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# On nonpositively curved Euclidean submanifolds: splitting results 

Luis A. Florit ${ }^{1}$ and Fangyang Zheng ${ }^{2}$


#### Abstract

In this article, we prove that a $n$-dimensional, non-positively curved Euclidean submanifold with codimension $p$ and with minimal index of relative nullity $\nu=n-2 p$ is (in an open dense subset) locally the product of $p$ hypersurfaces.


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Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$ be an isometric immersion from a Riemannian manifold into a complete simply connected Riemannian manifold of constant sectional curvature $c$ (superscripts will always denote dimensions). Denote by $\nu$ the index of relative nullity of $f$,

$$
\nu(x)=\operatorname{dim}\left\{X \in T_{x} M: \alpha_{f}(X, Y)=0, \forall Y \in T_{x} M\right\},
$$

where $\alpha_{f}$ stands for the vector valued second fundamental form of $f$. It is well known that having $\nu>0$ imposes strong restrictions on the manifold $M^{n}$ and on its isometric immersion $f$. In [F1], the first author proved the inequality $\nu \geq n-2 p$ when the sectional curvature of $M^{n}$ satisfies $K_{M} \leq c$ and gave several applications of this result. First let us show that this estimate is sharp.

Example. For each $i=1, \ldots, p$, let $S_{i} \subseteq \mathbb{R}^{3}$ be a negatively curved surface. Then the product $M^{2 p}=S_{1} \times \cdots \times S_{p} \subseteq \mathbb{R}^{3 p}$ satisfies the equality $\nu=n-2 p=0$.

More generically, let $M_{i}^{n_{i}} \subseteq \mathbb{R}^{n_{i}+1}$ be nowhere flat nonpositively curved hypersurfaces, $i=1, \ldots, p$. The Gauss equation tells us that the relative nullity $\nu_{i}$ of $M_{i}^{n_{i}}$ is $\nu_{i}=n_{i}-2$. Then, the product manifold $M^{n}=M_{1}^{n_{1}} \times \cdots \times M_{p}^{n_{p}} \subseteq \mathbb{R}^{n+p}$ also have $\nu=n-2 p$.

[^0]The first author proved in [F2] a general splitting theorem for Euclidean submanifolds $f$ of nonpositive sectional curvature, under the additional assumption that the normal bundle of $f$ is flat. The main purpose of this paper is to drop that assumption in the borderline case $\nu=n-2 p$ to prove that the above example is essentially the unique one with minimal relative nullity index.

Theorem 1. Let $f: M^{n} \rightarrow \mathbb{R}^{n+p}$ be an isometric immersion into Euclidean space of a Riemannian manifold with nonpositive sectional curvature. Assume that $\nu=n-2 p$ everywhere. Then there exists an open dense subset $\mathcal{U} \subset M^{n}$ such that $\left.f\right|_{\mathcal{U}}$ splits locally as a product of $p$ Euclidean hypersurfaces, that is, for any $x \in \mathcal{U}$, there exist a neighborhood $x \in \mathcal{V} \subseteq \mathcal{U}$ and $p$ nowhere flat Euclidean hypersurfaces $f_{i}: M_{i}^{n_{i}} \rightarrow \mathbb{R}^{n_{i}+1}$ of nonpositive sectional curvature, such that

$$
\mathcal{V}=M_{1} \times \cdots \times M_{p} \quad \text { and }\left.\quad f\right|_{\mathcal{V}}=f_{1} \times \cdots \times f_{p}
$$

split.
First of all, note that when $f$ is analytic, the splitting occurs on the entire $M$. In the general case, each $n_{i}$ is constant in a connected components of $\mathcal{U}$, in fact, the universal covering space of any component of of $\mathcal{U}$ is the product of $p$ Euclidean hypersurfaces. However, there are examples in which the $n_{i}$ 's are not constant in the entire $\mathcal{U}$. Secondly, it is interesting to observe that, from Theorem 1 of $[M]$ we have that $\left.f\right|_{\mathcal{V}}$ in the above is isometrically rigid if and only if each factor is rigid.

Corollary 2. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}, 2 p \leq n$, be an isometric immersion of $a$ connected Riemannian manifold $M^{n}$ with $K_{M} \leq c$ and Ricci curvature Ric $c_{M}<c$. Then $c=0, n=2 p$ and $f$ splits locally as a product of $p$ negatively curved surfaces of $\mathbb{R}^{3}$. Moreover, the splitting is global provided that $M^{n}$ is a Hadamard manifold.

The assumption on the Ricci curvature in the above can be replaced by the weaker one $\nu=0$. Also, the Hadamard condition can probably be relaxed a bit. Combining our results and $[\mathrm{Z}]$, we can state the complex analogue of the above:

Theorem 3. Let $X^{n}$ be an immersed complex submanifold of $\mathbb{C} \mathbb{Q}_{c}^{n+p}$, the complex space form of constant holomorphic sectional curvature c. Assume that $X^{n}$ has nonpositive extrinsic sectional curvature. Then the index of relative nullity of $X^{n}$ satisfies $\nu \geq n-p$ and:
(1) when $\nu=n-p=0$, we must have $c=0$;
(2) when $c=0$ and $\nu=n-p, X^{n}$ is locally holomorphically isometric to a product

$$
\mathbb{C}^{k} \times X^{n_{1}} \times \cdots \times X^{n_{p}} \subseteq \mathbf{X}^{n+p}, \quad n=k+\sum_{i=1}^{p} n_{i}
$$

for some $0 \leq k \leq \nu$, where each $X^{n_{i}} \subseteq \mathbb{C}^{n_{i}+1}$ is a nowhere flat nonpositively curved hypersurface.

Moreover, if $X^{n}$ is complete, then its universal covering is holomorphically isometric to the product $\mathbb{C}^{\nu} \times \Sigma_{1} \times \cdots \times \Sigma_{p}$, where each $\Sigma_{i} \hookrightarrow \mathbb{C}^{2}$ is a complete immersion of the unit disc. All dimensions here are the complex ones.

Notice that the real analyticity of $X^{n}$ prevented $k$ from jumping around. The last part of Theorem 3 is because, by a theorem of Abe in [A], any complete immersed complex submanifold of $\mathbb{C}^{m}$ with one dimensional Gauss image must be a cylinder.

Remark. Any Euclidean hypersurface $g: H^{m} \rightarrow \mathbb{R}^{m+1}$ of nonpositive sectional curvature without flat points can be described locally by means of the Gauss parametrization in the following way (see [DG] for details). Take a surface $\xi$ : $V^{2} \rightarrow \mathbb{S}^{m}$ in the Euclidean unit sphere and a smooth function $\gamma$ on $V^{2}$. The map $\Psi: T_{\xi}^{\perp} V \rightarrow \mathbb{R}^{m+1}$ given by

$$
\Psi(v)=\gamma \xi+\operatorname{grad} \gamma+v
$$

parametrizes $g$ over the normal bundle of $\xi$, in the open set of normal vectors $v$ which satisfies $\operatorname{det}\left(\gamma \mathrm{Id}+\mathrm{Hess}_{\gamma}-B_{v}\right)<0$. Here, $B_{v}$ denotes the second fundamental operator of $\xi$ in the direction $v$. In this parametrization, $\xi$ is the Gauss map of $g$ and $\gamma=\langle g, \xi\rangle$ its support function. For a discussion on the isometric deformations of those hypersurfaces see [DFT]. Observe that any isometric immersion $f$ as in Theorem 1 can now be explicitly parametrized locally along $\mathcal{U}$ using the Gauss parametrization for each factor.

## The flatness of the normal bundle

Let $\alpha: V^{n} \times V^{n} \rightarrow W^{p}$ be a symmetric bilinear map, where $V$ and $W$ are real vector spaces of dimension $n$ and $p$, respectively, and $W$ is equipped with an inner product $\langle$,$\rangle . Assume \alpha$ is nonpositive as defined in [F1], i.e.,

$$
K_{\alpha}(X, Y)=\langle\alpha(X, X), \alpha(Y, Y)\rangle-\|\alpha(X, Y)\|^{2} \leq 0
$$

for all $X, Y \in V$. Denote by $\nu$ the dimension of the null space $N$ of $\alpha$ :

$$
N=\{X \in V \mid \alpha(X, Y)=0, \forall Y \in V\} .
$$

Recall that a subspace $T \subseteq V$ is said to be asymptotic, if $\alpha(X, Y)=0$ for all $X, Y \in T$. We know from [F1] that, for the above $\alpha, \nu \geq n-2 p$. The main technical part of this article is the following diagonalization result for the borderline case $\nu=n-2 p$.

Proposition 4. Let $\alpha: V^{n} \times V^{n} \rightarrow W^{p}$ be a symmetric, nonpositive bilinear map. If $\nu=n-2 p$, then there exist a basis $\left\{e_{1}, \ldots, e_{n}\right\}$ of $V$ and an orthonormal
basis $\left\{w_{1}, \ldots, w_{p}\right\}$ of $W$ such that $\left\{e_{2 p+1}, \ldots, e_{n}\right\}$ is a basis of the null space $N$, and for each $i, j \leq 2 p$,

$$
\alpha\left(e_{i}, e_{j}\right)=\delta_{i j}(-1)^{i} w_{\left[\frac{i+1}{2}\right]}
$$

Proof.We will carry out the induction on $p$. When $p=1, \alpha$ is just a symmetric bilinear form, so it can always be diagonalized. The nonpositivity condition will force the rank of $\alpha$ to be less or equal than 2 , and when it equals 2 , the two nonzero eigenvalues must be of opposite sign. Now assume that the result holds when dim $W<p$, and consider the case $\operatorname{dim} W=p$.

By restricting $\alpha$ to a subspace $\widetilde{V}^{2 p}$ such that $V=N \oplus \widetilde{V}$, we may assume that $n=2 p$ and $\nu=0$. Denote by $\alpha_{X}$ the endomorphism $\alpha_{X}(Y)=\alpha(X, Y)$. By Proposition 6 of [F1] we know that there exists an asymptotic subspace $T^{p} \subseteq \widetilde{V}^{2 p}$ of $\alpha$. Set

$$
r=\min \left\{\operatorname{rank} \alpha_{X}: 0 \neq X \in T\right\}>0
$$

Fix a vector $X \in T$ with rank $\alpha_{X}=r$ and let $V^{\prime}=\operatorname{Ker}\left(\alpha_{X}\right) \supseteq T$. Thus, by the first claim in the proof of Proposition 6 of [F1], we know that the image $\alpha\left(V^{\prime} \times V^{\prime}\right)$ is perpendicular to the image subspace $\operatorname{Im}\left(\alpha_{X}\right)$, that is, we have the restriction map

$$
\left.\alpha\right|_{V^{\prime} \times V^{\prime}}: V^{\prime} \times V^{\prime} \rightarrow \operatorname{Im}\left(\alpha_{X}\right)^{\perp}
$$

Let $N^{\prime}$ be its null space. If there is $Y \in N^{\prime} \backslash T$, then $\operatorname{span}(T \cup\{Y\})$ would be an asymptotic subspace of $\alpha$ of dimension $p+1$. By Proposition 8 of [F1], we get $\nu \geq 1$, a contradiction to our assumption. Therefore, $N^{\prime} \subseteq T$.

For each $Y \in N^{\prime} \subseteq T$, we have $\operatorname{Ker}\left(\alpha_{Y}\right) \supseteq V^{\prime}=\operatorname{Ker}\left(\alpha_{X}\right)$, so rank $\alpha_{Y}=r$. Therefore,

$$
\begin{equation*}
V^{\prime}=\operatorname{Ker}\left(\alpha_{Y}\right), \quad \forall 0 \neq Y \in N^{\prime} \tag{1}
\end{equation*}
$$

Put $W_{0}=\operatorname{span}\left\{\operatorname{Im}\left(\alpha_{Y}\right): Y \in N^{\prime}\right\}$ which has dimension $r+s$, for some $s \geq$ 0. Again from the proof of Proposition 6 of [F1], we know that $\alpha\left(V^{\prime} \times V^{\prime}\right)$ is perpendicular to $W_{0}$, that is,

$$
\beta=\left.\alpha\right|_{V^{\prime} \times V^{\prime}}: V^{\prime} \times V^{\prime} \rightarrow W_{0}^{\perp}
$$

is itself a symmetric, nonpositive bilinear map, with $\operatorname{dim} V^{\prime}=2 p-r, \operatorname{dim} W_{0}^{\perp}=$ $p-r-s$. Write $q=\operatorname{dim} N^{\prime}$. Then by Proposition 9 of [F1] we have

$$
\begin{equation*}
q \geq(2 p-r)-2(p-r-s)=r+2 s \tag{2}
\end{equation*}
$$

On the other hand, if $\left\{Y_{1}, \ldots, Y_{q}\right\}$ is a basis of $N^{\prime}$ and $Z \in V \backslash V^{\prime}$, from (1) we obtain that the set of vectors $\left\{\alpha\left(Y_{1}, Z\right), \cdots, \alpha\left(Y_{q}, Z\right)\right\}$ in $W_{0}$ must be linearly independent. Thus

$$
\begin{equation*}
q \leq r+s \tag{3}
\end{equation*}
$$

We conclude from (2) and (3) that $s=0$ and $q=r$. So we can apply the induction hypothesis on $\beta$. However, we want to show first that $r=1$.

Assume the contrary, that is, $q>1$. Take a subspace $V_{1}^{r}$ such that $V_{1} \oplus V^{\prime}=V$. Choose any $Y \in N^{\prime}$ not collinear with $X$. Since $s=0$, (the restriction of) both $\alpha_{X}$ and $\alpha_{Y}$ give isomorphisms between $V_{1}$ and $W_{0}^{\perp}$. Fix an orthonormal basis $\left\{w_{1}, \ldots, w_{r}\right\}$ of $W_{0}^{\perp}$. Let $\left\{v_{1}, \ldots, v_{r}\right\}$ be the basis of $V_{1}$ such that $\alpha_{X}\left(v_{i}\right)=w_{i}$ and write $\alpha_{Y}\left(v_{i}\right)=\sum_{j=1}^{r} B_{i j} w_{j}$. That is, we identify $V_{1}$ and $W_{0}^{\perp}$ through $\alpha_{X}$, and use the matrix $B$ to represent $\alpha_{Y}$.

If $B$ has a real eigenvalue $\lambda$, then $\alpha_{Y-\lambda X}$ would have rank less than $r$, which contradicts (1). So the matrix $B$ has no real eigenvalues. By considering a complex eigenvector which corresponds to a complex eigenvalue of $B$, we obtain two 2planes $P \subseteq V_{1}, Q \subseteq W_{0}^{\perp}$, such that both $\alpha_{X}$ and $\alpha_{Y}$ give isomorphisms between $P$ and $Q$.

Now let us fix an orthonormal basis $\left\{w_{1}, w_{2}\right\}$ of $Q$, and let $\left\{e_{3}, e_{4}\right\}$ be the basis of $P$ such that $\alpha_{X}\left(e_{3}\right)=w_{1}, \alpha_{X}\left(e_{4}\right)=w_{2}$. Write

$$
\alpha_{Y}\left(e_{3}\right)=a w_{1}+b w_{2}, \quad \alpha_{Y}\left(e_{4}\right)=c w_{1}+d w_{2} .
$$

Replacing $Y$ by $Y-d X$, we may assume that

$$
d=0 .
$$

We know that the $2 \times 2$ real matrix with entries $a, b, c, 0$ can not have any real eigenvalue, or equivalently,

$$
4 b c+a^{2}<0
$$

Set $\epsilon_{1}=X, \epsilon_{2}=Y$. For arbitrary real constants $x$ and $y$, let us consider the vectors $Z=x e_{1}+x y e_{2}+x e_{3}-e_{4}$ and $Z^{\prime}=y e_{2}+e_{3}$. We have

$$
Z \wedge Z^{\prime}=x y e_{1} \wedge e_{2}+x e_{1} \wedge e_{3}+y e_{2} \wedge e_{4}+e_{3} \wedge e_{4}
$$

Define the symmetric bilinear form $R$ on $\Lambda^{2} V$, the curvature of $\alpha$, as

$$
\begin{equation*}
R\left(Z_{1} \wedge Z_{2}, Z_{3} \wedge Z_{4}\right)=\left\langle\alpha\left(Z_{1}, Z_{3}\right), \alpha\left(Z_{2}, Z_{4}\right)\right\rangle-\left\langle\alpha\left(Z_{1}, Z_{4}\right), \alpha\left(Z_{2}, Z_{3}\right)\right\rangle \tag{4}
\end{equation*}
$$

Hence, the matrix of $R$ under the partial basis $\left\{e_{1} \wedge e_{2}, e_{1} \wedge e_{3}, e_{2} \wedge e_{4}, e_{3} \wedge e_{4}\right\}$ is

$$
R=\left[\begin{array}{cccc}
0 & 0 & 0 & c-b \\
0 & -1 & -b & -f \\
0 & -b & -c^{2} & -g \\
c-b & -f & -g & -h
\end{array}\right] .
$$

Therefore $-R\left(Z \wedge Z^{\prime}, Z \wedge Z^{\prime}\right)=x^{2}+c^{2} y^{2}+h+2(2 b-c) x y+2 f x+2 g y$. Thus, the nonpositivity of $\alpha$ gives us

$$
c^{2} y^{2}+2((2 b-c) x+g) y+\left(x^{2}+2 f x+h\right) \geq 0 .
$$

Hence, the discriminant with respect to $y$ must be nonpositive, that is,
$\left.0 \leq c^{2}\left(x^{2}+2 f x+h\right)-((2 b-c) x+g)\right)^{2}=\left(4 b c-4 b^{2}\right) x^{2}+2\left(c^{2} f+c g-2 b g\right) x+\left(c^{2} h-g^{2}\right)$.
Since $a^{2}+4 b c<0$, the leading coefficient is negative, which is a contradiction for $x$ sufficiently large. This completes the proof of the claim that $q=r=1$.

Now applying the induction hypothesis on the restriction map $\beta$, we obtain an orthonormal basis $\left\{w_{1}, \ldots, w_{p}\right\}$ of $W$ and a basis $\left\{e_{1}^{\prime}, e_{2}, e_{2}^{\prime}, \ldots, e_{p}, \epsilon_{p}^{\prime}\right\}$ of $V^{\prime}=\operatorname{Ker}\left(\alpha_{X}\right)$ such that $X=e_{1}^{\prime}, \operatorname{Im}\left(\alpha_{X}\right)=\operatorname{span}\left\{w_{1}\right\}$,

$$
\alpha\left(e_{i}, e_{j}\right)=\delta_{i j} w_{i}, \quad \alpha\left(e_{i}^{\prime}, e_{j}^{\prime}\right)=-\delta_{i j} w_{i}, \quad \alpha\left(e_{i}, e_{j}^{\prime}\right)=0, \quad \forall 2 \leq i, j \leq p
$$

and of course $\alpha\left(e_{1}^{\prime}, e_{1}^{\prime}\right)=\alpha\left(e_{1}^{\prime}, e_{i}\right)=\alpha\left(e_{1}^{\prime}, e_{i}^{\prime}\right)=0$, for all $2 \leq i \leq p$.
Choose a vector $e_{1} \in V \backslash V^{\prime}$ such that $\alpha\left(e_{1}, e_{1}^{\prime}\right)=w_{1}$. Write $\alpha=\left(A^{1}, \ldots, A^{p}\right)$, where each $A_{a b}^{k}=\left\langle\alpha\left(e_{a}, e_{b}\right), w_{k}\right\rangle$ is a symmetric $2 p \times 2 p$ matrix. Here for convenience we adopt the notations $e_{i}^{\prime}=e_{p+i}$ and $i^{\prime}=i+p$, for $i \leq p$. Under the basis $\left\{e_{a} \wedge e_{b} ; 1 \leq a<b \leq 2 p\right\}$ of $\Lambda^{2} V$, the coordinate matrix of the bilinear form $R$ becomes

$$
R_{a b, c d}=\sum_{k=1}^{p}\left(A_{a c}^{k} A_{b d}^{k}-A_{a d}^{k} A_{b c}^{k}\right)
$$

The nonpositivity of $\alpha$ simply says that $R\left(Z_{1} \wedge Z_{2}, Z_{1} \wedge Z_{2}\right) \leq 0$. For any three vectors $Z_{i}, i=1,2,3$, by considering the nonpositivity at $Z_{1} \wedge\left(Z_{2}+x Z_{3}\right)$ for arbitrary $x$, we have

$$
\begin{equation*}
R\left(Z_{1} \wedge Z_{2}, Z_{1} \wedge Z_{2}\right) \cdot R\left(Z_{1} \wedge Z_{3}, Z_{1} \wedge Z_{3}\right) \geq\left(R\left(Z_{1} \wedge Z_{2}, Z_{1} \wedge Z_{3}\right)\right)^{2} \tag{5}
\end{equation*}
$$

For all $2 \leq i \leq p$ and $2 \leq a \neq i, i^{\prime}$, from the above and $R_{i a, i a}=0$ we have $R_{1 i, i a}=-A_{1 a}^{i}=0$. That is, $A_{1 j}^{i}=A_{1 j^{\prime}}^{i}=0$, for all $2 \leq i \neq j \leq p$. Replacing $e_{1}$ by $e_{1}-\sum_{i=2}^{p}\left(A_{1 i}^{i} e_{i}-A_{1 i}^{i}{ }^{\prime} e_{i}^{\prime}\right)$, we may assume that

$$
\begin{equation*}
A_{1 j}^{i} \equiv 0, \quad \forall i, j \geq 2 \tag{6}
\end{equation*}
$$

For $2 \leq i \leq p$, set

$$
b_{i}=A_{11}^{i}, \quad a_{i}=A_{1 i}^{1}, \quad c_{i}=A_{1 i^{\prime}}^{1}
$$

Thus,

$$
\begin{gathered}
R_{11^{\prime}, 11^{\prime}}=-1 \\
R_{1 i, 1 i}=b_{i}-a_{i}^{2}, \quad R_{11^{\prime}, 1 i}=-a_{i} \\
R_{1 i^{\prime}, 1 i^{\prime}}=b_{i}-c_{i}^{2}, \quad R_{11^{\prime}, 1 i^{\prime}}=-c_{i},
\end{gathered}
$$

since $A_{11^{\prime}}^{1}=1$. From (5) and $R_{11^{\prime}, 11^{\prime}} R_{1 i, 1 i} \geq\left(R_{11^{\prime}, 1 i}\right)^{2}$ we get $b_{i} \leq 0$. Similarly, replacing $i$ by $i^{\prime}$, we have $b_{i} \geq 0$. Therefore, all $b_{i}=0$.

Now we take any nonsingular $2 \times 2$ matrix

$$
\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]
$$

such that

$$
\left[\begin{array}{ll}
a & c \\
b & d
\end{array}\right]\left[\begin{array}{cc}
A_{11}^{1} & 1 \\
1 & 0
\end{array}\right]\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]
$$

and set

$$
\tilde{e}_{1}=a e_{1}+c e_{1}^{\prime}, \quad \tilde{e}_{1}^{\prime}=b e_{1}+d e_{1}^{\prime}, \quad \tilde{e}_{i}=e_{i}-a_{i} e_{1}^{\prime}, \quad \tilde{e}_{i}^{\prime}=e_{i}^{\prime}-c_{i} e_{1}^{\prime}, \quad 2 \leq i \leq p .
$$

Then under the new basis $\left\{\tilde{e}_{a}\right\}$ of $V$, we have $\alpha\left(\tilde{e}_{a}, \tilde{e}_{b}\right)=0$, if $a \neq b, b^{\prime}$, and

$$
\alpha\left(\tilde{e}_{i}, \tilde{e}_{i}\right)=w_{i}, \quad \alpha\left(\tilde{e}_{i}^{\prime}, \tilde{e}_{i}^{\prime}\right)=-w_{i}, \quad \forall 1 \leq i \leq p .
$$

This completes the proof of Proposition 4.
Let us examine the diagonalizing frame $\left\{w_{i}\right\}$ of Proposition 4. Set

$$
\mathcal{D}=\left\{X \in V: \operatorname{rank}\left(\alpha_{X}\right) \leq 1\right\} .
$$

This set of course depends only on $\alpha$. By Proposition 4, we know that $\mathcal{D}$ is the union of $p$ subspaces of dimension $\nu+2$, denoted by $\mathcal{D}_{i}, i=1, \ldots, p$, with $\mathcal{D}_{i} \cap \mathcal{D}_{j}=N$ for all $i \neq j$. If we choose a plane $V_{i} \subseteq \mathcal{D}_{i}$ which has trivial intersection with $N$, then $V$ is the direct sum

$$
V=N \oplus V_{1} \oplus \cdots \oplus V_{p}
$$

and $\alpha\left(\mathcal{D}_{i} \times \mathcal{D}_{j}\right)=0$ if $i \neq j$, while all $\alpha\left(\mathcal{D}_{i} \times \mathcal{D}_{i}\right)$ are one dimensional and mutually perpendicular. So the orthonormal frame $\left\{w_{i}\right\}$ is uniquely determined up to permutations.

It is interesting to note that $K \leq 0$ does not implies in general that the symmetric curvature operator $R$ is negative semidefinite. However, it is easy to see using Proposition 4 that, in our case, we really have $R \leq 0$. In fact, $\left\{e_{i} \wedge e_{i+p}: 1 \leq i \leq p\right\}$ is a basis of the orthogonal complement $F$ of the nullity space of $R$ in $\Lambda^{2} V$ formed by the unique (up to scaling) decomposable elements in $F$. Indeed, $e_{i} \wedge e_{i+p}$ is eigenvector of $R$ of eigenvalue $K\left(e_{i}, e_{i+p}\right) \neq 0$.

We are now in position to give the remaining proofs.
Proofs of Theorem 1 and Corollary 2. For each $x \in M^{n}$, consider $\alpha_{f}(x)$ the vector valued second fundamental form of $f$ at $x$. Since $K_{M} \leq 0$, the Gauss equation tells us that $\alpha_{f}(x)$ is nonpositive. Thus, we apply Proposition 4 to it to obtain the
special (smooth) orthonormal frame $\left\{w_{i}, 1 \leq i \leq p\right\}$. By Theorem 1 and Corollary 2 of [F2], we only need to prove that the normal bundle of $f$ is flat. We will show indeed that this frame is normal parallel.

For each $1 \leq i \leq p$, consider the shape tensor $A_{w_{i}}$ on $M^{n}$ defined by $\left\langle A_{w_{i}} X, Y\right\rangle=$ $\left\langle\alpha_{f}(X, Y), w_{i}\right\rangle$. By Proposition 4, $V_{i}=\operatorname{Im} A_{w_{i}}$ are two dimensional distributions on $M^{n}$ such that

$$
\begin{equation*}
V_{1} \oplus \cdots \oplus V_{p}=\Delta^{\perp} \tag{7}
\end{equation*}
$$

where $\Delta$ stands for the relative nullity distribution of $f$. Let $\psi_{i j}$ be the 1 -forms defined by $\psi_{i j}(X)=\left\langle\nabla \frac{1}{X} w_{i}, w_{j}\right\rangle$. We only need to show that $\psi_{i j}=0$, for all $i, j$.

Recall that the Codazzi equation for $A_{w_{i}}$ is

$$
\begin{equation*}
\nabla_{X}\left(A_{w_{i}} Y\right)-A_{w_{i}} \nabla_{X} Y-A_{\nabla \frac{1}{X} w_{i}} Y=\nabla_{Y}\left(A_{w_{i}} X\right)-A_{w_{i}} \nabla_{Y} X-A_{\nabla_{\frac{1}{Y}} w_{i}} X . \tag{8}
\end{equation*}
$$

Taking in (8) $X, Y \in V_{i}^{\perp}=\operatorname{Ker} A_{w_{i}}$ we easily obtain using (7) that

$$
A_{w_{j}}\left(\psi_{i j}(X) Y-\psi_{i j}(Y) X\right)=0, \quad \forall X, Y \in V_{i}^{\perp}, \quad 1 \leq j \leq p
$$

Suppose that there is $X_{0} \in V_{i}^{\perp}$, and $j \neq i$ such that $\psi_{i j}\left(X_{0}\right) \neq 0$. The above equation implies that $V_{i}^{\perp} \subset V_{j}^{\perp} \oplus \operatorname{span}\left\{X_{0}\right\}$, that is,

$$
T_{x} M \neq V_{i}^{\perp}+V_{j}^{\perp}=\left(V_{i} \cap V_{j}\right)^{\perp}
$$

which is a contradiction by (7). Thus $V_{i}^{\perp} \subset \operatorname{Ker} \psi_{i j}$, for all $i, j$. By the orthonormality of $\left\{w_{i}\right\}$ we have $\psi_{i j}=-\psi_{j i}$. Therefore, $T_{x} M=V_{i}^{\perp}+V_{j}^{\perp} \subset \operatorname{Ker} \psi_{i j}$. Notice that the Ricci equations imply that the $V_{i}$ 's are orthogonal. This concludes our proof.

The proof of Theorem 3 can be obtained by combining the diagonalization theorem of [Z] (together with the similar argument of the orthogonality of the special frame) and the proof of the Theorem 1 of [F2]. So we shall omit it here.

## Final comments

i) Let us explain Theorem 1 a little bit. We have everywhere on $M^{n}$ the orthogonal decomposition $T M=N \oplus V_{1} \oplus \cdots \oplus V_{p}$ of the tangent bundle into distributions. Let $\widetilde{V}_{i}$ be the distribution spanned by all vector fields in $V_{i}$ and all $\nabla_{X_{1}} \cdots \nabla_{X_{s}} X_{s+1}$, where all $X_{j} \in V_{i}$. It is shown in $[\mathrm{F} 2]$ that $\widetilde{V}_{i} \perp \widetilde{V}_{j}$ whenever $i \neq j$, and all $\widetilde{V}_{i}$ are parallel distributions (in the neighborhood where they have constant dimensions). Let $n_{i}(x)$ be the dimension of $\widetilde{V}_{i}$ at $x$. Each $n_{i}$ is a lower semicontinuous integervalued function. If $k=n-\sum_{i=1}^{p} n_{i}$, then $0 \leq k \leq \nu$. Let $\mathcal{U}$ be the open dense subset of $M^{n}$ which is the disjoint union of open subsets $\mathcal{U}_{j}$ in which $k(x)$ takes constant value $j$. All $n_{i}$ are necessarily constant in $\mathcal{U}_{j}$, and we have the desired
local splitting on $\mathcal{U}_{j}$. Observe that, using the Gauss parametrization, it is easy to construct examples of submanifolds with the functions $n_{i}$ nonconstant. Therefore, for $\nu>0$ we can only obtain the local splitting along an open dense subset. With this is mind, the same argument as in Corollary 2 of [F2] proves the following

Theorem 5. Let $f: M^{n} \rightarrow \mathbb{Q}_{c}^{2 n-r}, 2 \leq r \leq n / 2$, be an isometric immersion with flat normal bundle of a connected Riemannian manifold with $K_{M} \leq c$ and Ric $_{M}<c$. Then $c=0$ and $f$ splits locally as a product of $r$ nonpositively curved Euclidean submanifolds, that is, $f=f_{1} \times \cdots \times f_{r}$ locally, with $f_{i}: M_{i}^{n_{i}} \rightarrow \mathbb{R}^{2 n_{i}-1}$. The splitting is global provided $M^{n}$ is a Hadamard manifold.

Again, the assumption on the Ricci curvature can be replaced by $\nu=0$.
ii) We believe that the case $\nu=n-2 p>0$ for an isometric immersion $f$ : $M^{n} \rightarrow \mathbb{Q}_{c}^{n+p}$, with $c \neq 0$, cannot occur. It would be interesting either to prove its nonexistence or to construct such an example. The complex case should be similar.
iii) Taking the curvature tensor $R$ as a 4-tensor on $M^{n}$, it is defined the nullity space of $M^{n}$ at $x$ as the subspace $\Gamma(x)=\left\{X \in T_{x} M: R(X, Y, Z, W)=\right.$ $\left.0, \forall Y, Z, W \in T_{x} M\right\}$. This is an intrinsic subspace, so its dimension $\mu(x)$ called the nullity index of $M^{n}$ is an intrinsic function. For an isometric immersion $f$ of $M^{n}$ into Euclidean space we always have that the relative nullity distribution $\Delta$ of $f$ satisfies $\Delta \subset \Gamma$. Thus, our assumption on the relative nullity distribution in Theorem 1 can be replaced by the intrinsic one $\mu=n-2 p$. The same holds for Corollary 2.
iv) Now let us consider the more general situation discussed in Theorem 1 of [F2], namely, $\nu=n-p-r$, for some $2 \leq r \leq p$. It is natural to ask if it can be generalized by dropping the flatness of the normal bundle assumption as we did for the case $r=p$. The answer to this question seems to be negative, since the algebraic decomposition Proposition 4 does not generalizes, even for the case $r=p-1$, as the following example shows. Take $A_{i}$ defined as
$A_{1}=\left[\begin{array}{ccccc}1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0\end{array}\right], A_{2}=\left[\begin{array}{ccccc}0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0\end{array}\right], A_{3}=\left[\begin{array}{lllll}0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0\end{array}\right]$.
The bilinear form $\alpha=\left(A_{1}, A_{2}, A_{3}\right): \mathbb{R}^{5} \times \mathbb{R}^{5} \rightarrow \mathbb{R}^{3}$ is nonpositive, has $\nu=$ $n-p-r=0$ for $r=p-1=2$ but is not decomposable. It is easy to generalize this example for all $p$. Thus the analogous result to Proposition 4 is false for $\nu=n-p-r$ and $2 \leq r \leq p-1$.

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