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## Invariant currents on limit sets

John Lott

**Abstract.** We relate the  $L^2$ -cohomology of a complete hyperbolic manifold to the invariant currents on its limit set.

**Mathematics Subject Classification (2000).** 58G25, 57M50.

**Keywords.**  $L^2$ -cohomology, harmonic form, hyperbolic, limit set.

### 1. Introduction

Let  $M$  be a complete oriented connected  $n$ -dimensional hyperbolic manifold. We can write  $M = H^n/\Gamma$ , where  $\Gamma$  is a torsion-free discrete subgroup of  $\text{Isom}^+(H^n)$ , the group of orientation-preserving isometries of the hyperbolic space  $H^n$ . The action of  $\Gamma$  on  $H^n$  extends to a conformal action on  $S_\infty^{n-1}$ , the sphere at infinity. For basic notions of hyperbolic geometry, we refer to [2]. Unless otherwise indicated, we assume that  $\Gamma$  is nonelementary, i.e. does not have an abelian subgroup of finite index.

A major theme in the study of hyperbolic manifolds is the relationship between the properties of  $M$  and the action of  $\Gamma$  on  $S_\infty^{n-1}$ . For example, let  $\lambda_0(M) \in [0, \infty)$  be the infimum of the spectrum  $\sigma(\Delta)$  of the Laplacian on  $M$ . Let  $\Lambda \subset S_\infty^{n-1}$  be the limit set of  $\Gamma$  and let  $D(\Gamma)$  be its Hausdorff dimension. Sullivan [15] showed that if  $M$  is geometrically finite then

$$\lambda_0(M) = \begin{cases} (n-1)^2/4 & \text{if } D(\Gamma) \leq \frac{n-1}{2}, \\ D(\Gamma)(n-1-D(\Gamma)) & \text{if } D(\Gamma) \geq \frac{n-1}{2}. \end{cases} \quad (1.1)$$

Thus there is a strong relationship between the spectrum of the Laplacian, acting on functions on  $M$ , and the geometry of the limit set. There is also a Laplacian  $\Delta_p$  on  $p$ -forms on  $M$  (see, for example, [9]). The motivating question of this paper is: What, if any, is the relationship between the spectrum of  $\Delta_p$  and the geometry of the limit set?

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If  $p > 0$ , it is clear that the infimum of the spectrum of  $\Delta_p$  depends on more than just the limit set as a set. For example, let  $M$  be a closed hyperbolic 3-manifold. From Hodge theory,  $0 \in \sigma(\Delta_1)$  if and only if the first Betti number  $b_1(M)$  of  $M$  is nonzero. There are examples with  $b_1(M) = 0$  and examples with  $b_1(M) \neq 0$ . However, in either case,  $\Lambda = S_\infty^2$ .

In this paper, we address the question of whether  $\text{Ker}(\Delta_p) \neq 0$  for a hyperbolic manifold  $M$ . We show how the answer to the question is related to the existence of  $\Gamma$ -invariant  $p$ -currents on  $S_\infty^{n-1}$ , of a certain regularity. In some sense, these currents probe the finer geometry of the limit set.

In order to state our results, let us recall the notion of harmonic extension of  $p$ -forms. We use the hyperbolic ball model for  $H^n$ , with boundary  $S^{n-1}$ . The space of  $p$ -hyperforms on  $S^{n-1}$  is the dual space to the space of real-analytic  $(n - 1 - p)$ -forms on  $S^{n-1}$ . We think of a  $p$ -hyperform on  $S^{n-1}$  as a  $p$ -form whose coefficient functions are hyperfunctions. A  $p$ -current on  $S^{n-1}$  is a  $p$ -hyperform whose coefficient functions are distributions.

There is a Poisson transform  $\Phi_p$  from  $p$ -hyperforms on  $S^{n-1}$  to coclosed harmonic  $p$ -forms on  $H^n$  [6]. To describe  $\Phi_p$  in terms of visual extension, let  $\omega$  be a  $p$ -hyperform on  $S^{n-1}$ . Given  $x \in H^n$ , let  $S_x$  be the unit sphere in  $T_x H^n$  and let  $A_x : S_x \rightarrow S^{n-1}$  be the visual map. Given  $v \in T_x H^n \cong T_0(T_x H^n)$ , define a vector field  $V$  on  $S_x$  by saying that at  $y \in S_x$ ,  $V$  is the translation of  $v$  in  $T_x H^n$  from 0 to  $y$ , followed by orthogonal projection onto  $T_y S_x$ . Then for  $v_1, \dots, v_p \in T_x H^n$ ,

$$\langle \Phi_p(\omega), v_1 \wedge \dots \wedge v_p \rangle = \frac{1}{\text{vol}(S^{n-1})} \int_{S_x} \langle A_x^* \omega, V_1 \wedge \dots \wedge V_p \rangle d\text{vol}. \tag{1.2}$$

Equivalently, given  $x \in H^n$  and  $v \in T_x H^n$ , take an upper-half-space model

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\} \tag{1.3}$$

for  $H^n$  in which  $x = (0, \dots, 0, 1)$  and  $v = c \frac{\partial}{\partial x_n}$  for some  $c \in \mathbb{R}$ . Consider the Killing vector field  $c \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ . It restricts to a conformal vector field  $W$  on  $\partial H^n = S^{n-1}$ . Then for  $v_1, \dots, v_p \in T_x H^n$ ,

$$\langle \Phi_p(\omega), v_1 \wedge \dots \wedge v_p \rangle = \frac{1}{\text{vol}(S^{n-1})} \int_{S^{n-1}} \langle \omega, W_1 \wedge \dots \wedge W_p \rangle d\text{vol}. \tag{1.4}$$

By a result of Gaillard, for  $p > 0$ ,  $\Phi_p$  is an isomorphism from *exact*  $p$ -hyperforms on  $S^{n-1}$  to *closed and coclosed*  $p$ -forms on  $H^n$  [6, Théorème 2]. Following [6], we say that a  $p$ -form  $\alpha$  on  $H^n$  has slow growth if there are constants  $a, b > 0$  such that for some (or any)  $m_0 \in H^n$ ,

$$|\alpha(m)| \leq ae^{bd(m_0, m)} \tag{1.5}$$

for all  $m \in H^n$ . Then for  $p > 0$ ,  $\Phi_p$  is also an isomorphism from *exact*  $p$ -currents on  $S^{n-1}$  to *closed and coclosed*  $p$ -forms on  $H^n$  of slow growth [6, Théorème 3].

Let  $\pi : H^n \rightarrow H^n/\Gamma$  be the quotient map. Let  $\Omega = S^{n-1} - \Lambda$  be the domain of discontinuity.

By Gaillard's theorem, if  $p > 0$  then  $\Phi_p^{-1} \circ \pi^*$  induces an isomorphism between closed and coclosed  $p$ -forms on  $H^n/\Gamma$ , and  $\Gamma$ -invariant exact  $p$ -hyperforms on  $S^{n-1}$ . Let  $\alpha$  be an  $L^2$ -harmonic  $p$ -form on  $H^n/\Gamma$ . By Hodge theory,  $\alpha$  is closed and coclosed. Thus we can use results about the  $L^2$ -cohomology of  $H^n/\Gamma$  to construct  $\Gamma$ -invariant exact  $p$ -hyperforms on  $S^{n-1}$ , and vice versa. The questions that we address are :

1. What can we say about the regularity of these hyperforms?
2. Are they supported on the limit set?

Under Hodge duality, the space of  $L^2$ -harmonic  $p$ -forms on  $H^n/\Gamma$  is isomorphic to the space of  $L^2$ -harmonic  $(n - p)$ -forms. Without loss of generality, hereafter we assume that  $p \in [1, \frac{n}{2}]$ .

**Theorem 1.** *If  $n$  is even then up to a constant,  $\Phi_{\frac{n}{2}}$  is an isometric isomorphism between exact  $\frac{n}{2}$ -forms on  $S^{n-1}$  which are Sobolev  $H^{-\frac{1}{2}}$ -regular, and  $L^2$ -harmonic  $\frac{n}{2}$ -forms on  $H^n$ .*

From Theorem 1, we obtain that the  $\frac{n}{2}$ -hyperforms that we construct on  $S^{n-1}$  cannot be too regular.

**Corollary 1.** *Suppose that  $\alpha$  is a nonzero  $L^2$ -harmonic  $\frac{n}{2}$ -form on  $H^n/\Gamma$ . If  $\Gamma$  is infinite then  $\Phi_{\frac{n}{2}}^{-1}(\pi^*\alpha)$  is not Sobolev  $H^{-\frac{1}{2}}$ -regular.*

We now give some positive regularity results. Let us recall that  $\Gamma$  is said to be *cocompact* if  $H^n/\Gamma$  is compact. It is said to be *convex-cocompact* if there is a compact subset  $K$  of  $H^n/\Gamma$  such that all nontrivial closed geodesics in  $H^n/\Gamma$  lie in  $K$ . If  $\Gamma$  is convex-cocompact then  $H^n/\Gamma$  consists of  $K$  along with a finite number of flaring ends attached to  $K$ .

**Theorem 2. A.** *If  $\Gamma$  is cocompact then for any  $p \in [1, \frac{n}{2}]$ , there are isomorphisms between the following vector spaces :*

- $V_1 = \{\text{Harmonic } p\text{-forms on } H^n/\Gamma\}.$
- $V_2 = \{\Gamma\text{-invariant exact } p\text{-hyperforms on } S^{n-1}\}.$
- $V_3 = \{\Gamma\text{-invariant exact } p\text{-currents on } S^{n-1} \text{ which are Sobolev } H^{-p-\epsilon}\text{-regular for all } \epsilon > 0\}.$

$V_4 = H^p(H^n/\Gamma, \mathbb{R})$ , the  $p$ -dimensional real cohomology group of  $H^n/\Gamma$ .

**B.** *If  $\Gamma$  is convex-cocompact then for any  $p \in [1, \frac{n-1}{2})$ , there are isomorphisms between the following vector spaces :*

- $V_1 = \{L^2\text{-harmonic } p\text{-forms on } H^n/\Gamma\}.$
- $V_2 = \{\Gamma\text{-invariant exact } p\text{-hyperforms on } S^{n-1} \text{ which are supported on the limit set}\}.$

$V_3 = \{\Gamma\text{-invariant exact } p\text{-currents on } S^{n-1} \text{ which are supported on the limit set and which are Sobolev } H^{-p-\epsilon}\text{-regular for all } \epsilon > 0\}$ .  
 $V_4 = H_c^p(H^n/\Gamma, \mathbb{R})$ , the  $p$ -dimensional real compactly-supported cohomology group of  $H^n/\Gamma$ .

In Theorem 2, we show that the injection  $V_3 \rightarrow V_2$  is surjective and that  $\Phi_p$  induces an isomorphism from  $V_2$  to  $V_1$ . In case A, there is an isomorphism between  $V_4$  and  $V_1$  from standard Hodge theory. By [12], this is also true in case B.

There are extensions of Theorem 2 to hyperbolic manifolds with vanishing injectivity radius. We state one such extension here.

**Theorem 3.** *If  $n = 3$ , suppose that there is a positive lower bound to the lengths of the nontrivial closed geodesics on  $H^3/\Gamma$ . Let  $\alpha$  be an  $L^2$ -harmonic 1-form on  $H^3/\Gamma$ . Then for all  $\epsilon > 0$ , the hyperform  $\Phi_1^{-1}(\pi^*\alpha)$  is Sobolev  $H^{-1-\epsilon}$ -regular.*

We show that the regularity estimate in Theorem 2 is sharp in some cases. We find an interesting distinction between cocompact groups, and convex-cocompact groups which are not cocompact.

**Theorem 4.** *A. Suppose that  $\Gamma$  is cocompact. Let  $\alpha$  be a nonzero harmonic 1-form on  $H^n/\Gamma$ . Then  $\Phi_1^{-1}(\pi^*\alpha)$  is not Sobolev  $H^{-1}$ -regular.  
 B. Let  $\Gamma$  be a convex-cocompact group which is not cocompact. Let  $\alpha$  be an  $L^2$ -harmonic 1-form on  $H^n/\Gamma$ . Then  $\Phi_1^{-1}(\pi^*\alpha)$  is Sobolev  $H^{-1}$ -regular.*

We look at what our general results become in the case of surfaces and 3-manifolds. In the case of surfaces, we obtain results about the actions of Fuchsian groups on certain function spaces on  $S^1$ . Let  $\mathcal{A}'(S^1)$  denote the hyperfunctions on  $S^1$  and let  $\mathcal{A}'_0(S^1)$  denote those which vanish on constant functions. Let  $\mathcal{D}'(S^1)$  denote the distributions on  $S^1$  and let  $\mathcal{D}'_0(S^1)$  denote those which vanish on constant functions. Recall that a Zygmund function on  $S^1$  is a function  $f : S^1 \rightarrow \mathbb{C}$  such that

$$\sup_{x \in S^1, h \in \mathbb{R}^+} \frac{|f(x+h) + f(x-h) - 2f(x)|}{h} < \infty. \tag{1.6}$$

A Zygmund function is continuous and lies in the Sobolev space  $H^{1-\epsilon}(S^1)$  for all  $\epsilon > 0$ . Let  $\mathcal{DZ}(S^1)$  denote the generalized functions on  $S^1$  which are derivatives of Zygmund functions, plus constants. If  $\Gamma$  is a subgroup of  $\text{PSL}(2, \mathbb{R})$ , let  $(\mathcal{A}'_0(S^1))^\Gamma$  denote the  $\Gamma$ -invariant subspace of  $\mathcal{A}'_0(S^1)$ , and similarly for  $(\mathcal{D}'_0(S^1))^\Gamma$  and  $(\mathcal{DZ}(S^1)/\mathbb{C})^\Gamma$ .

**Theorem 5.** *A. Let  $\Gamma$  be a torsion-free uniform lattice in  $\text{Isom}^+(H^2)$ , with  $H^2/\Gamma$  a closed surface of genus  $g$ . Then*  
 1.  $\dim (\mathcal{A}'_0(S^1))^\Gamma = 2g$ .

- 2.  $\dim(\mathcal{D}'_0(S^1))^\Gamma = 2g.$
- 3.  $\dim(\mathcal{D}Z(S^1)/\mathbb{C})^\Gamma = 2g.$
- 4.  $\dim(L^2(S^1)/\mathbb{C})^\Gamma = 0.$

B. Let  $\Gamma$  be a torsion-free nonuniform lattice in  $\text{Isom}^+(H^2)$ , with  $H^2/\Gamma$  the complement of  $k$  points in a closed surface  $S$  of genus  $g$ . Then

- 1.  $\dim(\mathcal{A}'_0(S^1))^\Gamma = \infty.$
- 2.  $\dim(\mathcal{D}'_0(S^1))^\Gamma = \max(2g, 2g + 2k - 2).$
- 3.  $\dim(H^{-\frac{1}{2}}(S^1)/\mathbb{C})^\Gamma = 2g.$
- 4.  $\dim(\mathcal{D}Z(S^1)/\mathbb{C})^\Gamma = 2g.$
- 5.  $\dim(L^2(S^1)/\mathbb{C})^\Gamma = 0.$

Parts A.2 and B.2 of Theorem 5 are due to Haefliger and Banghe [8].

Next, we look at the case of quasi-Fuchsian 3-manifolds. We follow the philosophy of Connes and Sullivan [5, Section IV.3.γ]. Let  $S$  be a closed oriented surface of genus  $g > 1$ . Let  $\Gamma$  be a quasi-Fuchsian subgroup of  $\text{Isom}^+(H^3)$  which is isomorphic to  $\pi_1(S)$ . Then  $H^3/\Gamma$  is diffeomorphic to  $\mathbb{R} \times S$  and  $H^3_c/\Gamma; \mathbb{C} = \mathbb{C}$ . Thus there is a nonzero  $L^2$ -harmonic 1-form  $\alpha$  on  $H^3/\Gamma$ .

We show that  $\Phi_1^{-1}(\pi^*\alpha)$  is a  $\Gamma$ -invariant exact 1-current supported on the limit set  $\Lambda \subset S^2$ . The domain of discontinuity  $\Omega \subset S^2$  is the union of two 2-disks  $D_+$  and  $D_-$ , with  $D_+/\Gamma$  and  $D_-/\Gamma$  homeomorphic to  $S$ . Let  $\chi_{D_+} \in L^2(S^2)$  be the characteristic function of  $D_+$ . We show that  $\Phi_1^{-1}(\pi^*\alpha)$  is proportionate to the exact 1-current  $d\chi_{D_+}$  on  $S^2$ .

Let  $Z : D^2 \rightarrow D_+$  be a uniformization of  $D_+$ . By Carathéodory's theorem,  $Z$  extends to a continuous homeomorphism  $\bar{Z} : \bar{D}^2 \rightarrow \bar{D}_+$ . The restriction of  $\bar{Z}$  to  $\partial\bar{D}^2$  gives a homeomorphism  $\partial\bar{Z} : S^1 \rightarrow \Lambda$ .

The 1-current  $d\chi_{D_+}$  defines a cyclic 1-cocycle  $\tau$  on the algebra  $C^1(S^2)$  by

$$\tau(F^0, F^1) = \int_{S^2} d\chi_{D_+} \wedge F^0 dF^1. \tag{1.7}$$

**Lemma 1.** *The function space  $H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$  is a Banach algebra with the norm*

$$\|f\| = \left( \int_{\mathbb{R}^+} \int_{S^1} \frac{|f(\theta + h) - f(\theta)|^2}{h^2} d\theta dh \right)^{\frac{1}{2}} + \|f\|_\infty. \tag{1.8}$$

Given  $f^0, f^1 \in H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$ , let

$$f^i(\theta) = \sum_{j \in \mathbb{Z}} c_j^i e^{\sqrt{-1}j\theta} \tag{1.9}$$

be the Fourier expansion. Define a bilinear function

$$\bar{\tau} : \left( \mathbb{H}^{\frac{1}{2}}(S^1) \cap L^\infty(S^1) \right) \times \left( \mathbb{H}^{\frac{1}{2}}(S^1) \cap L^\infty(S^1) \right) \rightarrow \mathbb{C} \tag{1.10}$$

by

$$\bar{\tau}(f^0, f^1) = -2\pi i \sum_{j \in \mathbb{Z}} j c_j^0 c_{-j}^1. \tag{1.11}$$

Then  $\bar{\tau}$  is a continuous cyclic 1-cocycle on  $\mathbb{H}^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$ .

We relate the function-theoretic 1-cocycle  $\bar{\tau}$  to the 1-cocycle  $\tau$ .

**Theorem 6.** *Given  $F^0, F^1 \in C^1(S^2)$ , put  $f^i = (\partial\bar{Z})^* F^i$ ,  $i \in \{1, 2\}$ . Then  $f^i \in \mathbb{H}^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$  and*

$$\tau(F^0, F^1) = -\bar{\tau}(f^0, f^1). \tag{1.12}$$

In Subsection 5.2 we give examples of discrete subgroups  $\Gamma$  of  $\text{Isom}^+(H^3)$  with limit set  $S^2$  such that for all  $\epsilon > 0$ , the  $\Gamma$ -invariant subspace of  $H^{-\epsilon}(S^2)/\mathbb{C}$  is infinite-dimensional. This contrasts with the fact that from ergodicity, the  $\Gamma$ -invariant subspace of  $L^2(S^2)/\mathbb{C}$  vanishes.

Let us remark that our results could be extended to eigenfunctions of  $\Delta_p$  with nonzero eigenvalue. In this paper we only deal with  $L^2$ -harmonic forms since the dimension of the space of such forms can often be computed in terms of topological data, such as when  $M$  is a geometrically-finite hyperbolic manifold [12].

## 2. Regularity

Let  $p$  be an integer in  $[1, \frac{n}{2}]$ . Take coordinates  $(r, \theta) \in (0, 1) \times S^{n-1}$  for  $H^n - \{0\}$ , with metric

$$ds^2 = \frac{4(dr^2 + r^2 d\theta^2)}{(1-r^2)^2}. \tag{2.1}$$

For  $k \geq 0$ , consider the hypergeometric function

$$F_{p,k}(z) = F\left(1+p - \frac{n}{2}, 1+p+k; 1+\frac{n}{2}+k; z\right). \tag{2.2}$$

Put

$$c_{p,k} = \frac{2^{p+1} \Gamma(n-p+k) \Gamma(\frac{n}{2}+1)}{n \Gamma(n-p) \Gamma(\frac{n}{2}+k+1)} = \frac{2^{p+1}}{n} \frac{(n-p)(n-p+1) \dots (n-p+k-1)}{(\frac{n}{2}+1)(\frac{n}{2}+2) \dots (\frac{n}{2}+k)}. \tag{2.3}$$

Let  $\{\alpha_i\}_{i=1}^\infty$  be a sequence of coclosed  $(p-1)$ -forms on  $S^{n-1}$  such that

1.  $\alpha_i$  is an eigenvector for the Laplacian with eigenvalue  $(k_i + p)(k_i + n - p)$ ,  $k_i \in \mathbb{Z} \cap [0, \infty)$ .
2.  $\{d\alpha_i\}_{i=1}^\infty$  is an orthonormal basis of the exact  $p$ -forms on  $S^{n-1}$ .

Then

$$\|\alpha_i\|_{L^2}^2 = \frac{1}{(k_i + p)(k_i + n - p)}. \tag{2.4}$$

Given an exact  $p$ -hyperform  $\omega$  on  $S^{n-1}$ , let

$$\omega = \sum_{i=1}^\infty c_i d\alpha_i \tag{2.5}$$

be its Fourier expansion. Gaillard [6, p. 599] showed that the Poisson transform of  $\omega$  is

$$\begin{aligned} \Phi_p(\omega) = \sum_{i=1}^\infty c_i \frac{(k_i + p)(k_i + n - p)}{2} c_{p,k_i} r^{p-1+k_i} \\ \left[ \frac{r}{k_i + p} F_{p-1,k_i}(r^2) d\alpha_i + (1 - r^2) F_{p,k_i}(r^2) dr \wedge \alpha_i \right]. \end{aligned} \tag{2.6}$$

Put  $S^{n-1}(r) = \{(r, \theta) : \theta \in S^{n-1}\} \subset H^n$ . Given  $\eta \in \Omega^{p-1}(S^{n-1})$ , we can think of  $d\eta$  and  $dr \wedge \eta$  as  $p$ -forms on  $H^n - \{0\}$ . Their pointwise norms on  $S^{n-1}(r)$  are

$$|d\eta|_{S^{n-1}(r)} = \left(\frac{1 - r^2}{2r}\right)^p |d\eta|_{S^{n-1}} \tag{2.7}$$

and

$$|dr \wedge \eta|_{S^{n-1}(r)} = \frac{1 - r^2}{2} \left(\frac{1 - r^2}{2r}\right)^{p-1} |\eta|_{S^{n-1}}. \tag{2.8}$$

**Theorem 1.** *If  $n$  is even then up to a constant,  $\Phi_{\frac{n}{2}}$  is an isometric isomorphism between exact  $\frac{n}{2}$ -forms on  $S^{n-1}$  which are Sobolev  $H^{-\frac{1}{2}}$ -regular, and  $L^2$ -harmonic  $\frac{n}{2}$ -forms on  $H^n$ .*

*Proof.* We have

$$F_{\frac{n}{2},k}(z) = F\left(1, 1 + \frac{n}{2} + k; 1 + \frac{n}{2} + k; z\right) = (1 - z)^{-1}, \tag{2.9}$$

$$F_{\frac{n}{2}-1,k}(z) = F\left(0, \frac{n}{2} + k; 1 + \frac{n}{2} + k; z\right) = 1 \tag{2.10}$$

and

$$c_{\frac{n}{2},k} = \frac{2^{\frac{n}{2}}}{k + \frac{n}{2}}. \tag{2.11}$$

Then

$$\Phi_{\frac{n}{2}}(\omega) = \sum_{i=1}^{\infty} c_i 2^{\frac{n}{2}-1} \left[ r^{\frac{n}{2}+k_i} d\alpha_i + \left(k_i + \frac{n}{2}\right) r^{\frac{n}{2}+k_i-1} dr \wedge \alpha_i \right]. \tag{2.12}$$

Thus

$$\begin{aligned} \int_{H^n} |\Phi_{\frac{n}{2}}(\omega)|^2 d\text{vol} &= \sum_{i=1}^{\infty} |c_i|^2 2^{n-2} \text{vol}(S^{n-1}) \int_0^1 \left[ r^{n+2k_i} \left(\frac{1-r^2}{2r}\right)^n + \right. \\ &\quad \left. r^{n+2k_i-2} \left(\frac{1-r^2}{2}\right)^2 \left(\frac{1-r^2}{2r}\right)^{n-2} \right] \left(\frac{2r}{1-r^2}\right)^{n-1} \frac{2}{1-r^2} dr \\ &= \sum_{i=1}^{\infty} |c_i|^2 2^{n-1} \text{vol}(S^{n-1}) \int_0^1 r^{2k_i+n-1} dr \\ &= 2^{n-2} \text{vol}(S^{n-1}) \sum_{i=1}^{\infty} \frac{1}{k_i + \frac{n}{2}} |c_i|^2. \end{aligned} \tag{2.13}$$

The theorem follows. □

**Corollary 1.** *Suppose that  $\alpha$  is a nonzero  $L^2$ -harmonic  $\frac{n}{2}$ -form on  $H^n/\Gamma$ . If  $\Gamma$  is infinite then  $\Phi_{\frac{n}{2}}^{-1}(\pi^*\alpha)$  is not Sobolev  $H^{-\frac{1}{2}}$ -regular.*

*Proof.* If  $\Phi_{\frac{n}{2}}^{-1}(\pi^*\alpha)$  were Sobolev  $H^{-\frac{1}{2}}$ -regular then Theorem 1 would imply that  $\pi^*\alpha$  is  $L^2$ , contradicting the assumption that  $\Gamma$  is infinite. □

The following is the main technical result of the paper.

**Theorem 7.** *If  $\omega$  is an exact  $p$ -hyperform on  $S^{n-1}$  and if  $\Phi_p(\omega)$  is  $L^\infty$ -bounded on  $H^n$  then  $\omega$  is Sobolev  $H^{-p-\epsilon}$ -regular for all  $\epsilon > 0$ .*

*Proof.* By the assumptions,  $\frac{1}{\text{vol}(S^{n-1}(r))} \int_{S^{n-1}(r)} |\Phi_p(\omega)|^2 d\text{vol}$  is uniformly bounded in  $r \in (0, 1)$ . Thus for  $\epsilon > 0$ ,

$$\int_0^1 r(1-r^2)^{-1+2\epsilon} \frac{1}{\text{vol}(S^{n-1}(r))} \int_{S^{n-1}(r)} |\Phi_p(\omega)|^2 d\text{vol} dr < \infty. \tag{2.15}$$

In particular, just looking at the  $dr \wedge \alpha$  component of  $\Phi_p(\omega)$  in (2.6) gives

$$\sum_{i=1}^{\infty} (k_i + p)^2 (k_i + n - p)^2 c_{p,k_i}^2 |c_i|^2 \tag{2.16}$$

$$\int_0^1 r(1-r^2)^{-1+2\epsilon} r^{2p-2+2k_i} (1-r^2)^2 F_{p,k_i}^2(r^2) \left(\frac{1-r^2}{2}\right)^2 \left(\frac{1-r^2}{2r}\right)^{2p-2} \frac{1}{(k_i+p)(k_i+n-p)} dr < \infty, \tag{2.17}$$

or

$$\sum_{i=1}^{\infty} (k_i+p)(k_i+n-p) c_{p,k_i}^2 |c_i|^2 \int_0^1 z^{k_i} (1-z)^{2p+1+2\epsilon} F_{p,k_i}^2(z) dz < \infty. \tag{2.18}$$

For the regularity question, it is the regime of large  $k_i$  and  $z$  near 1 which is relevant. Thus our main problem is to derive uniform estimates for  $F_{p,k_i}^2(z)$ , for large  $k_i$  and  $z$  near 1.

Substituting  $z = \frac{w-1}{w+1}$  gives

$$\sum_{i=1}^{\infty} (k_i+p)(k_i+n-p) c_{p,k_i}^2 |c_i|^2 \int_1^{\infty} (w-1)^{k_i} (w+1)^{-2p-k_i-3-2\epsilon} F_{p,k_i}^2\left(\frac{w-1}{w+1}\right) dw < \infty. \tag{2.19}$$

Restricting the summation to  $k_i > 0$ , the further substitution  $w = k_i x$  gives

$$\sum_i (k_i+p)(k_i+n-p) c_{p,k_i}^2 |c_i|^2 k_i^{-2p-2-2\epsilon} \int_{k_i^{-1}}^{\infty} x^{-2p-3-2\epsilon} \left(1 - \frac{1}{k_i x}\right)^{k_i} \left(1 + \frac{1}{k_i x}\right)^{-2p-k_i-3-2\epsilon} F_{p,k_i}^2\left(\frac{k_i x - 1}{k_i x + 1}\right) dx < \infty. \tag{2.20}$$

In order to estimate  $F_{p,k_i}$ , we use the transformation [1, 15.3.4]

$$\begin{aligned} F_{p,k}(z) &= F\left(1+p-\frac{n}{2}, 1+p+k; 1+\frac{n}{2}+k; z\right) \\ &= (1-z)^{\frac{n}{2}-p-1} F\left(1+p-\frac{n}{2}, \frac{n}{2}-p; 1+\frac{n}{2}+k; \frac{z}{z-1}\right). \end{aligned} \tag{2.21}$$

Then

$$F_{p,k}\left(\frac{w-1}{w+1}\right) = \left(\frac{2}{w+1}\right)^{\frac{n}{2}-p-1} F\left(1+p-\frac{n}{2}, \frac{n}{2}-p; 1+\frac{n}{2}+k; \frac{1}{2}-\frac{w}{2}\right). \tag{2.22}$$

From [11, (4) p. 246 and (15) p. 248],

$$P_{\frac{n}{2}-p-1}^{-\frac{n}{2}-k}(w) = \frac{1}{\Gamma\left(1+\frac{n}{2}+k\right)} \left(\frac{w+1}{w-1}\right)^{-\frac{n}{4}-\frac{k}{2}} F\left(1+p-\frac{n}{2}, \frac{n}{2}-p; 1+\frac{n}{2}+k; \frac{1}{2}-\frac{w}{2}\right) \tag{2.23}$$

and

$$\int_0^\infty e^{-wt} t^{\frac{n}{2}+k-\frac{1}{2}} K_{\frac{n}{2}-p-\frac{1}{2}}(t) dt = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\Gamma(k+p+1)\Gamma(n+k-p)}{(w^2-1)^{\frac{n+k}{4}+\frac{k}{2}}} P_{\frac{n}{2}-p-1}^{-\frac{n}{2}-k}(w). \tag{2.24}$$

We obtain

$$F_{p,k}\left(\frac{w-1}{w+1}\right) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} 2^{\frac{n}{2}-p-1} \frac{\Gamma(1+\frac{n}{2}+k)}{\Gamma(k+p+1)\Gamma(n+k-p)} (w+1)^{k+p+1} \int_0^\infty e^{-wt} t^{\frac{n}{2}+k-\frac{1}{2}} K_{\frac{n}{2}-p-\frac{1}{2}}(t) dt, \tag{2.25}$$

so

$$F_{p,k_i}\left(\frac{k_i x - 1}{k_i x + 1}\right) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} 2^{\frac{n}{2}-p-1} \frac{k_i^{k_i+p+1} \Gamma(1+\frac{n}{2}+k_i)}{\Gamma(k_i+p+1)\Gamma(n+k_i-p)} x^{k_i+p+1} \left(1+\frac{1}{k_i x}\right)^{k_i+p+1} \int_0^\infty e^{-k_i x t} t^{\frac{n}{2}+k_i-\frac{1}{2}} K_{\frac{n}{2}-p-\frac{1}{2}}(t) dt. \tag{2.26}$$

(Recall that for large  $t$  [1, 9.7.2 and 10.2.17],

$$K_{\frac{n}{2}-p-\frac{1}{2}}(t) \sim \sqrt{\frac{\pi}{2t}} e^{-t}.) \tag{2.27}$$

Then from (2.20),

$$\sum_i (k_i+p)(k_i+n-p) c_{p,k_i}^2 |c_i|^2 \frac{k_i^{2k_i-2\epsilon} \Gamma^2(1+\frac{n}{2}+k_i)}{\Gamma^2(k_i+p+1)\Gamma^2(n+k_i-p)} \int_{k_i^{-1}}^\infty x^{2k_i-1-2\epsilon} \int_0^\infty \int_0^\infty \left(1-\frac{1}{k_i x}\right)^{k_i} \left(1+\frac{1}{k_i x}\right)^{k_i-1-2\epsilon} e^{-k_i x(t+t')} (tt')^{\frac{n}{2}+k_i-\frac{1}{2}} K_{\frac{n}{2}-p-\frac{1}{2}}(t) K_{\frac{n}{2}-p-\frac{1}{2}}(t') dt dt' dx < \infty, \tag{2.28}$$

or

$$\sum_i (k_i+p)(k_i+n-p) c_{p,k_i}^2 |c_i|^2 \frac{k_i^{2k_i-2\epsilon} \Gamma^2(1+\frac{n}{2}+k_i)}{\Gamma^2(k_i+p+1)\Gamma^2(n+k_i-p)} \int_{k_i^{-1}}^\infty x^{2k_i-1-2\epsilon} \int_0^\infty \int_0^\infty \left(1-\frac{1}{k_i^2 x^2}\right)^{k_i} \left(1+\frac{1}{k_i x}\right)^{-1-2\epsilon} e^{-k_i x(t+t')} (tt')^{\frac{n}{2}+k_i-\frac{1}{2}} K_{\frac{n}{2}-p-\frac{1}{2}}(t) K_{\frac{n}{2}-p-\frac{1}{2}}(t') dt dt' dx < \infty. \tag{2.29}$$

Formally taking  $k_i$  large, we obtain

$$\sum_i (k_i + p)(k_i + n - p)c_{p,k_i}^2 |c_i|^2 \frac{k_i^{2k_i - 2\epsilon} \Gamma^2(1 + \frac{n}{2} + k_i)}{\Gamma^2(k_i + p + 1)\Gamma^2(n + k_i - p)} \int_0^\infty \int_0^\infty \int_0^\infty x^{2k_i - 1 - 2\epsilon} e^{-k_i x(t+t')} (tt')^{\frac{n}{2} + k_i - \frac{1}{2}} K_{\frac{n}{2} - p - \frac{1}{2}}(t) K_{\frac{n}{2} - p - \frac{1}{2}}(t') dx dt dt' < \infty, \tag{2.30}$$

or

$$\sum_i (k_i + p)(k_i + n - p)c_{p,k_i}^2 |c_i|^2 \frac{\Gamma(2k_i - 2\epsilon)\Gamma^2(1 + \frac{n}{2} + k_i)}{\Gamma^2(k_i + p + 1)\Gamma^2(n + k_i - p)} \int_0^\infty \int_0^\infty (t + t')^{-2k_i + 2\epsilon} (tt')^{\frac{n}{2} + k_i - \frac{1}{2}} K_{\frac{n}{2} - p - \frac{1}{2}}(t) K_{\frac{n}{2} - p - \frac{1}{2}}(t') dt dt' < \infty. \tag{2.31}$$

That is,

$$\sum_i (k_i + p)(k_i + n - p)c_{p,k_i}^2 |c_i|^2 \frac{\Gamma(2k_i - 2\epsilon)\Gamma^2(1 + \frac{n}{2} + k_i)}{\Gamma^2(k_i + p + 1)\Gamma^2(n + k_i - p)} 4^{-k_i} \int_0^\infty \int_0^\infty \left(\frac{t + t'}{2\sqrt{tt'}}\right)^{-2k_i} (tt')^{\frac{n}{2} - \frac{1}{2}} (t + t')^{2\epsilon} K_{\frac{n}{2} - p - \frac{1}{2}}(t) K_{\frac{n}{2} - p - \frac{1}{2}}(t') dt dt' < \infty. \tag{2.32}$$

Making the change of variables  $t = e^u v$  and  $t' = e^{-u} v$ , we have

$$\sum_i (k_i + p)(k_i + n - p)c_{p,k_i}^2 |c_i|^2 \frac{\Gamma(2k_i - 2\epsilon)\Gamma^2(1 + \frac{n}{2} + k_i)}{\Gamma^2(k_i + p + 1)\Gamma^2(n + k_i - p)} 4^{-k_i} \int_{-\infty}^\infty (\cosh u)^{-2k_i} (\cosh u)^{2\epsilon} \int_0^\infty v^{n+2\epsilon} K_{\frac{n}{2} - p - \frac{1}{2}}(e^u v) K_{\frac{n}{2} - p - \frac{1}{2}}(e^{-u} v) dv du < \infty. \tag{2.33}$$

From [11, (8) p. 325],

$$\int_0^\infty v^{n+2\epsilon} K_{\frac{n}{2} - p - \frac{1}{2}}(e^u v) K_{\frac{n}{2} - p - \frac{1}{2}}(e^{-u} v) dv = \frac{2^{n+2\epsilon+1} e^{-(2n-2p+2\epsilon)u} \Gamma(n-p+\epsilon) \Gamma^2(\frac{1+n+2\epsilon}{2}) \Gamma(1+p+\epsilon)}{8\Gamma(1+n+2\epsilon)} F(n-p+\epsilon, \frac{1+n+2\epsilon}{2}; 1+n+2\epsilon; 1-e^{-4u}). \tag{2.34}$$

Using the asymptotics of the hypergeometric function from [1, 15.3.6], one finds that for large  $u$ ,

$$(\cosh u)^{2\epsilon} \int_0^\infty v^{n+2\epsilon} K_{\frac{n}{2} - p - \frac{1}{2}}(e^u v) K_{\frac{n}{2} - p - \frac{1}{2}}(e^{-u} v) dv = O(e^{-2|u|}). \tag{2.35}$$

Thus we can apply steepest descent methods to (2.33) to obtain

$$\sum_i (k_i + p)(k_i + n - p)c_{p,k_i}^2 |c_i|^2 \frac{\Gamma(2k_i - 2\epsilon)\Gamma^2(1 + \frac{n}{2} + k_i)}{\Gamma^2(k_i + p + 1)\Gamma^2(n + k_i - p)} 4^{-k_i} \tag{2.36}$$

$$k_i^{-\frac{1}{2}} \int_0^\infty v^{n+2\epsilon} K_{\frac{n}{2}-p-\frac{1}{2}}^2(v) dv < \infty.$$

Using the asymptotics of the gamma function [1, 6.1.39], we find

$$\sum_i k_i^{-2p-2\epsilon} |c_i|^2 < \infty. \tag{2.37}$$

Recalling (2.5), this is equivalent to saying that  $\omega$  is Sobolev  $H^{-p-\epsilon}$ -regular.

To justify passing from (2.29) to (2.30), it is enough to note that if  $x \geq k_i^{-1}$  then

$$(1 - \frac{1}{k_i^2 x^2})^{k_i} (1 + \frac{1}{k_i x})^{-1-2\epsilon} < 1. \tag{2.38}$$

Thus we have uniform bounds in the preceding arguments. □

**Corollary 2.** *Suppose that  $H^n/\Gamma$  has positive injectivity radius. Suppose that  $\alpha$  is an  $L^2$ -harmonic  $p$ -form on  $H^n/\Gamma$ ,  $p \in [1, \frac{n}{2}]$ . Then  $\Phi_p^{-1}(\pi^*\alpha)$  is Sobolev  $H^{-p-\epsilon}$ -regular for all  $\epsilon > 0$ .*

*Proof.* By elliptic theory [4, Prop. 1.3], there is a constant  $r > 0$  such that for all  $m \in H^n/\Gamma$ ,  $|\alpha(m)|$  is bounded in terms of the  $L^2$ -norm of  $\alpha$  on the ball  $B_r(m) \subset H^n/\Gamma$ . Then  $\pi^*\alpha$  is uniformly bounded on  $H^n$ . The corollary follows from Theorem 7. □

**Theorem 2.A.** *If  $\Gamma$  is cocompact then for any  $p \in [1, \frac{n}{2}]$ , there are isomorphisms between the following vector spaces :*

$$V_1 = \{\text{Harmonic } p\text{-forms on } H^n/\Gamma\}.$$

$$V_2 = \{\Gamma\text{-invariant exact } p\text{-hyperforms on } S^{n-1}\}.$$

$$V_3 = \{\Gamma\text{-invariant exact } p\text{-currents on } S^{n-1} \text{ which are Sobolev } H^{-p-\epsilon}\text{-regular for all } \epsilon > 0\}.$$

$$V_4 = H^p(H^n/\Gamma, \mathbb{R}), \text{ the } p\text{-dimensional real cohomology group of } H^n/\Gamma.$$

*Proof.* By standard Hodge theory,  $V_1 \cong V_4$ . In particular,  $V_1$  is finite-dimensional. By Corollary 2, there is an injection  $V_1 \rightarrow V_3$ . There is an evident injection  $V_3 \rightarrow V_2$ . By Gaillard's theorem [6, Théorème 2], if  $\omega \in V_2$  then  $\Phi_p(\omega)$  is a  $\Gamma$ -invariant closed and coclosed  $p$ -form on  $H^n$ . Hence  $\Phi_p(\omega) = \pi^*\alpha$  for some closed and coclosed  $p$ -form  $\alpha$  on  $H^n/\Gamma$ . Hence there is an injection  $V_2 \rightarrow V_1$ . The theorem follows. □

**Corollary 3.** *Suppose that there is a positive lower bound to the lengths of the nontrivial closed geodesics on  $H^n/\Gamma$ . Suppose that all of the cusps of  $H^n/\Gamma$  have rank  $n - 1$ . If  $\alpha$  is an  $L^2$ -harmonic  $p$ -form on  $H^n/\Gamma$ ,  $p \in \{\frac{n-1}{2}, \frac{n}{2}\}$ , then for all  $\epsilon > 0$ , the hyperform  $\Phi_p^{-1}(\pi^*\alpha)$  is Sobolev  $H^{-p-\epsilon}$ -regular.*

*Proof.* For some  $\mu > 0$  less than the Margulis constant of  $H^n$ , the  $\mu$ -thin part of  $H^n/\Gamma$  has a finite number of compact components. By the proof of Corollary 2,  $\alpha$  is bounded on the  $\mu$ -thick part of  $H^n/\Gamma$ . It follows from [12, Theorem 4.12] that  $\alpha$  is bounded on the cusps of  $H^n/\Gamma$ . The corollary follows from Theorem 7.  $\square$

**Theorem 3.** *In the case  $n = 3$ , suppose that there is a positive lower bound to the lengths of the nontrivial closed geodesics on  $H^3/\Gamma$ . Let  $\alpha$  be an  $L^2$ -harmonic 1-form on  $H^3/\Gamma$ . Then for all  $\epsilon > 0$ , the hyperform  $\Phi_1^{-1}(\pi^*\alpha)$  is Sobolev  $H^{-1-\epsilon}$ -regular.*

*Proof.* Following the line of proof of Corollary 3, it suffices to analyze the asymptotics of an  $L^2$ -harmonic 1-form  $\omega$  on a rank-1 cusp. We can take a neighborhood of such a cusp to be the quotient of

$$\{(x, y, z) : y^2 + z^2 \geq R, z \geq 0\} \tag{2.39}$$

by the group generated by  $x \rightarrow x + 2\pi$ , for some  $R > 0$ . We follow the analysis of [12, Section 4], with care for constants. Make a change of coordinates to  $y = r \cos \theta$ ,  $z = r \sin \theta$ , with  $r \in [R, \infty)$ ,  $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ . The Riemannian metric in these coordinates is

$$ds^2 = \frac{dx^2}{r^2 \cos^2 \theta} + \frac{dr^2}{r^2 \cos^2 \theta} + \frac{d\theta^2}{\cos^2 \theta}, \tag{2.40}$$

with volume form  $d\text{vol} = \frac{dx dr d\theta}{r^2 \cos^3 \theta}$ .

Let

$$\omega = \alpha_0 d\theta + \alpha_1 dx + \beta_0 dr \tag{2.41}$$

be an  $L^2$ -harmonic 1-form on the cusp. Then

$$\int \left( r^{-2} |\alpha_0|^2 + |\alpha_1|^2 + |\beta_0|^2 \right) \frac{dx dr d\theta}{\cos \theta} < \infty. \tag{2.42}$$

The equations  $d\omega = d^*\omega = 0$  become

$$\begin{aligned} 0 &= \partial_x \alpha_0 - \partial_\theta \alpha_1 = \partial_r \alpha_0 - \partial_\theta \beta_0 = \partial_r \alpha_1 - \partial_x \beta_0 \\ &= \cos \theta \partial_\theta \left( \frac{\alpha_0}{\cos \theta} \right) + r^2 \partial_x \alpha_1 + r^2 \partial_r \beta_0. \end{aligned} \tag{2.43}$$

From these equations, one obtains the Laplacian-type equations

$$\begin{aligned} -\partial_r^2 \alpha_0 - \partial_x^2 \alpha_0 - \frac{1}{r^2} \partial_\theta \left( \cos \theta \partial_\theta \left( \frac{\alpha_0}{\cos \theta} \right) \right) &= 0, \\ -\partial_r^2 \alpha_1 - \partial_x^2 \alpha_1 - \frac{1}{r^2} \cos \theta \partial_\theta \left( \frac{1}{\cos \theta} \partial_\theta \alpha_1 \right) &= 0, \\ -\partial_r^2 \beta_0 - \partial_x^2 \beta_0 - \frac{1}{r^2} \cos \theta \partial_\theta \left( \frac{1}{\cos \theta} \partial_\theta \beta_0 \right) &= -\frac{2}{r^3} \cos \theta \partial_\theta \left( \frac{\alpha_0}{\cos \theta} \right). \end{aligned} \tag{2.44}$$

We first analyze the second equation in (2.44). Given a function  $f \in C^\infty(-\frac{\pi}{2}, \frac{\pi}{2})$ , put

$$Lf = -\cos \theta \partial_\theta \left( \frac{1}{\cos \theta} \partial_\theta f \right) \tag{2.45}$$

Then  $L$  is the self-adjoint operator coming from the Dirichlet form on  $L^2\left(-\frac{\pi}{2}, \frac{\pi}{2}, \frac{1}{\cos \theta} d\theta\right)$ . Making the change of variable  $u = \sin \theta$ , the eigenfunction equation  $Lf = \lambda f$  becomes

$$-(1 - u^2)f''(u) = \lambda f. \tag{2.46}$$

The square-integrable solutions to this have  $\lambda = (q + 1)(q + 2)$  with  $q \in \mathbb{Z} \cap [0, \infty)$ . The corresponding eigenfunction is given in terms of ultraspherical polynomials [1, 22.6.6] by

$$f_q(u) = (1 - u^2)C_q^{3/2}(u). \tag{2.47}$$

Explicitly,  $f_q(u)$  is proportionate to  $\frac{d^q}{du^q}((1 - u^2)^{q+1})$ .

Performing separation of variables on the second equation in (2.44), suppose that

$$\alpha_1(x, r, \theta) = e^{imx} g(r) f_q(\theta), \tag{2.48}$$

with  $m \in \mathbb{Z}$ . Then

$$-g'' + m^2 g + \frac{(q + 1)(q + 2)}{r^2} g = 0. \tag{2.49}$$

If  $m \neq 0$  then  $g$  decreases exponentially fast in  $r$ . Suppose that  $m = 0$ . One finds that for large  $r$ ,  $g(r) \sim r^{q+2}$  or  $g(r) \sim r^{-q-1}$ . For  $\omega$  to be square-integrable, one must have  $g(r) \sim r^{-q-1}$ . If  $q > 0$  then  $|\alpha_1 dx| = r \cos \theta |g(r)| |f_q(\theta)|$  decays polynomially fast in  $r$ . In the critical case  $q = 0$ ,  $|\alpha_1 dx|$  remains bounded in  $r$ .

Next, put

$$L'f = -\partial_\theta \left( \cos \theta \partial_\theta \left( \frac{f}{\cos \theta} \right) \right). \tag{2.50}$$

and  $\widehat{L} = \frac{1}{\cos \theta} \circ L' \circ \cos \theta$ . Then  $\widehat{L}$  is the self-adjoint operator coming from the Dirichlet form on  $L^2\left(-\frac{\pi}{2}, \frac{\pi}{2}, \cos \theta d\theta\right)$ . It has a nonnegative discrete spectrum starting at 0, and hence so does  $L'$ . Suppose that  $f(\theta)$  is an eigenfunction of  $L'$

with eigenvalue  $\lambda \geq 0$ . Performing separation of variables on the first equation in (2.44), suppose that

$$\alpha_0(x, r, \theta) = e^{imx}g(r)f(\theta), \tag{2.51}$$

with  $m \in \mathbb{Z}$ . Then

$$-g'' + m^2g + \frac{\lambda}{r^2}g = 0. \tag{2.52}$$

If  $m \neq 0$  then  $g$  decreases exponentially fast in  $r$ . Suppose that  $m = 0$ . One finds that for large  $r$ ,  $g(r) \sim r^{\frac{1 \pm \sqrt{1+4\lambda}}{2}}$ . For  $\omega$  to be square-integrable, one must have  $g(r) \sim r^{\frac{1-\sqrt{1+4\lambda}}{2}}$ . If  $\lambda > 0$  then  $|\alpha_0 d\theta| = \cos \theta |g(r)| |f(\theta)|$  decays like a power in  $r$ . In the critical case  $\lambda = 0$ ,  $|\alpha_0 d\theta|$  remains bounded in  $r$ .

Finally, one can analyze the third equation in (2.44), an inhomogeneous equation, by similar methods. The upshot is that  $|\omega|$  is bounded on the rank-1 cusp.  $\square$

**Proposition 1.** *Suppose that there is a positive lower bound to the lengths of the nontrivial closed geodesics on  $H^n/\Gamma$ . Let  $\alpha$  be an  $L^2$ -harmonic  $p$ -form on  $H^n/\Gamma$ ,  $p \in [1, \frac{n}{2}]$ . Then  $\Phi_p^{-1}(\pi^*\alpha)$  is a current.*

*Proof.* For some  $\mu > 0$  less than the Margulis constant of  $H^n$ , the  $\mu$ -thin part of  $H^n/\Gamma$  has a finite number of compact components. As in the proof of Corollary 2, there is a uniform upper bound for  $|\alpha|$  on the  $\mu$ -thick part of  $H^n/\Gamma$ . On each cuspidal component of the  $\mu$ -thin part,  $|\alpha|$  has at most exponential growth, with a uniform exponential constant [12, Section 4]. The result follows from [6, Théorème 3].  $\square$

**Proposition 2.** *For  $r \in (0, 1)$ , let  $i_r : S^{n-1} \rightarrow S^{n-1}(r)$  be the embedding of  $S^{n-1}$  as the  $r$ -sphere around 0 in the ball model of  $H^n$ . As in [6, p. 586], put*

$$C_p = \frac{2^p \Gamma(n - 2p + 1) \Gamma(\frac{n}{2} + 1)}{n \Gamma(n - p) \Gamma(\frac{n}{2} - p + 1)}. \tag{2.53}$$

*Let  $\omega$  be an exact  $p$ -current on  $S^{n-1}$ . Then as  $r \rightarrow 1$ , the forms  $i_r^* \Phi_p(\omega)$  converge to  $C_p \omega$  in the sense of convergence of currents.*

*Proof.* From (2.6),

$$i_r^* \Phi_p(\omega) = \sum_{i=1}^{\infty} c_i \frac{(k_i + p)(k_i + n - p)}{2} c_{p, k_i} r^{p-1+k_i} \frac{r}{k_i + p} F_{p-1, k_i}(r^2) d\alpha_i. \tag{2.54}$$

Given a smooth form  $\eta \in \Omega^p(S^{n-1})$ , let  $\Pi(\eta)$  be the projection of  $\eta$  onto the square-integrable exact  $p$ -forms on  $S^{n-1}$ . Then  $\Pi(\eta)$  is also smooth and has a Fourier expansion

$$\Pi(\eta) = \sum_{i=1}^{\infty} a_i d\alpha_i, \tag{2.55}$$

with  $\sum_{i=1}^\infty k_i^N |a_i|^2 < \infty$  for all  $N \in \mathbb{Z}^+$ . The pairing

$$\langle i_r^* \Phi_p(\omega), \eta \rangle = \int_{S^{n-1}} i_r^* \Phi_p(\omega) \wedge \overline{* \eta} \tag{2.56}$$

is given by

$$\langle i_r^* \Phi_p(\omega), \eta \rangle = \sum_{i=1}^\infty \overline{a_i} c_i \frac{(k_i + p)(k_i + n - p)}{2} c_{p, k_i} r^{p-1+k_i} \frac{r}{k_i + p} F_{p-1, k_i}(r^2). \tag{2.57}$$

Then

$$\begin{aligned} \langle i_1^* \Phi_p(\omega), \eta \rangle &= \sum_{i=1}^\infty \overline{a_i} c_i \frac{(k_i + p)(k_i + n - p)}{2} c_{p, k_i} \frac{1}{k_i + p} \\ &\quad \frac{\Gamma(1 + \frac{n}{2} + k_i) \Gamma(1 - 2p + n)}{\Gamma(1 - p + n + k_i) \Gamma(1 - p + \frac{n}{2})} \\ &= C_p \sum_{i=1}^\infty \overline{a_i} c_i \\ &= C_p \langle \omega, \eta \rangle. \end{aligned} \tag{2.58}$$

As  $\omega$  is a current,  $\sum_{i=1}^\infty k_i^N |a_i| |c_i| < \infty$  for all  $N \in \mathbb{Z}^+$ .

**Lemma 2.** *As  $r$  increases from 0 to 1, the expression  $r^{p-1+k_i} \frac{r}{k_i+p} F_{p-1, k_i}(r^2)$  increases monotonically from 0 to  $\frac{1}{k_i+p} \frac{\Gamma(1+\frac{n}{2}+k_i)\Gamma(1-2p+n)}{\Gamma(1-p+n+k_i)\Gamma(1-p+\frac{n}{2})}$ .*

*Proof.* The fact that the right-hand-side of (2.6) is closed implies that

$$\frac{d}{dr} \left( r^{p-1+k_i} \frac{r}{k_i+p} F_{p-1, k_i}(r^2) \right) = r^{p-1+k_i} (1-r^2) F_{p, k_i}(r^2). \tag{2.59}$$

(Of course, this can be checked directly.) From [1, 15.3.3],

$$\begin{aligned} F_{p, k_i}(r^2) &= F\left(1 + p - \frac{n}{2}, 1 + p + k_i; 1 + \frac{n}{2} + k_i; r^2\right) \\ &= (1 - r^2)^{n-1-2p} F\left(n + k_i - p, \frac{n}{2} - p; 1 + \frac{n}{2} + k_i; r^2\right). \end{aligned} \tag{2.20}$$

As the arguments of  $F(n + k_i - p, \frac{n}{2} - p; 1 + \frac{n}{2} + k_i; r^2)$  are all nonnegative, the lemma follows. □

Proposition 2 now follows from dominated convergence. □

**Proposition 3.** *Suppose that  $\alpha$  is an  $L^2$ -harmonic  $p$ -form on  $H^n/\Gamma$ ,  $p \in [1, \frac{n}{2})$ . Suppose that  $\Phi_p^{-1}(\pi^*\alpha)$  is a current. Then  $\Phi_p^{-1}(\pi^*\alpha)$  is supported on the limit set  $\Lambda$  of  $\Gamma$ .*

*Proof.* Given a smooth form  $\phi \in \Omega^p(S^{n-1})$  with relatively compact support in  $S^{n-1} - \Lambda$ , Proposition 2 implies that

$$\lim_{r \rightarrow 1} \langle i_r^* \pi^* \alpha, \phi \rangle = C_p \langle \Phi_p^{-1}(\pi^* \alpha), \phi \rangle. \tag{2.61}$$

If  $\Lambda = \emptyset$ , we assume that  $\text{supp}(\phi) \neq S^{n-1}$ ; this is sufficient for the argument. Then we can use an upper-half-space model for  $H^n$ , with  $\text{supp}(\phi) \subset \mathbb{R}^{n-1}$ . Put  $V = \text{supp}(\phi) \times (0, \infty) \subset H^n$ . Using the coordinates  $(x_1, \dots, x_{n-1}, y)$  for  $H^n$ , let us write  $\tilde{\alpha} = a(x, y) + dy \wedge b(x, y)$ . Then [12, Theorem 4.3] states that on  $V$ , as  $y \rightarrow 0$ ,

$$a = \begin{cases} a_{00}(x)y^{n-2p-1} + O(y^{n-2p} \log(y)) & \text{if } p < \frac{n-1}{2}, \\ a_{01}(x)y^2 \log(y) + O(y^2) & \text{if } p = \frac{n-1}{2} \end{cases} \tag{2.62}$$

and

$$b = \begin{cases} b_{01}(x)y^{n-2p} \log(y) + O(y^{n-2p}) & \text{if } p < \frac{n-1}{2}, \\ b_{00}(x)y + O(y^2 \log(y)) & \text{if } p = \frac{n-1}{2}. \end{cases} \tag{2.63}$$

(The statement of [12, Theorem 4.3] should read “ $y \rightarrow 0$ ”.) As  $r \rightarrow 1$ , the intersections  $S^{n-1}(r) \cap V$  asymptotically approach the horosphere pieces

$$\{(x_1, \dots, x_{n-1}, y) \in H^n : y = \frac{1-r}{1+r}\} \cap V. \tag{2.64}$$

It follows that  $\langle \Phi_p^{-1}(\pi^*\alpha), \phi \rangle = 0$  for all such  $\phi$ , from which the proposition follows.  $\square$

**Remark.** The analog of Proposition 3 is false if  $p = \frac{n}{2}$ . This can be seen in the case  $\Gamma = \{e\}$  using Theorem 1.

We give a partial converse to Proposition 3, in the case of convex-cocompact groups.

**Theorem 2.B.** *If  $\Gamma$  is convex-cocompact then for any  $p \in [1, \frac{n-1}{2})$ , there are isomorphisms between the following vector spaces :*

$$V_1 = \{L^2\text{-harmonic } p\text{-forms on } H^n/\Gamma\}.$$

$$V_2 = \{\Gamma\text{-invariant exact } p\text{-hyperforms on } S^{n-1} \text{ which are supported on the limit set}\}.$$

$$V_3 = \{\Gamma\text{-invariant exact } p\text{-currents on } S^{n-1} \text{ which are supported on the limit set and which are Sobolev } H^{-p-\epsilon}\text{-regular for all } \epsilon > 0\}.$$

$$V_4 = H_c^p(H^n/\Gamma, \mathbb{R}), \text{ the } p\text{-dimensional real compactly-supported cohomology group of } H^n/\Gamma.$$

*Proof.* By [12],  $V_1 \cong V_4$ . In particular,  $V_1$  is finite-dimensional. By Gaillard’s theorem [6, Théorème 2], Corollary 2 and Proposition 3, there are injections  $V_1 \rightarrow V_3 \rightarrow V_2$ . It remains to show that there is an injection  $V_2 \rightarrow V_1$ . In view of Gaillard’s theorem, it suffices to show that if  $\omega \in V_2$  then  $\Phi_p(\omega)$  descends to a form which is square-integrable on  $H^n/\Gamma$ . If  $\Gamma$  is cocompact then this is automatic, so assume that  $\Gamma$  is not cocompact. As  $\Omega/\Gamma$  is compact, we can find a fundamental domain  $F$  for the action of  $\Gamma$  on  $H^n$  such that  $\overline{F} \cap S^{n-1}$  is disjoint from  $\Lambda$ . Take an upper-half-space model for  $H^n$  with  $\infty \in \Omega$ . In terms of the upper-half-space coordinates  $(x_1, \dots, x_{n-1}, y)$ , [6, Lemme 3] implies that near  $y = 0$ ,

$$\Phi_p(\omega)|_F = y^{n-2p-1} \phi(x, y), \tag{2.65}$$

where the  $p$ -form  $\phi(x, y)$  is continuous up to  $y = 0$ . It follows that  $\int_F |\Phi_p(\omega)|^2 d\text{vol} < \infty$ .  $\square$

### 3. 1-Forms

In this section we look in more detail at the case of  $L^2$ -harmonic 1-forms on convex-cocompact hyperbolic manifolds. If the hyperbolic manifold is compact, we show that the Sobolev regularity estimate of Theorem 2.A is sharp. If the hyperbolic manifold is convex-cocompact but not compact, we show how to construct its  $L^2$ -harmonic 1-forms explicitly in terms of the harmonic extension of functions. In this case, we show that the Sobolev regularity estimate of Corollary 2 can be slightly improved.

**Proposition 4.** *Suppose that  $\Gamma$  is cocompact. For  $\epsilon > 0$ , let  $V_\epsilon^\Gamma$  be the  $\Gamma$ -invariant subspace of the function space  $H^{-\epsilon}(S^{n-1})/\mathbb{C}$ . Then  $V_\epsilon^\Gamma$  is isomorphic to  $H^1(\Gamma; \mathbb{C})$ .*

*Proof.* We first define linear maps  $I : H^1(\Gamma; \mathbb{C}) \rightarrow V_\epsilon^\Gamma$  and  $J : V_\epsilon^\Gamma \rightarrow H^1(\Gamma; \mathbb{C})$ . To define  $I$ , given  $x \in H^1(\Gamma; \mathbb{C}) = H^1(H^n/\Gamma; \mathbb{C})$ , let  $\alpha \in \Omega^1(H^n/\Gamma)$  be the harmonic 1-form which represents  $x$ . Put  $\tilde{\alpha} = \pi^* \alpha$ . By Theorem 7,  $\Phi_1^{-1}(\tilde{\alpha})$  is an exact  $H^{-1-\epsilon}$ -regular  $\Gamma$ -invariant 1-form on  $S^{n-1}$ . Choose  $f \in H^{-\epsilon}(S^{n-1})$  so that  $\Phi_1^{-1}(\tilde{\alpha}) = df$ . Then for all  $\gamma \in \Gamma$ ,

$$d(f - \gamma \cdot f) = df - \gamma \cdot df = 0. \tag{3.1}$$

Thus

$$f - \gamma \cdot f = c(\gamma) \tag{3.2}$$

for some  $c(\gamma) \in \mathbb{C}$ . Put  $I(x) = f \pmod{\mathbb{C}}$ .

To define  $J$ , given  $\bar{f} \in V_\epsilon^\Gamma$ , let  $f \in H^{-\epsilon}(S^{n-1})$  be a representative of  $\bar{f}$ , not necessarily  $\Gamma$ -invariant. As  $\bar{f}$  is  $\Gamma$ -invariant, for each  $\gamma \in \Gamma$  there is a  $c(\gamma) \in \mathbb{C}$  such that  $f - \gamma \cdot f = c(\gamma)$ . As

$$c(\gamma_1 \gamma_2) = f - (\gamma_1 \gamma_2) \cdot f = (f - \gamma_1 \cdot f) + \gamma_1 \cdot (f - \gamma_2 \cdot f) = c(\gamma_1) + \gamma_1 \cdot c(\gamma_2) = c(\gamma_1) + c(\gamma_2), \tag{3.3}$$

we have a cocycle  $c : \Gamma \rightarrow \mathbb{C}$ . Put  $J(\bar{f}) = [c]$ .

We show that  $J \circ I$  is the identity. It suffices to show that the cocycle  $c$  of (3.2) represents  $x \in H^1(\Gamma; \mathbb{C})$ . For this, it suffices to show that for all  $\gamma \in \Gamma$ ,

$$c(\gamma) = \int_{C_\gamma} \alpha, \tag{3.4}$$

where  $C_\gamma$  is a closed curve on  $H^n/\Gamma$  in the homotopy class of  $\gamma \in \pi_1(H^n/\Gamma)$  and  $\alpha \in \Omega^1(H^n/\Gamma)$  is the harmonic representative of  $x$ . Let  $\tilde{C}_\gamma$  be a lift of  $C_\gamma$  to  $H^n$ , ending at a point  $m \in H^n$  and starting at  $\gamma^{-1} \cdot m$ . Then

$$\begin{aligned} \int_{C_\gamma} \alpha &= \int_{\tilde{C}_\gamma} \tilde{\alpha} = \int_{\tilde{C}_\gamma} \Phi_1(df) = \int_{\tilde{C}_\gamma} d\Phi_0(f) = (\Phi_0(f))(m) - (\Phi_0(f))(\gamma^{-1} \cdot m) \\ &= (\Phi_0(f) - \gamma \cdot \Phi_0(f))(m) = (\Phi_0(f - \gamma \cdot f))(m) = (\Phi_0(c(\gamma)))(m) = c(\gamma). \end{aligned} \tag{3.5}$$

This shows that  $J \circ I$  is the identity. To see that  $I \circ J$  is the identity, given  $\bar{f} \in V_\epsilon^\Gamma$ , let  $f \in H^{-\epsilon}(S^{n-1})$  be a representative of  $\bar{f}$ , not necessarily  $\Gamma$ -invariant. Define  $\tilde{\alpha} = \Phi_1(df)$ . Then  $\tilde{\alpha}$  is a smooth  $\Gamma$ -invariant harmonic 1-form on  $H^n$  and projects to a harmonic 1-form  $\alpha \in \Omega^1(H^n/\Gamma)$ . By the same sort of calculation as in (3.5), one finds that  $J(\bar{f}) = [\alpha]$  in  $H^1(\Gamma; \mathbb{C})$ . By construction,  $I([\alpha]) = \bar{f}$ . Thus  $I \circ J$  is the identity.  $\square$

**Theorem 4.A.** *Suppose that  $\Gamma$  is cocompact. Let  $\alpha$  be a nonzero harmonic 1-form on  $H^n/\Gamma$ . Then  $\Phi_1^{-1}(\pi^*\alpha)$  is not Sobolev  $H^{-1}$ -regular.*

*Proof.* Suppose that  $\Phi_1^{-1}(\pi^*\alpha)$  is Sobolev  $H^{-1}$ -regular. Then  $\Phi_1^{-1}(\pi^*\alpha) = df$  for some  $f \in L^2(S^{n-1})$ . Extending the proof of Proposition 4 to the case  $\epsilon = 0$ , the equivalence class  $\bar{f}$  of  $f$  in  $L^2(S^{n-1})/\mathbb{C}$  is  $\Gamma$ -invariant and satisfies  $J(\bar{f}) = [\alpha]$ . As  $\Gamma$  acts ergodically on  $S^{n-1}$ , we must have  $\bar{f} = 0$  and hence  $[\alpha]$  vanishes in  $H^1(H^n/\Gamma; \mathbb{C})$ , which is a contradiction.  $\square$

We now consider groups  $\Gamma$  which are convex-cocompact but not compact. First, we prove some generalities about the relationship between compactly-supported cohomology and  $L^2$ -cohomology.

Let  $M$  be a complete connected oriented Riemannian manifold. Let  $H_{(2)}^p(M)$  be the  $p$ -th (reduced)  $L^2$ -cohomology group of  $M$ . It is isomorphic to  $\text{Ker}(\Delta_p)$ . There is a map  $i : H_c^p(M; \mathbb{C}) \rightarrow H_{(2)}^p(M)$ . In general,  $i$  is not injective; think of  $M = \mathbb{R}^n$ . However, it is true, and well-known, that  $i$  always induces an injection of  $\text{Im}(H_c^p(M; \mathbb{C}) \rightarrow H^p(M; \mathbb{C}))$  into  $H_{(2)}^p(M)$  [9, Prop. 4]. The next result gives a sufficient condition for  $i$  to be injective on all of  $H_c^1(M; \mathbb{C})$ . Recall that there is a notion of the space of ends of  $M$ , and of an end being contained in an open set  $U \subset M$ ; see, for example, [3, §1.2].

**Proposition 5.** *Suppose that for every end  $e$  of  $M$ , every open set  $U$  containing  $e$  has infinite volume. Suppose that  $M$  has a Green's operator  $G : C_0^\infty(M) \rightarrow L^2(M)$  such that  $\Delta \circ G = \text{Id}$ . Then  $i : H_c^1(M; \mathbb{C}) \rightarrow H_{(2)}^1(M)$  is injective.*

*Proof.* We have the decomposition

$$H_c^1(M; \mathbb{C}) = \left( \text{Ker}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C})) \right) \oplus \left( \text{Im}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C})) \right). \tag{3.6}$$

We first show that  $i$  is injective on  $\text{Ker}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C}))$ . A representative of  $\text{Ker}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C}))$  is a closed compactly-supported 1-form  $\alpha$  such that  $\alpha = df$  for some function  $f$ . By construction,  $f$  is locally constant outside of a compact subset of  $M$  and so gives a function on the space of ends of  $M$ . Now  $d(f - G\Delta f)$  is a harmonic 1-form on  $M$ . As

$$\langle dG\Delta f, dG\Delta f \rangle = \langle G\Delta f, \Delta f \rangle, \tag{3.7}$$

we have that  $d(f - G\Delta f)$  is square-integrable. The map  $\alpha \rightarrow d(f - G\Delta f)$  describes  $i$  on  $\text{Ker}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C}))$ . To see that it is injective, suppose that  $d(f - G\Delta f) = 0$ . Then  $f - G\Delta f$  is constant. As  $G\Delta f \in L^2(M)$ , the volume assumption implies that  $f$ , as a function on the space of ends of  $M$ , is a constant  $c$ . Then  $f - c$  is compactly-supported on  $M$ , with  $d(f - c) = \alpha$ , so  $[\alpha] = 0$  in  $H_c^1(M; \mathbb{C})$ . In summary, we have realized an injection of  $\text{Ker}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C}))$  into  $H_{(2)}^1(M)$ .

It remains to show that

$$i \left( \text{Ker}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C})) \right) \cap i \left( \text{Im}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C})) \right) = 0. \tag{3.8}$$

Suppose that  $d(f - G\Delta f)$  is nonzero and lies in the image, under  $i$ , of  $\text{Im}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C}))$ . Then  $d(f - G\Delta f) = \omega \pmod{\overline{\text{Im}(d)}}$  for some closed compactly-supported 1-form  $\omega$ . Furthermore, by assumption, there is a closed compactly-supported  $(\dim(M) - 1)$ -form  $\eta$  such that  $\int_M \omega \wedge \eta = 1$ . However,  $\int_M d(f - G\Delta f) \wedge \eta = 0$ . It follows that

$$i \left( \text{Ker}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C})) \right) \cap i \left( \text{Im}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C})) \right) = 0. \tag{3.9}$$

This proves the proposition. □

Suppose that  $\Gamma$  is convex-cocompact but not cocompact. Then  $H^n/\Gamma$  satisfies the hypotheses of Proposition 5 and so  $i : H_c^1(H^n/\Gamma; \mathbb{C}) \rightarrow H_{(2)}^1(H^n/\Gamma)$  is injective. For the rest of this section, we assume that  $n > 2$ . It follows from [12, Theorem 3.13] that  $i$  is an isomorphism. This essentially comes from the fact that given an  $L^2$ -harmonic 1-form  $\omega$  on  $H^n/\Gamma$ , one can apply the Poincaré Lemma from infinity

to homotop  $\omega$  to something with compact support. We show how to construct the  $L^2$ -harmonic 1-forms on  $H^n/\Gamma$  explicitly.

**Lemma 3.** *There is an isomorphism between  $H_c^1(H^n/\Gamma; \mathbb{C})$  and the quotient space*

$$W = \{(f, c) \in C^\infty(\Omega) \times H^1(\Gamma; \mathbb{C}) : f \text{ is locally-constant and for all } \gamma \in \Gamma, \quad (3.10)$$

$$f - \gamma \cdot f = c(\gamma)\} / \mathbb{C}.$$

(Here  $\mathbb{C}$  acts by addition on  $C^\infty(\Omega)$  and fixes  $H^1(\Gamma; \mathbb{C})$ .)

*Proof.* Given  $x \in H_c^1(H^n/\Gamma; \mathbb{C})$ , represent it by a smooth closed compact-supported 1-form  $\alpha \in \Omega^1(H^n/\Gamma)$ . Put  $\tilde{\alpha} = \pi^*\alpha$ . As  $\alpha$  is compactly-supported, we can extend  $\tilde{\alpha}$  continuously by zero to become a closed 1-form on  $H^n \cup \Omega$ . Fix a point  $s \in \Omega$ . Define  $f : \Omega \rightarrow \mathbb{C}$  by

$$f(z) = \int_{\tilde{C}} \tilde{\alpha}, \tag{3.11}$$

where  $\tilde{C}$  is a curve in  $H^n \cup \Omega$  from  $s$  to  $z$ . Then

$$(f - \gamma \cdot f)(z) = \int_{\tilde{C}'} \tilde{\alpha}, \tag{3.12}$$

where  $\tilde{C}'$  is a curve in  $H^n \cup \Omega$  from  $\gamma^{-1} \cdot z$  to  $z$ . Now  $\tilde{C}'$  projects to a closed curve  $C'$  on the compact manifold-with-boundary  $(H^n \cup \Omega)/\Gamma$ . Then

$$(f - \gamma \cdot f)(z) = \int_{C'} \alpha. \tag{3.13}$$

It follows that  $f - \gamma \cdot f = c(\gamma)$ , where  $c$  is the image of  $x$  in  $H^1((H^n \cup \Omega)/\Gamma; \mathbb{C}) \cong H^1(\Gamma; \mathbb{C})$ . A different choice of  $s$  changes  $f$  by a constant.

Conversely, given  $(f, c) \in W$ , fix a point  $m_0 \in H^n/\Gamma$ . Let  $R$  be large enough that the convex core of  $H^n/\Gamma$  lies within  $B_R(m_0)$ . Let  $\phi : [0, \infty) \rightarrow \mathbb{R}$  be a smooth function which is monotonically nonincreasing, identically one on  $[0, R]$  and identically zero on  $[R+1, \infty)$ . Let  $\eta \in C^\infty(H^n)$  be the lift to  $H^n$  of  $\phi(d(m_0, \cdot)) \in C^\infty(H^n/\Gamma)$ . Extend  $f$  inward to a locally-constant smooth function  $F : (H^n - \pi^{-1}(B_R(m_0))) \rightarrow \mathbb{C}$ . Put  $\tilde{\alpha} = d((1 - \eta)F)$  on  $H^n - \pi^{-1}(B_R(m_0))$  and extend it by zero to  $H^n$ . Then  $\tilde{\alpha}$  is a closed  $\Gamma$ -invariant 1-form on  $H^n$  which descends to a closed 1-form  $\alpha \in \Omega^1(H^n/\Gamma)$  with support in  $B_{R+1}(m_0)$ , and hence an element  $[\alpha] \in H_c^1(H^n/\Gamma; \mathbb{C})$ .

One can check that these two maps are inverses. We omit the details. □

The map  $W \rightarrow H^1(H^n/\Gamma; \mathbb{C})$  induced from  $(f, c) \rightarrow c$  is the same as the map  $H_c^1(H^n/\Gamma; \mathbb{C}) \rightarrow H^1(H^n/\Gamma; \mathbb{C})$ . Its kernel can be identified with the  $\Gamma$ -invariant

locally-constant functions on  $\Omega$ , modulo  $\mathbb{C}$ . This has dimension equal to the number of ends of  $H^n/\Gamma$  minus one, as it should.

Choose  $x \in H_c^1(H^n/\Gamma; \mathbb{C})$ . Define the locally-constant function  $f : \Omega \rightarrow \mathbb{C}$  as in the proof of Lemma 3. As  $\Lambda$  has measure zero, we can think of  $f$  as a measurable function on  $S^{n-1}$ .

**Proposition 6.**  *$f$  lies in  $L^p(S^{n-1})$  for all  $p \in [1, \infty)$ .*

*Proof.* Let  $K$  be the convex core of  $H^n/\Gamma$  and let  $\partial K$  be its boundary. Put  $\tilde{K} = \pi^{-1}(K)$ , the convex hull of  $\Lambda$ , and put  $\tilde{\partial K} = \pi^{-1}(\partial K)$ . As  $\tilde{K}$  is convex and  $K$  is compact, it follows that  $\tilde{\partial K}$  is quasi-convex, meaning that there is an  $R > 0$  such that if  $y_1, y_2 \in \tilde{\partial K}$  then the geodesic from  $y_1$  to  $y_2$ , in  $H^n$ , lies in an  $R$ -neighborhood of  $\tilde{\partial K}$ . We take a ball model  $B^n$  for  $H^n$  such that  $x_0 = \pi(0)$  lies in  $K$ .

If  $\Omega \subset S^{n-1}$  is connected then the result is trivial, so we assume that  $\Omega$  has more than one connected component. Let  $D$  be a connected component of  $\Omega$ . We first estimate the spherical volume of  $D$ . There is an end  $e$  of  $H^n/\Gamma$  such that if a curve  $c$  in  $H^n$  goes to  $D$  then  $\pi \circ c$  exits  $e$ . Let  $\partial_e K$  be the connected component of  $\partial K$  corresponding to  $e$ . Then there is a component  $\tilde{\partial}_D K$  of  $\pi^{-1}(\partial_e K)$  such that  $D$  retracts onto  $\tilde{\partial}_D K$  under the nearest-point retraction. Furthermore, the closure of  $\tilde{\partial}_D K$  in  $\tilde{B}^n$  separates  $D$  from  $K - \tilde{\partial}_D K$ . Let  $r_D$  be the hyperbolic distance from 0 to  $\tilde{\partial}_D K$ . Then  $\tilde{\partial}_D K \subset H^n - B_{r_D}(0)$ . We are interested in what happens when  $r_D$  is large. If  $z_1, z_2 \in \tilde{\partial}_D K$  then the geodesic from  $z_1$  to  $z_2$  cannot enter  $B_{r_D-R}(0)$ , as this would violate the quasi-convexity of  $\tilde{\partial}_D K$ . Quantitatively, this implies that the spherical distance from  $z_1$  to  $z_2$  cannot exceed  $2 \sin^{-1} \left( \frac{1}{\cosh(r_D-R)} \right)$ . Thus  $D$  lies within a spherical ball of radius  $r_0 = 4 \sin^{-1} \left( \frac{1}{\cosh(r_D-R)} \right)$ . As the volume of this spherical ball is bounded above by a constant times  $r_0^{n-1}$ , we conclude that there is a constant  $C > 0$  such that  $\text{vol}(D) \leq C e^{-(n-1)r_D}$ , uniformly in the choice of  $D$ .

The connected components of  $\Omega$  are in one-to-one correspondence with the set  $\pi_1(K, \partial K)$ . Fix an end  $e$  of  $M$ , with associated connected component  $\partial_e K$  of  $\partial K$ . Take the ball model so that  $x_0 \in \partial_e K$ . The connected components  $D$  of  $\Omega$  corresponding to  $e$  form the preimage of  $\partial_e K$  under the map  $\pi_1(K, \partial K) \rightarrow \pi_0(\partial K)$ . Given  $D$ , let  $c(s), 0 \leq s \leq r_D$ , be a normalized minimal geodesic from 0 to  $\tilde{\partial}_D K$ . Consider a loop  $L_D$  in  $H^n/\Gamma$  which starts at  $x_0$ , follows  $\pi \circ c$  to  $\pi(c(r_D)) \in \partial_e K$  and then returns to  $x_0$  by a length-minimizing path in  $\partial_e K$ . The length of  $L_D$  will be bounded above by  $r_D + \text{diam}(\partial_e K)$ . On the other hand,  $L_D$  describes a class  $[L_D] \in \pi_1(K, x_0)$ . It follows that  $d(0, [L_D] \cdot 0) \leq \text{length}(L_D)$ . Also, as  $c$  is

minimal from 0 to  $c(r_D)$ , we have  $r_D \leq d(0, [L_D] \cdot 0) + \text{diam}(\partial_e K)$ . Thus

$$d(0, [L_D] \cdot 0) \leq \text{length}(L_D) \leq r_D + \text{diam}(\partial_e K) \leq d(0, [L_D] \cdot 0) + 2\text{diam}(\partial_e K). \tag{3.14}$$

In terms of the homotopy sequence

$$\pi_1(K, x_0) \xrightarrow{\alpha} \pi_1(K, \partial K) \xrightarrow{\beta} \pi_0(\partial K), \tag{3.15}$$

we have defined a map  $s : \beta^{-1}(\partial_e K) \rightarrow \pi_1(K, x_0)$  which sends  $D$  to  $[L_D]$ , with  $\alpha \circ s = \text{Id}$  on  $\beta^{-1}(\partial_e K)$ . Thus  $s$  is injective. By the construction of  $f$ , there is a bound

$$|f(D)| \leq A \text{length}(L_D) + B \leq A d(0, [L_D] \cdot 0) + B' \tag{3.16}$$

for  $D \in \beta^{-1}(\partial_e K)$ . Then

$$\sum_{D \in \beta^{-1}(\partial_e K)} |f(D)|^p \text{vol}(D) \leq \sum_{D \in \beta^{-1}(\partial_e K)} (A d(0, [L_D] \cdot 0) + B')^p. \tag{3.17}$$

$$C e^{-(n-1)(d(0, [L_D] \cdot 0) - \text{diam}(\partial_e K))}.$$

By [14], there is an  $\epsilon > 0$  such that

$$\sum_{\gamma \in \Gamma} e^{-(n-1-\epsilon)d(0, \gamma \cdot 0)} < \infty. \tag{3.18}$$

It follows that  $f$  is  $L^p$  on  $\bigcup\{D \in \beta^{-1}(\partial_e K)\}$ . Considering together the finite number of ends of  $H^n/\Gamma$ , the proposition follows.  $\square$

**Lemma 4.** For  $f \in L^2(S^{n-1})$ , let  $\Phi_0 f \in C^\infty(H^n)$  be its harmonic extension. For  $1 \leq j \leq n$ , let  $x_j$  be the restriction to  $S^{n-1}$  of the  $j$ -th coordinate function on  $\mathbb{R}^n$ . Then

$$|\nabla(\Phi_0 f)|^2(0) = (n-1)^2 \sum_{j=1}^n \left| \frac{\int_{S^{n-1}} x_j f d\text{vol}}{\text{vol}(S^{n-1})} \right|^2. \tag{3.19}$$

*Proof.* Let  $\{\beta_i\}_{i=1}^\infty$  be an orthonormal basis of  $L^2(S^{n-1})$  consisting of eigenvectors of  $\Delta_{S^{n-1}}$  with eigenvalue  $(k_i+1)(k_i+n-1)$ ,  $k_i \in \mathbb{Z} \cap [-1, \infty)$ . Let  $f = \sum_{i=1}^\infty a_i \beta_i$  be the Fourier expansion of  $f$ . Then from [6, p. 599],

$$(\Phi_0 f)(r, \theta) = \frac{\Gamma(\frac{n}{2})}{\Gamma(n-1)} \sum_{i=1}^\infty a_i \frac{\Gamma(n+k_i)}{\Gamma(\frac{n}{2}+k_i+1)} r^{1+k_i} F(1-\frac{n}{2}, 1+k_i; 1+\frac{n}{2}+k_i; r^2) \beta_i(\theta). \tag{3.20}$$

It follows that

$$|\nabla(\Phi_0 f)|^2(0) = \frac{(n-1)^2}{n^2} \sum_{k_i=0} |a_i|^2 \left( |\beta_i|^2 + |\nabla_{S^{n-1}} \beta_i|^2 \right). \tag{3.21}$$

We can take the  $\beta_i$ 's with  $k_i = 0$  to be the functions  $\left\{ \left( \frac{n}{\text{vol}(S^{n-1})} \right)^{\frac{1}{2}} x_j \right\}_{j=1}^n$ . In this case, one can verify that  $|\beta_i|^2 + |\nabla_{S^{n-1}}\beta_i|^2$  is constant on  $S^{n-1}$ . Its integral is

$$\int_{S^{n-1}} \left( |\beta_i|^2 + |\nabla_{S^{n-1}}\beta_i|^2 \right) d\text{vol} = \langle \beta_i, \beta_i \rangle + \langle \beta_i, \Delta_{S^{n-1}}\beta_i \rangle = 1 + (n-1) = n. \tag{3.22}$$

Hence

$$|\beta_i|^2 + |\nabla_{S^{n-1}}\beta_i|^2 = \frac{n}{\text{vol}(S^{n-1})} \tag{3.23}$$

and so

$$\begin{aligned} |\nabla(\Phi_0 f)|^2(0) &= \frac{(n-1)^2}{n \text{vol}(S^{n-1})} \sum_{k_i=0} |a_i|^2 \tag{3.24} \\ &= \frac{(n-1)^2}{n \text{vol}(S^{n-1})} \sum_{j=1}^n \left| \int_{S^{n-1}} \left( \frac{n}{\text{vol}(S^{n-1})} \right)^{\frac{1}{2}} x_j f d\text{vol} \right|^2 \\ &= (n-1)^2 \sum_{j=1}^n \left| \frac{\int_{S^{n-1}} x_j f d\text{vol}}{\text{vol}(S^{n-1})} \right|^2. \end{aligned}$$

The lemma follows. □

**Proposition 7.**  *$d(\Phi_0 f)$  is a  $\Gamma$ -invariant harmonic 1-form on  $H^n$ . It descends to an  $L^2$ -harmonic 1-form on  $H^n/\Gamma$ .*

*Proof.* As  $f$  is  $L^2$ ,  $\Phi_0 f$  is well-defined. As  $\Phi_0 f$  is harmonic,  $\Delta_1 d(\Phi_0 f) = d(\Delta_0 \Phi_0 f) = 0$ . Thus  $d(\Phi_0 f)$  is harmonic. Furthermore, for all  $\gamma \in \Gamma$ ,

$$d(\Phi_0 f) - \gamma \cdot d(\Phi_0 f) = d(\Phi_0(f - \gamma \cdot f)) = d(\Phi_0 c_\gamma) = dc_\gamma = 0. \tag{3.25}$$

Thus  $d(\Phi_0 f)$  is  $\Gamma$ -invariant. It remains to show that the descent of  $d(\Phi_0 f)$  to  $H^n/\Gamma$  is  $L^2$ .

Let  $m$  be a point in the connected component of  $H^n/\Gamma - K$  corresponding to an end  $e$ . Take a ball model  $B^n$  of  $H^n$  with  $\pi(0) = m$ . Let  $D$  be the connected component of  $\Omega$  adjacent, in  $\overline{B^n}$ , to the connected component of  $H^n - \tilde{K}$  containing 0. Changing  $f$  by a constant, we may assume that  $f$  vanishes on  $D$ . The method of proof of Proposition 6 implies that the  $L^1$ -norm of  $f$ , as seen in the visual sphere at  $m$ , is  $O(e^{-(n-1)d(m,K)})$  with respect to  $m$ . Then by Lemma 4,

$$|\nabla(\Phi_0 f)|^2(0) = O(e^{-2(n-1)d(m,K)}). \tag{3.26}$$

On the other hand, the volume of  $\{m \in H^n/\Gamma : d(m, K) \in [j, j+1]\}$  is  $O(e^{(n-1)j})$ . The proposition follows. □

Thus we have constructed  $\dim(\mathbb{H}_c^1(H^n/\Gamma; \mathbb{C}))$  linearly-independent  $L^2$ -harmonic 1-forms on  $H^n/\Gamma$ .

**Theorem 4.B.** *Let  $\Gamma$  be a convex-cocompact group which is not cocompact. Let  $\alpha$  be a nonzero  $L^2$ -harmonic 1-form on  $H^n/\Gamma$ . Then  $\Phi_1^{-1}(\pi^*\alpha)$  is Sobolev  $H^{-1}$ -regular.*

*Proof.* We know that  $\pi^*\alpha = d(\Phi_0 f)$  for some  $f \in L^2(S^{n-1})$  constructed as in Lemma 3. Then  $\pi^*\alpha = \Phi_1(df)$ , with  $df$  being Sobolev  $H^{-1}$ -regular.  $\square$

#### 4. Surfaces

**Theorem 5.A.** *Let  $\Gamma$  be a torsion-free uniform lattice in  $\text{Isom}^+(H^2)$ , with  $H^2/\Gamma$  a closed surface of genus  $g$ . Then*

1.  $\dim(\mathcal{A}'_0(S^1))^\Gamma = 2g$ .
2.  $\dim(\mathcal{D}'_0(S^1))^\Gamma = 2g$ .
3.  $\dim(\mathcal{D}Z(S^1)/\mathbb{C})^\Gamma = 2g$ .
4.  $\dim(L^2(S^1)/\mathbb{C})^\Gamma = 0$ .

*Proof.* The proof is similar to the proof of Theorem 2.A. If  $F \in (\mathcal{A}'_0(S^1))^\Gamma$  then  $dF$  is a  $\Gamma$ -invariant exact hyperform on  $S^1$  and  $\Phi_1(dF)$  is a  $\Gamma$ -invariant closed and coclosed 1-form on  $H^2$ . Thus  $\Phi_1(dF) = \pi^*\alpha$  for a harmonic 1-form on  $H^2/\Gamma$ . In terms of the complex coordinate  $z$  on  $D^2$ , we can write  $\Phi_1(dF) = h_1(z)dz + h_2(\bar{z})d\bar{z}$  where  $h_1(z)$  and  $h_2(z)$  are holomorphic functions. Let  $k_1(z)$  and  $k_2(z)$  satisfy  $h_i(z) = k'_i(z)$  for  $i \in \{1, 2\}$ . Then

$$d(\Phi_0 F) = \Phi_1(dF) = d(k'_1(z) + k'_2(\bar{z})), \tag{4.1}$$

so  $\Phi_0 F = k_1(z) + k_2(\bar{z}) + \text{const}$ . As  $\alpha$  is bounded,  $\Phi_1(dF)$  is uniformly bounded on  $H^2$  and so

$$\sup_{z \in D^2} (1 - |z|^2) |k''_i(z)| < \infty. \tag{4.2}$$

That is,  $k'_i$  is an element of the Bloch space and so  $k_i$  has a boundary value in the Zygmund functions  $\mathcal{Z}$  [7, p. 282,442]. Thus  $F(\theta) = k'_1(e^{i\theta}) + k'_2(e^{-i\theta}) + \text{const.}$ , showing that  $F$  has the required regularity.

Part (4) follows from the fact that  $\Gamma$  acts ergodically on  $S^1$ .  $\square$

**Theorem 5.B.** *Let  $\Gamma$  be a torsion-free nonuniform lattice in  $\text{Isom}^+(H^2)$ , with  $H^2/\Gamma$  the complement of  $k$  points in a closed surface  $S$  of genus  $g$ . Then*

1.  $\dim(\mathcal{A}'_0(S^1))^\Gamma = \infty$ .

2.  $\dim(\mathcal{D}'_0(S^1))^\Gamma = \max(2g, 2g + 2k - 2)$ .
3.  $\dim\left(H^{-\frac{1}{2}}(S^1)/\mathbb{C}\right)^\Gamma = 2g$ .
4.  $\dim(\mathcal{D}Z(S^1)/\mathbb{C})^\Gamma = 2g$ .
5.  $\dim(L^2(S^1)/\mathbb{C})^\Gamma = 0$ . □

*Proof.* Sending  $f \in (\mathcal{A}'_0(S^1))^\Gamma$  to  $\Phi_1(df)$ , we see that  $(\mathcal{A}'_0(S^1))^\Gamma$  is isomorphic to the space of closed and coclosed 1-forms on  $H^2/\Gamma$ . Let  $p$  be a puncture point in  $S$  and let  $\mathbb{Z}$  be the subgroup of  $\Gamma$  generated by a loop around  $p$ . Then the cusp of  $H^2/\Gamma$  corresponding to  $p$  embeds in  $H^2/\mathbb{Z}$ . We model the latter by the upper-half-plane quotiented by  $z \rightarrow z + 1$ . Consider the pullback of  $\Phi_1(df)$  under the quotient map  $H^2/\mathbb{Z} \rightarrow H^2/\Gamma$ . As in [8], such a 1-form on  $H^2/\mathbb{Z}$  can be written as  $h_1(z)dz + h_2(\bar{z})d\bar{z}$ , where  $h_i(z) = h_i(z + 1)$ . Each  $h_i$  has a Fourier expansion

$$h_i(z) = \sum_{j \in \mathbb{Z}} c_{i,j} e^{2\pi\sqrt{-1}jz}. \tag{4.3}$$

If  $c_{1,j} = 0$  for  $j < -J$  then a change of variable  $w = e^{2\pi\sqrt{-1}z}$  gives

$$h_1(z)dz = \sum_{j \geq -J} c_{1,j} w^{j-1} \frac{dw}{2\pi\sqrt{-1}}, \tag{4.4}$$

and similarly for  $h_2(\bar{z})d\bar{z}$ .

To each puncture point  $p_l \in S$ ,  $1 \leq l \leq k$ , assign an integer  $J_l$  and let  $i\left(-\sum_{l=1}^k (J_l + 1)p_l\right)$  denote the space of holomorphic differentials on  $S$  whose Laurent expansion around each  $p_l$  has the form of the right-hand-side of (4.4) with  $J = J_l$ . By the Riemann-Roch theorem,  $i(D) \geq g - 1 + \sum_{l=1}^k (J_l + 1)$ . Taking the numbers  $\{J_l\}_{l=1}^k$  large, part (1) follows.

Part (2) was proven in [8]. For completeness, we repeat the argument. On the upper-half-plane,  $|h_1(z)dz| = |h_1(x + iy)|y$ . As  $d(i, iy) = |\ln(y)|$ , if  $h_1(z)dz$  has slow growth as  $y \rightarrow \infty$  then we must have  $c_{1,j} = 0$  for  $j < 0$ . The space of such holomorphic differentials on  $S$  has dimension  $i\left(-\sum_{l=1}^k p_l\right)$ . The Riemann-Roch theorem implies that  $i\left(-\sum_{l=1}^k p_l\right) = \max(g + k, g + k - 1)$ . Part (2) follows.

Suppose that  $f \in \left(H^{-\frac{1}{2}}(S^1)/\mathbb{C}\right)^\Gamma$ . Then  $df$  is  $H^{-\frac{3}{2}}$ -regular. Considering  $\Phi_1(df)$ , we know that on a cusp,  $h_1(z)$  has an expansion (4.3) with  $c_{1,j} = 0$  for  $j < 0$ . If  $c_{1,0} \neq 0$  then as  $y \rightarrow \infty$ ,  $h_1(z)dz \sim c_{1,0}dz$ . To analyze the singularity at a cusp point on  $S^1$ , we consider the 1-form  $c_{1,0}dz$  on the upper-half-plane and perform the reflection  $z \rightarrow \frac{z}{|z|^2}$ . On the boundary of the upper-half-plane, this restricts to  $x \rightarrow \frac{1}{x}$  and so  $c_{1,0}dx \rightarrow -c_{1,0}\frac{dx}{x^2}$ . The point  $i\infty$  gets mapped

to 0 and so it is enough to look at the singularity of  $-c_{1,0} \frac{dx}{x^2}$  near  $x = 0$ . The Fourier transform of  $\frac{1}{x^2}$  is proportionate to  $|k|$ . Hence  $\frac{1}{x^2}$  lies in  $H^s$  if and only if  $\int_{\mathbb{R}} (1+k^2)^s |k|^2 dk < \infty$ , i.e. if  $s < -\frac{3}{2}$ . This contradicts the assumption that  $df$  is  $H^{-\frac{3}{2}}$ -regular. Thus  $c_{1,0} = 0$ . Then  $\Phi_1(df)$  is bounded and as in the proof of Theorem 5.A,  $f \in (\mathcal{DZ}(S^1)/\mathbb{C})^\Gamma$ . Furthermore,  $h_1(z)dz$  extends smoothly over the puncture points to give a holomorphic differential on  $S$ . We conclude that both  $(H^{-\frac{1}{2}}(S^1)/\mathbb{C})^\Gamma$  and  $(\mathcal{DZ}(S^1)/\mathbb{C})^\Gamma$  are isomorphic to two copies of the space of holomorphic differentials on  $S$ , the dimension of which is  $g$ . Parts (3) and (4) follow.

Finally, part (5) follows from the ergodicity of the  $\Gamma$ -action on  $S^1$ . □

### 5. 3-Manifolds

#### 5.1. Quasi-Fuchsian groups

Let  $S$  be a closed oriented surface of genus  $g > 1$ . Let  $\Gamma$  be a quasi-Fuchsian subgroup of  $\text{Isom}^+(H^3)$  which is isomorphic to  $\pi_1(S)$ . Then  $H^3/\Gamma$  is diffeomorphic to  $\mathbb{R} \times S$  and  $H_c^1(H^3/\Gamma; \mathbb{C}) = \mathbb{C}$ . (In terms of the projection  $p : \mathbb{R} \times S \rightarrow \mathbb{R}$ , a proper map, one has  $H_c^1(H^3/\Gamma; \mathbb{C}) = p^*(H_c^1(\mathbb{R}; \mathbb{C}))$ ). Thus there is a nonzero  $L^2$ -harmonic 1-form  $\alpha$  on  $H^3/\Gamma$ .

By Corollary 2 and Proposition 3,  $\Phi_1^{-1}(\pi^*\alpha)$  is a  $\Gamma$ -invariant exact 1-current supported on the limit set  $\Lambda \subset S^2$ . The domain of discontinuity  $\Omega \subset S^2$  is the union of two 2-disks  $D_+$  and  $D_-$ , with  $D_+/\Gamma$  and  $D_-/\Gamma$  homeomorphic to  $S$ . Let  $\chi_{D_+} \in L^2(S^2)$  be the characteristic function of  $D_+$ . By Proposition 7,  $\Phi_1^{-1}(\pi^*\alpha)$  is proportionate to the exact 1-current  $d\chi_{D_+}$  on  $S^2$ .

In order to write  $d\chi_{D_+}$  more directly on  $\Lambda$ , we follow the general scheme of [5, Section IV.3.γ]. Let  $Z : D^2 \rightarrow D_+$  be a uniformization of  $D_+$ . By Carathéodory's theorem,  $Z$  extends to a continuous homeomorphism  $\bar{Z} : \bar{D}^2 \rightarrow \bar{D}_+$ . The restriction of  $\bar{Z}$  to  $\partial\bar{D}^2$  gives a homeomorphism  $\partial\bar{Z} : S^1 \rightarrow \Lambda$ .

From a general construction [5, Theorem 2, p. 208], the 1-current  $d\chi_{D_+}$  defines a cyclic 1-cocycle  $\tau$  on the algebra  $C^1(S^2)$  by

$$\tau(F^0, F^1) = \int_{S^2} d\chi_{D_+} \wedge F^0 dF^1. \tag{5.1}$$

**Lemma 1.** *The function space  $H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$  is a Banach algebra with the norm*

$$\|f\| = \left( \int_{\mathbb{R}_+} \int_{S^1} \frac{|f(\theta+h) - f(\theta)|^2}{h^2} d\theta dh \right)^{\frac{1}{2}} + \|f\|_\infty. \tag{5.2}$$

Given  $f^0, f^1 \in H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$ , let

$$f^i(\theta) = \sum_{j \in \mathbb{Z}} c_j^i e^{\sqrt{-1}j\theta} \tag{5.3}$$

be the Fourier expansion. Define a bilinear function

$$\bar{\tau} : \left( H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1) \right) \times \left( H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1) \right) \rightarrow \mathbb{C} \tag{5.4}$$

by

$$\bar{\tau}(f^0, f^1) = -2\pi i \sum_{j \in \mathbb{Z}} j c_j^0 c_{-j}^1. \tag{5.5}$$

Then  $\bar{\tau}$  is a continuous cyclic 1-cocycle on  $H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$ .

*Proof.* It is straightforward to check that  $H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$  is a Banach algebra with the given norm. It is also easy to check that  $\bar{\tau}$  is continuous. If  $f^0, f^1 \in C^\infty(S^1)$  then

$$\bar{\tau}(f^0, f^1) = \int_{S^1} f^0 df^1. \tag{5.6}$$

As in [5, p. 182], put

$$(b\bar{\tau})(f^0, f^1, f^2) = \bar{\tau}(f^0 f^1, f^2) - \bar{\tau}(f^0, f^1 f^2) + \bar{\tau}(f^2 f^0, f^1). \tag{5.7}$$

If  $f^0, f^1, f^2 \in C^\infty(S^1)$  then  $(b\bar{\tau})(f^0, f^1, f^2) = 0$ . As  $C^\infty(S^1)$  is dense in  $H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$  and  $b\bar{\tau}$  is continuous in its arguments, it follows that  $b\bar{\tau} = 0$ .  $\square$

**Theorem 6.** Given  $F^0, F^1 \in C^1(S^2)$ , put  $f^i = (\partial\bar{Z})^* F^i$ ,  $i \in \{1, 2\}$ . Then  $f^i \in H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$  and

$$\tau(F^0, F^1) = -\bar{\tau}(f^0, f^1). \tag{5.8}$$

*Proof.* Consider  $S^2$  as  $\mathbb{C} \cup \infty$  with  $\infty \in D_-$ . For  $r \in (0, 1)$ , let  $i_r : S^1 \rightarrow D^2$  be the embedding of  $S^1$  as the circle of radius  $r$  around  $0 \in D^2$ . Thinking of  $Z$  as a map from  $D^2$  to  $\mathbb{C}$ , let

$$Z(z) = \sum_{k=0}^{\infty} c_k z^k \tag{5.9}$$

be its Taylor's series. Then

$$\frac{i}{2} \int_{B_r(0)} dZ \wedge dZ^* = \frac{i}{2} \int_{S^1} i_r^* Z d(i_r^* Z^*) = \pi \sum_{k=0}^{\infty} k r^{2k} |c_k|^2. \tag{5.10}$$

As  $Z$  is univalent,

$$\frac{i}{2} \int_{D^2} dZ \wedge dZ^* = \text{area}(Z(D^2)) < \infty. \tag{5.11}$$

It follows that

$$\lim_{r \rightarrow 1} i_r^* Z = \partial \bar{Z} \tag{5.12}$$

in  $H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$ . Then  $f^i \in H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$ .

We have

$$\begin{aligned} \tau(F^0, F^1) &= \int_{S^2} d\chi_{D^+} \wedge F^0 dF^1 \\ &= - \int_{S^2} \chi_{D^+} dF^0 \wedge dF^1 \\ &= - \int_{D^+} dF^0 \wedge dF^1 \\ &= - \int_{D^2} d(Z^* F^0) \wedge d(Z^* F^1). \end{aligned} \tag{5.13}$$

Then

$$\begin{aligned} \tau(F^0, F^1) &= \lim_{r \rightarrow 1} - \int_{B_r(0)} d(Z^* F^0) \wedge d(Z^* F^1) \\ &= \lim_{r \rightarrow 1} - \int_{S^1} i_r^* Z^* F^0 \wedge d(i_r^* Z^* F^1) \\ &= \lim_{r \rightarrow 1} -\bar{\tau}(i_r^* Z^* F^0, i_r^* Z^* F^1). \end{aligned} \tag{5.14}$$

From (5.12),

$$\lim_{r \rightarrow 1} i_r^* Z^* F^i = f^i \tag{5.15}$$

in  $H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$ . The theorem follows. □

**Example.** Let  $\Sigma$  be a closed oriented surface of genus  $g > 2$ , let  $\phi \in \text{Diff}(\Sigma)$  be an orientation-preserving pseudo-Anosov diffeomorphism and let  $M$  be the mapping torus of  $\phi$ . Then  $M$  is a 3-manifold which fibers over the circle and admits a hyperbolic structure [16, 13]. Let  $\widehat{M} = H^3/\Gamma$  be the corresponding cyclic cover of  $M$ , with the pullback hyperbolic metric. The group  $\Gamma$  is isomorphic to  $\pi_1(\Sigma)$ . From [10, Proposition 9],  $\widehat{M}$  has no nonzero  $L^2$ -harmonic 1-forms. This contrasts with the quasi-Fuchsian case.

**5.2. Covering spaces**

If  $M$  is a closed 3-manifold then  $M$  has nontrivial  $L^2$ -harmonic 1-forms if and only if  $b_1(M) > 0$ . There are many examples of hyperbolic manifolds 3-manifolds  $M$  with  $b_1(M) > 0$ , such as those which fiber over a circle. It is less obvious that there are infinite normal covers  $\widehat{M} = H^3/\Gamma$  of closed hyperbolic 3-manifolds such that  $\widehat{M}$  has nonzero  $L^2$ -harmonic 1-forms. We give some examples. The limit sets will be all of  $S^2$ .

Let  $M$  be a closed oriented hyperbolic 3-manifold with a surjective homomorphism  $\alpha : \pi_1(M) \rightarrow F_r$  onto a free group with  $r > 1$  generators. Let  $\widehat{M} = H^3/\Gamma$  be the corresponding cover with  $\Gamma \cong \text{Ker}(\alpha)$ . The space of ends of  $\widehat{M}$  is a Cantor set. As  $F_r$  is nonamenable, Proposition 5 applies to show that  $\widehat{M}$  has an infinite-dimensional space of  $L^2$ -harmonic 1-forms. Thus for all  $\epsilon > 0$ ,  $(H^{-\epsilon}(S^2)/\mathbb{C})^\Gamma$  is infinite-dimensional.

For another example, let  $\Sigma$  be a closed oriented surface of genus  $g > 2$ . Let  $\rho$  be a nonzero element of  $H^1(\Sigma; \mathbb{Z}) = \mathbb{Z}^{2g}$ . Let  $\widehat{\Sigma}$  be the cyclic cover of  $\Sigma$  coming from the homomorphism  $\pi_1(\Sigma) \rightarrow H_1(\Sigma; \mathbb{Z}) \xrightarrow{\rho} \mathbb{Z}$ . It is an infinite-genus surface.

Let  $\phi$  be an orientation-preserving pseudo-Anosov diffeomorphism of  $\Sigma$  which acts trivially on  $H^1(\Sigma; \mathbb{Z})$ ; it is a surprising fact that such diffeomorphisms exist [17]. It lifts to a diffeomorphism  $\widehat{\phi}$  of  $\widehat{\Sigma}$ . Let  $M$  be the mapping torus of  $\phi$ , with its hyperbolic metric. It follows from the Wang sequence that  $H^1(M; \mathbb{Z}) = \mathbb{Z}^{2g} \oplus \mathbb{Z}$ . Let  $\widehat{M} = H^3/\Gamma$  be the cyclic covering of  $M$  coming from  $\rho \oplus 0 \in H^1(M; \mathbb{Z})$ . Equivalently,  $\widehat{M}$  is the mapping torus of  $\widehat{\phi}$ .

Given  $e^{i\theta} \in U(1)$ , let  $\rho_\theta : \mathbb{Z} \rightarrow U(1)$  be the representation  $\rho_\theta(n) = e^{in\theta}$ . Let  $E_\theta$  be the flat unitary line bundle on  $\Sigma$  coming from the representation  $\pi_1(\Sigma) \rightarrow H_1(\Sigma; \mathbb{Z}) \xrightarrow{\rho} \mathbb{Z} \xrightarrow{\rho_\theta} U(1)$ . Let  $F_\theta$  be the flat unitary line bundle on  $M$  coming from the representation  $\pi_1(M) \rightarrow H_1(M; \mathbb{Z}) \xrightarrow{\rho \oplus 0} \mathbb{Z} \xrightarrow{\rho_\theta} U(1)$ ; it is the mapping torus for the action of  $\phi$  on  $E_\theta$ . As in [10, Section 4], it follows from Fourier analysis that  $\widehat{M}$  has a nonzero  $L^2$ -harmonic 1-form if and only if  $H^1(M; F_\theta) \neq 0$  for all  $\theta$ . Furthermore, because of the  $\mathbb{Z}$ -action on  $\widehat{M}$ , if there is one nonzero  $L^2$ -harmonic 1-form then there is an infinite-dimensional space.

From the Euler characteristic identity and Poincaré duality,

$$2 - 2g = 2 \dim H^0(\Sigma; E_\theta) - \dim H^1(\Sigma; E_\theta). \tag{5.16}$$

As  $\dim H^0(\Sigma; E_\theta) \leq 1$ , it follows that

$$\dim H^1(\Sigma; E_\theta) = 2g - 2 \left( 1 - \dim H^0(\Sigma; E_\theta) \right) > 0. \tag{5.17}$$

From the Wang sequence,

$$H^1(M; F_\theta) \cong H^0(\Sigma; E_\theta) \oplus H^1(\Sigma; E_\theta) \neq 0. \tag{5.18}$$

Thus  $\widehat{M}$  has nonzero  $L^2$ -harmonic 1-forms and for all  $\epsilon > 0$ ,  $(H^{-\epsilon}(S^2)/\mathbb{C})^\Gamma$  is infinite-dimensional. The  $L^2$ -harmonic 1-forms on  $\widehat{M}$  arise from the fact that  $\text{Im}\left(H_c^1(\widehat{M}; \mathbb{C}) \rightarrow H^1(\widehat{M}; \mathbb{C})\right)$  is nonzero.

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