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Commentarii Mathematici Helvetici

On a problem of Nazarova and Roiter

Bangming Deng

Abstract. In the present paper we introduce the notion of representations of a bush which is a generalization of matrix problems (self–reproducing systems) introduced by Nazarova and Roiter. We show that the problem of classifying representations of clannish algebras come down to such generalized matrix problems. Based on the classification of Crawley–Boevey, we provide a description of indecomposable representations of bushes over any field. The proof is based on a categorical formulation of the matrix reduction of Nazarova and Roiter.

Mathematics Subject Classification (2000). 16G10, 16G20, 18A25.

Keywords. Rods, bushes, clannish algebras, self–reproducing systems.

Introduction

In the present article, we consider a generalization of matrix problems (self–reproducing systems) introduced by Nazarova and Roiter [8]. Their motivation was to solve a problem posed by Gelfand [6]: classify the indecomposable representations of the quiver

$$\begin{tikzcd} \bullet \arrow[r, shift left, "a_1"] & \bullet \arrow[l, shift left, "b_1"] \arrow[r, shift right, "b_2"'] & \bullet \arrow[l, shift right, "a_2"'] \end{tikzcd}$$

subjected to the relation $a_1b_1 = a_2b_2$.

In [2] Crawley–Boevey reconsiders the problem and introduces a new class of matrix problems called “clans”. The approach used in [2] is the functorial filtration method. It seems to us that both the notion of a clan and the functorial method are not well adapted to the problem treated by Crawley–Boevey.

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Our aim here is to replace clans by a generalization of self–reproducing systems and to use the method presented in [8] instead of the functorial one. Our method also works for fields of cardinality 2, a case which Crawley–Boevey was unable to handle with his method. Our classification however is based on that of Crawley–Boevey. For the proof of our classification theorem we use a categorical formulation of the matrix reduction of Nazarova and Roiter (see [3]).

After the completion of a preliminary version of the present paper, Prof. Sergejchuk pointed out to me that the matrix problems considered here have been studied by Bondarenko [1].

Throughout the paper, k denotes an arbitrary field.

The terminology used throughout the paper is taken from [5].

1. Tangles and Bushes

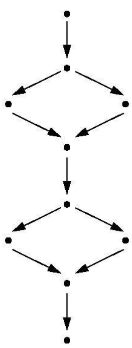
1.1. Let $\mathcal{A}$ be an aggregate over a field k with spectroid $\mathcal{S}$. A *tangle* over $\mathcal{A}$ is a pair (M^-, M^+) formed by sequences $M^- = (M_1^-, \cdots, M_n^-)$ and $M^+ = (M_1^+, \cdots, M_n^+)$ of pointwise finite left $\mathcal{A}$–modules. Given such a tangle, we denote by $\operatorname{rep}(M^-, M^+)$ the aggregate whose objects are the *representations* of (M^-, M^+), i.e. the sequences $(X; f_1, \cdots, f_n)$ where $X \in \mathcal{A}$ and $f_i \in \operatorname{Hom}_k(M_i^-(X), M_i^+(X))$, $i = 1, \cdots, n$. A *morphism* from $(X; f_1, \cdots, f_n)$ to $(X'; f_1', \cdots, f_n')$ is given by a morphism $\mu \in \mathcal{A}(X, X')$ such that $f_i' M_i^-(\mu) = M_i^+(\mu) f_i$ for $i = 1, \cdots, n$.

1.2. Our aim is to classify the indecomposables of $\operatorname{rep}(M^-, M^+)$ for particular tangles (M^-, M^+) which we describe now.

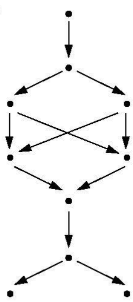
By definition, a *rod* is a finite ordered set R such that each $x \in R$ admits at most one $y \in R$ satisfying $y \lessgtr x$ (i.e. incomparable with x).

Examples. The ordered sets with the following Hasse–quivers are rods:

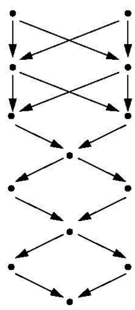
R_1



R_2



R_3



A tangle (M^-, M^+) is called *rodded* if the following conditions are satisfied:
(R1) For each i, the lattices of submodules of M_i^- and M_i^+ are rods;
(R2) For each $s \in \mathcal{S}$,

$$\sum_{i=1}^{n} (\dim_k M_i^-(s) + \dim_k M_i^+(s)) \leq 2;$$

(R3) If the submodules of M_i^ε generated by elements $a \in M_i^\varepsilon(s)$ and $b \in M_i^\varepsilon(t)$ are incomparable for some i, some $s, t \in \mathcal{S}$ and some $\varepsilon \in \{-, +\}$, then

$$\sum_{j,\eta} \dim_k M_j^\eta(s) = 1 = \sum_{j,\eta} \dim_k M_j^\eta(t);$$

(R4) For any $s, t \in \mathcal{S}$, the canonical map

$$\theta(s,t) : \mathcal{R}_{\mathcal{S}}(s,t) \longrightarrow \prod_{i=1}^{n} \mathcal{R}_i^-(s,t) \times \mathcal{R}_i^+(s,t)$$

is surjective.

Here $\mathcal{R}_{\mathcal{S}}$ denotes the radical of $\mathcal{S}$ and $\mathcal{R}_i^\varepsilon(s,t)$ the set of all $f \in \mathrm{Hom}_k(M_i^\varepsilon(s), M_i^\varepsilon(t))$ satisfying $f(N(s)) \subset \mathcal{R}N(t)$ for each submodule N of M_i^ε.

1.3. Given a tangle (M^-, M^+) over $\mathcal{A}$, we denote by $\mathcal{I}$ the intersection of the annihilators of all M_i^- and M_i^+. The tangle is called *faithful* if $\mathcal{I} = 0$. In the case of a faithful rodded tangle, the maps $\theta(s,t)$ are bijective. Our purpose is to give a concrete construction of faithful rodded tangles.

Let S be a pair formed by two sequences of disjoint rods $S^- = (S_1^-, \cdots, S_n^-)$ and $S^+ = (S_1^+, \cdots, S_n^+)$. We then equip the union $|S| = \cup_{i=1}^n (S_i^- \cup S_i^+)$ with the smallest order relation containing the order relations of the rods S_i^- and S_i^+. If there is no risk of confusion, we simply write S instead of $|S|$. By kS we denote the spectroid whose objects are the elements of S, whose morphism–spaces $kS(x,y)$ are one–dimensional with basis $(y|x)$ if $y \geq x$, or else are 0. The composition is such that $(z|y) \circ (y|x) = (z|x)$ [5]. Each interval I of S gives rise to a module k_I over kS such that $k_I(x) = 0$ if $x \notin I$ and $k_I(y) = k$, $k_I(z|y) = \mathbb{1}_k$ if $y, z \in I$ and $y \leq z$ [5]. We set $L_i^- = k_I$ if $I = S_i^-$ and $L_i^+ = k_I$ if $I = S_i^+$.

Let further $\sim$ be an equivalence relation on S such that:
(E1) Each equivalence class contains at most two elements;
(E2) In case x, $y \in S_i^\varepsilon$ and $x \lessgtr y$, the equivalence class of x consists of x only.

The S together with the equivalence relation is called a *bush*.

Let $\mathcal{S}$ denote the spectroid whose objects are the equivalence classes of S, whose spaces of radical morphisms are $\mathcal{R}_{\mathcal{S}}(a,b) = \oplus_{x \in a, y \in b, y > x} k(y|x)$, whose composition is such that $(z|y') \circ (y|x)$ is $(z|x)$ if $y' = y$ and 0 otherwise. Let further $\mathcal{A} := \oplus\mathcal{S}$ denote the *additive hull* of $\mathcal{S}$, whose objects are sequences $(X_1, \cdots, X_l)$ of objects of $\mathcal{S}$, whose morphisms $(X_1, \cdots, X_l) \to (Y_1, \cdots, Y_m)$ are identified with the "matrices" $\mu = [\mu_{ji}] \in \oplus_{i,j} \mathcal{S}(X_i, Y_j)$. The composition of morphisms obeys the rules of matrix multiplication. Then each module L over kS provides a module $\tilde{L}$ over $\mathcal{A}$ such that $\tilde{L}(a) = \oplus_{x \in a} L(x)$ for each $a \in \mathcal{S}$; the action of $\tilde{L}(y|t)$ on $m \in L(x) \subset \tilde{L}(a)$ coincides with that of $L(y|x)$ if $x = t$ or else is 0. In case $L = L_i^-$ (resp. L_i^+), we set $\tilde{L} = M_i^-$ (resp. M_i^+), thus obtaining a tangle (M^-, M^+) over $\mathcal{A}$. This tangle is faithful and rodded.

In the sequel, the representations of (M^-, M^+) will be simply called *representations* of S.

Proposition. *For each faithful rodded tangle (N^-, N^+) over an aggregate $\mathcal{B}$ with spectroid $\mathcal{T}$, there is a bush S as above and an equivalence $\Phi : \mathcal{A} \to \mathcal{B}$ such that $N_i^\varepsilon \Phi \cong M_i^\varepsilon$ for all $i \in \{1, \cdots, n\}$ and $\varepsilon \in \{-, +\}$.*

Proof. Let the points of S_i^ε be given by the submodules X of N_i^ε with simple top $X/\mathcal{R}X$, $i \in \{1, \cdots, n\}$, $\varepsilon \in \{-, +\}$. We equip S_i^ε with an order relation such that $X \leq Y$ is equivalent to $X \supseteq Y$. By (R1), S_i^ε is a rod.

Set $S = (S_1^-, \cdots, S_n^-; S_1^+, \cdots, S_n^+)$ and equip S with an equivalence relation such that $X \sim Y \iff X/\mathcal{R}X \cong Y/\mathcal{R}Y$. By (R2) and (R3), this relation satisfies (E1) and (E2), i.e. S is a bush.

For each $X \in \mathcal{S}$ (the spectroid associated with the bush S), we denote by $t_X \in \mathcal{T}$ the point supporting $X/\mathcal{R}X$, and we choose a generator $e_X \in N_i^\varepsilon(t_X)$ of X. Then $N_i^\varepsilon(t_X) = \oplus_{X' \sim X, X' \in S_i^\varepsilon} k e_{X'}$.

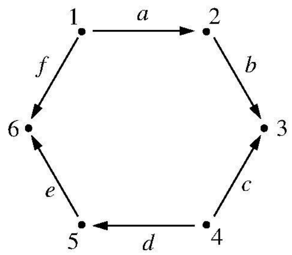
The map $X \longmapsto t_X$ gives rise to a functor $\phi : \mathcal{S} \to \mathcal{T}$ such that $\phi(Y|X) = \theta(t_X, t_Y)^{-1}(f)$, where $f \in \mathrm{Hom}_k(N_i^\varepsilon(t_X), N_i^\varepsilon(t_Y))$ maps e_X to e_Y and annihilates $e_{X'}$ whenever $X' \neq X$. The functor ϕ is an isomorphism and induces an equivalence $\Phi : \mathcal{A} \to \mathcal{B}$. The k–linear maps

$$\begin{aligned} \mu_i^\varepsilon(X) : N_i^\varepsilon \Phi(X) = \bigoplus_{X' \sim X, X' \in S_i^\varepsilon} k e_{X'} &\longrightarrow \bigoplus_{X' \sim X, X' \in S_i^\varepsilon} k X' = M_i^\varepsilon(X) \\ \sum_{X'} \lambda_{X'} e_{X'} &\longmapsto \sum_{X'} \lambda_{X'} X' \end{aligned}$$

define an isomorphism between $N_i^\varepsilon \Phi$ and M_i^ε, $i \in \{1, \cdots, n\}$, $\varepsilon \in \{-, +\}$.

1.4. Example 1. In [8] Nazarova and Roiter examine the particular case of one pair of rods. The classification of representations in [7], [4], and [9] can be reduced to that of bushes.

Example 2. *Representations of* $\widetilde{A}_n$. We illustrate the general construction with the following example:

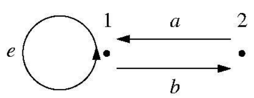


Set $S_1^- = \{a^-\}, S_1^+ = \{a^+\}, \cdots, S_6^- = \{f^-\}, S_6^+ = \{f^+\}$ and equip $S = \cup_{i=1}^6 (S_i^- \cup S_i^+) = \{a^-, a^+, \cdots, f^-, f^+\}$ with the equivalence relation $a^- \sim f^-$, $a^+ \sim b^-$, $b^+ \sim c^+$, $c^- \sim d^-$, $d^+ \sim e^-$, $e^+ \sim f^+$. Then rep$\widetilde{A}_6$ is equivalent to rep(M^-, M^+), where (M^-, M^+) is the rodded tangle associated with the bush S (see 1.3).

Example 3. *Clannish algebras*[2]: Let Q be a quiver and Sp a set of loops in Q. The arrows in Sp are called "special" and the others "ordinary". Let further $R = Z \cup \{e^2 - e : e \in Sp\}$ be a set of "relations" of Q, where Z consists of compositions $\mu\nu$ of ordinary arrows μ, ν. The algebra $A = k[Q]/R$, where $k[Q]$ denotes the algebra of the quiver Q, is called *clannish* if the following conditions hold:
(C1) At most two arrows start at each vertex, at most two stop;
(C2) For each ordinary arrow a, there is at most one arrow b with $ba \notin Z$ and at most one c with $ac \notin Z$;
(C3) Without real loss of generality, we further suppose that R is minimal with respect to (C2).

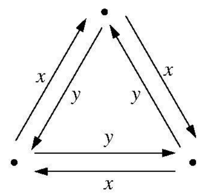
Examples. The algebras with the following data are clannish:

a)



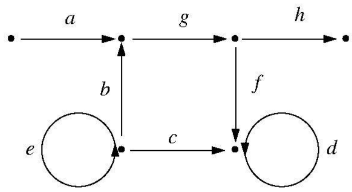
$Sp = \{e\}$
$Z = \{ba\}$

b)



$Sp = \emptyset$
$Z = \{\text{all } xy, yx\}$

c)



$Sp = \{e\}$
$Z = \{ba, cb, fc, d^2, hg\}$

With each clannish algebra $k[Q]/R$ we will associate a tangle.

For the sake of simplicity, we identify the set Q_v of vertices with $\{1, 2, \cdots, n\}$. We further set $x^- = (x, -)$, $x^+ = (x, +)$ whenever x is a vertex or an ordinary arrow. To each $i \in Q_v$, we then attach a set A_i consisting of i^- and i^+, of special loops at i, and of all x^- (resp. x^+) where x is an ordinary arrow starting (resp. stopping) at i. Finally, we construct two disjoint rods S_i^-, S_i^+ such that:
a) $i^- \in S_i^-$, $i^+ \in S_i^+$ and $A_i = S_i^- \cup S_i^+$,
b) Each S_i^ε has one of the following forms:
1) $\{i^\varepsilon \lessgtr e\}$, where e is a special loop at i,
2) $\{a^- < i^\varepsilon\}$, where a is an ordinary arrow starting at i,
3) $\{i^\varepsilon < b^+\}$, where b is an ordinary arrow stopping at i,
4) $\{a^- < i^\varepsilon < b^+\}$, where a (resp. b) is an ordinary arrow starting (resp. stopping) at i and $ab \in Z$.

Of course, if $A_i \neq \{i^-, i^+\}$, there are exactly two possible choices for S_i^- and S_i^+. For instance, in case

$$Q \qquad \begin{tikzcd} 1 \arrow[dr, "b"] & & \\ & 3 \arrow[r, "a"] & 4 \\ 2 \arrow[ur, "c"'] & & \end{tikzcd}, \qquad Z = \{ab\},$$

we obtain $S_3^- = \{a^- < 3^- < b^+\}$, $S_3^+ = \{3^+ < c^+\}$, or reversely, $S_3^- = \{3^- < c^+\}$, $S_3^+ = \{a^- < 3^+ < b^+\}$.

We equip $S = \cup_{i=1}^n (S_i^- \cup S_i^+)$ with an equivalence relation such that $a^- \sim a^+$ for each ordinary arrow a.

We denote by (M^-, M^+) the tangle associated with S and by $\mathrm{rep}_b(M^-, M^+)$ the full subcategory of $\mathrm{rep}(M^-, M^+)$ fromed by representations $(X; f_1, \cdots, f_n)$ such that all f_i are bijective.

Proposition. *$rep(Q, R)$ is equivalent to $rep_b(M^-, M^+)$.*

Proof. For each arrow $a \notin Sp$ with $a^- \in S_i^\varepsilon$ and $a^+ \in S_j^\eta$, and each $X \in \mathcal{A}$, we denote by ξ_X^a the canonical isomorphism

$$M_i^\varepsilon(X)/\mathcal{R}M_i^\varepsilon(X) \to \mathcal{R}^{h_j^\eta} M_j^\eta(X),$$

where $h_j^\eta = 1$ (resp. 2) if S_j^η consists of 2 (resp. 3) elements.

For each $e \in Sp$ with $e \in S_i^\varepsilon$, we denote by J_i^ε and L_i^ε the simple submodules of M_i^ε supported by i^ε and e respectively.

With each object $(X; f_1, \cdots, f_n)$ in $\mathrm{rep}_b(M^-, M^+)$ we attach an object $V = (V(i), V(\alpha))$ in $\mathrm{rep}(Q, R)$ as follows:
1) $V(i) = M_i^+(X)$, $i = 1, \cdots, n$.
2) For each arrow $\alpha : i \longrightarrow j$, in order to define $V(\alpha) : M_i^+(X) \to M_j^+(X)$ we consider two cases.

Case 1. $\alpha \notin Sp$, $\alpha^- \in S_i^\varepsilon$, $\alpha^+ \in S_j^\varepsilon$. If $\varepsilon = \eta = +$, $V(\alpha)$ is the composition

$$M_i^+(X) \xrightarrow{\mathrm{pr}} M_i^+(X)/\mathcal{R}M_i^+(X) \xrightarrow{\xi_X^\alpha} \mathcal{R}^{h_j^+} M_j^+(X) \xrightarrow{\mathrm{im}} M_j^+(X);$$

if $\varepsilon = +$ and $\eta = -$, $V(\alpha)$ is the composition

$$M_i^+(X) \xrightarrow{\mathrm{pr}} M_i^+(X)/\mathcal{R}M_i^+(X) \xrightarrow{\xi_X^\alpha} \mathcal{R}^{h_j^-} M_j^-(X) \xrightarrow{\mathrm{im}} M_j^-(X) \xrightarrow{f_j} M_j^+(X);$$

if $\varepsilon = -$ and $\eta = +$, $V(\alpha)$ is the composition

$$M_i^+(X) \xrightarrow{f_i^{-1}} M_i^-(X) \xrightarrow{\mathrm{pr}} M_i^-(X)/\mathcal{R}M_i^-(X) \xrightarrow{\xi_X^\alpha} \mathcal{R}^{h_j^+} M_j^+(X) \xrightarrow{\mathrm{im}} M_j^+(X);$$

if $\varepsilon = \eta = -$, $V(\alpha)$ is the composition

$$\begin{aligned} M_i^+(X) \xrightarrow{f_i^{-1}} M_i^-(X) &\xrightarrow{\mathrm{pr}} M_i^-(X)/\mathcal{R}M_i^-(X) \\ &\xrightarrow{\xi_X^\alpha} \mathcal{R}^{h_j^-} M_j^-(X) \xrightarrow{\mathrm{im}} M_j^-(X) \xrightarrow{f_j} M_j^+(X). \end{aligned}$$

(By pr we denote the canonical projection, by im the canonical immersion.)

Case 2. $\alpha \in Sp$ and $\alpha \in S_i^\varepsilon$ for some $i \in \{1, \cdots, n\}$, $\varepsilon \in \{-,+\}$. If $\varepsilon = +$, $V(\alpha)$ is identified with

$$0 \oplus \mathbb{1}_{L_i^+(X)} : M_i^+(X) = J_i^+(X) \oplus L_i^+(X) \longrightarrow J_i^+(X) \oplus L_i^+(X) = M_i^+(X);$$

if $\varepsilon = -$, $V(\alpha)$ is the composition

$$\begin{aligned} M_i^+(X) \xrightarrow{f_i^{-1}} M_i^-(X) &= J_i^-(X) \oplus L_i^-(X) \\ &\xrightarrow{0 \oplus \mathbb{1}_{L_i^-(X)}} J_i^-(X) \oplus L_i^-(X) = M_i^-(X) \xrightarrow{f_i} M_i^+(X). \end{aligned}$$

Thus we obtain a functor

$$\begin{aligned} F : \mathrm{rep}_b(M^-, M^+) &\longrightarrow \mathrm{rep}(Q, R) \\ (X; f_1, \cdots, f_n) &\longmapsto V = (V(i), V(\alpha)) \end{aligned}$$

which maps a morphism $\mu : (X; f_1, \cdots, f_n) \longrightarrow (X'; f_1', \cdots, f_n')$ to the morphism $F(\mu) := (M_i^+(\mu))_{i \in Q_v}$.

Since (M^-, M^+) is rodded and faithful, the functor F is fully faithful.

Let $V = (V(i), V(\alpha)) \in \mathrm{rep}(Q, R)$. For each arrow α from i to j, we set $K_i^\alpha = \ker V(\alpha)$, $I_j^\alpha = \mathrm{Im} V(\alpha)$ and denote by $\bar{V}(\alpha)$ the isomorphism $V(i)/\ker V(\alpha) \longrightarrow \mathrm{Im} V(\alpha)$ induced by $V(\alpha)$.

In case $S_i^\varepsilon = \{i^\varepsilon\}$, we set $P_i^\varepsilon = V_i^\varepsilon = V(i)$ and denote by Φ_i^ε the identity $\mathbb{1}_{V(i)}$.

In case $S_i^\varepsilon = \{i^\varepsilon \lessgtr e\}$, we set $P_i^\varepsilon = K_i^e$, $V_i^\varepsilon = K_i^e \oplus I_i^e$, and denote by Φ_i^ε the canonical isomorphism $K_i^e \oplus I_i^e \longrightarrow V(i)$, $(x,y) \longmapsto x+y$.

In case $S_i^\varepsilon = \{a^- < i^\varepsilon\}$, we set $P_i^\varepsilon = K_i^a$, $V_i^\varepsilon = I_j^a \oplus K_i^a$, and choose a section t_i of the canonical projection $V(i) \longrightarrow V(i)/K_i^a$, we then denote by Φ_i^ε the composition

$$V_i^\varepsilon = I_j^a \oplus K_i^a \overset{(V(\bar{a}))^{-1}\oplus 1\!\!1_{K_i^a}}{\longrightarrow} V(i)/K_i^a \oplus K_i^a \overset{[t_i \text{ im}]}{\longrightarrow} V(i).$$

In case $S_i^\varepsilon = \{i^\varepsilon < b^+\}$, we set $P_i^\varepsilon = V(i)/I_i^b$, $V_i^\varepsilon = V(i)/I_i^b \oplus I_i^b$, and choose a section s_i of the canonical projection $V(i) \longrightarrow V(i)/I_i^b$, we then denote by Φ_i^ε the isomorphism $[s_i \text{ im}] : V_i^\varepsilon = V(i)/I_i^b \oplus I_i^b \longrightarrow V(i)$.

In case $S_i^\varepsilon = \{a^- < i^\varepsilon < b^+\}$, we set $P_i^\varepsilon = K_i^a/I_i^b$, $V_i^\varepsilon = I_j^a \oplus K_i^a/I_i^b \oplus I_i^b$, and choose a section u_i of the canonical projection $K_i^a \longrightarrow K_i^a/I_i^b$ and a section v_i of the canonical projection $V(i) \longrightarrow V(i)/K_i^a$, we then denote by Φ_i^ε the composition

$$V_i^\varepsilon = I_j^a \oplus K_i^a/I_i^b \oplus I_i^b \overset{(V(\bar{a}))^{-1}\oplus 1\!\!1 \oplus 1\!\!1}{\longrightarrow} V(i)/K_i^a \oplus K_i^a/I_i^b \oplus I_i^b$$
$$\overset{1\!\!1 \oplus [u_i \text{ im}]}{\longrightarrow} V(i)/K_i^a \oplus K_i^a \overset{[v_i \text{ im}]}{\longrightarrow} V(i).$$

Finally, we set

$$X = \bigoplus_{a \in Q_a \setminus Sp} \bar{a} \otimes I_j^a \oplus (\bigoplus_{e \in Sp} e \otimes I_i^e) \oplus (\bigoplus_{i^\varepsilon \in S_i^\varepsilon} i^\varepsilon \otimes P_i^\varepsilon) \in \mathcal{A}$$

and denote by f_i the composition

$$M_i^-(X) \overset{can.}{\cong} V_i^- \overset{\Phi_i^-}{\longrightarrow} V(i) \overset{(\Phi_i^+)^{-1}}{\longrightarrow} V_i^+ \overset{\Psi_i}{\cong} M_i^+(X),$$

where $\bar{a}$ denotes the equivalence class of a.

Thus we obtain an object $(X; f_1, \cdots, f_n)$ in $\text{rep}_b(M^-, M^+)$. By Ψ_i we denote the canonical isomorphism $V_i^+ \to M_i^+(X)$. Then $(\Psi_i(\Phi_i^+)^{-1})_{i \in Q_v}$ defines an isomorphism from $V = (V(i), V(\alpha))$ to $F(X; f_1, \cdots, f_n)$. Therefore, F hits each isoclass in $\text{rep}(Q, R)$.

1.5. Remark. With each tangle (M^-, M^+) over $\mathcal{A}$ we can associate as follows a tangle $(\bar{M}^-, \bar{M}^+)$ over a new aggregate $\bar{\mathcal{A}}$.

Let $\bar{\mathcal{S}}$ denote the spectroid obtained from the spectroid $\mathcal{S}$ of $\mathcal{A}$ by adding objects s_i and t_i for $i \in \{1, \cdots, n\}$, whose spaces of radical morphisms $\mathcal{R}_{\bar{\mathcal{S}}}(x,y)$ are $\mathcal{R}_{\mathcal{S}}(x,y)$ if $x,\ y \in \mathcal{S}$, $\text{Hom}_k(k, M_i^-(y))$ if $y \in \mathcal{S},\ x = s_i$, $\text{Hom}_k(M_i^+(x), k)$ if $x \in \mathcal{S},\ y = t_i$, and 0 otherwise. The composition $g \circ f$ of $f \in \mathcal{R}_{\bar{\mathcal{S}}}(x,y)$ and $g \in \mathcal{R}_{\bar{\mathcal{S}}}(y,z)$ is gf if $x,\ y,\ z \in \mathcal{S}$, $M_i^-(g)f$ if $y,\ z \in \mathcal{S},\ x = s_i$, $gM_i^+(f)$ if $x,\ y \in \mathcal{S},\ z = t_i$, and 0 in all the remaining cases.

Let $\bar{\mathcal{A}}$ denote the additive hull of $\bar{\mathcal{S}}$ and $\bar{M}_i^-$ (resp. $\bar{M}_i^+$) the module over $\bar{\mathcal{A}}$ such that the value $\bar{M}_i^-(x)$ (resp. $\bar{M}_i^+(x)$) at $x \in \bar{\mathcal{S}}$ is k if $x = s_i$ (resp. $x = t_i$), is $M_i^-(x)$ (resp. $M_i^+(x)$) if $x \in \mathcal{S}$, and 0 otherwise. If $f \in \mathcal{R}_{\bar{\mathcal{S}}}(x,y)$, the morphism $\bar{M}_i^-(f) \in \mathrm{Hom}_k(\bar{M}_i^-(x), \bar{M}_i^-(y))$ (resp. $\bar{M}_i^+(f) \in \mathrm{Hom}_k(\bar{M}_i^+(x), \bar{M}_i^+(y))$) is $M_i^-(f)$ (resp. $M_i^+(f)$) if $x,\ y \in \mathcal{S}$, is f if $x = s_i$ (resp. $y = t_i$), and 0 otherwise.

By the construction, the tangle $(\bar{M}^-, \bar{M}^+)$ is rodded if so is (M^-, M^+).

Let $\Psi : \bar{\mathcal{A}} \longrightarrow \mathcal{A}$ be the natural functor which maps $\bar{X} \in \bar{\mathcal{A}}$ onto the "largest summand" $\Psi(\bar{X})$ belonging to $\mathcal{A}$. Then M_i^- provides a submodule $M_i^-\Psi$ of $\bar{M}_i^-$, M_i^+ provides a subquotient $M_i^+\Psi$ of $\bar{M}_i^+$, $i \in \{1, \cdots, n\}$, and Ψ gives rise to a functor

$$\begin{aligned} F : \mathrm{rep}_b(\bar{M}^-, \bar{M}^+) &\longrightarrow \mathrm{rep}(M^-, M^+) \\ (\bar{X}; \bar{f}_1, \cdots, \bar{f}_n) &\longmapsto (\Psi(\bar{X}); f_1, \cdots, f_n), \end{aligned}$$

where f_i is the composition

$$M_i^-(\Psi(\bar{X})) = M_i^-\Psi(\bar{X}) \xrightarrow{\text{im}} \bar{M}_i^-(\bar{X}) \xrightarrow{\bar{f}_i} \bar{M}_i^+(\bar{X}) \xrightarrow{\text{pr}} M_i^+\Psi(\bar{X}) = M_i^+(\Psi(\bar{X})).$$

Proposition. *The functor F is quasisurjective, and the indecomposables annihilated by F are those isomorphic to $(s_i \oplus t_i; 0, \cdots, 0, \mathbb{1}, 0, \cdots, 0)$, $i = 1, \cdots, n$.*

Proof. Let $(X; f_1, \cdots, f_n)$ be an object in $\mathrm{rep}(M^-, M^+)$. Consider the sequence

$$K_i := \ker f_i \xrightarrow{\text{im}} M_i^-(X) \xrightarrow{f_i} M_i^+(X) \xrightarrow{\text{pr}} \mathrm{Coker} f_i =: C_i.$$

Choose a retraction p_i of the canonical immersion and a section μ_i of the canonical projection above, then

$$\begin{bmatrix} \mu_i & f_i \\ 0 & p_i \end{bmatrix} : C_i \oplus M_i^-(X) \longrightarrow M_i^+(X) \oplus K_i$$

is bijective.

Set $\bar{X} = (\oplus_{i=1}^n s_i \otimes C_i) \oplus X \oplus (\oplus_{i=1}^n t_i \otimes K_i) \in \bar{\mathcal{A}}$, and denote by $\bar{f}_i$ the composition

$$\bar{M}_i^-(\bar{X}) \overset{can.}{\cong} C_i \oplus M_i^-(X) \xrightarrow{\begin{bmatrix} \mu_i & f_i \\ 0 & p_i \end{bmatrix}} M_i^+(X) \oplus K_i \overset{can.}{\cong} \bar{M}_i^+(\bar{X}).$$

Then $(\bar{X}; \bar{f}_1, \cdots, \bar{f}_n) \in \mathrm{rep}_b(\bar{M}^-, \bar{M}^+)$ and $F(\bar{X}; \bar{f}_1, \cdots, \bar{f}_n) \cong (X; f_1, \cdots, f_n)$.

It is not difficult to see that each $(\bar{X}; \bar{f}_1, \cdots, \bar{f}_n) \in \mathrm{rep}_b(\bar{M}^-, \bar{M}^+)$ is isomorphic to the direct sum of objects of the form $(s_i \oplus t_i; 0, \cdots, 0, \mathbb{1}, 0, \cdots, 0)$ and of the form $\bar{Y} = (\oplus_{i=1}^{n} s_i \otimes S_i) \oplus Y \oplus (\oplus_{i=1}^{n} t_i \otimes T_i); \bar{g}_1, \cdots, \bar{g}_n)$, where S_i, $T_i \in \operatorname{mod} k$, $Y \in \mathcal{A}$, and $\bar{g}_i$ has the form

$$\begin{bmatrix} a_i & g_i \\ 0 & b_i \end{bmatrix} : S_i \oplus M_i^-(Y) \longrightarrow M_i^+(Y) \oplus T_i,$$

where a_i, b_i and g_i are k–linear maps.

Let $(\bar{X}; \bar{f}_1, \cdots, \bar{f}_n)$ and $(\bar{X}'; \bar{f}_1', \cdots, \bar{f}_n')$ be objects in $\operatorname{rep}_b(\bar{M}^-, \bar{M}^+)$, and $\mu \in \mathcal{A}(\Psi(\bar{X}), \Psi(\bar{X}'))$ a morphism from $F(\bar{X}; \bar{f}_1, \cdots, \bar{f}_n)$ to $F(\bar{X}'; \bar{f}_1', \cdots, \bar{f}_n')$

Since F annihilates $(s_i \oplus t_i; 0, \cdots, 0, \mathbb{1}, 0, \cdots, 0)$, we may assume that $\bar{X}$ and $\bar{X}'$ have respectively the forms $(\oplus_{i=1}^{n} s_i \otimes S_i) \oplus X \oplus (\oplus_{i=1}^{n} t_i \otimes T_i)$ and $(\oplus_{i=1}^{n} s_i \otimes S_i') \oplus X' \oplus (\oplus_{i=1}^{n} t_i \otimes T_i')$, where S_i, T_i, S_i', $T_i' \in \operatorname{mod} k$ and X, $X' \in \mathcal{A}$, and that $\bar{f}_i$ and $\bar{f}_i'$ are of the forms:

$$\bar{f}_i = \begin{bmatrix} a_i & f_i \\ 0 & b_i \end{bmatrix} : S_i \oplus M_i^-(X) \longrightarrow M_i^+(X) \oplus T_i,$$

$$\bar{f}_i' = \begin{bmatrix} a_i' & f_i' \\ 0 & b_i' \end{bmatrix} : S_i' \oplus M_i^-(X') \longrightarrow M_i^+(X') \oplus T_i',$$

where a_i, b_i, f_i, a_i', b_i' and f_i' are k–linear maps, $i \in \{1, \cdots, n\}$.

Thus $F(\bar{X}; \bar{f}_1, \cdots, \bar{f}_n) = (X; f_1, \cdots, f_n)$ and $F(\bar{X}'; \bar{f}_1', \cdots, \bar{f}_n') = (X'; f_1', \cdots, f_n')$.

Consider the following commutative diagram

$$\begin{tikzcd}
 & T_i & S_i \arrow[d, "a_i"] \arrow[dr, "\pi_i a_i", "\sim"'] & \\
\operatorname{Ker} f_i \arrow[r, "\kappa_i"] \arrow[ur, "b_i\kappa_i", "\sim"'] & M_i^-(X) \arrow[r, "f_i"] \arrow[u, "b_i"] \arrow[d, "M_i^-(\mu)"] & M_i^+(X) \arrow[r, "\pi_i"] \arrow[d, "M_i^+(\mu)"] & \operatorname{Coker} f_i \\
\operatorname{Ker} f_i' \arrow[r, "\kappa_i'"] \arrow[dr, "b_i'\kappa_i'"', "\sim"] & M_i^-(X') \arrow[r, "f_i'"] \arrow[d, "b_i'"] & M_i^+(X') \arrow[r, "\pi_i'"] & \operatorname{Coker} f_i' \\
 & T_i' & S_i' \arrow[u, "a_i'"] \arrow[ur, "\pi_i' a_i'"', "\sim"] &
\end{tikzcd}$$

where κ_i and κ_i' denote the canonical immersions, π_i and π_i' the canonical projections.

The bijectivity of $\bar{f}_i$ and $\bar{f}_i'$ implies that $b_i\kappa_i$, $b_i'\kappa_i'$, $\pi_i a_i$ and $\pi_i' a_i'$ are bijective. Set $u_i = (\pi_i' a_i')^{-1}\pi_i' M_i^+(\mu) a_i$ and $v_i = b_i' M_i^-(\mu)\kappa_i(b_i\kappa_i)^{-1}$. It is easy to see that there exist a $w_i : S_i \longrightarrow M_i^-(X')$ and a $z_i : M_i^+(X) \longrightarrow T_i'$ such that $z_i a_i = 0$,

$b_i' w_i = 0$, $M_i^+(\mu) a_i = a_i' u_i + f_i' w_i$, and $b_i' M_i^-(\mu) = z_i f_i + v_i b_i$, i.e. the following square commutes.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
S_i \oplus M_i^-(X) \arrow[r, "\begin{bmatrix} a_i & f_i \\ 0 & b_i \end{bmatrix}"] \arrow[d, "\begin{bmatrix} u_i & 0 \\ w_i & M_i^-(\mu) \end{bmatrix}"'] & M_i^+(X) \oplus T_i \arrow[d, "\begin{bmatrix} M_i^+(\mu) & 0 \\ z_i & v_i \end{bmatrix}"] \\
S_i' \oplus M_i^-(X') \arrow[r, "\begin{bmatrix} a_i' & f_i' \\ 0 & b_i' \end{bmatrix}"] & M_i^+(X) \oplus T_i'
\end{tikzcd}$$

Set

$$\bar{\mu} = \begin{bmatrix} u_1 & 0 & & & & & \\ 0 & \ddots & & & 0 & & \\ & & u_n & & & & \\ w_1 & \cdots & w_n & \mu & & & \\ & & & z_1 & v_1 & 0 & \\ & 0 & & \vdots & & \ddots & \\ & & & z_n & & 0 & v_n \end{bmatrix} : \bar{X} = (\oplus_{i=1}^n s_i \otimes S_i) \oplus X \oplus (\oplus_{i=1}^n t_i \otimes T_i)$$
$$\longrightarrow (\oplus_{i=1}^n s_i \otimes S_i') \oplus X' \oplus (\oplus_{i=1}^n t_i \otimes T_i') = \bar{X}'.$$

Then $F(\bar{\mu}) = \mu$, that is, F is full.

This finishes the proof of the proposition.

2. The classification

2.1. Terminology. Let $S = (S_1^-, \cdots, S_n^-; S_1^+, \cdots, S_n^+; \sim)$ be a bush.

In the sequel, we write $x \wedge y$ if x and y belong to the same rod and are incomparable, and we write $x|y$ if $(x, y) \in \cup_{i=1}^n ((S_i^- \times S_i^+) \cup (S_i^+ \times S_i^-))$. We further set

$$x^\sim = \begin{cases} y & \text{if } y \sim x \text{ and } y \neq x \\ x & \text{if the equivalence class of } x \text{ contains only } x \end{cases}$$
$$x^\wedge = \begin{cases} y & \text{if } y \wedge x \\ x & \text{if } x \text{ is comparable with all points of its rod} \end{cases}$$
$$\text{and} \quad x^* = (x^\sim)^\wedge.$$

We call *catenation* of S a sequence $w = w_1 w_2 \cdots w_t$ of points of S such that $w_i^* | w_{i+1}$ for all $i < t$. The *reverse catenation* is the sequence $w^* = w_t^* \cdots w_2^* w_1^*$. Two catenations $w = w_1 w_2 \cdots w_s$ and $w' = w'_1 w'_2 \cdots w'_t$ are called equivalent if $s = t$ and $w'_i = w_i$ or $w_i^\wedge$ for all $1 \leq i \leq s$. For each catenation w, we then denote by $[w]$ the equivalence class of w. Then the set of equivalence classes of all catenations of S is equipped with an order relation such that

$$[v] = [v_1 \cdots v_s] < [w_1 \cdots w_t] = [w]$$

$$\Longleftrightarrow \begin{cases} & \text{if } w = v'w',\ [v'] = [v],\ w'_1 = w_{s+1} \in S^- \\ \text{or} & \text{if } v = w'v',\ [w'] = [w],\ v'_1 = v_{t+1} \in S^+ \\ \text{or} & \text{if } v = uxv',\ w = u'yw',\ [u] = [u'],\ x < y. \end{cases}$$

The equivalence classes of catenations which start in a fixed rod are pairwise comparable.

2.2. From now onwards, we suppose that S is *complete*, i.e. that $x \neq x^*$ for all $x \in S$. [This is no real restriction. Otherwise, we replace S by a completed bush S^o obtained from S by adding new rods $S_{ix}^- = \{x^o\}$, $S_{ix}^+ = \emptyset$ and by agreeing that $x \sim x^o$ for each point x of S such that $x = x^\wedge = x^\sim$. The new bush S^o is complete, and rep$S$ is equivalent to repS^o.]

If S is complete, we attach a representation $(X_w; f_{w1}, \cdots, f_{wn})$ of S to each catenation $w = w_1 w_2 \cdots w_t$. First we set

$$X_w = \widehat{w_1} \oplus \widehat{w_2} \oplus \cdots \oplus \widehat{w_t},$$

where $\widehat{w_i} = \{w_i,\ w_i^*\} \in \mathcal{S}$ (=the spectroid attached to S in 1.3) if $w_i \neq w_i^\sim$ and $\widehat{w_i} = \{w_i\} \oplus \{w_i^*\} \in \mathcal{A}$ (=the additive hull of S) if $w_i \neq w_i^\wedge$. Thus each term x of the sequence $w_1 w_1^* w_2 w_2^* \cdots w_t w_t^*$ contributes a one–dimensional summand $k\underline{x}$ to the space $M_i^\varepsilon(X_w)$ associated with the rod S_i^ε containing x. Accordingly, $M_i^-(X_w)$ and $M_i^+(X_w)$ have the form:

$$(*) \qquad \begin{cases} M_i^-(X_w) = \oplus_p k\underline{w}_p \oplus \oplus_l k\underline{w}_l^* \\ M_i^+(X_w) = \oplus_m k\underline{w}_m^* \oplus \oplus_q k\underline{w}_q \end{cases}$$

where $p,\ q,\ l$ and m are subjected to the conditions $w_p,\ w_l^* \in S_i^-$ and $w_q,\ w_m^* \in S_i^+$. The structure maps are defined as sums

$$f_{wi} = \sum_r f_{wir} : M_i^-(X_w) \longrightarrow M_i^+(X_w)$$

where each $r > 1$ satisfying $w_r \in S_i^- \cup S_i^+$ provides a contribution

$$f_{wir} = h_{wir} g_{wir} : M_i^-(X_w) \xrightarrow{g_{wir}} k \xrightarrow{h_{wir}} M_i^+(X_w).$$

To define the factors g_{wir} and h_{wir}, we distinguish two cases:

1) *Case* $w^*_{r-1} \in S_i^-$, $w_r \in S_i^+$. *Construction of* g_{wir}: If $w_{r-1} = w^\wedge_{r-1}$, we set $g_{wir}(\underline{w}^*_{r-1}) = 1$ and let g_{wir} vanish on the remaining basis vectors.

If $w_{r-1} \neq w^\wedge_{r-1}$, the map g_{wir} is the composition

$$M_i^-(X_w) \xrightarrow{\text{pr}} k\underline{w}_{r-1} \oplus k\underline{w}^*_{r-1} \xrightarrow{[0\ 1]} k$$

provided $[w^*_{r-2} \cdots w^*_1] \leq [w_r \cdots w_t]$. Otherwise, it is the composition

$$M_i^-(X_w) \xrightarrow{\text{pr}} k\underline{w}_{r-1} \oplus k\underline{w}^*_{r-1} \xrightarrow{[1\ 1]} k$$

(By pr we denote the projection which annihilates the basis vectors $\neq \underline{w}_{r-1}$, $\underline{w}^*_{r-1}$.)

Construction of h_{wir}: If $w_r = w^\wedge_r$, we define $h_{wir}(1) = \underline{w}_r$. If $w_r \neq w^\wedge_r$, h_{wir} is the composition

$$k \xrightarrow{[1\ 0]^T} k\underline{w}_r \oplus k\underline{w}^*_r \xrightarrow{\text{im}} M_i^+(X_w)$$

provided $[w^*_{r-1} \cdots w^*_1] \leq [w_{r+1} \cdots w_t]$. Otherwise, it is the composition

$$k \xrightarrow{[1\ 1]^T} k\underline{w}_r \oplus k\underline{w}^*_r \xrightarrow{\text{im}} M_i^+(X_w).$$

(By im we denote the canonical immersion.)

2) *Case* $w^*_{r-1} \in S_i^+$, $w_r \in S_i^-$. *Construction of* g_{wir}: If $w_r = w^\wedge_r$, we set $g_{wir}(\underline{w}_r) = 1$ and let g_{wir} vanish on the remaining basis vectors. If $w_r \neq w^\wedge_r$, the map g_{wir} is the composition

$$M_i^-(X_w) \xrightarrow{\text{pr}} k\underline{w}_r \oplus k\underline{w}^*_r \xrightarrow{[1\ 0]} k$$

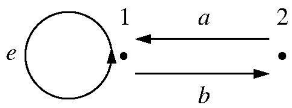
provided $[w^*_{r-1} \cdots w^*_1] > [w_{r+1} \cdots w_t]$. Otherwise, [0 1] is replaced by [1 1].

Construction of h_{wir}: If $w_{r-1} = w^\wedge_{r-1}$, we define $h_{wir}(1) = \underline{w}^*_{r-1}$. If $w_{r-1} \neq w^\wedge_{r-1}$, h_{wir} is the composition

$$k \xrightarrow{[0\ 1]^T} k\underline{w}_{r-1} \oplus k\underline{w}^*_{r-1} \xrightarrow{\text{im}} M_i^+(X_w)$$

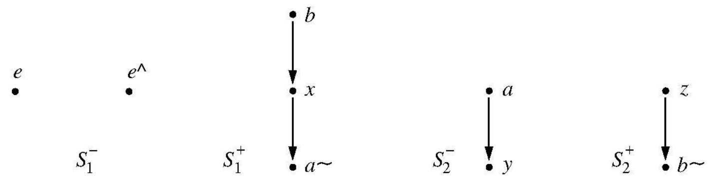
provided $[w^*_{r-2} \cdots w^*_1] > [w_r \cdots w_t]$. Otherwise, $[0\ 1]^T$ is replaced by $[1\ 1]^T$.

2.3. Example. The clannish algebra $k[Q]/(ba, e^2 - e)$, where Q denotes the quiver

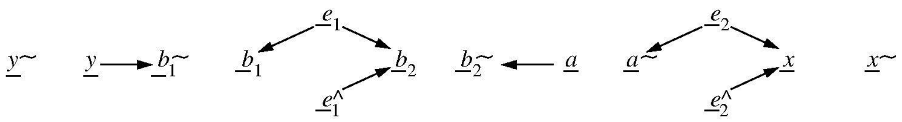


, $Sp = \{e\}$, $Z = \{ba\}$

gives rise to the (non–completed) bush



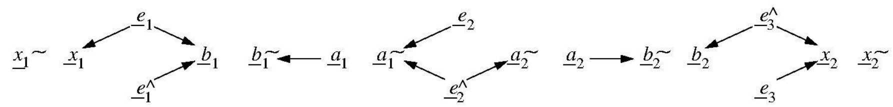
As a typical example, we choose the catenation $w = y^{\sim}b^{\sim}ebaex$ of the completed bush. The maps f_{wi} then behave as follows:



The matrices of the representation of the non–completed bush associated with w – or, more precisely, the matrices of the linear maps f_{w1} and f_{w2} – are displayed as follows:

$$\begin{array}{c} \\ b \\ \\ x \\ a^{\sim} \end{array}\begin{array}{c} \begin{array}{cc} e & e^{\wedge} \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \end{array}\right] \end{array} \qquad \begin{array}{c} z \\ b^{\sim} \end{array}\begin{array}{c} \begin{array}{cc} y & a \end{array} \\ \left[\begin{array}{c|c} \hline 1 & 0 \\ 0 & 1 \end{array}\right] \end{array}$$

Similarly, the maps f_{vi} of the representation associated with the catenation $v = x^{\sim}ebaea^{\sim}b^{\sim}e^{\wedge}x$ are



2.4. In the first example considered above, the catenation w is *asymmetric*, i.e. $[w] \neq [w^*]$. The matrices of the representation associated with w^* are

$$\begin{array}{c} \\ b \\ \\ x \\ a^{\sim} \end{array}\begin{array}{c} \begin{array}{cc} e & e^{\wedge} \end{array} \\ \left[\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right] \end{array} \qquad \begin{array}{c} z \\ b^{\sim} \end{array}\begin{array}{c} \begin{array}{cc} y & a \end{array} \\ \left[\begin{array}{c|c} \hline 0 & 1 \\ 1 & 0 \end{array}\right] \end{array}$$

By “permissible” row and column transformations, these matrices can obviously be converted into the matrices associated with w. The representations associated with w and w^* are therefore isomorphic.

In general, we choose a set Ω_1 of asymmetric catenations which contains one representative of each class $[w]\coprod[w^*]$ of asymmetric catenations. For each $w \in \Omega_1$, we denote by $R(w)$ the corresponding representation of S. Representations isomorphic to such an $R(w)$ will be called *asymmetric strings.*

In the second example of 2.3, the catenation v is *symmetric*, i.e. $[v] = [v^*]$. In this case, the representation associated with v is clearly the direct sum of two representations $R(v, 0)$ and $R(v, 1)$.

Of course, this is a general fact (This fact will be shown in Section 4). For each symmetric catenation v, the associated representation in 2.2 decomposes into the direct sum of two representations $R(v, 0)$ and $R(v, 1)$. These representations are indexed by $\Omega_2 \times \{0, 1\}$, where Ω_2 denotes the set of symmetric catenations which contains one representative of each class $[w]$ of symmetric catenations. Representations of S isomorphic to some $R(v, i)$, $(v, i) \in \Omega_2 \times \{0, 1\}$, will be called *dimidiate strings.*

2.5. Besides finite catenations, we consider *periodic catenations.* These are sequences $u = (u_i)_{i\in\mathbb{Z}}$ which satisfy $u_i^*|u_{i+1}$ for all $i \in \mathbb{Z}$ and admit a natural number $\pi \geq 1$ such that $u_{i+\pi} = u_i$ or $u_i^\wedge$ for all i. The smallest π satisfying these conditions is the *period* of u. Each periodic catenation u is consorted with a *reverse* u^* such that $(u^*)_i = (u_{-i})^*$ and with *translates* $u\{p\}$ such that $u\{p\}_i = u_{p+i}$. It is called *symmetric* if $[u^*] = [u\{p\}]$ for some p and *asymmetric* if not.

To each asymmetric period catenation u we shall attach a family of representations of S which are indexed by the powers

$$Q = P^l = X^{ml} - a_1X^{ml-1} - a_2X^{ml-2} - \cdots - a_{ml}, \quad l \geq 1$$

of the irreducible unitary polynomials P in one determinate X with coefficients in k. The index set formed by the powers $Q = P^l$ with $P \neq X$ is denoted by $\mathcal{P}$. To each $Q \in \mathcal{P}$ we attach the invertible matrix

$$A(Q) = \begin{bmatrix} 0 & 0 & \dots & 0 & a_{ml} \\ 1 & 0 & \dots & 0 & a_{ml-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & a_2 \\ 0 & 0 & \dots & 1 & a_1 \end{bmatrix}$$

The representation $(Y_u^Q; \xi_{u1}^Q, \cdots, \xi_{un}^Q)$ associated with an asymmetric periodic catenation u of period π and a polynomial $Q \in \mathcal{P}$ of degree d is obtained as follows. First we consider the representation $(X_w; f_{w1}, \cdots, f_{wn})$ attached to the catenation

$$\begin{aligned} w &= w_1w_2\cdots w_\pi w_{\pi+1}\cdots w_{2\pi}w_{2\pi+1}\cdots w_{3\pi} \\ &= u_{-\pi}u_{-\pi+1}\cdots u_{-1}u_0u_1\cdots u_{\pi-1}u_\pi\cdots u_{2\pi-1}. \end{aligned}$$

With the notations of 2.2 we then set

$$Y_u^Q = \widehat{w_{\pi+1}}^d \oplus \widehat{w_{\pi+2}}^d \oplus \cdots \oplus \widehat{w_{2\pi}}^d = \widehat{u_0}^d \oplus \widehat{u_1}^d \oplus \cdots \oplus \widehat{u_{\pi-1}}^d$$

and thus obtain

$$\begin{aligned} M_i^-(Y_u^Q) &= (\oplus_p k\underline{u}_p \oplus \oplus_l k\underline{u}_l^*) \otimes_k k^d \\ M_i^+(Y_u^Q) &= (\oplus_m k\underline{u}_m^* \oplus \oplus_q k\underline{u}_q) \otimes_k k^d \end{aligned}$$

where l, m, p and q belong to $\{0, 1, \cdots, \pi - 1\}$ and satisfy $u_p,\ u_l^* \in S_i^-$ and $u_q,\ u_m^* \in S_i^+$. The structure maps are defined as sums

$$\xi_{ui}^Q = \lambda_{ui}^Q + \mu_{ui}^Q : M_i^-(Y_u^Q) \longrightarrow M_i^+(Y_u^Q),$$

where λ_{ui}^Q is the composition

$$M_i^-(Y_u^Q) \xrightarrow{\mathrm{im}} M_i^-(X_w) \otimes_k k^d \xrightarrow{f_{wi} \otimes \mathbb{1}_d} M_i^+(X_w) \otimes_k k^d \xrightarrow{\mathrm{pr}} M_i^+(Y_u^Q)$$

(By im we denote the canonical immersion, and by pr the canonical projection.), and where μ_{ui}^Q is the composition

$$\begin{aligned} M_i^-(Y_u^Q) \xrightarrow{\mathrm{im}} M_i^-(X_w) \otimes_k k^d \xrightarrow{g_{wi\,2\pi+1} \otimes A(Q)} k \otimes_k k^d \xrightarrow{h_{wi\,\pi+1} \otimes \mathbb{1}_d} M_i^+(X_w) \otimes_k k^d \\ \xrightarrow{\mathrm{pr}} M_i^+(Y_u^Q) \end{aligned}$$

if $u_0 \in S_i^+$;

$$\begin{aligned} M_i^+(Y_u^Q) \xleftarrow{\mathrm{pr}} M_i^+(X_w) \otimes_k k^d \xleftarrow{h_{wi\,2\pi+1} \otimes \mathbb{1}_d} k \otimes_k k^d \xleftarrow{g_{wi\,\pi+1} \otimes A(Q)^{-1}} M_i^-(X_w) \otimes_k k^d \\ \xleftarrow{\mathrm{im}} M_i^-(Y_u^Q) \end{aligned}$$

if $u_0 \in S_i^-$; and zero if $u_0 \notin S_i^- \cup S_i^+$.

As a typical example, we consider the case where S is as in 2.3, and

$$u_0 u_1 \cdots u_{\pi-1} = aebae^\wedge a^\sim b^\sim eb.$$

The structure maps g_{ui}^Q are then visualized by the following diagram

$$\begin{tikzcd}
(k\underline{a}_1)^d \arrow[rrrrrrrrd, bend right, "A(Q)^{-1}"'] & (k\underline{e}_1)^d \arrow[r] \arrow[dl] & (k\underline{b}_1)^d & (k\underline{a}_2)^d \arrow[dl] & (k\underline{e}_2^\wedge)^d \arrow[r] \arrow[dl] & (k\underline{a}_3^\sim)^d & (k\underline{b}_2^\sim)^d & (k\underline{e}_3)^d \arrow[r] \arrow[dl] & (k\underline{b}_3)^d \\
(k\underline{a}_1^\sim)^d & (k\underline{e}_1^\wedge)^d \arrow[ur] & (k\underline{b}_1^\sim)^d & (k\underline{a}_2^\sim)^d & (k\underline{e}_2)^d \arrow[ur] & (k\underline{a}_3)^d \arrow[ur] & (k\underline{b}_2)^d & (k\underline{e}_3^\wedge)^d \arrow[ur] & (k\underline{b}_3^\sim)^d
\end{tikzcd}$$

The corresponding matrices are

$$
\begin{array}{c}
\\ \\ b \\ x \\ a^{\sim} \\ \\
\end{array}
\begin{array}{c}
\begin{array}{ccc} \qquad e \qquad & & \qquad e^{\wedge} \qquad \end{array} \\
\left[\begin{array}{ccc|ccc}
\mathbb{1}_d & 0 & 0 & \mathbb{1}_d & 0 & 0 \\
0 & 0 & \mathbb{1}_d & 0 & 0 & 0 \\
0 & 0 & \mathbb{1}_d & 0 & 0 & \mathbb{1}_d \\
\hline\hline
\mathbb{1}_d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbb{1}_d & 0 \\
0 & \mathbb{1}_d & 0 & 0 & \mathbb{1}_d & 0
\end{array}\right]
\end{array}
\qquad
\begin{array}{c}
\\ z \\ \\ b^{\sim} \\ \\
\end{array}
\begin{array}{c}
\begin{array}{cc} y \qquad & \qquad a \end{array} \\
\left[\begin{array}{c|ccc}
 & & & \\
\hline
 & 0 & \mathbb{1}_d & 0 \\
 & 0 & 0 & \mathbb{1}_d \\
 & A(Q)^{-1} & 0 & 0
\end{array}\right]
\end{array}
$$

In view of the required classification, we now choose a set Ω_3 of asymmetric periodic catenations which contains one representative of each class $\coprod_{p\in\mathbb{Z}}([u\{p\}] \coprod [u\{p\}^*])$. For each $(u, Q) \in \Omega_3 \times \mathcal{P}$, we denote the representation constructed above by $R(u, Q)$. A representation of S isomorphic to such an $R(u, Q)$ will be called *an asymmetric band.*

2.6. We finally turn to the case of a symmetric periodic catenation u. It is easy to prove that $u_0 u_1 \cdots u_{\pi-1}$ then has the form

$$u_0 u_1 \cdots u_{\pi-1} = a_1 \cdots a_r e b_r \cdots b_1 c_1 \cdots c_s f d_s \cdots d_1$$

where $[a_r^* \cdots a_1^*] = [b_r \cdots b_1]$, $[c_s^* \cdots c_1^*] = [d_s \cdots d_1]$, $e \neq e^{\wedge}$, and $f \neq f^{\wedge}$. Setting

$$
\begin{aligned}
w &= w_1 w_2 \cdots w_\pi w_{\pi+1} \cdots w_{2\pi} w_{2\pi+1} \cdots w_{3\pi} \\
&= u_{-\pi} u_{-\pi+1} \cdots u_{-1} u_0 u_1 \cdots u_{\pi-1} u_\pi \cdots u_{2\pi-1}.
\end{aligned}
$$

as in 2.5. We shall associate a representation $(Z_u^K; \eta_{u1}^K, \cdots, \eta_{un}^K)$ with each matrix

$$K = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in k^{(m+m')\times(l+l')}$$

belonging to $\mathcal{Q}$. By $\mathcal{Q}$ we denote the set of the following matrices ($q \geq 0$):

$$
1)\ \left[\begin{array}{c|c} 0 & \\ \mathbb{1}_q & \mathbb{1}_{q+1} \\ \hline \mathbb{1}_q & \mathbb{1}_q\ 0 \end{array}\right],\
\left[\begin{array}{c|c} \mathbb{1}_q\ 0 & \mathbb{1}_q \\ \hline & 0 \\ \mathbb{1}_{q+1} & \mathbb{1}_q \end{array}\right],\
\left[\begin{array}{c|c} \mathbb{1}_{q+1} & \mathbb{1}_{q+1} \\ \hline \mathbb{1}_{q+1} & J_{q+1} \end{array}\right],\
\left[\begin{array}{c|c} J_{q+1} & \mathbb{1}_{q+1} \\ \hline \mathbb{1}_{q+1} & \mathbb{1}_{q+1} \end{array}\right],
$$

$$
2)\ \left[\begin{array}{c|c} & 0 \\ \mathbb{1}_{q+1} & \mathbb{1}_q \\ \hline \mathbb{1}_q\ 0 & \mathbb{1}_q \end{array}\right],\
\left[\begin{array}{c|c} \mathbb{1}_q & \mathbb{1}_q\ 0 \\ \hline 0 & \\ \mathbb{1}_q & \mathbb{1}_{q+1} \end{array}\right],\
\left[\begin{array}{c|c} \mathbb{1}_{q+1} & \mathbb{1}_{q+1} \\ \hline J_{q+1} & \mathbb{1}_{q+1} \end{array}\right],\
\left[\begin{array}{c|c} \mathbb{1}_{q+1} & J_{q+1} \\ \hline \mathbb{1}_{q+1} & \mathbb{1}_{q+1} \end{array}\right],
$$

$$
3)\ \left[\begin{array}{c|c} A(Q) & \mathbb{1}_{q+1} \\ \hline \mathbb{1}_{q+1} & \mathbb{1}_{q+1} \end{array}\right],
$$

where

$$J_{q+1} = \begin{bmatrix} 0 & 1 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \cdot & 0 & 1 \\ 0 & 0 & 0 & \cdot & 0 & 0 \end{bmatrix},$$

and where $Q = P^r$ is a power of an irreducible unitary polynomial $P \neq X,\ X-1$ (see 2.5).

For this sake, we consider the following summands of X_w

$$E = \widehat{a_1} \oplus \widehat{a_2} \oplus \cdots \oplus \widehat{a_r} \oplus \{e\}, \qquad E' = \{e^\wedge\} \oplus \widehat{b_r} \oplus \cdots \widehat{b_2} \oplus \widehat{b_1},$$
$$F = \widehat{c_1} \oplus \widehat{c_2} \oplus \cdots \oplus \widehat{c_r} \oplus \{f\}, \qquad F' = \{f^\wedge\} \oplus \widehat{d_r} \oplus \cdots \widehat{d_2} \oplus \widehat{d_1},$$

and set $Z_u^K = E^l \oplus E'^{l'} \oplus F^{m'} \oplus F'^m$.

The structure maps $\eta_{ui}^K : M_i^-(Z_u^K) \longrightarrow M_i^+(Z_u^K)$ are defined as sums

$$\eta_{ui}^K = \eta_{uiE}^K + \eta_{uiE'}^K + \eta_{uiF}^K + \eta_{uiF'}^K + \nu_{ui}^K,$$

where the first four summands are induced by $f_{wi} : M_i^-(X_w) \longrightarrow M_i^+(X_w)$. For instance, η_{uiE}^K is the composition

$$M_i^-(Z_u^K) \xrightarrow{\mathrm{pr}} M_i^-(E)^l \xrightarrow{\mathrm{im}} M_i^-(X_w)^l \xrightarrow{f_{wi}^l} M_i^+(X_w)^l \xrightarrow{\mathrm{pr}} M_i^+(E)^l \xrightarrow{\mathrm{im}} M_i^+(Z_u^K).$$

The last summand ν_{ui}^K is also a composition, namely,

$$M_i^-(Z_u^K) \xrightarrow{\mathrm{pr}} M_i^-(E)^l \oplus M_i^-(E')^{l'} \xrightarrow{\mathrm{im}} M_i^-(X_w)^l \oplus M_i^-(X_w)^{l'}$$
$$\xrightarrow{g_{wi\,\pi+1}^l \oplus g_{wi\,\pi+2r+2}^{l'}} k^l \oplus k^{l'} \xrightarrow{K} k^m \oplus k^{m'} \xrightarrow{h_{wi\,\pi+2r+2}^m \oplus h_{wi\,2\pi+1}^{m'}}$$
$$M_i^+(X_w)^m \oplus M_i^+(X_w)^{m'} \xrightarrow{\mathrm{pr}} M_i^+(F')^m \oplus M_i^+(F)^{m'} \xrightarrow{\mathrm{im}} M_i^+(Z_u^K)$$

if $a_1 \in S_i^-$;

$$M_i^+(Z_u^K) \xleftarrow{\mathrm{im}} M_i^+(E)^l \oplus M_i^+(E')^{l'} \xleftarrow{\mathrm{pr}} M_i^+(X_w)^l \oplus M_i^+(X_w)^{l'}$$
$$\xleftarrow{h_{wi\,\pi+1}^l \oplus h_{wi\,\pi+2r+2}^{l'}} k^l \oplus k^{l'} \xleftarrow{K'} k^m \oplus k^{m'} \xleftarrow{g_{wi\,\pi+2r+2}^m \oplus g_{wi\,2\pi+1}^{m'}}$$
$$M_i^-(X_w)^m \oplus M_i^-(X_w)^{m'} \xleftarrow{\mathrm{im}} M_i^-(F')^m \oplus M_i^-(F)^{m'} \xleftarrow{\mathrm{pr}} M_i^-(Z_u^K)$$

if $a_1 \in S_i^+$, where $K' = \left[\begin{array}{c|c} D & C \\ \hline B & A \end{array}\right]$ if K is one of matrices listed in 1), and $K' = K$ otherwise. The ν_{ui}^K is zero if $a_1 \notin S_i^- \cup S_i^+$.

As an example, we consider the bush of 2.3 and the case

$$u_0u_1\cdots u_{\pi-1} = aea^{\sim}b^{\sim}e^{\wedge}b.$$

The structure maps are then visualized by the following diagram:

$$\begin{tikzcd}
(ke_1)^l \arrow[r] & (ka_1^{\sim})^l & (ka_1)^l \arrow[r, "A"] \arrow[dr, "C"'] & (kb_1^{\sim})^m & (kb_1)^m & (ke_2)^m \arrow[l] \\
(ke_1^{\wedge})^{l'} \arrow[r] & (ka_2^{\sim})^{l'} & (ka_2)^{l'} \arrow[r, "D"] \arrow[ur, "B"'] & (kb_2^{\sim})^{m'} & (kb_2)^{m'} & (ke_2^{\wedge})^{m'} \arrow[l]
\end{tikzcd}$$

and the corresponding matrices are

$$\begin{array}{c c}
 & \begin{array}{cccc} & e & e^{\wedge} & \end{array} \\
\begin{array}{c} b \\ x \\ \\ a^{\sim} \end{array} & \left[\begin{array}{cc|cc} 0 & \mathbb{1}_m & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{m'} \\ \hline\hline \mathbb{1}_l & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1}_{l'} & 0 \end{array}\right]
\end{array}
\qquad\qquad
\begin{array}{c c}
 & \begin{array}{ccc} y & a & \end{array} \\
\begin{array}{c} z \\ \\ b^{\sim} \end{array} & \left[\begin{array}{c|cc} & & \\ \hline & A & B \\ & C & D \end{array}\right]
\end{array}$$

In view of our classification, we finally choose a set Ω_4 of symmetric periodic catenations which contains one representative of each class $\coprod_{p\in\mathbb{Z}}[u\{p\}]$. For each $(u,K)\in\Omega_4\times\mathcal{Q}$ the preceding construction then provides a representation $R(u,K)$. A representation of S isomorphic to such an $R(u,K)$ will be called a *dimidiate band.*

2.7. Main Theorem. *Each indecomposable representation of a (completed) bush S is a string (asymmetric or dimidiate) or a band (asymmetric or dimidiate). The represetations $R(\delta)$, where*

$$\delta\in\Omega_1\amalg\Omega_2\times\{0,1\}\amalg\Omega_3\times\mathcal{P}\amalg\Omega_4\times\mathcal{Q}\,,$$

are indecomposable and pairwise non–isomorphic.

The proof of the main theorem is based on the reduction in section 3 and will be given in section 4.

2.8. Remark. (a) Let $w = w_1w_2\cdots w_t$ be an asymmetric catenation. then the reverse catenation $w^* = w_t^*\cdots w_2^*w_1^*$ is also asymmetric. By the construction of $R(w)$ and $R(w^*)$, we may identify X_w with X_{w^*} by identifying $\widehat{w_i}$ with $\widehat{w_{t-i+1}^*}$

for $1 \leq i \leq t$. In fact, one sees easily that such an identification induces an isomorphism between $R(w)$ and $R(w^*)$.

(b) Let $w = w_1 w_2 \cdots w_t$ be a symmetric catenation. Then w is of the form

$$w = a_1 a_2 \cdots a_s e b_s \cdots b_2 b_1,$$

where $[a_s^* \cdots a_2^* a_1^*] = [b_s \cdots b_2 b_1]$ and $e \neq e^\wedge$. Set

$$v = a_1 a_2 \cdots a_s e a_s^* \cdots a_2^* a_1^*.$$

Then v is a symmetric catenation and equivalent to w. By the construction in 2.2, one easily sees that $(X_v; f_{v1}, \cdots, f_{vn})$ is decompose into a direct sum of two representations.

(c) Let $u = (u_i)_{i\in\mathbb{Z}}$ be an asymmetric periodic catenations of period π. Let $v = (v_i)_{\mathbb{Z}}$ be such that $v_{k\pi+i} = u_i$ for all $k \in \mathbb{Z}$ and $0 \leq i \leq \pi - 1$. Then v is also an asymmetric periodic catenation of period π which is eqiuvalent to u. By the construction in 2.5, there holds that $R(u, Q) \cong R(v, Q)$ for each $Q \in \mathcal{P}$. Moreover, by changing basis vectors, one can easily prove that $R(v, Q) \cong R(v\{p\}, Q)$ for all $p \in \mathbb{Z}$. Thus u and $v\{p\}$ $(p \in \mathbb{Z})$ provide the same family of isoclasses of representations of S.

Further, the reverse catenation v^* of v is asymmetric. By the construction in 2.5, for each

$$Q = P^l = X^{ml} - a_1 X^{ml-1} - a_2 X^{ml-2} - \cdots - a_{ml}, \quad l \geq 1$$

in $\mathcal{P}$, we set $Q'(X) = (-1)\frac{1}{a_{ml}} X^{ml} Q(\frac{1}{X}) \in \mathcal{P}$, then there holds that $R(v, Q) \cong R(v^*, Q')$ since $A(Q') = A(Q)^{-1}$. Conversely, $R(v^*, Q) \cong R(v, Q')$. Thus v and v^* provide the same family of isoclasses of representations of S.

(d) Let $u = (u_i)_{i\in\mathbb{Z}}$ be a symmetric periodic catenations of period π. As in (c), let $v = (v_i)_{i\in\mathbb{Z}}$ be such that $v_{k\pi+i} = u_i$ for all $k \in \mathbb{Z}$ and $0 \leq i \leq \pi - 1$. Then v is also a symmetric periodic catenation of period π which is eqiuvalent to u. By changing basis vectors, there holds that $R(u, K) \cong R(v\{p\}, K)$ for each $K \in \mathcal{Q}$ and each $p \in \mathbb{Z}$. Hence u and $v\{p\}$ $(p \in \mathbb{Z})$ provide the same family of isoclasses of representations of S.

3. A reduction of representations of bushes

In this section, we shall formulate the algorithm in [3] with respect to tangles formed by sequences of modules. We shall see in next section that such an algorithm will lead us to an efficient reduction of representations of bushes. All the proofs are analogous to those in [3]. We omit them.

3.1. Let $S = (S_1^-, \cdots, S_n^-; S_1^+, \cdots S_n^+; \sim)$ ba a bush as in 1.3 and (M^-, M^+) the tangle associated with S. For each representation $(X; f_1, \cdots, f_n)$ of (M^-, M^+), we denote by f the sequence $(f_1, \cdots, f_n)$ and simply write $(X; f)$ for $(X; f_1, \cdots, f_n)$.

Let $1 \leq i \leq n$. We start with submodules $K^- \subseteq L^-$ of M_i^- and submodules $K^+ \subseteq L^+$ of M_i^+. We are interested in the representations $(X; f) = (X; f_1, \cdots, f_i, \cdots f_n)$ of (M^-, M^+) which satisfy $f_i(K^-(X)) \subseteq K^+(X)$ and $f_i(L^-(X)) \subseteq L^+(X)$.

From now on, for each $X \in \mathcal{A}$, we fix subspaces $U_1^-(X)$, $U_2^-(X)$ of $M_i^-(X)$ and subspaces $U_1^+(X)$, $U_2^+(X)$ of $M_i^+(X)$ such that

$$L^-(X) = K^-(X) \oplus U_1^-(X), \qquad M_i^-(X) = L^-(X) \oplus U_2^-(X),$$
$$L^+(X) = U_1^+(X) \oplus K^+(X), \qquad M_i^+(X) = U_2^+(X) \oplus L^+(X).$$

Then for each representation $(X; f)$ of (M^-, M^+), the f_i can be written as the form:

$$f_i = \begin{bmatrix} f_{i1} & f_{i2} & f_{i3} \\ f_{i4} & f_{i5} & f_{i6} \\ f_{i7} & f_{i8} & f_{i9} \end{bmatrix} : M_i^-(X) = K^-(X) \oplus U_1^-(X) \oplus U_2^-(X)$$
$$\longrightarrow U_2^+(X) \oplus U_1^+(X) \oplus K^+(X) = M_i^+(X).$$

To the tangle (M^-, M^+) we now attach a new tangle as follows. First, we denote by $\mathcal{B}$ the full subcategory of $\mathrm{rep}(M^-, M^+)$ formed by representations $(X; \tilde{\rho})$, where $\rho : U_1^-(X) \to U_1^+(X)$ is a k–linar map and $\tilde{\rho}$ denote the sequence $(0, \cdots, 0, \rho_0, 0, \cdots, 0)$ with ρ_0 of the form:

$$\rho_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 0 \end{bmatrix} : M_i^-(X) = K^-(X) \oplus U_1^-(X) \oplus U_2^-(X)$$
$$\longrightarrow U_2^+(X) \oplus U_1^+(X) \oplus K^+(X) = M_i^+(X).$$

Further, for each $(X; \tilde{\rho}) \in \mathcal{B}$, we define

$$N_j^-(X; \tilde{\rho}) := M_j^-(X),\ N_j^+(X, \tilde{\rho}) := M_j^+(X), \ \text{ for all } \ j \neq i,$$
$$N_i^-(X; \tilde{\rho}) := \mathrm{Ker}\tilde{\rho} = K^-(X) \oplus \mathrm{Ker}\rho \oplus U_2^-(X),$$
$$\text{and} \quad N_i^+(X; \tilde{\rho}) := \mathrm{Coker}\tilde{\rho} = U_2^+(X) \oplus \mathrm{Coker}\rho \oplus K^+(X).$$

For a morphism $\mu : (X; \tilde{\rho}) \to (X'; \tilde{\rho'})$, we denote by $N_j^-(\mu)$ and $N_j^+(\mu)$ the k–linear maps induced respectively by $M_j^-(\mu)$ and $M_j^+(\mu)$ for $1 \leq j \leq n$. Then we obtain two sequences of modules $N^- = (N_1^-, \cdots, N_n^-)$ and $N^+ = (N_1^+, \cdots, N_n^+)$ over $\mathcal{B}$, that is, a tangle (N^-, N^+) over $\mathcal{B}$. Moreover, the modules N_i^- and N_i^+ admit respectively submodules J^- and J^+ such that

$$J^-(X, \tilde{\rho}) = K^-(X) \oplus \mathrm{Ker}\rho, \qquad J^+(X, \tilde{\rho}) = K^+(X).$$

Finally, we denote by $\mathcal{M}$ the full subcategory of rep(M^-, M^+) formed by representations $(X; f)$ satisfying $f_i(K^-(X)) \subseteq K^+(X)$ and $f_i(L^-(X)) \subseteq L^+(X)$, and by $\mathcal{N}$ the full subcategory of rep(N^-, N^+) formed by representations $((X, \tilde{\rho}); h) = ((X; \tilde{\rho}); h_1, \cdots h_n)$ satisfying $h_i(J^-(X, \tilde{\rho})) \subseteq J^+(X, \tilde{\rho})$. Our purpose is to build up a relation between categories $\mathcal{M}$ and $\mathcal{N}$.

3.2. From now on, we suppose that $K^- = \mathcal{R}^{i_1} M_i^-$ and $L^- = \mathcal{R}^{i_2} M_i^-$ for some $i_1 \geq i_2$, and that $K^+ = \mathcal{R}^{j_1} M_i^+$ and $L^+ = \mathcal{R}^{j_2} M_i^+$ for some $j_1 \geq j_2$.

In order to establish a reduction from objects of $\mathcal{M}$ to those of $\mathcal{N}$, for each $(X; \tilde{\rho})$ in $\mathcal{B}$, we choose a supplement $T^-(X; \tilde{\rho})$ of Kerρ in $U_1^-(X)$ and a supplement $T^+(X; \tilde{\rho})$ of Imρ in $U_1^+(X)$. Then ρ_0 can be written in the form:

$$\rho_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\rho} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : M_i^-(X) = K^-(X) \oplus \mathrm{Ker}\rho \oplus T^-(X; \tilde{\rho}) \oplus U_2^-(X)$$
$$\longrightarrow U_2^+(X) \oplus \mathrm{Im}\rho \oplus T^+(X; \tilde{\rho}) \oplus K^+(X) = M_i^+(X),$$

where $\bar{\rho} : T^-(X; \tilde{\rho}) \to \mathrm{Im}\rho$ is induced by ρ.

Further, for each object $(X; f)$ in $\mathcal{M}$, the f_i is of the form:

$$f_i = \begin{bmatrix} 0 & 0 & f_{i1} \\ 0 & f_{i2} & f_{i3} \\ f_{i4} & f_{i5} & f_{i6} \end{bmatrix} : M_i^-(X) = K^-(X) \oplus U_1^-(X) \oplus U_2^-(X)$$
$$\longrightarrow U_2^+(X) \oplus U_1^+(X) \oplus K^+(X) = M_i^+(X).$$

In such a way, $(X; f)$ gives rise to an object $(X; \tilde{f_{i2}})$ in $\mathcal{B}$. By further decomposing $U_1^-(X)$ and $U_1^+(X)$, we infer that f_i has the form:

$$f_i = \begin{bmatrix} 0 & 0 & 0 & f_{i1} \\ 0 & 0 & \bar{f_{i2}} & f''_{i3} \\ 0 & 0 & 0 & f'_{i3} \\ f_{i4} & f'_{i5} & f''_{i5} & f_{i6} \end{bmatrix} : M_i^-(X) = K^-(X) \oplus \mathrm{Ker} f_2 \oplus T^-(X; \tilde{f_{i2}}) \oplus U_2^-(X)$$
$$\longrightarrow U_2^+(X) \oplus \mathrm{Im} f_2 \oplus T^+(X; \tilde{f_{i2}}) \oplus K^+(X) = M_i^+(X).$$

Since the tangle (M^-, M^+) is rodded, $(X; f)$ is isomorphic to the object $(X; f') = (X; f_1, \cdots, f_{i-1}, f'_i, f_{i+1}, \cdots, f_n)$ with f'_i of the form:

$$f'_i = \begin{bmatrix} 0 & 0 & 0 & f_{i1} \\ 0 & 0 & \bar{f_{i2}} & 0 \\ 0 & 0 & 0 & f'_{i3} \\ f_{i4} & f'_{i5} & 0 & f'_{i6} \end{bmatrix} : M_i^-(X) = K^-(X) \oplus \mathrm{Ker} f_2 \oplus T^-(X; \tilde{f_{i2}}) \oplus U_2^-(X)$$
$$\longrightarrow U_2^+(X) \oplus \mathrm{Im} f_2 \oplus T^+(X; \tilde{f_{i2}}) \oplus K^+(X) = M_i^+(X),$$

where $f'_{i6} = f_{i6} - f''_{i5}\bar{f_{i2}}^{-1}f''_{i3}$. Finally, we denote by $\eta_{(X;f)}$ an isomorphism from $(X;f)$ to $(X;f')$.

Thus each object $(X;f)$ then gives rise to an object $((X;\tilde{f_{i2}}),\hat{f})$ in $\mathcal{N}$ with $\hat{f}=(f_1,\cdots,f_{i-1},\hat{f_i},f_{i+1},\cdots,f_n)$, where $\hat{f_i}$ is of the form

$$\hat{f_i}=\begin{bmatrix} 0 & 0 & f_{i1} \\ 0 & 0 & p_{(X;\tilde{f_{i2}})}f'_{i3} \\ f_{i4} & f'_{i5} & f'_{i6} \end{bmatrix} : N_i^-(X;\tilde{f_{i2}}) = K^-(X)\oplus \mathrm{Ker}f_2 \oplus U_2^-(X)$$
$$\longrightarrow U_2^+(X)\oplus \mathrm{Coker}f_2 \oplus K^+(X) = N_i^+(X;\tilde{f_{i2}}).$$

(Here $p_{(X;\tilde{f_{i2}})}$ denotes the restriction of the canonical projection $\pi_{(X;\tilde{f_{i2}})} : U_1^+(X) \to \mathrm{Coker}f_2 = U_1^+(X)/\mathrm{Im}f_2$ to $T^+(X;\tilde{f_{i2}})$.)

3.3. Remark. Up to isomorphisms, the representation $((X;\tilde{f_{i2}}),\hat{f})$ induced by $(X;f)$ is independent on the choice of supplements $T^-(X;\tilde{f_{i2}})$ and $T^+(X;\tilde{f_{i2}})$.

3.4. In view of Remark 3.3, for each $(X;\tilde{\rho})$ in $\mathcal{B}$, we may fix a supplement $T^-(X;\tilde{\rho})$ of $\mathrm{Ker}\rho$ in $U_1^-(X)$ and a supplement $T^+(X;\tilde{\rho})$ of $\mathrm{Im}\rho$ in $U_1^+(X)$. By the discussion in 3.2, each object $(X;f)$ in $\mathcal{M}$ then gives rise uniquely to an object $((X;\tilde{f_{i2}});\hat{f})$ in $\mathcal{N}$.

Let $(X;f)$ and $(Y;g)$ be objects in $\mathcal{M}$ and μ a morphism from $(X;f)$ to $(Y;g)$. With μ we now associate a morphism from $((X;\tilde{f_{i2}});\hat{f})$ to $((Y;\tilde{g_{i2}});\hat{g})$.

Again by 3.2, one has that $\tilde{\mu} =: \eta_{(Y;g)}\mu(\eta_{(X;f)})^{-1}$ is a morphism from $(X;f') = (X;f_1,\cdots,f'_i,\cdots,f_n)$ to $(Y;g') = (Y;g_1,\cdots,g'_i,\cdots,g_n)$, where f'_i and g'_i are of the forms:

$$f'_i=\begin{bmatrix} 0 & 0 & 0 & f_{i1} \\ 0 & 0 & \bar{f_{i2}} & 0 \\ 0 & 0 & 0 & f'_{i3} \\ f_{i4} & f'_{i5} & 0 & f'_{i6} \end{bmatrix} : M_i^-(X) = K^-(X)\oplus \mathrm{Ker}f_2 \oplus T^-(X;\tilde{f_{i2}})\oplus U_2^-(X)$$
$$\longrightarrow U_2^+(X)\oplus \mathrm{Im}f_2 \oplus T^+(X;\tilde{f_{i2}})\oplus K^+(X) = M_i^+(X)$$

and

$$g'=\begin{bmatrix} 0 & 0 & 0 & g_{i1} \\ 0 & 0 & \bar{g_{i2}} & 0 \\ 0 & 0 & 0 & g'_{i3} \\ g_{i4} & g_{i5} & 0 & g'_{i6} \end{bmatrix} : M_i^-(Y) = K^-(Y)\oplus \mathrm{Ker}g_2 \oplus T^-(Y;\tilde{g_{i2}})\oplus S_2^-(Y)$$
$$\longrightarrow S_2^+(Y)\oplus \mathrm{Im}g_2 \oplus T^+(Y;\tilde{g_{i2}})\oplus K^+(Y) = M_i^+(Y).$$

Further, the maps $M_i^-(\tilde{\mu})$ and $M_i^+(\tilde{\mu})$ can be written as the following forms:

$$M_i^-(\tilde{\mu}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} : M_i^-(X)$$
$$= K^-(X) \oplus \mathrm{Ker} f_2 \oplus T^-(X; \tilde{f_{i2}}) \oplus U_2^-(X)$$
$$\longrightarrow K^-(Y) \oplus \mathrm{Ker} g_2 \oplus T^-(Y; \tilde{g_{i2}}) \oplus S_2^-(Y) = M_i^-(Y)$$

and

$$M_i^+(\tilde{\mu}) = \begin{bmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} : M_i^+(X)$$
$$= U_2^+(X) \oplus \mathrm{Im} f_2 \oplus T^+(X; \tilde{f_{i2}}) \oplus K^+(X)$$
$$\longrightarrow S_2^+(Y) \oplus \mathrm{Im} g_2 \oplus T^+(Y; \tilde{g_{i2}}) \oplus K^+(Y) = M_i^+(Y).$$

Then there holds that

$$\begin{bmatrix} 0 & 0 & 0 & g_{i1} \\ 0 & 0 & \bar{g_{i2}} & 0 \\ 0 & 0 & 0 & g'_{i3} \\ g'_{i4} & g'_{i5} & 0 & g'_{i6} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$
$$= \begin{bmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & f_{i1} \\ 0 & 0 & \bar{f_{i2}} & 0 \\ 0 & 0 & 0 & f'_{i3} \\ f_{i4} & f'_{i5} & 0 & f'_{i6} \end{bmatrix} \tag{1}$$

since $\tilde{\mu}$ is a morphism.

It then follows that $a_{32} = 0$ and $b_{32} = 0$. Since (M^-, M^+) is rodded, there is a morphism $\mu' \in \mathcal{R}_{\mathcal{A}}(X, Y)$ such that

$$(M_j^-(\mu'), M_j^+(\mu') = (0,0) \quad \text{for all } j \neq i,$$
$$(M_i^-(\mu'), M_i^+(\mu')) = \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b_{42} & 0 & 0 \end{bmatrix} \right).$$

We then set $\hat{\mu} = \tilde{\mu} - \mu' : X \to Y$. It is easy to show that $\hat{\mu}$ is a morphism from $(X; \tilde{f_{i2}})$ to $(Y; \tilde{g_{i2}})$. By (1) there also holds that

$$\hat{g} N_i^-(\hat{\mu}) = N_i^+(\hat{\mu}) \hat{f},$$

that is, $\hat{\mu}$ is a morphism from $((X;\tilde{f_{i2}});\hat{f})$ to $((Y;\tilde{g_{i2}}),\hat{g})$.

As a conclusion, we obtain two correspondences $(X;f) \longmapsto ((X;\tilde{f_{i2}});\hat{f})$ and $\mu \longmapsto \hat{\mu}$ which induce a functor

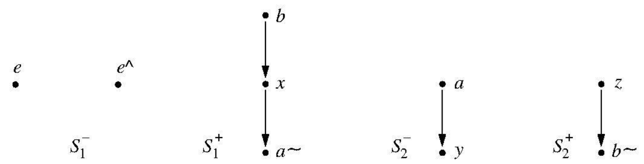
$$\Phi : \mathcal{M} \longrightarrow \mathcal{N}/I$$

such that $\Phi(X;f) = ((X;\tilde{f_{i2}});\hat{f})$ and $\Phi(\mu) = \hat{\mu} + I$, where I denotes the ideal of $\mathcal{N}$ generated by $\hat{\nu\mu} - \hat{\nu}\hat{\mu}$ for $\mu : (X;f) \to (Y;g)$ and $\nu : (Y;g) \to (Z;h)$ in $\mathcal{M}$.

Proposition. *(1) The ideal I lies in the radical of $\mathcal{N}$.*

(2) The functor Φ is an epivalence, i.e. Φ is full, hits each isoclass, and detects isomorphisms.

3.5. For the practical application, in certain situations it imports us to translate the reduction into the language of matrix problems. We illustrate the translation with an example: Let S be the (non–complete) bush in 2.3, i.e. S is formed by the following pairs of rods:



The associated tangle consists of two pairs of modules (M_1^-, M_1^+) and (M_2^-, M_2^+). Let $(X; f_1, f_2)$ be a representation of S. If X is fixed, the chosen bases of $M_i^-(X)$ and of $M_i^+(X)$ provide us a matrix problem given by a pair of partitioned matrices

$$\begin{array}{c} \\ b \\ x \\ a^{\sim} \end{array}\begin{array}{c} \begin{array}{cc} e & e^{\wedge} \end{array} \\ \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \\ \hline A_5 & A_6 \end{array}\right] \end{array} \qquad \begin{array}{c} \\ z \\ b^{\sim} \end{array}\begin{array}{c} \begin{array}{cc} y & a \end{array} \\ \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right] \end{array}$$

together with the following admissible transformations

(a) arbitrary row transformations within stripes x and z and abitrary column transformations within stripes e, $e^{\wedge}$ and y;

(b) row transformations within stripe $a^{\sim}$ coupled with the conjugate column transformations within stripe a, row transformations within stripe b coupled with the same row transformations within stripe $b^{\sim}$ (Note that the number of rows in stripe $a^{\sim}$ equals to the number of columns in stripe a and that the number of rows in stripe b equals to the number of rows in stripe $b^{\sim}$);

(c) additions of multiples of rows between different stripes are allowed from b to x and $a^{\sim}$, from x to $a^{\sim}$, and from z to $b^{\sim}$, additions of multiples of columns between different stripes are only allowed from y to a.

Thanks to the algorithm, we first reduce $[A_1|A_2]$ to the following form:

$$\left[\begin{array}{c:c:c|c:c:c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & \mathbb{1} & 0 \\ \hdashline 0 & \mathbb{1} & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & \mathbb{1} & 0 & 0 & \mathbb{1} \end{array}\right]$$

By performing admissible transformations, we reduce A_3, A_4, A_5 and A_6 to the following forms (the row partition of stripe b induces a partition of stripe $b^{\sim}$):

$$\begin{array}{c} \\ \\ b \\ \\ \\ x \\ a^{\sim} \end{array} \begin{array}{c} \begin{array}{ccc} & e & \end{array} \qquad\qquad \begin{array}{ccc} & e^{\wedge} & \end{array} \\ \left[\begin{array}{c:c:c|c:c:c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & \mathbb{1} & 0 \\ \hdashline 0 & \mathbb{1} & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & \mathbb{1} & 0 & 0 & \mathbb{1} \\ \hline A_{31} & 0 & 0 & A_{41} & 0 & A_{42} \\ \hline A_{51} & 0 & 0 & A_{61} & 0 & A_{62} \end{array}\right] \end{array} \qquad \begin{array}{c} \\ z \\ \\ b^{\sim} \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} y & a \end{array} \\ \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_{31} & B_{41} \\ \hdashline B_{32} & B_{42} \\ \hdashline B_{33} & B_{43} \\ \hdashline B_{34} & B_{44} \end{array}\right] \end{array}$$

Thus we are reduced to the matrix problems described by the following matrices:

$$\begin{array}{c|ccc} & e_0 & e_0^{\wedge} & e' \\ \hline b_0 & 0 & 0 & 0 \\ x & A_{31} & A_{41} & A_{42} \\ a^{\sim} & A_{51} & A_{61} & A_{62} \end{array} \qquad \begin{array}{c|cc} & y & a \\ \hline z & B_1 & B_2 \\ b_0^{\sim} & B_{31} & B_{41} \\ b_1' & B_{32} & B_{42} \\ b_2' & B_{33} & B_{43} \\ b_3' & B_{34} & B_{44} \end{array}$$

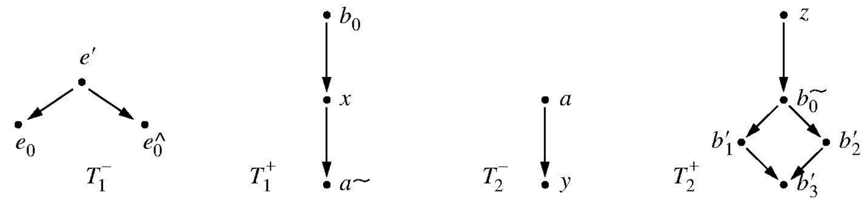
Without spoiling the reduced form of $[A_1|A_2]$, we can perform the following transformations to the matrices above:

(a') arbitrary row transformations within stripes x, z, $b_0^{\sim}$, b_1', and b_2', and abitrary column transformations within stripes e_0, $e_0^{\wedge}$, and y;

(b') row transformations within stripe $a^{\sim}$ (resp. b_2) coupled with the conjugate column transformations within stripe a (resp. e'),

(c') additions of multiples of rows and columns between different stripes are illustrated by the arrows in the figure above.

Thus the reduced matrices can be viewed as a matrix representation of a new bush T given by the following pair of rods:



together with the equivalence relation such that $a \sim a^{\sim}$, $b_0 \sim b_0^{\sim}$ and $e' \sim b'_3$. This matrix problem coincides with that obtained from the algorithm. For further reduction, one can reduce matrix $[A_{31}|A_{41}]$, and so on.

4. The proof of the main theorem

4.1. In this section, we shall keep all the notations in the preceding Sections. Let S be a bush and $(X; f_1, \cdots, f_n)$ a representation of S. By definition, the *dimension* of $(X; f_1, \cdots, f_n)$ is

$$\sum_{i=1}^{n} (\dim_k M_i^-(X) + \dim_k M_i^+(X)),$$

where M_i^- and M_i^+ are the modules associated with S (see 1.3).

Let us now return and stick to complete bushes. We start from a complete bush $S = (S_1^-, \cdots, S_n^-; S_1^+, \cdots, S_n^+; \sim)$.

By abuse of notations in Section 2 , we call a representation $(X; f_1, \cdots, f_n)$ of S an asymmetric string if it is isomorphic to $R(w)$ for some asymmetric catenation w, a dimidiate string if it is isomorphic to a non–trivial summand of $(X_v; f_{v1}, \cdots, f_{vn})$ for some symmetric catenation v, and an asymmetric (resp. a dimidiate) band if it is isomorphic to $R(u, Q)$ (resp. $R(u, K)$) for some asymmetric (resp. symmetric) periodic catenation u and some $Q \in \mathcal{P}$ (resp. $K \in \mathcal{Q}$).

Let $(X; f) = (X; f_1, \cdots, f_n)$ be an indecomposable representation of S with dimension d. Our objective is to prove by induction on d that $(X; f_1, \cdots, f_n)$ is a string or a band.

If $d = 1$, it is clear that $(X; f_1, \cdots, f_n)$ is a dimidiate string. We now suppose that $d > 1$ and that every indecomposable representation of an arbitrary complete bush T with dimension $< d$ is a string or a band.

If all f_i vanish, X is indecomposable in $\mathcal{A}$ and $(X; f_1, \cdots, f_n) = (X; 0, \cdots, 0)$ is an asymmetric string (since $d > 1$). Otherwise, let $1 \leq i \leq n$ be such that $f_i \neq 0$. Then there are m^-, $m^+ \in \mathbb{N}$ such that

i) $f_i(\mathcal{R}^{m^-+1} M_i^-(X)) \subseteq \mathcal{R}^{m^++1} M_i^+(X)$,

ii) $f_i(\mathcal{R}^{m^-} M_i^-(X)) \subseteq \mathcal{R}^{m^+} M_i^+(X)$,

iii) the induced map

$$\bar{f}_i : \mathcal{R}^{m^-} M_i^-(X)/\mathcal{R}^{m^-+1} M_i^-(X) \longrightarrow \mathcal{R}^{m^+} M_i^+(X)/\mathcal{R}^{m^++1} M_i^+(X)$$

is not zero, where $\mathcal{R}$ denotes the radical of $\mathcal{A}$.

We then set $K^- = \mathcal{R}^{m^-+1}M_i^-$, $L^- = \mathcal{R}^{m^-}M_i^-$, $K^+ = \mathcal{R}^{m^++1}M_i^+$, and $L^+ = \mathcal{R}^{m^+}M_i^+$.

By Proposition 3.4, we may reduce $(X; f)$ to a representation $((X; \tilde{f_{i2}}); \hat{f})$ of a new tangle (N^-, N^+) over the aggregate $\mathcal{B}$ such that $((X; \tilde{f_{i2}}); \hat{f})$ is indecomposable and has dimension strictly less than d.

Since the lattices of submodules of M_i^- and M_i^+ are rods, both the supports of L^-/K^- and of L^+/K^+ contain one or two elements in $\mathcal{S}$. We examine the various cases seperately.

4.2. Case I. *supp*$(L^-/K^-) = \{\bar{x}\}$, *supp*$(L^+/K^+) = \{\bar{y}\}$ *and* $\bar{x} \neq \bar{y}$ *for some* $x \in S_i^-$ *and* $y \in S_i^+$.

By way of example, we may suppose that $x^\sim \in S_{j_1}^-$, $y^\sim \in S_{j_2}^-$ for some $j_1 \neq i$, $j_2 \neq i$. All the other situations can be treated similarly.

In order to apply the algorithm described in 3.2–3.4, we choose the supplements U_1^-, U_2^-, U_1^+ and U_2^+ in the following canonical way: For each $a \in \mathcal{S}$, we set

$$U_1^-(a) = \begin{cases} 0 & \text{if } a \neq \bar{x} \\ kx & \text{if } a = \bar{x} \end{cases} \qquad U_2^-(a) = \oplus_{u \in a, u \in P^-, u < x} ku$$

and

$$U_1^+(a) = \begin{cases} 0 & \text{if } a \neq \bar{y} \\ ky & \text{if } a = \bar{y} \end{cases} \qquad U_2^+(a) = \oplus_{v \in a, v \in P^+, v < y} kv$$

where $\bar{x}$ and $\bar{y}$ denote the equivalence classes of x and y in S, respectively. Finally, for each $X \overset{\mu}{\cong} \oplus_{a \in \mathcal{S}} a^{n(a)} \in \mathcal{A}$, we set

$$\begin{aligned} &U_i^-(X) = M_i^-(\mu)^{-1}(\oplus_{a \in \mathcal{S}} U_i^-(a)^{n(a)}) \\ &\text{and} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2. \\ &U_i^+(X) = M_i^+(\mu)^{-1}(\oplus_{a \in \mathcal{S}} U_i^+(a)^{n(a)}) \end{aligned}$$

The representations $(a; 0) = (a; 0, \cdots, 0)$, $a \in \mathcal{S}$, and $(\bar{x} \oplus \bar{y}; \eta)$, furnish a complete list of indecomposables in the aggregate $\mathcal{B}$, where η denotes the sequence $(0, \cdots, 0, \eta_i = 1, 0, \cdots, 0)$. Then there holds that

$$N_{j_1}^-(Y; g) = \begin{cases} M_{j_1}^-(Y) & \text{if } (Y; g) = (a; 0) \\ kx^\sim & \text{if } (Y; g) = (\bar{x} \oplus \bar{y}, \eta) \end{cases}$$

$$N_{j_2}^-(X; f) = \begin{cases} M_{j_2}^-(Y) & \text{if } (Y; g) = (a; 0) \\ ky^\sim & \text{if } (Y; g) = (\bar{x} \oplus \bar{y}, \eta) \end{cases}$$

We denote by $\bar{\mathcal{S}}$ the spectroid of $\mathcal{B}$ formed by representations $(a; 0) = (a; 0, \cdots, 0)$, $a \in \mathcal{S}$ and $(\bar{x} \oplus \bar{y}; \eta)$.

By $T=(T_1^-,\cdots,T_n^-;T_1^+,\cdots,T_n^+;\sim)$ we denote the bush formed by rods $T_{j_1}^-=S_{j_1}^-\amalg\{x_1\}$, $T_{j_2}^-=S_{j_2}^-\amalg\{y_1\}$, and $T_l^\varepsilon=S_l^\varepsilon$ for $(l,\varepsilon)\neq(j_1,-)$, $(j_2,-)$. The union $\cup_{i=1}^n(T_i^-\cup T_i^+)$ is equipped with the smallest order relation which contains that of S induced by S_k^ε and is such that $x^\sim>x_1$, $y^\sim<y_1$ and $z\gtrless x_1$ (resp. $z\gtrless y_1$) iff $z\gtrless x^\sim$ (resp. $z\gtrless y^\sim$). Finally we equip T with the equivalence relation induced by S and extended by $x_1\sim y_1$. The spectroid associated with T (see 1.3) is denoted by $\mathcal{T}$.

An easy observation shows that the correspondence

$$(\bar z;0)\longmapsto\bar z,\quad z\in S,\qquad(\bar x\oplus\bar y;\eta)\longmapsto\bar x_1=\bar y_1$$

gives rise to an isomorphism from $\bar{\mathcal{S}}$ to $\mathcal{T}$. Therefore, by identifying $\bar{\mathcal{S}}$ with $\mathcal{T}$, the reduced form $((X;\tilde{f_{i2}}),\hat f)$ of $(X;f)$ can be considered as a representation of the new bush T.

By induction hypothesis, $((X;\tilde{f_{i2}});\hat f)$ is a string or a band which is associated with a catenation v (finite or periodic) of T.

We denote by w the catenation of S obtained from v by replacing each term x_1 by $x^\frown y$ and y_1 by $y^\frown x$.

We first consider the case where v is an asymmetric catenation. Then w is also an asymmetric catenation and $R(w)=(X_w;f_{w1},\cdots,f_{wn})$ is an asymmetric string. By the construction of $R(w)$, one sees that each part $x^\frown y$ or $y^\frown x$ in w provides a summand $(\bar x\oplus\bar y;\eta)$ of $(X_w;\tilde{f}_{wi2})$. Thus $(X_w;\tilde{f}_{wi2})$ and X_v considered as objects in $\mathcal{B}$ are isomorphic. By identifying $(X_w;\tilde{f}_{wi2})$ with X_v, the action of $\hat f_{wi}$ coincides with that of f_{vi}, so the representation $((X_w;\tilde{f}_{wi2});\hat f_w)$ is isomorphic to $R(v)$. By Proposition 3.4, we infer that $(X;f)\cong R(w)$, that is, (X,f) is an asymmetric string (The decisive point is the following: If a term w_r of w arises from some term $v_q\neq v_q^\wedge$, then $[w_{r-1}^*w_{r-2}^*\cdots]\le[w_{r+1}w_{r+2}\cdots]$ is equivalent to $[v_{q-1}^*v_{q-2}^*\cdots]\le[v_{q+1}v_{q+2}\cdots]$).

If v is a symmetric catenation, so is w. One then obtains that $(X;f)$ is isomorphic to a non–trivial summand of $(X_w;f_{w1},\cdots,f_{wn})$ since $(X;\tilde{f_{i2}});\hat f)$ is isomorphic to a non–trivial summand of $(X_v;f_{v1},\cdots,f_{vn})$, that is, $(X;f)$ is a dimidiate string.

In the case where v is a periodic catenation, one can similarly prove that $(X;f)\cong R(w,Q)$ (resp. $R(w,K)$) according as $(X;\tilde{f_{i2}});\hat f)\cong R(v,Q)$ (resp. $R(v,K)$) for some $Q\in\mathcal{P}$ (resp. $K\in\mathcal{P}$), that is, an asymmetric (resp. a dimidiate) band.

4.3. Case II. *supp*$(L^-/K^-)=$ *supp*$(L^+/K^+)=\{\bar x\}$ *for some* $x\in S_i^-$ *with* $x^\sim\in S_i^+$.

In this case, one can easily see that the representations $(a;0)$, $a\in\mathcal{S}$ and $((\bar x)^t;\eta(t))$, $t>1$, furnish a complete list of indecomposables in $\mathcal{B}$ which are not annihilated by $\mathcal{J}$, where $\mathcal{J}$ denotes the intersection of annihilators of all N_j^- and N_j^+, and where $\eta(t)$ denotes the sequence $(0,\cdots,0,\eta(t)_i,0,\cdots,0)$ with $\eta(t)_i$ of the form:

$$\eta(t)_i = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} : M_i^-((\bar{x})^t) = (kx)^t \longrightarrow (kx^\sim)^t = M_i^+((\bar{x})^t).$$

(Note that the supplements S_1^-, S_2^-, S_1^+ and S_2^+ are chosen in a cononical way similar to the case I.)

Since $(X; f)$ is finite dimensional, there exists an $L > 0$ such that the induced representations $(X; \tilde{f_{i2}})$ does not contain a summand isomorphic to some $((\bar{x})^t, \eta(t))$ for $t > L$.

Let $\bar{\mathcal{S}}$ denote the spectroid formed by $(a; 0)$, $a \in \mathcal{S}$, and $((\bar{x})^t; \eta(t))$, $1 < t \leq L+1$, and $\mathcal{T}$ the spectroid associated with the bush $T = T(L) = (T_1^-, \cdots, T_n^-; T_1^+, \cdots, T_n^+; \sim)$, where the order relation on the union of the sets $T_i^- = S_i^- \amalg \{x_1, \cdots, x_L\}$, $T_i^+ = S_i^+ \amalg \{x_1^\sim, \cdots, x_L^\sim\}$, and $T_l^\varepsilon = S_l^\varepsilon$, $(l, \varepsilon) \neq (i, -)$, $(i, +)$, are defined as in case I (in particular, $x < x_1 < \cdots < x_L$, $x_L^\sim < \cdots < x_1^\sim < x^\sim$). The equivalence relation equipped with T is induced by that of S and extended by $x_j \sim x_j^\sim$ for $j = 1, \cdots, L$.

Then the correspondence

$$(\bar{z}; 0) \longmapsto \bar{z}, \quad z \in S, \qquad ((\bar{x})^t; \eta(t)) \longmapsto \bar{x}_{t-1}, \quad 1 < t \leq L+1$$

defines an isomorphism from $\bar{\mathcal{S}}$ to $\mathcal{T}$.

If $(X; \tilde{f_{i2}})$ contains a non–zero summand annihilated by $\mathcal{J}$, $(X; f)$ is isomorphic to (x, Q) for some $Q \in \mathcal{P}$ because of the indecomposability of $(X; f)$, thus is an asymmetric band.

If $(X; \tilde{f_{i2}})$ does not contain a non–zero summand annihilated by $\mathcal{J}$, the reduced form $((X; \tilde{f_{i2}}), \hat{f})$ of $(X; f)$ can be considered as a representation of the bush T.

By induction hypothesis, $((X; \tilde{f_{i2}}); \hat{f})$ is a string or a band associated with a catenation v of T. We denote by w the catenation of S obtained from v by replacing each term x_j $(j \geq 1)$ by $\underbrace{x \cdots x}_{j+1}$ and each term $x_j^\sim$ by $\underbrace{x^\sim \cdots x^\sim}_{j+1}$.

By a similar argument in case I, there holds that $(X; f)$ is a string or a band according as $((X; \tilde{f_{i2}}); \hat{f})$ is a string or a band.

4.4. Case III. *supp*$(L^-/K^-) = \{\{x\}, \{x^\wedge\}\}$ *and supp*$(L^+/K^+) = \{\bar{y}\}$ *for some* $x, x^\wedge \in S_i^-$ *with* $x \lessgtr x^\wedge$ *and some* $y \in S_i^+$.

By way of example, we suppose that $y^\sim$ lies in S_j^- for some $j \neq i$. In this case, the representations $(a; 0)$, $a \in \mathcal{S}$, $(\{x\} \oplus \bar{y}; \eta(1))$, $(\{x^\wedge\} \oplus \bar{y}; \eta(2))$ and $(\{x\} \oplus \{x^\wedge\} \oplus \bar{y}; \eta(3))$, furnish a complete list of indecomposables in $\mathcal{B}$, where $\eta(1) = \eta(2)$ denotes the sequence $(0, \cdots, 0, 1, 0, \cdots, 0)$, and $\eta(3)$ the sequence

$= (0, \cdots, 0, [1\ 1], 0, \cdots, 0)$. By $\bar{\mathcal{S}}$ we denote the spectroid formed by these representations.

Let $\mathcal{T}$ be the spectroid associated with the bush $T = (T_1^-, \cdots, T_n^-; T_1^+, \cdots, T_n^+; \sim)$, where the union of the sets $T_i^- = S_i^- \amalg \{x_1'\}$, $T_j^- = S_j^- \amalg \{y_1', y_2', y_3'\}$, and $T_l^\varepsilon = S_l^\varepsilon$ $((l, \varepsilon) \neq (i, -),\ (j, -))$ is equipped with the order relation defined as in case I (in particular, $x > x_1',\ x^\wedge > x_1',\ y_3' > y_1' > y^\sim,\ y_3' > y_2' > y^\sim$). Finally, we equip T with the equivalence relation induced by that of S and extended by $x_1' \sim y_3'$.

Then the correspondence

$$
\begin{aligned}
(\bar{z}; 0) &\longmapsto \bar{z}, \quad z \in S, & (\{x\} \oplus \bar{y}; \eta(1)) &\longmapsto \{y_1'\}, \\
(\{x^\wedge\} \oplus \bar{y}; \eta(2)) &\longmapsto \{y_2'\}, & (\{x\} \oplus \{x^\wedge\} \oplus \bar{y}; \eta(3)) &\longmapsto \bar{x}_1' = \bar{y}_3'
\end{aligned}
$$

induces an isomorphism from $\bar{\mathcal{S}}$ and $\mathcal{T}$. Hence $(X; \tilde{f_{i2}}); \hat{f})$ can be viewed as a representation of the new bush T.

By induction hypothesis, $((X; \tilde{f_{i2}}); \hat{f})$ is a string or a band associated with a catenation v of T. We denote by w the catenation of S obtained from v by substituting xy for each term x_1', $y^\sim x^\wedge$ for y_3', $y^\sim xy$ for y_1', and $y^\sim x^\wedge y$ for y_2'.

First we suppose that $v = v_1 v_2 \cdots v_s$ is an asymmetric catenation, thus $w = w_1 w_2 \cdots w_t$ is also an asymmetric catenation. We consider the following parts in w.

a) $w_r w_{r+1} = xy$ (obtained from a term $v_q = x_1'$ in v). By construction of $R(w)$, the maps g_{wir} and f_{wir+1} associated with w (2.2) behave as follows:

$$
\begin{tikzcd}
 & k\underline{w}_r \arrow[r, "1"] \arrow[dl, "1"'] & k\underline{w}_{r+1} \\
k & & \\
 & k\underline{w}^*_r \arrow[uur, "1"] & k\underline{w}^*_{r+1}
\end{tikzcd}
$$

Note that $[w^*_{r-1} \dots w^*_1] > [w_{r+1} \dots w_t]$ since $v^*_{q-1} = w^*_{r-1} > x$ (This follows from the fact that $((X; \tilde{f_{i2}}); \hat{f})$ satisfies $\hat{f}_i(J^-(X; \tilde{f_{i2}})) \subseteq J^+(X; \tilde{f_{i2}}))$ (see 3.1)). Thus every part xy in w provides a summand $(\{x\} \oplus \{x^\wedge\} \oplus \bar{y}; \eta(3))$ in $(X_w; \tilde{f}_{wi2})$. Such a summand contributes a one–dimensional subspace $k(\underline{x} - \underline{x}^\wedge)$ in $\mathrm{Ker} f_{wi2} \subseteq N_i^-(X_w; \tilde{f}_{wi2})$ and a one–dimensional subspace $ky^\sim$ in $N_j^-(X_w; \tilde{f}_{wi2})$.

Similarly, each part $y^\sim x^\wedge$ also provides a summand $(\{x\} \oplus \{x^\wedge\} \oplus \bar{y}); \eta(3))$ in $(X_w; \tilde{f}_{wi2})$ which contributes a one–dimensional subspace both in $N_i^-(X_w; \tilde{f}_{wi2})$ and in $N_j^-(X_w; \tilde{f}_{wi2})$.

b) $w_r w_{r+1} w_{r+2} = y^\sim xy$ (obtained from $v_q = x_1'$). By construction of $R(w)$, the maps f_{wir+1} and f_{wir+2} behave as follows:

$$\begin{tikzcd}
 & k\underline{w}_r \arrow[dl, "1"'] & k\underline{w}_{r+1} \arrow[r, "1"] \arrow[dl, "1"] & k\underline{w}_{r+2} & \\
k & & & & k \\
 & k\underline{w}^*_r & k\underline{w}^*_{r+1} \arrow[ur, "1"] & k\underline{w}^*_{r+2} \arrow[ur, "1"] &
\end{tikzcd}$$

provided $[v^*_{q-1} \dots v^*_1] > [v_{q+1} \dots v_s]$ (thus $[w^*_r \dots w^*_1] > [w_{r+2} \dots w_t]$).

In this case, the part $y^\sim xy$ provides a summand $(\{x\} \oplus \{x^\wedge\} \oplus (\bar{y})^2; \eta)$, where η is of the form

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} : M^-(\{x\} \oplus \{x^\wedge\} \oplus (\bar{y})^2) = kx \oplus kx^\wedge \longrightarrow ky \oplus ky = M^+(\{x\} \oplus \{x^\wedge\} \oplus (\bar{y})^2).$$

Such a summand contributes a two–dimensional subspace $(ky^\sim)^2$ of $N^-_j(X_w; \tilde{f}_{wi2})$.

It is easy to see that the morphism

$$\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} : \{x\} \oplus \{x^\wedge\} \oplus (\bar{y})^2 \longrightarrow \{x\} \oplus \{x^\wedge\} \oplus (\bar{y})^2$$

is an isomorphism from $(\{x\} \oplus \{x^\wedge\} \oplus (\bar{y})^2; \eta)$ to $(\{x\} \oplus \bar{y}; \eta(1)) \oplus (\{x^\wedge\} \oplus \bar{y}; \eta(2))$.

In case $[v^*_{q-1} \dots v^*_1] \leq [v_{q+1} \dots v_s]$ (thus $[w^*_r \dots w^*_1] \leq [w_{r+2} \dots w_t]$), the maps f_{wir+1} and f_{wir+2} behave as follows:

$$\begin{tikzcd}
 & k\underline{w}_r \arrow[dl, "1"'] & k\underline{w}_{r+1} \arrow[dl, "1"'] & k\underline{w}_{r+2} & \\
k & & & & k \\
 & k\underline{w}^*_r & k\underline{w}^*_{r+1} \arrow[l, "1"'] \arrow[ur, "1"] & k\underline{w}^*_{r+2} \arrow[ur, "1"] &
\end{tikzcd}$$

In this situation, the part $y^\sim xy$ provides a summand $(\{x\} \oplus \{x^\wedge\} \oplus (\bar{y})^2); \eta')$, where η' is of the form

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} : M^-(\{x\} \oplus \{x^\wedge\} \oplus (\bar{y})^2) = kx \oplus kx^\wedge \longrightarrow ky \oplus ky = M^+(\{x\} \oplus \{x^\wedge\} \oplus (\bar{y})^2).$$

This summand also contributes a two-dimensional subspace $(ky^\sim)^2$ of $N^-_j(X_w; \tilde{f}_{wi2})$.

It is easy to see that the morphism

$$\mu' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} : (\{x\} \oplus \{x^\wedge\} \oplus (\bar{y})^2) \longrightarrow (\{x\} \oplus \{x^\wedge\} \oplus (\bar{y})^2)$$

is an isomorphism from $(\{x\}\oplus\{x^\wedge\}\oplus(\bar{y})^2;\eta')$ to $(\{x\}\oplus\bar{y};\eta(1))\oplus(\{x^\wedge\}\oplus\bar{y};\eta(2))$.

Therefore, every part $y^\sim xy$ in w provides a summand in $(X_w;\tilde{f}_{wi2})$ which is isomorphic to $(\{x\}\oplus\bar{y};\eta(1))\oplus(\{x^\wedge\}\oplus\bar{y};\eta(2))$.

Similarly, every part $y^\sim x^\wedge y$ in w also provides a summand isomorphic to $(\{x\}\oplus\bar{y};\eta(1))\oplus(\{x^\wedge\}\oplus\bar{y};\eta(2))$.

c) Each term w_r (obtained from some term v_q in v) in w provides a summand $(\bar{w}_r;0)$ in $(X_w;\tilde{f}_{wi2})$ if $w_r\neq w_r^\sim$, and a summand $(\{w^r\}\oplus\{w_r^\wedge\};0)$ if $w_r\neq w_r^\wedge$.

Form the observations in a)–c), it follows that $(X_w;\tilde{f}_{wi2})$ and X_v viewed as objects in $\mathcal{B}$ are isomorphic.

Furthermore, by suitably choosing basis vectors of $N_j^\varepsilon(X_w;\tilde{f}_{wi2})$ for $1\leq j\leq n$ and $\varepsilon=-,+$, one can show that $((X_w;\tilde{f}_{wi2});\hat{f}_w)$ is isomorphic to $(X_v;f_{v1},\cdots,f_{vn})$. This implies that $(X;f)$ is isomorphic to $R(w)=(X_w;f_{w1},\cdots,f_{wn})$, that is, $(X;f)$ is an asymmetric string.

Similarly, one gets that $(X;f)$ is a dimidiate string if so is $((X;\tilde{f_{i2}});\hat{f})$ and that $(X;f)\cong R(w,Q)$ (resp. $R(w,K)$) according as $((X;\tilde{f_{i2}});\hat{f})\cong R(v,Q)$ (resp. $R(v,K)$) for some $Q\in\mathcal{P}$ (resp. $K\in\mathcal{P}$).

4.5. Case IV. *supp*$(L^-/K^-)=\{\bar{x}\}$ *and supp*$(L^+/K^+)=\{\{y\},\{y^\wedge\}\}$ *for some* $x\in S_i^-$ *and some* $y,y^\wedge\in S_i^+$ *with* $y\lessgtr y^\wedge$. This is an anologue to Case III.

4.6. Case V. *supp*$(L^-/K^-)=\{\{x\},\{x^\wedge\}\}$ *and supp*$(L^+/K^+)=\{\{y\},\{y^\wedge\}\}$ *for some* $x,x^\wedge\in S_i^-$ *and* $y,y^\wedge\in S_i^+$ *with* $x\lessgtr x^\wedge$ *and* $y\lessgtr y^\wedge$.

In this case, one can show that the representatons $(a;0)$, $a\in\mathcal{S}$, and $\mathcal{R}(E):=((\{x\})^{s_1}\oplus(\{x^\wedge\})^{s_2}\oplus(\{y\})^{t_1}\oplus(\{y^\wedge\})^{t_2};\eta(E))$, furnish a complete list of indecomposables in $\mathcal{B}$ which are not annihilated by $\mathcal{J}$, where $\eta(E)$ denotes the sequences $(0,\cdots,0,E,0,\cdots,0)$ and E ranges over the following matrices ($m\geq 1$)(see Sect. 11 in [GKR]):

$$P_{2m-1}=\left[\begin{array}{c|c} \mathbb{1}_m & \begin{matrix}\mathbb{1}_{m-1}\\ 0\end{matrix} \\ \hline \mathbb{1}_m & \begin{matrix}0\\ \mathbb{1}_{m-1}\end{matrix}\end{array}\right]\quad \left(\begin{matrix}s_2=m-1\\ s_1=t_1=t_2=m\end{matrix}\right),$$

$$P^\wedge_{2m-1}=\left[\begin{array}{c|c} \begin{matrix}0\\ \mathbb{1}_{m-1}\end{matrix} & \mathbb{1}_m \\ \hline \begin{matrix}\mathbb{1}_{m-1}\\ 0\end{matrix} & \mathbb{1}_m\end{array}\right]\quad \left(\begin{matrix}s_1=m-1\\ s_2=t_1=t_2=m\end{matrix}\right),$$

$$P_{2m}=\left[\begin{array}{c|c} \begin{matrix}0\\ \mathbb{1}_m\end{matrix} & \begin{matrix}\mathbb{1}_m\\ 0\end{matrix} \\ \hline \mathbb{1}_m & \mathbb{1}_m\end{array}\right]\quad \left(\begin{matrix}s_1=s_2=t_2=m\\ t_1=m+1\end{matrix}\right),$$

$$P_{2m}^{\wedge} = \left[\begin{array}{c|c} \mathbb{1}_m & \mathbb{1}_m \\ \hline \mathbb{1}_m & 0 \\ 0 & \mathbb{1}_m \end{array}\right] \quad (\begin{smallmatrix} s_1 = s_2 = t_1 = m \\ t_2 = m+1 \end{smallmatrix}),$$

$$I_{2m-1} = \left[\begin{array}{c|c} \mathbb{1}_m & \mathbb{1}_m \\ \hline \mathbb{1}_{m-1}\ 0 & 0\ \mathbb{1}_{m-1} \end{array}\right] \quad (\begin{smallmatrix} t_2 = m-1 \\ s_1 = s_2 = t_1 = m \end{smallmatrix}),$$

$$I_{2m-1}^{\wedge} = \left[\begin{array}{c|c} 0\ \mathbb{1}_{m-1} & \mathbb{1}_{m-1}\ 0 \\ \hline \mathbb{1}_m & \mathbb{1}_m \end{array}\right] \quad (\begin{smallmatrix} t_1 = m-1 \\ s_1 = s_2 = t_2 = m \end{smallmatrix}),$$

$$I_{2m} = \left[\begin{array}{c|c} 0\ \mathbb{1}_m & \mathbb{1}_m \\ \hline \mathbb{1}_m\ 0 & \mathbb{1}_m \end{array}\right] \quad (\begin{smallmatrix} s_2 = t_1 = t_2 = m \\ s_1 = m+1 \end{smallmatrix}),$$

$$I_{2m}^{\wedge} = \left[\begin{array}{c|c} \mathbb{1}_m & \mathbb{1}_m\ 0 \\ \hline \mathbb{1}_m & 0\ \mathbb{1}_m \end{array}\right] \quad (\begin{smallmatrix} s_1 = t_1 = t_2 = m \\ s_2 = m+1 \end{smallmatrix}),$$

$$T_m^1 = \left[\begin{array}{c|c} \begin{matrix} & \mathbb{1}_m \\ + & J_m \end{matrix} & \mathbb{1}_m \\ \hline \mathbb{1}_m & \mathbb{1}_m \end{array}\right] \quad (s_1 = s_2 = t_1 = t_2 = m).$$

By the finite–dimensionality of $(X; f)$, there exists an $L > 0$ such that the induced representation $(X; \tilde{f_{i2}})$ does not contain a summand isomorphic to some $\mathcal{R}(E)$ for $E = P_t, P_t^{\wedge}, I_t, I_t^{\wedge}$ or T_t^1 with $t > L$. We then denote by $\bar{\mathcal{S}}$ the spectroid formed by $(a; 0)$, $a \in \mathcal{S}$, and $\mathcal{R}(E)$ for $E = P_t, P_t^{\wedge}, I_t, I_t^{\wedge}$ and T_t^1 wiht $1 \le t \le L$.

By $\mathcal{T}$ we denote the spectroid associated with the bush $T = T(L) = (T_1^-, \dots, T_n^-; T_1^+, \dots, T_n^+; \sim)$, where the sets $T_i^- = S_i^- \amalg \{x_{-1}, \cdots, x_{-L}, x_1, x_1^{\wedge}, \cdots, x_L, x_L^{\wedge}\}$, $T_i^+ = S_i^+ \amalg \{y_{-1}, \cdots y_{-L}, y_1, y_1^{\wedge}, \cdots, y_L, y_L^{\wedge}\}$ and $T_l^{\varepsilon} = S_l^{\varepsilon}$ $((l, \varepsilon) \neq (i, -), (i, +))$ are equipped with order relations defined as in case I. In particular, we require that the induced order relations on $\{x_{-1}, \cdots, x_{-L}, x, x^{\wedge}, x_1, x_1^{\wedge}, \cdots, x_L, x_L^{\wedge}\}$ and $\{y_{-1}, \cdots, y_{-L}, y, y^{\wedge}, y_1, y_1^{\wedge}, \cdots, y_L, y_L^{\wedge}\}$ admit respectively the following Hasse-quivers:

$$
\begin{tikzcd}
 & & & & x_L \arrow[r] \arrow[ddr] & \cdots \arrow[r] \arrow[ddr] & x_1 \arrow[r] \arrow[ddr] & x \\
x_{-1} \arrow[r] & x_{-2} \arrow[r] & \cdots \arrow[r] & x_{-L} \arrow[ur] \arrow[dr] & & & & \\
 & & & & x^\wedge_L \arrow[r] \arrow[uur] & \cdots \arrow[r] \arrow[uur] & x^\wedge_1 \arrow[r] \arrow[uur] & x^\wedge \\
y \arrow[r] \arrow[ddr] & y_1 \arrow[r] \arrow[ddr] & \cdots \arrow[r] \arrow[ddr] & y_L \arrow[dr] & & & & \\
 & & & & y_{-L} \arrow[r] & \cdots \arrow[r] & y_{-2} \arrow[r] & y_{-1} \\
y^\wedge \arrow[r] \arrow[uur] & y^\wedge_1 \arrow[r] \arrow[uur] & \cdots \arrow[r] \arrow[uur] & y^\wedge_L \arrow[ur] & & & &
\end{tikzcd}
$$

We then equip T with the equivalence relation induced by S and extended by $x_{-m} \sim y_{-m}$, $m = 1, \dots, L$.

Then the correspondence

$$
\begin{array}{lll}
(\bar{z}; 0) \longmapsto \bar{z}, \quad z \in S, & \mathcal{R}(I_t) \longmapsto \{x_t\}, & \mathcal{R}(I_t^\wedge) \longmapsto \{x_t^\wedge\}, \\
 & & \quad 1 \le t \le L \\
\mathcal{R}(T_t^1) \longmapsto \bar{x}_{-t} = \bar{y}_{-t}, & \mathcal{R}(P_t) \longmapsto \{y_t\}, & \mathcal{R}(P_t^\wedge) \longmapsto \{y_t^\wedge\}
\end{array}
$$

defines an isomorphism from $\bar{\mathcal{S}}$ to $\mathcal{T}$.

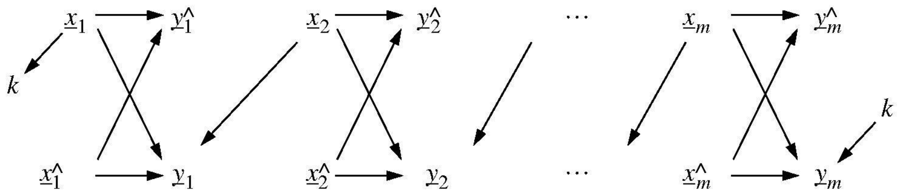
If $(X; \tilde{f_{i2}})$ contains a non–zero summand annihilated by $\mathcal{J}$, $((X; \tilde{f_{i2}}); \hat{f})$ is isomorphic to $R(xy, K)$ for some $K \in \mathcal{Q}$, thus is a dimidiate band.

If $(X; \tilde{f_{i2}})$ does not contain a non–zero summand annihilated by $\mathcal{J}$, $((X; \tilde{f_{i2}}); \hat{f})$ can be considered as a representation of the bush T. In the following we simply identify $\bar{S}$ and $\mathcal{T}$.

By induction hypothesis, the representation $((X; \tilde{f_{i2}}); \hat{f})$ is a string or a band associated with a catenation v of T. We denote by w the catenation of S obtained from v by replacing each term x_m by $\underbrace{xy^\wedge \cdots y^\wedge x}_{m}\, y\, \underbrace{x^\wedge y \cdots yx^\wedge}_{m}$ (resp. $\underbrace{xy^\wedge \cdots xy^\wedge}_{m}\, x\, \underbrace{yx^\wedge \cdots yx^\wedge}_{m}$) if m is odd (resp. even), $x_m^\wedge$ by $\underbrace{x^\wedge y \cdots yx^\wedge}_{m}\, y^\wedge\, \underbrace{xy^\wedge \cdots y^\wedge x}_{m}$ (resp. $\underbrace{x^\wedge y \cdots x^\wedge y}_{m}\, x^\wedge\, \underbrace{y^\wedge x \cdots y^\wedge x}_{m}$) if m is odd (resp. even), y_m by $\underbrace{yx^\wedge \cdots x^\wedge y}_{m}\, x\, \underbrace{y^\wedge x \cdots xy^\wedge}_{m}$ (resp. $\underbrace{yx^\wedge \cdots y^\wedge x}_{m}\, y\, \underbrace{xy^\wedge \cdots xy^\wedge}_{m}$) if m is odd (resp. even)$y_m^\wedge$ by $\underbrace{y^\wedge x \cdots xy^\wedge}_{m}\, x^\wedge\, \underbrace{yx^\wedge \cdots x^\wedge y}_{m}$ (resp. $\underbrace{y^\wedge x \cdots y^\wedge x}_{m}\, y^\wedge\, \underbrace{x^\wedge y \cdots x^\wedge y}_{m}$) if m is odd (resp. even), x_{-m} by $\underbrace{xy^\wedge \cdots xy^\wedge}_{2m}$, and y_{-m} by $\underbrace{yx^\sim \cdots yx^\sim}_{2m}$.

First we suppose that $v = v_1 v_2 \cdots v_s$ is finite and asymmetric, and set $w = w_1 w_2 \cdots w_t$. We consider the following parts in w:

a) $w_r w_{r+1} \cdots w_{r+2m-1} = xy^\wedge \cdots xy^\wedge$ (obtained from a term x_{-m} in v). By construction of $R(w)$, the map f_{wi} associated with w acts as follows on basis vectors $\underline{w}_r = \underline{x}_1$, $\underline{w}_r^* = \underline{x}_1^\wedge, \cdots\cdots, \underline{w}_{r+2m-2} = \underline{x}_m$, $\underline{w}_{r+2m-2}^* = \underline{x}_m^\wedge$:



since $[w_j^* \ldots w_1^*] > [w_{j+2} \ldots w_t]$, for $j = r-1, \ldots, r+2m-2$.

The matrix describing the action of f_{wi} on the basis vectors $\underline{x}_i$, $\underline{x}_i^\wedge, 1 \le i \le m$ in the figure above is

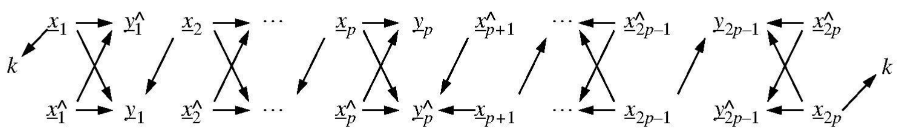
$$\left[\begin{array}{ccccc|ccccc}
1 & 1 & \cdot & 0 & 0 & 1 & 0 & \cdot & 0 & 0 \\
0 & 1 & \cdot & 0 & 0 & 0 & 1 & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \cdot & 1 & 1 & 0 & 0 & \cdot & 1 & 0 \\
0 & 0 & \cdot & 0 & 1 & 0 & 0 & \cdot & 0 & 1 \\
\hline
1 & 0 & \cdot & 0 & 0 & 1 & 0 & \cdot & 0 & 0 \\
0 & 1 & \cdot & 0 & 0 & 0 & 1 & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \cdot & 1 & 0 & 0 & 0 & \cdot & 1 & 0 \\
0 & 0 & \cdot & 0 & 1 & 0 & 0 & \cdot & 0 & 1
\end{array}\right] = T_m^1$$

Thus every part $\underbrace{xy^\wedge \cdots xy^\wedge}_{2m}$ in w provides a summand in $(X_w; \tilde{f}_{wi2})$ which is isomorphic to $\mathcal{R}(T_m^1)$, and such a summand contributes a one–dimensional subspace both in $N_i^-(X_w; \tilde{f}_{wi2})$ and in $N_i^+(X_w; \tilde{f}_{wi2})$.

Similarly, every part $\underbrace{y^\wedge x \cdots y^\wedge x}_{2m}$ (obtained from some term y_{-m} in v) provides a summand isomorphic to $\mathcal{R}(T_m^1)$, too.

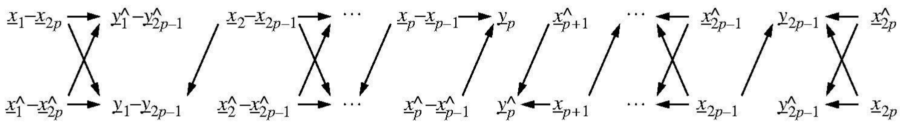
b) $w_r w_{r+1} \cdots w_{r+2m} = xy^\wedge \cdots yx^\wedge$ (obtained from a term $v_q = x_m$).

Case 1. $m = 2p-1$. By construction of $R(w)$, the map f_{wi} associated with w acts as follows on basis vectors $\underline{w}_r = \underline{x}_1$, $\underline{w}_r^* = \underline{x}_1^\wedge, \cdots\cdots, \underline{w}_{r+2m} = \underline{x}_{2p}^\wedge$, $\underline{w}_{r+2m}^* = \underline{x}_{2p}$:



provided $[v^*_{q-1}\cdots v^*_1] > [v_{q+1}\cdots v_s]$ (Hence $[w^*_{r+2p-1}\cdots w^*_1] > [w_{r+2p+1}\cdots w_t]$).

We start with the following change of basis vectors.



Thus, the part $w_r w_{r+1}\cdots w_{r+2m} = \underbrace{xy^\wedge\cdots y^\wedge x}_{m}\, y\, \underbrace{x^\wedge y\cdots yx^\wedge}_{m}$ porvides a summand of $(X_w;\tilde f_{wi2})$ which is isomorphic to $\mathcal{R}(Y)\oplus\mathcal{R}(Y^\wedge)$, where Y and $Y^\wedge$ are the following matrices:

$$Y = \left[\begin{array}{ccccc|ccccc}
1 & 1 & \cdot & 0 & 0 & 1 & 0 & \cdot & 0 & 0\\
0 & 1 & \cdot & 0 & 0 & 0 & 1 & \cdot & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & \cdot & 1 & 1 & 0 & 0 & \cdot & 1 & 0\\
0 & 0 & \cdot & 0 & 1 & 0 & 0 & \cdot & 0 & 1\\
\hline
1 & 0 & \cdot & 0 & 0 & 1 & 0 & \cdot & 0 & 0\\
0 & 1 & \cdot & 0 & 0 & 0 & 1 & \cdot & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & \cdot & 1 & 0 & 0 & 0 & \cdot & 1 & 0
\end{array}\right]$$

$$Y^\wedge = \left[\begin{array}{ccccc|ccccc}
1 & 1 & \cdot & 0 & 0 & 1 & 0 & \cdot & 0 & 0\\
0 & 1 & \cdot & 0 & 0 & 0 & 1 & \cdot & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & \cdot & 1 & 1 & 0 & 0 & \cdot & 1 & 0\\
\hline
1 & 0 & \cdot & 0 & 0 & 1 & 0 & \cdot & 0 & 0\\
0 & 1 & \cdot & 0 & 0 & 0 & 1 & \cdot & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & \cdot & 1 & 0 & 0 & 0 & \cdot & 1 & 0\\
0 & 0 & \cdot & 0 & 1 & 0 & 0 & \cdot & 0 & 1
\end{array}\right]$$

Set

$$U_p = \begin{bmatrix} 0 & \exp R_{p-1} \\ & H_p \end{bmatrix} \in k^{p\times p}$$

where

$$R_p = \begin{bmatrix} 0 & 0 & \cdot & 0 & 0\\ -1 & 0 & \cdot & 0 & 0\\ 0 & -2 & \cdot & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & \cdot & -p+1 & 0 \end{bmatrix} \in k^{p\times p}$$

and H_p denotes the last row of the matrix $\exp R_p$. Then there holds that

$$\left[\begin{array}{c|c} U_p & 0 \\ \hline 0 & \exp R_{p-1} \end{array}\right] I_{2p-1} = Y \left[\begin{array}{c|c} \exp R_p & 0 \\ \hline 0 & U_p \end{array}\right]$$

and

$$\left[\begin{array}{c|c} \exp R_{p-1} & 0 \\ \hline 0 & \exp R_p \end{array}\right] I^{\wedge}_{2p-1} = Y^{\wedge} \left[\begin{array}{c|c} \exp R_p & 0 \\ \hline 0 & \exp R_p \end{array}\right]$$

that is, the morphism

$$\begin{aligned} \exp R_p \oplus U_p \oplus U_p \oplus \exp R_{p-1} :& \{x\}^p \oplus \{x^{\wedge}\}^p \oplus \{y\}^p \oplus \{y^{\wedge}\}^{p-1} \\ &\longrightarrow \{x\}^p \oplus \{x^{\wedge}\}^p \oplus \{y\}^p \oplus \{y^{\wedge}\}^{p-1} \end{aligned}$$

defines an ismorphism from $\mathcal{R}(Y)$ to $\mathcal{R}(I_{2p-1})$, and the morphism

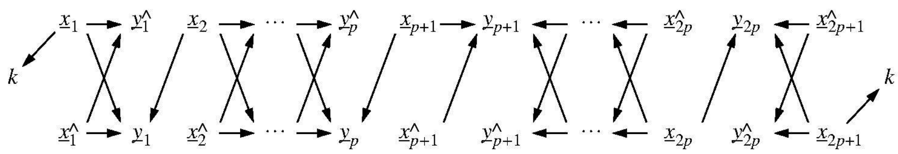
$$\begin{aligned} \exp R_p \oplus \exp R_p \oplus \exp R_{p-1} \oplus \exp R_p :& \{x\}^p \oplus \{x^{\wedge}\}^p \oplus \{y\}^{p-1} \oplus \{y^{\wedge}\}^p \\ &\longrightarrow \{x\}^p \oplus \{x^{\wedge}\}^p \oplus \{y\}^{p-1} \oplus \{y^{\wedge}\}^p \end{aligned}$$

defines an ismorphism from $\mathcal{R}(Y^{\wedge})$ to $\mathcal{R}(I^{\wedge}_{2p-1})$.

Therefore, every part $\underbrace{xy^{\wedge}\cdots y^{\wedge}x}_{m}\, y\, \underbrace{x^{\wedge}y\cdots yx^{\wedge}}_{m}$ in w porvides a summand of $(X;\tilde{f_{i2}})$ which is isomorphic to $\mathcal{R}(I_{2p-1}) \oplus \mathcal{R}(I^{\wedge}_{2p-1})$ and which contributes a two–dimensional subspace in $N_i^-(X_w;\tilde{f}_{wi2})$.

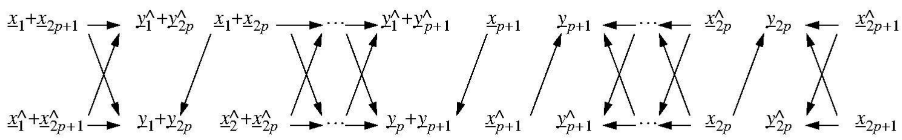
The case $[v^*_{q-1}\cdots v^*_1] \leq [v_{q+1}\cdots v_s]$ can be treated similarly.

Case 2. $m = 2p$. By construction of $R(w)$, the map f_{wi} acts as follows on basis vectors $\underline{w}_r = \underline{x}_1$, $\underline{w}^*_r = \underline{x}^{\wedge}_1, \cdots\cdots, \underline{w}_{r+2m} = \underline{x}^{\wedge}_{2p+1}$, $\underline{w}^*_{r+2m} = \underline{x}_{2p+1}$:



provided $[v^*_{q-1}\cdots v^*_1] > [v_{q+1}\cdots v_s]$.

We start with the following change of basis vectors.



Thus, in this case, the part $w_r w_{r+1} \cdots w_{r+2m} = \underbrace{xy^{\wedge} \cdots xy^{\wedge}}_{m}\, x\, \underbrace{yx^{\wedge} \cdots yx^{\wedge}}_{m}$ provides a summand of $(X_w; \tilde{f}_{wi2})$ which is isomorphic to $\mathcal{R}(Z) \oplus \mathcal{R}(Z^{\wedge})$, where Z and $Z^{\wedge}$ are the following matrices:

$$Z = \left[\begin{array}{cccccc|ccccc}
1 & 1 & \cdot & 0 & 0 & 0 & 1 & 0 & \cdot & 0 & 0 \\
0 & 1 & \cdot & 0 & 0 & 0 & 0 & 1 & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \cdot & 1 & 1 & 0 & 0 & 0 & \cdot & 1 & 0 \\
0 & 0 & \cdot & 0 & 1 & 1 & 0 & 0 & \cdot & 0 & 1 \\
\hline
1 & 0 & \cdot & 0 & 0 & 0 & 1 & 0 & \cdot & 0 & 0 \\
0 & 1 & \cdot & 0 & 0 & 0 & 0 & 1 & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \cdot & 1 & 0 & 0 & 0 & 0 & \cdot & 1 & 0 \\
0 & 0 & \cdot & 0 & 1 & 0 & 0 & 0 & \cdot & 0 & 1
\end{array}\right]$$

$$Z^{\wedge} = \left[\begin{array}{ccccc|cccccc}
1 & 1 & \cdot & 0 & 0 & 1 & 0 & \cdot & 0 & 0 & 0 \\
0 & 1 & \cdot & 0 & 0 & 0 & 1 & \cdot & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \cdot & 1 & 1 & 0 & 0 & \cdot & 1 & 0 & 0 \\
0 & 0 & \cdot & 0 & 1 & 0 & 0 & \cdot & 0 & 1 & 1 \\
\hline
1 & 0 & \cdot & 0 & 0 & 1 & 0 & \cdot & 0 & 0 & 0 \\
0 & 1 & \cdot & 0 & 0 & 0 & 1 & \cdot & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \cdot & 1 & 0 & 0 & 0 & \cdot & 1 & 0 & 0 \\
0 & 0 & \cdot & 0 & 1 & 0 & 0 & \cdot & 0 & 1 & 0
\end{array}\right]$$

then there hold that

$$\left[\begin{array}{c|c} \exp R_p & 0 \\ \hline 0 & \exp R_p \end{array}\right] I_{2p} = Z \left[\begin{array}{c|c} \exp R_{p+1} & 0 \\ \hline 0 & \exp R_p \end{array}\right]$$

$$\left[\begin{array}{c|c} U_p & 0 \\ \hline 0 & \exp R_p \end{array}\right] I_{2p}^{\wedge} = Z^{\wedge} \left[\begin{array}{c|c} \exp R_p & 0 \\ \hline 0 & V_{p+1} \end{array}\right]$$

where U_p and R_p are defined as before and V_{p+1} has the form

$$V_{p+1} = \begin{bmatrix} 0 & \exp R_p \\ & -H_{p+1} \end{bmatrix} \in k^{(p+1)\times(p+1)}$$

This implies that the morphism

$$\begin{aligned} \exp R_{p+1} \oplus \exp R_p \oplus \exp R_p \oplus \exp R_p :& \{x\}^{p+1} \oplus \{x^{\wedge}\}^p \oplus \{y\}^p \oplus \{y^{\wedge}\}^p \\ \longrightarrow & \{x\}^{p+1} \oplus \{x^{\wedge}\}^p \oplus \{y\}^p \oplus \{y^{\wedge}\}^p \end{aligned}$$

defines an ismorphism from $\mathcal{R}(Z)$ to $\mathcal{R}(I_{2p})$, and the morphism

$$\begin{aligned}\exp R_p \oplus V_{p+1} \oplus U_p \oplus \exp R_p :& \{x\}^p \oplus \{x^\wedge\}^{p+1} \oplus \{y\}^p \oplus \{y^\wedge\}^p \\ \longrightarrow & \{x\}^p \oplus \{x^\wedge\}^{p+1} \oplus \{y\}^p \oplus \{y^\wedge\}^p\end{aligned}$$

defines an ismorphism from $\mathcal{R}(Z^\wedge)$ to $\mathcal{R}(I_{2p}^\wedge)$.

Hence each part $\underbrace{xy^\wedge \cdots xy^\wedge}_{m} x \underbrace{yx^\wedge \cdots yx^\wedge}_{m}$ in w porvides a summand of $(X_w; \tilde{f}_{wi2})$ which is isomorphic to $\mathcal{R}(I_{2p}) \oplus \mathcal{R}(I_{2p}^\wedge)$ and which contributes a two–dimensional subspace in $N_i^-(X_w; \tilde{f}_{wi2})$.

The case $[v_{q-1}^* \cdots v_1^*] \leq [v_{q+1} \cdots v_s]$ is similar.

b$'$) $w_r w_{r+1} \cdots w_{r+2m} = x^\wedge y \cdots y^\wedge x$ (obtained from a term $v_q = x_m^\wedge$). As in b), one has that every part $\underbrace{x^\wedge y \cdots y^\wedge x}_{2m+1}$ provides a summand in $(X_w; \tilde{f}_{wi2})$ which is isomorphic to $\mathcal{R}(I_m) \oplus \mathcal{R}(I_m^\wedge)$.

c) By a similar argument in b), one can show that both the parts $\underbrace{yx^\wedge \cdots xy^\wedge}_{2m+1}$ (obtained from a term $v_q = y_m$) and $\underbrace{y^\wedge x \cdots x^\wedge y}_{2m+1}$ (obtained from a term $v_q^\wedge = y_m$) provides a summand in $(X_w; \tilde{f}_{wi2})$ which is isomorphic to $\mathcal{R}(P_m) \oplus \mathcal{R}(P_m^\wedge)$.

d) Each term w_r (obtained from some term v_q in v) in w provides a summand $(\bar{w}_r; 0)$ in $(X_w; \tilde{f}_{wi2})$ if $w_r \neq w_r^\sim$, and a summand $(\{w^r\} \oplus \{w_r^\wedge\}; 0)$ if $w_r \neq w_r^\wedge$.

From the observations in a)–d), one gets that $(X_w; \tilde{f}_{wi2})$ and X_v viewed as objects in $\mathcal{B}$ are isomorphic.

Furthermore, by checking each summand described in a)–d) and suitably choosing basis vectors of $N_j^\varepsilon(X_w; \tilde{f}_{wi2})$ for $1 \leq j \leq n$ and $\varepsilon = -, +$, one obtains that $((X_w; \tilde{f}_{wi2}); \hat{f}_w)$ is isomorphic to $(X_v; f_{v1}, \cdots, f_{vn})$. Thus $(X; f)$ is isomorphic to $R(w)$, that is, an asymmetric string.

Similarly, one can show the following:

(1) If v is a symmetric catenation, there holds that $(X; f)$ is isomorphic to a non–trivial summand of $(X_w; f_{w1}, \cdots, f_{wn})$, that is, a dimidiate string.

(2) If v is an asymmetric periodic catenation of period π and $((X; \tilde{f_{i2}}); \hat{f}) \cong R(v, Q)$ for some $Q \in \mathcal{P}$, there holds that $(X; f) \cong R(w, \bar{Q})$, where $\bar{Q}$ denotes the polynomial $\bar{Q}(X) = (-1)^{\iota \deg Q} Q((-1)^\iota X)$, and ι is the number of terms x_m, $x_m^\wedge$, y_m, and $x_m^\wedge$ in $v = v_0 v_1 \cdots v_{\pi-1}$ with m an odd number.

(3) If v is symmetric periodic and $((X; \tilde{f_{i2}}); \hat{f}) \cong R(v, K)$ for some $K \in \mathcal{Q}$, there holds that $(X; f) \cong R(u, K)$.

4.7. As a conclusion of 4.2–4.6, we have the following

Proposition. *Each indecomposable representation of the bush S is a string or a band.*

4.8. Throughout the reduction above, by substituting $(X; f_1, \cdots, f_n)$ for representations associated with catenations, one can prove inductively the following propositions.

Proposition 1. *(1) The representation $R(w)$ associated with each asymmetric catenation w is indecomposable, and the representation $(X_v; f_{v1}, \cdots, f_{vn})$ associated with each symmetric catenation v is a direct sum of two non–isomorphic indecomposables.*

(2) For each asymmetric (resp. a symmetric) periodic catenation and each $Q \in \mathcal{P}$ (resp. $K \in \mathcal{Q}$), the representation $R(u, Q)$ (resp. $R(u, K)$) is indecomposable.

(3) The representaions $R(\delta)$, where

$$\delta \in \Omega_1 \amalg \Omega_2 \times \{0, 1\} \amalg \Omega_3 \times \mathcal{P} \amalg \Omega_4 \times \mathcal{Q}\,,$$

are pairwise non–isomorphic.

Proposition 2. *The equivalent catenations (finite or periodic) of S define the same family of isoclasses of indecomposables.*

Proof. Let $w = w_1 w_2 \cdots w_t$ and $w' = w'_1 w'_2 \cdots w'_t$ be equivalent catenations. We denote by $d(w, w')$ the number of indices i $(1 \leq i \leq t)$ such that $w'_i \neq w_i$. If $d(w, w') = 0$, the proposition holds. If $d(w, w') > 1$, there exists a sequence of equivalent catenations $W_1 = w, W_2, \cdots, W_d = w'$ such that $d(W_i, W_{i+1}) = 1$ for $1 \leq i \leq d - 1$. So we may suppose that $d(w, w') = 1$. Applying the reduction in 4.2–4.6, the representations $(X_w; f_{w1}, \cdots, f_{wn})$ and $(X_{w'}; f_{w'1}, \cdots, f_{w'n})$ are respectively reduced to representations $(X_v; f_{v1}, \cdots, f_{vn})$ and $(X_{v'}; f_{v'1}, \cdots, f_{v'n})$ of a new bush T, where v and v' are equivalent catenations of T such that $d(v, v') \leq 1$. By induction, we may suppose that v and v' define the same family of isoclasses of indecomposables of T. By Proposition 3.4, this implies that w and w' defines the same family of isoclasses of indecomposables of S.

By a similar argument, the proposition holds for periodic catenations.

4.9. Remark. By Proposition 2 in 4.8 and Remark 2.8, one obtains the following statements.

(a) For each asymmetric catenation w, catenations in the class $[w] \coprod [w^*]$ define isomorphic representations.

(b) For each symmetric catenation w, catenations in the class $[w]$ define isomorphic representations.

(c) For each asymmetric periodic catenation u, catenations in the class $\coprod_{p \in \mathbb{Z}}([u\{p\}] \coprod [u\{p\}^*])$ provide the same family of isoclasses of indecomposables.

(d) For each symmetric catenation w, catenations in the class $\coprod_{p \in \mathbb{Z}}[u\{p\}]$ provide the same family of isoclasses of indecomposables.

The main theorem then follows from Propositions in 4.7, 4.8, and Remark 4.9.

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