Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 75 (2000)

Artikel: Universal octonary diagonal forms over some real quadratic fields
Autor: Kim, Byeong Moon
DOI: https://doi.org/10.5169/seals-56626

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 06.02.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Comment. Math. Helv. 75 (2000) 410–414
0010-2571/00/030410-5 $ 1.50+0.20/0

© 2000 Birkhäuser Verlag, Basel

Commentarii Mathematici Helvetici

Universal octonary diagonal forms over some real quadratic fields

Byeong Moon Kim

Abstract. In this paper, we will prove there are infinitely many integers n such that n^2-1 is square-free and $\mathbb{Q}(\sqrt{n^2-1})$ admits universal octonary diagonal quadratic forms.

Mathematics Subject Classification (2000). Primary 11E12, Secondary 11E20.

Keywords. Universal quadratic forms, real quadratic fields.

1. Introduction

A universal integral form over totally real number field K is a positive definite quadratic form over the ring of integers of K which represents all the totally positive integers of K. For example, the sum of four squares is universal integral over $\mathbb{Q}$. In 1917, Ramanujan [8] found there are exactly 54 universal positive diagonal integral quadratic forms over $\mathbb{Q}$. More concretely, he showed there are 54 diagonal quaternary quadratic forms $f(x,y,z,w)=ax^2+by^2+cz^2+dw^2$ such that $a,b,c,d\in\mathbb{Z}^+$ and the equation $f=n$ is solvable for all $n\in\mathbb{Z}^+$. In 1947, M. Willerding [10] proved there are exactly 178 classic universal integral forms. More concretely, she showed there are 178 quaternary quadratic forms $f(x,y,z,w)$ up to equivalence such that f is positive definite integral quadratic form, the coefficients of cross terms of f are always even and the equation $f=n$ is solvable for all $n\in\mathbb{Z}^+$. On the other hand, the study of positive universal quadratic integral forms over totally real number fields was initiated by F. Götzky [3]. In 1928, he proved that the sum of four squares is universal over $\mathbb{Q}(\sqrt{5})$. H. Maass [6] improved this result. In 1941, he proved the sum of three squares is positive universal over $\mathbb{Q}(\sqrt{5})$. Four years later, Siegel [9] proved $\mathbb{Q}(\sqrt{5})$ is the only totally real number field, other than $\mathbb{Q}$, over which every (totally) positive integer is a sum of squares. In other words, he showed if a totally real number field K is different from $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{5})$, there is a totally positive algebraic integer α of K which cannot be represented by the sum of any number of squares. For example, if $K=\mathbb{Q}(\sqrt{2})$, $\alpha=2+\sqrt{2}$. In 1996, W. K. Chan, M.-H. Kim and S. Raghavan [1]

classified all (totally) positive universal integral ternary lattices over real quadratic fields. Only $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$ admit universal integral ternary lattices and total number of universal integral ternary lattices over real quadratic fields is 11. Recently, the author [5] proved there are only finitely many real quadratic fields which admit universal integral septenary diagonal forms. The content of this paper is to prove if n^2-1 is square-free, there are universal octonary diagonal forms over $\mathbb{Q}(\sqrt{n^2-1})$. So we can prove there are infinitely many real quadratic fields which admit universal integral octonary diagonal forms. Obviously 8 is the minimal rank with this property.

2. Main Theorem

Throughout this chapter, we let $m = n^2 - 1$ be a positive square free integer, $K = \mathbb{Q}(\sqrt{m})$ and $\mathcal{O}_K$ be the ring of algebraic integers of K. Note that $\epsilon = n+\sqrt{m}$ is the fundamental unit of $\mathcal{O}_K$ and is totally positive.

Theorem 1. *The octonary diagonal form* $x_1^2+x_2^2+x_3^2+x_4^2+\epsilon x_5^2+\epsilon x_6^2+\epsilon x_7^2+\epsilon x_8^2$ *is universal over* $\mathcal{O}_K$.

This Theorem is a consequence of following Lemmas.

Lemma 1. *Let* $1 \le b < 2n$. $\alpha = a + b\sqrt{m}$ *is totally positive algebraic integer in* K *if and only if* $nb \le a$.

Proof. As $nb + b\sqrt{m} = b(n+\sqrt{m})$ is totally positive, the necessity is trivial. For the sufficiency, it suffices to prove $nb - 1 - (b\sqrt{m}) < 0$. This follows from

$$(nb-1)^2 - (b\sqrt{m})^2 = n^2b^2 - 2nb + 1 - b^2(n^2-1)$$

$$= (b-n)^2 - n^2 + 1 \le (n-1)^2 - n^2 + 1 < 0.$$

□

Lemma 2. *If* $\alpha \in \mathcal{O}_K^+$, α *belongs to*

$$S = \{a_0\epsilon^k + a_1\epsilon^{k+1} + \ldots + a_l\epsilon^{k+l} \mid k, l \in \mathbb{Z},\ a_0, a_1, \ldots, a_l \in \mathbb{N}\}.$$

Proof. Suppose $\alpha = a + b\sqrt{m} \notin S$. We may assume that $b > 0$ and $\mathrm{tr}_{K/\mathbb{Q}}(\alpha) \le \mathrm{tr}_{K/\mathbb{Q}}(\beta)$ for all elements $\beta \notin S$. Then, by Lemma 1, we have $b \ge 2n$. Since

$$bn - 1 + b\sqrt{m} = \epsilon^2 + (b-2n)\epsilon \in S,$$

we also have $a \leq bn - 1$. Then,

$$\alpha\epsilon^{-1} = (a + b\sqrt{m})(n - \sqrt{m}) = an - bm + (bn - a)\sqrt{m}.$$

So

$$\begin{aligned}\mathrm{tr}_{K/\mathbb{Q}}(\alpha\epsilon^{-1}) &= 2(an - bm) \leq 2(n(bn-1) - b(n^2-1)) \\ &= 2(b - n) < 2a = \mathrm{tr}_{K/\mathbb{Q}}(\alpha).\end{aligned}$$

So $\alpha\epsilon^{-1} \in S$. Thus $\alpha \in S$. Contradiction. □

Lemma 3. *For $l \geq 2$,$\epsilon^l = -1 + b_1\epsilon + b_2\epsilon^2 + \ldots + b_{l-1}\epsilon^{l-1}$ where $b_1 \geq 2n - 1$ and $b_2, \ldots, b_{l-1} \geq 2n - 2$.*

Proof. We use induction on l. As $\epsilon^2 = 2n\epsilon - 1$, the assertion holds for $l = 2$. If this Lemma is true for $l = s \geq 2$,

$$\begin{aligned}\epsilon^{s+1} = \epsilon\epsilon^s &= \epsilon(-1 + b_1\epsilon + b_2\epsilon^2 + \ldots + b_{s-1}\epsilon^{s-1}) \\ &= -\epsilon + \epsilon^2 + (b_1 - 1)\epsilon^2 + b_2\epsilon^2 + \ldots + b_{s-1}\epsilon^s \\ &= -1 + (2n-1)\epsilon + (b_1 - 1)\epsilon^2 + b_2\epsilon^2 + \ldots + b_{s-1}\epsilon^s.\end{aligned}$$

This proves the Lemma. □

Lemma 4. *If $\alpha \in \mathcal{O}_K^+$, $\alpha = p\epsilon^k + q\epsilon^{k+1}$ for some $p, q \in \mathbb{N}$ and $k \in \mathbb{Z}$.*

Proof. By Lemma 2, $\alpha = a_k\epsilon^k + \ldots + a_{k+l}\epsilon^{k+l}$ with $a_k, \ldots, a_{k+l} \geq 0$.

If $l \geq 2$ and $a_{k+l} \leq a_k$,

$$\begin{aligned}\alpha &= a_k\epsilon^k + \ldots + a_{k+l-1}\epsilon^{k+l-1} + a_{k+l}\epsilon^k(-1 + b_1\epsilon + \ldots + b_{l-1}\epsilon^{l-1}) \\ &= (a_k - a_{k+l})\epsilon^k + (a_{k+1} + a_{k+l}b_1)\epsilon^{k+1} + \ldots + (a_{k+l-1} + a_{k+l}b_{l-1})\epsilon^{k+l-1}.\end{aligned}$$

If $l \geq 2$ and $a_{k+l} \geq a_k$,

$$\begin{aligned}\alpha &= a_k\epsilon^k + \ldots + a_{k+l-1}\epsilon^{k+l-1} + (a_{k+l} - a_k)\epsilon^{k+l} + a_k\epsilon^k(-1 + b_1\epsilon + \ldots + b_{l-1}\epsilon^{l-1}) \\ &= (a_k + a_{k+l}b_1)\epsilon^{k+1} + \ldots + (a_k + a_{k+l}b_{l-1})\epsilon^{k+l-1} + (a_{k+l} - a_k)\epsilon^{k+l}.\end{aligned}$$

Repeating the same process, we can obtain the desired expression of α. □

Proof of Theorem 1. If $\alpha \in \mathcal{O}_K^+$, by Lemma 4, $\alpha = a\epsilon^k + b\epsilon^{k+1}$ for some a, $b \in \mathbb{N}$ and $k \in \mathbb{Z}$. If k is even, by Lagrange's four square theorem, $a\epsilon^k$ is represented by $x_1^2 + x_2^2 + x_3^2 + x_4^2$ and $b\epsilon^{k+1}$ is represented by $\epsilon x_5^2 + \epsilon x_6^2 + \epsilon x_7^2 + \epsilon x_8^2$. So f represents α. Similarly f represents α for odd k. Thus f is universal integral over K. □

Lemma 5. *There are infinitely many square free integers of the form* $n^2 - 1$.

Proof. If n is even, $n^2 - 1$ is square free if and only if both $n+1$ and $n-1$ are square free. It is known that [4] the number of positive square free integers which do not exceed x is $\frac{6x}{\pi^2} + O(\sqrt{x})$. So the number of positive integer n such that $n \leq x$ and both $n+1$ and $n-1$ are square free is larger than

$$(\frac{6x}{\pi^2} + O(\sqrt{x})) + (\frac{6x}{\pi^2} + O(\sqrt{x})) - x = \frac{12 - \pi^2}{\pi^2}x + O(\sqrt{x}).$$

Since $\frac{12-\pi^2}{\pi^2} > 0$, there are infinitely many n such that $n \leq x$ and $n^2 - 1$ is square free. □

Theorem 2. *There are infinitely many real quadratic fields that admit octonary universal forms.*

Proof. This is an immediate consequence of Theorem 1 and Lemma 5. □

Acknowledgement

The content of this paper is a part of author's thesis. The author wishes to represent his greatest thanks to his advisor Prof. M.-H. Kim of Seoul National University for his kind advice and careful revision of manuscript.

References

[1] Chan, W. K., Kim, M-H., Raghavan,S., Ternary Universal Quadratic Forms over Real Quadratic Fields, *Japanese J. Math.* **22** (1996), 263-273.
[2] Dixon, L. E., Quaternary Quadratic Forms Representing All integers, *Amer. J. Math.* **49** (1927), 39-56.
[3] Götzky, F., Über eine Zahlentheoretische Anwendung von Modulfunktionen einer Veränderlichen, *Math. Ann.* **100** (1928), 411-437.
[4] Hardy, G. H., *An introduction to the theory of numbers*, fifth edition, Oxford, 1979.
[5] Kim, B. M., Finiteness of Real Quadratic Fields which admit a Positive Integral Diagonal Septanary Universal Forms, preprint.
[6] Maass, H., Über die Darstellung total positiver des Körpers $R(\sqrt{5})$ als Summe von drei Quadraten, *Abh. Math. Sem. Hamburg* **14** (1941), 185-191.
[7] O'Meara, O. T., *Introduction to quadratic forms*, Springer Verlag, 1973.
[8] Ramanujan, S., On the Expression of a Number in the Form $ax^2 + by^2 + cz^2 + dw^2$, *Proc. Cambridge Phil. Soc.* **19** (1917), 11-21.
[9] Siegel, C. L., Sums of m-th Powers of Algebraic Integers, *Ann. Math.* **46** (1945), 313-339.
[10] Willerding, M. F., Determination of all classes of positive quaternary quadratic forms which represent all (positive) integers, *Bull. Amer. Math. Soc.* **54**, 334-337.

Byeong Moon Kim
Department of Mathematics
College of Natural Science
Kangnung National University
123 Chibyon-Dong Kangnung
Kangwon-do 210-702
Korea
e-mail: kbm@knusun.kangnung.ac.kr

(Received: November 2, 1998)