Also note that an isolated fixed point of  $\partial\alpha$  is singular if and only if it belongs to the limit set of an  $\alpha$ -invariant cyclic subgroup (for the “only if” direction, simply observe that  $\alpha$  leaves invariant the stabilizer of  $X$  for the action of  $F_n$  on  $\partial F_n$ ). In particular, the natural action of  $\text{Fix } \alpha$  on the set of regular fixed points of  $\partial\alpha$  is free. This action has finitely many orbits [2], indeed it follows from [5] that the number of orbits is at most  $4n$ . It is not clear to us whether there is a bound depending only on  $G$  when  $G$  is an arbitrary hyperbolic group.

Now recall that the boundary of  $F_n$  (of any hyperbolic group, in fact) has a canonical *Hölder structure* (see [3], [7]). It may be viewed as a collection  $\mathcal{D}$  of distance functions on  $\partial F_n$  that are pairwise bi-Hölder equivalent: Given  $d, d' \in \mathcal{D}$ , there exist  $C > 0$  and  $\beta \in (0, 1]$  such that  $\frac{1}{C} d^{\frac{1}{\beta}} \leq d' \leq C d^\beta$ . This Hölder structure is preserved by  $\partial\alpha$  for every  $\alpha \in \text{Aut } F_n$ . If  $J \subset F_n$  has finite rank, the inclusion  $\partial J \hookrightarrow \partial F_n$  is bi-Hölder.

We represent the Hölder structure by the visual metrics  $d_\varepsilon(X, Y) = \exp(-\varepsilon c_X Y)$ .

Let  $X \in \partial F_n$  be a fixed point of  $\partial\alpha$ , and  $d = d_\varepsilon$  a visual metric. If  $X$  is regular, attracting, it follows from (1) that

$$\lim_{Y \rightarrow X} \frac{d(\partial\alpha(Y), X)}{d(Y, X)} = 0. \quad (2)$$

If  $X$  is repelling or singular, however, the above quotient is bounded away from 0 on a neighborhood of  $X$  (if  $X$  is singular,  $c_X(\partial\alpha(Y)) - c_X Y$  is bounded near  $X$  because  $\text{Fix } \alpha$  is quasiconvex and  $\alpha$  is a quasi-isometry).

Thus (2) is a metric characterization of attracting regular fixed points, similar to the characterization of a superattracting fixed point  $c$  of a holomorphic map

$f : \mathbf{C} \rightarrow \mathbf{C}$  by  $f'(c) = 0$ . For this reason, we call an attracting regular point *superattracting* (and a repelling regular point *superrepelling*).

Of course the map  $\partial\alpha$  is a homeomorphism, and superattracting fixed points may exist only because  $\partial\alpha$  is bi-Hölder but in general not bi-Lipschitz. For instance, if  $t$  is any lift to the Poincaré disc of a pseudo-Anosov diffeomorphism of a closed hyperbolic surface, then the homeomorphism induced by  $t$  on the circle at infinity is never bi-Lipschitz (see Remark (22.14) in [12]).

Characterization (2) above does not depend on the chosen visual metric  $d$ , but it is not valid for arbitrary metrics in  $\mathcal{D}$ . The following characterization will apply to every  $d \in \mathcal{D}$ .

**Proposition 4.1.** *Let  $\alpha \in \text{Aut } F_n$ . A fixed point  $X$  of  $\partial\alpha$  is superattracting if and only if*

$$\lim_{p \rightarrow +\infty} \frac{1}{p} \log d(\partial\alpha^p(Y), X) = -\infty$$

*for  $Y \in \partial F_n$  close to  $X$ , where  $d$  is any metric on  $\partial F_n$  defining the Hölder structure.*

This equation means that  $\partial\alpha^p(Y)$  converges to  $X$  super-exponentially as  $p \rightarrow \infty$ . Unlike (2), it is true for every metric in  $\mathcal{D}$  if it is true for one.

*Proof.* We may assume that  $d$  is a visual metric. Suppose  $X$  is superattracting. We have to prove  $\lim_{p \rightarrow \infty} \frac{1}{p} c_X(\partial\alpha^p(Y)) = +\infty$  for  $Y$  close to  $X$ . Given  $A > 0$ , let  $m$  be as in (1). If  $\lim_{p \rightarrow \infty} \partial\alpha^p(Y) = X$ , there exists  $n_0$  such that  $c_X(\partial\alpha^p(Y)) \geq m$  for  $p \geq n_0$ . For  $p$  large, we then have

$$c_X(\partial\alpha^p(Y)) \geq A(p - n_0) + m,$$

and the result follows.

Conversely, if  $X$  is singular, then  $c_X(\partial\alpha(Z)) - c_X Z$  is bounded in a neighborhood of  $X$ , and therefore  $\frac{1}{p} \log d(\partial\alpha^p(Y), X)$  is bounded from below as  $p \rightarrow \infty$ .  $\square$

### Speed of convergence

We consider  $\alpha \in \text{Aut } F_n$ , and the associated  $\Phi \in \text{Out } F_n$ . Recall that  $\Lambda(\Phi) \subset (1, \infty)$  is the set of nontrivial growth rates. It may also be viewed as a set of  $\lambda_J$  (see Proposition 3.3).

**Theorem 4.2.** *Let  $\alpha \in \text{Aut } F_n$ . Let  $X \in \partial F_n$  be a superattracting fixed point of  $\partial\alpha$ . There exists  $\lambda = \lambda(\alpha, X) \in \Lambda(\Phi) \cup \{1\}$  such that*

$$\lim_{p \rightarrow +\infty} \frac{1}{p} \log \left( -\log d(\partial\alpha^p(Y), X) \right) = \log \lambda \quad (3)$$

for  $Y \in \partial F_n$  close to  $X$  (and any metric  $d$  on  $\partial F_n$  defining the Hölder structure).

Conversely, given  $\mu \in \Lambda(\Phi)$ , there exist  $q \geq 1$ , an automorphism  $\beta \in \Phi^q$ , and a superattracting fixed point  $X$  of  $\partial\beta$  with  $\lambda(\beta, X) = \mu^q$ .

It follows that the set  $\Lambda_h(\Phi)$  of Hölder exponents defined in the introduction equals  $\Lambda(\Phi)$ . Note that replacing  $d$  by a metric bi-Hölder equivalent to  $d$  does not affect the validity of (3).

*Proof of Theorem 4.2.* We fix a basis of  $F_n$  and we consider the corresponding Cayley tree  $\Gamma$ .

Let  $X$  be a superattracting fixed point of  $\partial\alpha$ . We need to prove

$$\lim_{p \rightarrow +\infty} \frac{1}{p} \log c_X(\partial\alpha^p(Y)) = \log \lambda.$$

We will bound the left-hand side, first from above and then from below.

**Lemma 4.3.** *Suppose  $X \in \partial J$ , with  $J \subset F_n$  a finitely generated  $\alpha$ -invariant malnormal subgroup. Then*

$$\limsup_{p \rightarrow +\infty} \frac{1}{p} \log c_X(\partial\alpha^p(Y)) \leq \log \lambda_J$$

for all  $Y \in \partial F_n$ .

Recall that  $\lambda_J$  is the top growth rate of  $\alpha|_J$ .

*Proof.* Let  $x_{t_p}$  be the projection of  $\partial\alpha^p(Y)$  onto the geodesic from  $1 \in F_n$  to  $X$  in  $\Gamma$ . By quasiconvexity of  $J$ , we can find  $j_p \in J$  within a fixed distance of  $x_{t_p}$ . We need to prove  $\limsup_{p \rightarrow +\infty} \frac{1}{p} \log |j_p| \leq \log \lambda_J$ . We will work with word length  $|j_p|_J$  in  $J$ , which is comparable to  $|j_p|$ .

Define  $w_p \in J$  by  $j_p = \alpha(j_{p-1})w_p$ . Since  $\alpha$  is a quasi-isometry, there is a uniform bound for  $|w_p|$ , hence also for  $|w_p|_J$  because  $J$  is quasiconvex. Now write

$$j_p = \alpha^p(j_0)\alpha^{p-1}(w_1) \cdots \alpha(w_{p-1})w_p.$$

For  $\nu > \lambda_J$  we have

$$|j_p|_J \leq C\nu^p |j_0|_J + C\nu^{p-1} |w_1|_J + \cdots + C\nu |w_{p-1}|_J + |w_p|_J,$$

with  $C$  given by Remark 3.1. Thus  $|j_p|_J = O(\nu^p)$  for all  $\nu > \lambda_J$ , and the lemma is proved.  $\square$

**Corollary 4.4.** *Theorem 4.2 holds if  $\alpha$  is polynomially growing (i.e.  $\lambda(\alpha) = 1$ ), with  $\lambda(\alpha, X) = 1$ .  $\square$*

Fix a subgroup  $J$  as in Lemma 4.3, and consider an  $\mathbf{R}$ -tree  $T$  with an action of  $J$  satisfying the conditions of Theorem 1.3 with respect to  $\alpha|_J$ . Using Proposition 3.2, we assume that the stretching factor of the homothety  $H$  equals  $\lambda_J$ . Suppose furthermore  $\lambda_J > 1$ .

**Lemma 4.5.** *Suppose  $X = j(\rho)$ , where  $\rho$  is an eigenray of  $H : T \rightarrow T$  (in particular,  $X \in \partial J$ ). Then*

$$\liminf_{p \rightarrow +\infty} \frac{1}{p} \log c_X(\partial \alpha^p(Y)) \geq \log \lambda_J$$

for  $Y \in \partial F_n$  close enough to  $X$ .

*Proof.* With the notations of Section 1, let  $Q \in \overline{T}$  be the fixed point of  $H$  (i.e. the origin of  $\rho$ ). Choose  $j_p$  as in the proof of Lemma 4.3 and define  $d_p$  as  $\overline{d}(Q, j_p Q)$  (where  $\overline{d}$  denotes the distance in  $\overline{T}$ ). Note that

$$\overline{d}(Q, \alpha(j_p)Q) = \overline{d}(Q, \alpha(j_p)HQ) = \overline{d}(Q, H j_p Q) = \lambda_J \overline{d}(Q, j_p Q).$$

On the other hand, recall that the distance in  $J$  from  $\alpha(j_p)$  to  $j_{p+1}$  is bounded independently of  $p$  (and of  $Y$ ). Thus we obtain an inequality of the form

$$d_{p+1} \geq \lambda_J d_p - A,$$

with  $A$  independent of  $p$  and  $Y$ .

If  $Y$  is close enough to  $X$  in  $\partial F_n$ , then  $j_0$  is close to  $X$  in  $J \cup \partial J$ , and therefore  $d_0$  is large (by bounded backtracking, see section 3 of [5]). This implies

$$\liminf_{p \rightarrow +\infty} \frac{1}{p} \log d_p \geq \log \lambda_J.$$

Finally, we observe that  $d_p = \overline{d}(Q, j_p Q)$  is bounded above by a constant times  $|j_p|_J$ , hence by a constant times  $|j_p|$ .  $\square$

Now we complete the proof of Theorem 4.2. If  $\lambda(\alpha) = 1$ , then we are done by Corollary 4.4. Assume  $\lambda(\alpha) > 1$ , and consider a tree  $T$  as in Proposition 3.2, with stretching factor  $\lambda(\alpha)$ . If  $X = j(\rho)$  as in Lemma 4.5, we are done, with  $\lambda = \lambda(\alpha)$ . If not, then by Proposition 4.3 of [5] we have  $X \in \partial \text{Stab } Q$ , where  $Q \in T$  is the fixed point of  $H$  (recall that points of  $\overline{T} \setminus T$  have trivial stabilizer).

The subgroup  $\text{Stab } Q$  is  $\alpha$ -invariant, malnormal, and has rank  $< n$  (see section 3). Repeat the argument, working with  $\alpha|_{\text{Stab } Q}$ . After a finite number of steps we find that  $X \in \partial J$  (with  $J$  invariant, malnormal, of rank  $< n$ ), and either  $\lambda_J = 1$  or  $X = j(\rho)$ . It follows from Lemmas 4.3 and 4.5 that Theorem 4.2 holds, with  $\lambda(\alpha, X) = \lambda_J$ .

Conversely, consider  $\mu \in \Lambda(\Phi)$ . First suppose  $\mu = \lambda(\alpha)$ . Consider an  $\mathbf{R}$ -tree  $T$  as in Theorem 1.3, with  $\lambda = \lambda(\alpha)$ . By Theorem 1.4 and Proposition 1.5, there exist  $m \in F_n$  and  $q \geq 1$  such that  $mH^q$  has an eigenray  $\rho$ . Let  $\beta = i_m \circ \alpha^q$ , with  $i_m(g) = mgm^{-1}$ . Then  $X = j(\rho)$  is a fixed point of  $\partial\beta$ , and  $\lambda(\beta, X) = \lambda(\beta) = \mu^q$ .

For arbitrary  $\mu = \lambda_J \in \Lambda(\Phi)$ , let  $\alpha' \in \Phi^r$  leave  $J$  invariant. The previous argument yields  $\beta \in \Phi^{rq}$  and a fixed point  $X$  of  $\partial\beta$  in  $\partial J$  such that  $\lambda(\beta|_J, X) = \mu^{rq}$ . Since the inclusion  $\partial J \hookrightarrow \partial F_n$  is bi-Hölder,  $\lambda(\beta, X) = \lambda(\beta|_J, X)$  has the required form.  $\square$

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