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# On nonpositively curved Euclidean submanifolds: splitting results

Luis A. Florit<sup>1</sup> and Fangyang Zheng<sup>2</sup>

**Abstract.** In this article, we prove that a n-dimensional, non-positively curved Euclidean submanifold with codimension p and with minimal index of relative nullity  $\nu=n-2p$  is (in an open dense subset) locally the product of p hypersurfaces.

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Let  $f: M^n \to \mathbb{Q}_c^{n+p}$  be an isometric immersion from a Riemannian manifold into a complete simply connected Riemannian manifold of constant sectional curvature c (superscripts will always denote dimensions). Denote by  $\nu$  the *index of relative nullity* of f,

$$\nu(x) = \dim\{X \in T_xM : \alpha_f(X, Y) = 0, \forall Y \in T_xM\},\$$

where  $\alpha_f$  stands for the vector valued second fundamental form of f. It is well known that having  $\nu>0$  imposes strong restrictions on the manifold  $M^n$  and on its isometric immersion f. In [F1], the first author proved the inequality  $\nu\geq n-2p$  when the sectional curvature of  $M^n$  satisfies  $K_M\leq c$  and gave several applications of this result. First let us show that this estimate is sharp.

**Example.** For each  $i=1,\ldots,p,$  let  $S_i\subseteq\mathbb{R}^3$  be a negatively curved surface. Then the product  $M^{2p}=S_1\times\cdots\times S_p\subseteq\mathbb{R}^{3p}$  satisfies the equality  $\nu=n-2p=0$ . More generically, let  $M_i^{n_i}\subseteq\mathbb{R}^{n_i+1}$  be nowhere flat nonpositively curved hypersurfaces,  $i=1,\ldots,p.$  The Gauss equation tells us that the relative nullity  $\nu_i$  of  $M_i^{n_i}$  is  $\nu_i=n_i-2$ . Then, the product manifold  $M^n=M_1^{n_1}\times\cdots\times M_p^{n_p}\subseteq\mathbb{R}^{n+p}$  also have  $\nu=n-2p$ .

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The first author proved in [F2] a general splitting theorem for Euclidean submanifolds f of nonpositive sectional curvature, under the additional assumption that the normal bundle of f is flat. The main purpose of this paper is to drop that assumption in the borderline case  $\nu = n - 2p$  to prove that the above example is essentially the unique one with minimal relative nullity index.

**Theorem 1.** Let  $f: M^n \to \mathbb{R}^{n+p}$  be an isometric immersion into Euclidean space of a Riemannian manifold with nonpositive sectional curvature. Assume that  $\nu = n-2p$  everywhere. Then there exists an open dense subset  $\mathcal{U} \subset M^n$  such that  $f|_{\mathcal{U}}$  splits locally as a product of p Euclidean hypersurfaces, that is, for any  $x \in \mathcal{U}$ , there exist a neighborhood  $x \in \mathcal{V} \subseteq \mathcal{U}$  and p nowhere flat Euclidean hypersurfaces  $f_i: M_i^{n_i} \to \mathbb{R}^{n_i+1}$  of nonpositive sectional curvature, such that

$$\mathcal{V} = M_1 \times \cdots \times M_p$$
 and  $f|_{\mathcal{V}} = f_1 \times \cdots \times f_p$ 

split.

First of all, note that when f is analytic, the splitting occurs on the entire M. In the general case, each  $n_i$  is constant in a connected components of  $\mathcal{U}$ , in fact, the universal covering space of any component of of  $\mathcal{U}$  is the product of p Euclidean hypersurfaces. However, there are examples in which the  $n_i$ 's are not constant in the entire  $\mathcal{U}$ . Secondly, it is interesting to observe that, from Theorem 1 of [M] we have that  $f|_{\mathcal{V}}$  in the above is isometrically rigid if and only if each factor is rigid.

Corollary 2. Let  $f: M^n \to \mathbb{Q}_c^{n+p}$ ,  $2p \le n$ , be an isometric immersion of a connected Riemannian manifold  $M^n$  with  $K_M \le c$  and Ricci curvature  $Ric_M < c$ . Then c = 0, n = 2p and f splits locally as a product of p negatively curved surfaces of  $\mathbb{R}^3$ . Moreover, the splitting is global provided that  $M^n$  is a Hadamard manifold.

The assumption on the Ricci curvature in the above can be replaced by the weaker one  $\nu=0$ . Also, the Hadamard condition can probably be relaxed a bit. Combining our results and [Z], we can state the complex analogue of the above:

**Theorem 3.** Let  $X^n$  be an immersed complex submanifold of  $\mathbb{CQ}_c^{n+p}$ , the complex space form of constant holomorphic sectional curvature c. Assume that  $X^n$  has nonpositive extrinsic sectional curvature. Then the index of relative nullity of  $X^n$  satisfies  $\nu \geq n-p$  and:

- (1) when  $\nu = n p = 0$ , we must have c = 0;
- (2) when c=0 and  $\nu=n-p$ ,  $X^n$  is locally holomorphically isometric to a product

$$\mathbb{C}^k \times X^{n_1} \times \cdots \times X^{n_p} \subseteq \mathbf{X}^{n+p}, \quad n = k + \sum_{i=1}^p n_i,$$

for some  $0 \le k \le \nu$ , where each  $X^{n_i} \subseteq \mathbb{C}^{n_i+1}$  is a nowhere flat nonpositively curved hypersurface.

Moreover, if  $X^n$  is complete, then its universal covering is holomorphically isometric to the product  $\mathbb{C}^{\nu} \times \Sigma_1 \times \cdots \times \Sigma_p$ , where each  $\Sigma_i \hookrightarrow \mathbb{C}^2$  is a complete immersion of the unit disc. All dimensions here are the complex ones.

Notice that the real analyticity of  $X^n$  prevented k from jumping around. The last part of Theorem 3 is because, by a theorem of Abe in [A], any complete immersed complex submanifold of  $\mathbb{C}^m$  with one dimensional Gauss image must be a cylinder.

**Remark.** Any Euclidean hypersurface  $g:H^m\to\mathbb{R}^{m+1}$  of nonpositive sectional curvature without flat points can be described locally by means of the Gauss parametrization in the following way (see [DG] for details). Take a surface  $\xi:V^2\to\mathbb{S}^m$  in the Euclidean unit sphere and a smooth function  $\gamma$  on  $V^2$ . The map  $\Psi:T^\perp_{\mathcal{E}}V\to\mathbb{R}^{m+1}$  given by

$$\Psi(v) = \gamma \xi + \operatorname{grad} \gamma + v$$

parametrizes g over the normal bundle of  $\xi$ , in the open set of normal vectors v which satisfies  $\det(\gamma \operatorname{Id} + \operatorname{Hess}_{\gamma} - B_v) < 0$ . Here,  $B_v$  denotes the second fundamental operator of  $\xi$  in the direction v. In this parametrization,  $\xi$  is the Gauss map of g and  $\gamma = \langle g, \xi \rangle$  its support function. For a discussion on the isometric deformations of those hypersurfaces see [DFT]. Observe that any isometric immersion f as in Theorem 1 can now be explicitly parametrized locally along  $\mathcal U$  using the Gauss parametrization for each factor.

## The flatness of the normal bundle

Let  $\alpha: V^n \times V^n \to W^p$  be a symmetric bilinear map, where V and W are real vector spaces of dimension n and p, respectively, and W is equipped with an inner product  $\langle \, , \, \rangle$ . Assume  $\alpha$  is *nonpositive* as defined in [F1], i.e.,

$$K_{\alpha}(X,Y) = \langle \alpha(X,X), \alpha(Y,Y) \rangle - \| \alpha(X,Y) \|^2 \le 0,$$

for all  $X, Y \in V$ . Denote by  $\nu$  the dimension of the null space N of  $\alpha$ :

$$N = \{X \in V \mid \alpha(X, Y) = 0, \ \forall \ Y \in V\}.$$

Recall that a subspace  $T \subseteq V$  is said to be asymptotic, if  $\alpha(X,Y) = 0$  for all  $X,Y \in T$ . We know from [F1] that, for the above  $\alpha$ ,  $\nu \geq n-2p$ . The main technical part of this article is the following diagonalization result for the borderline case  $\nu = n-2p$ .

**Proposition 4.** Let  $\alpha: V^n \times V^n \to W^p$  be a symmetric, nonpositive bilinear map. If  $\nu = n-2p$ , then there exist a basis  $\{e_1, \ldots, e_n\}$  of V and an orthonormal

basis  $\{w_1, \ldots, w_p\}$  of W such that  $\{e_{2p+1}, \ldots, e_n\}$  is a basis of the null space N, and for each  $i, j \leq 2p$ ,

$$\alpha(e_i, e_j) = \delta_{ij}(-1)^i w_{\left[\frac{i+1}{2}\right]}.$$

*Proof.*We will carry out the induction on p. When p=1,  $\alpha$  is just a symmetric bilinear form, so it can always be diagonalized. The nonpositivity condition will force the rank of  $\alpha$  to be less or equal than 2, and when it equals 2, the two nonzero eigenvalues must be of opposite sign. Now assume that the result holds when dim W < p, and consider the case dim W = p.

By restricting  $\alpha$  to a subspace  $\widetilde{V}^{2p}$  such that  $V = N \oplus \widetilde{V}$ , we may assume that n = 2p and  $\nu = 0$ . Denote by  $\alpha_X$  the endomorphism  $\alpha_X(Y) = \alpha(X,Y)$ . By Proposition 6 of [F1] we know that there exists an asymptotic subspace  $T^p \subseteq \widetilde{V}^{2p}$  of  $\alpha$ . Set

$$r = \min\{\operatorname{rank} \alpha_X : 0 \neq X \in T\} > 0.$$

Fix a vector  $X \in T$  with rank  $\alpha_X = r$  and let  $V' = Ker(\alpha_X) \supseteq T$ . Thus, by the first claim in the proof of Proposition 6 of [F1], we know that the image  $\alpha(V' \times V')$  is perpendicular to the image subspace  $Im(\alpha_X)$ , that is, we have the restriction map

$$\alpha \mid_{V' \times V'} : V' \times V' \to \operatorname{Im}(\alpha_X)^{\perp}.$$

Let N' be its null space. If there is  $Y \in N' \setminus T$ , then  $\operatorname{span}(T \cup \{Y\})$  would be an asymptotic subspace of  $\alpha$  of dimension p+1. By Proposition 8 of [F1], we get  $\nu \geq 1$ , a contradiction to our assumption. Therefore,  $N' \subseteq T$ .

For each  $Y \in N' \subseteq T$ , we have  $Ker(\alpha_Y) \supseteq V' = Ker(\alpha_X)$ , so rank  $\alpha_Y = r$ . Therefore,

$$V' = \text{Ker}(\alpha_Y), \quad \forall \ 0 \neq Y \in N'. \tag{1}$$

Put  $W_0 = \text{span}\{\text{Im}(\alpha_Y) : Y \in N'\}$  which has dimension r + s, for some  $s \ge 0$ . Again from the proof of Proposition 6 of [F1], we know that  $\alpha(V' \times V')$  is perpendicular to  $W_0$ , that is,

$$\beta = \alpha \mid_{V' \times V'} : V' \times V' \to W_0^{\perp}$$

is itself a symmetric, nonpositive bilinear map, with dim V' = 2p - r, dim  $W_0^{\perp} = p - r - s$ . Write  $q = \dim N'$ . Then by Proposition 9 of [F1] we have

$$q \ge (2p - r) - 2(p - r - s) = r + 2s. \tag{2}$$

On the other hand, if  $\{Y_1,\ldots,Y_q\}$  is a basis of N' and  $Z\in V\setminus V'$ , from (1) we obtain that the set of vectors  $\{\alpha(Y_1,Z),\cdots,\alpha(Y_q,Z)\}$  in  $W_0$  must be linearly independent. Thus

$$q \le r + s. \tag{3}$$

We conclude from (2) and (3) that s = 0 and q = r. So we can apply the induction hypothesis on  $\beta$ . However, we want to show first that r = 1.

Assume the contrary, that is, q > 1. Take a subspace  $V_1^r$  such that  $V_1 \oplus V' = V$ . Choose any  $Y \in N'$  not collinear with X. Since s = 0, (the restriction of) both  $\alpha_X$  and  $\alpha_Y$  give isomorphisms between  $V_1$  and  $W_0^{\perp}$ . Fix an orthonormal basis  $\{w_1,\ldots,w_r\}$  of  $W_0^{\perp}$ . Let  $\{v_1,\ldots,v_r\}$  be the basis of  $V_1$  such that  $\alpha_X(v_i) = w_i$  and write  $\alpha_Y(v_i) = \sum_{j=1}^r B_{ij}w_j$ . That is, we identify  $V_1$  and  $W_0^{\perp}$  through  $\alpha_X$ , and use the matrix B to represent  $\alpha_Y$ .

If B has a real eigenvalue  $\lambda$ , then  $\alpha_{Y-\lambda X}$  would have rank less than r, which contradicts (1). So the matrix B has no real eigenvalues. By considering a complex eigenvector which corresponds to a complex eigenvalue of B, we obtain two 2-planes  $P \subseteq V_1$ ,  $Q \subseteq W_0^{\perp}$ , such that both  $\alpha_X$  and  $\alpha_Y$  give isomorphisms between P and Q.

Now let us fix an orthonormal basis  $\{w_1, w_2\}$  of Q, and let  $\{e_3, e_4\}$  be the basis of P such that  $\alpha_X(e_3) = w_1$ ,  $\alpha_X(e_4) = w_2$ . Write

$$\alpha_Y(e_3) = aw_1 + bw_2, \quad \alpha_Y(e_4) = cw_1 + dw_2.$$

Replacing Y by Y - dX, we may assume that

$$d = 0$$

We know that the  $2 \times 2$  real matrix with entries a, b, c, 0 can not have any real eigenvalue, or equivalently,

$$4bc + a^2 < 0$$
.

Set  $e_1 = X$ ,  $e_2 = Y$ . For arbitrary real constants x and y, let us consider the vectors  $Z = xe_1 + xye_2 + xe_3 - e_4$  and  $Z' = ye_2 + e_3$ . We have

$$Z \wedge Z' = xye_1 \wedge e_2 + xe_1 \wedge e_3 + ye_2 \wedge e_4 + e_3 \wedge e_4.$$

Define the symmetric bilinear form R on  $\Lambda^2 V$ , the curvature of  $\alpha$ , as

$$R(Z_1 \wedge Z_2, Z_3 \wedge Z_4) = \langle \alpha(Z_1, Z_3), \alpha(Z_2, Z_4) \rangle - \langle \alpha(Z_1, Z_4), \alpha(Z_2, Z_3) \rangle. \tag{4}$$

Hence, the matrix of R under the partial basis  $\{e_1 \land e_2, e_1 \land e_3, e_2 \land e_4, e_3 \land e_4\}$  is

$$R = \begin{bmatrix} 0 & 0 & 0 & c - b \\ 0 & -1 & -b & -f \\ 0 & -b & -c^2 & -g \\ c - b & -f & -q & -h \end{bmatrix}.$$

Therefore  $-R(Z \wedge Z', Z \wedge Z') = x^2 + c^2y^2 + h + 2(2b-c)xy + 2fx + 2gy$ . Thus, the nonpositivity of  $\alpha$  gives us

$$c^{2}y^{2} + 2((2b - c)x + g)y + (x^{2} + 2fx + h) \ge 0.$$

Hence, the discriminant with respect to y must be nonpositive, that is,

$$0 \leq c^2(x^2 + 2fx + h) - ((2b - c)x + g))^2 = (4bc - 4b^2)x^2 + 2(c^2f + cg - 2bg)x + (c^2h - g^2).$$

Since  $a^2 + 4bc < 0$ , the leading coefficient is negative, which is a contradiction for x sufficiently large. This completes the proof of the claim that q=r=1.

Now applying the induction hypothesis on the restriction map  $\beta$ , we obtain an orthonormal basis  $\{w_1,\ldots,w_p\}$  of W and a basis  $\{e'_1,e_2,e'_2,\ldots,e_p,e'_p\}$  of  $V'=Ker(\alpha_X)$  such that  $X=e'_1,\ \operatorname{Im}(\alpha_X)=\operatorname{span}\{w_1\},$ 

$$\alpha(e_i, e_j) = \delta_{ij} w_i, \quad \alpha(e_i', e_j') = -\delta_{ij} w_i, \quad \alpha(e_i, e_j') = 0, \quad \forall \ 2 \le i, j \le p,$$

and of course  $\alpha(e'_1,e'_1)=\alpha(e'_1,e_i)=\alpha(e'_1,e'_i)=0$ , for all  $2\leq i\leq p$ . Choose a vector  $e_1\in V\setminus V'$  such that  $\alpha(e_1,e'_1)=w_1$ . Write  $\alpha=(A^1,\ldots,A^p)$ , where each  $A^k_{ab}=\langle \alpha(e_a,e_b),w_k\rangle$  is a symmetric  $2p\times 2p$  matrix. Here for convenience we adopt the notations  $e'_i=e_{p+i}$  and i'=i+p, for  $i\leq p$ . Under the basis  $\{e_a\wedge e_b;\ 1\leq a< b\leq 2p\}$  of  $\Lambda^2V$ , the coordinate matrix of the bilinear form R becomes

$$R_{ab,cd} = \sum_{k=1}^{p} (A_{ac}^{k} A_{bd}^{k} - A_{ad}^{k} A_{bc}^{k}).$$

The nonpositivity of  $\alpha$  simply says that  $R(Z_1 \wedge Z_2, Z_1 \wedge Z_2) \leq 0$ . For any three vectors  $Z_i$ , i = 1, 2, 3, by considering the nonpositivity at  $Z_1 \wedge (Z_2 + xZ_3)$  for arbitrary x, we have

$$R(Z_1 \wedge Z_2, Z_1 \wedge Z_2) \cdot R(Z_1 \wedge Z_3, Z_1 \wedge Z_3) \ge (R(Z_1 \wedge Z_2, Z_1 \wedge Z_3))^2.$$
 (5)

For all  $2 \leq i \leq p$  and  $2 \leq a \neq i, i'$ , from the above and  $R_{ia,ia} = 0$  we have  $R_{1i,ia} = -A^i_{1a} = 0$ . That is,  $A^i_{1j} = A^i_{1j'} = 0$ , for all  $2 \leq i \neq j \leq p$ . Replacing  $e_1$  by  $e_1 - \sum_{i=2}^p (A^i_{1i}e_i - A^i_{1i'}e_i')$ , we may assume that

$$A_{1j}^i \equiv 0, \quad \forall \ i, j \ge 2. \tag{6}$$

For  $2 \le i \le p$ , set

$$b_i = A_{11}^i, \ a_i = A_{1i}^1, \ c_i = A_{1i'}^1.$$

Thus,

$$R_{11',11'}=-1,$$
  $R_{1i,1i}=b_i-a_i^2, \quad R_{11',1i}=-a_i,$   $R_{1i',1i'}=b_i-c_i^2, \quad R_{11',1i'}=-c_i,$ 

since  $A_{11'}^1=1$ . From (5) and  $R_{11',11'}R_{1i,1i}\geq (R_{11',1i})^2$  we get  $b_i\leq 0$ . Similarly, replacing i by i', we have  $b_i\geq 0$ . Therefore, all  $b_i=0$ .

Now we take any nonsingular  $2 \times 2$  matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} A_{11}^1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and set

$$\tilde{e}_1 = ae_1 + ce'_1, \quad \tilde{e}'_1 = be_1 + de'_1, \quad \tilde{e}_i = e_i - a_i e'_1, \quad \tilde{e}'_i = e'_i - c_i e'_1, \quad 2 \le i \le p.$$

Then under the new basis  $\{\tilde{e}_a\}$  of V, we have  $\alpha(\tilde{e}_a,\tilde{e}_b)=0$ , if  $a\neq b,b',$  and

$$\alpha(\tilde{e}_i, \tilde{e}_i) = w_i, \quad \alpha(\tilde{e}'_i, \tilde{e}'_i) = -w_i, \quad \forall \ 1 \le i \le p.$$

This completes the proof of Proposition 4.

Let us examine the diagonalizing frame  $\{w_i\}$  of Proposition 4. Set

$$\mathcal{D} = \{ X \in V : \operatorname{rank}(\alpha_X) < 1 \}.$$

This set of course depends only on  $\alpha$ . By Proposition 4, we know that  $\mathcal{D}$  is the union of p subspaces of dimension  $\nu+2$ , denoted by  $\mathcal{D}_i$ ,  $i=1,\ldots,p$ , with  $\mathcal{D}_i\cap\mathcal{D}_j=N$  for all  $i\neq j$ . If we choose a plane  $V_i\subseteq\mathcal{D}_i$  which has trivial intersection with N, then V is the direct sum

$$V = N \oplus V_1 \oplus \cdots \oplus V_n$$

and  $\alpha(\mathcal{D}_i \times \mathcal{D}_j) = 0$  if  $i \neq j$ , while all  $\alpha(\mathcal{D}_i \times \mathcal{D}_i)$  are one dimensional and mutually perpendicular. So the orthonormal frame  $\{w_i\}$  is uniquely determined up to permutations.

It is interesting to note that  $K \leq 0$  does not implies in general that the symmetric curvature operator R is negative semidefinite. However, it is easy to see using Proposition 4 that, in our case, we really have  $R \leq 0$ . In fact,  $\{e_i \land e_{i+p} : 1 \leq i \leq p\}$  is a basis of the orthogonal complement F of the nullity space of R in  $\Lambda^2 V$  formed by the unique (up to scaling) decomposable elements in F. Indeed,  $e_i \land e_{i+p}$  is eigenvector of R of eigenvalue  $K(e_i, e_{i+p}) \neq 0$ .

We are now in position to give the remaining proofs.

Proofs of Theorem 1 and Corollary 2. For each  $x \in M^n$ , consider  $\alpha_f(x)$  the vector valued second fundamental form of f at x. Since  $K_M \leq 0$ , the Gauss equation tells us that  $\alpha_f(x)$  is nonpositive. Thus, we apply Proposition 4 to it to obtain the

special (smooth) orthonormal frame  $\{w_i, 1 \leq i \leq p\}$ . By Theorem 1 and Corollary 2 of [F2], we only need to prove that the normal bundle of f is flat. We will show indeed that this frame is normal parallel.

For each  $1 \leq i \leq p$ , consider the shape tensor  $A_{w_i}$  on  $M^n$  defined by  $\langle A_{w_i}X,Y\rangle = \langle \alpha_f(X,Y),w_i\rangle$ . By Proposition 4,  $V_i = \operatorname{Im} A_{w_i}$  are two dimensional distributions on  $M^n$  such that

$$V_1 \oplus \cdots \oplus V_p = \Delta^{\perp}, \tag{7}$$

where  $\Delta$  stands for the relative nullity distribution of f. Let  $\psi_{ij}$  be the 1-forms defined by  $\psi_{ij}(X) = \langle \nabla_X^{\perp} w_i, w_j \rangle$ . We only need to show that  $\psi_{ij} = 0$ , for all i, j. Recall that the Codazzi equation for  $A_{w_i}$  is

$$\nabla_X (A_{w_i} Y) - A_{w_i} \nabla_X Y - A_{\nabla_X^{\perp} w_i} Y = \nabla_Y (A_{w_i} X) - A_{w_i} \nabla_Y X - A_{\nabla_X^{\perp} w_i} X. \quad (8)$$

Taking in (8)  $X, Y \in V_i^{\perp} = \text{Ker } A_{w_i}$  we easily obtain using (7) that

$$A_{w_i}(\psi_{ij}(X)Y - \psi_{ij}(Y)X) = 0, \ \forall \ X, Y \in V_i^{\perp}, \ 1 \le j \le p.$$

Suppose that there is  $X_0 \in V_i^{\perp}$ , and  $j \neq i$  such that  $\psi_{ij}(X_0) \neq 0$ . The above equation implies that  $V_i^{\perp} \subset V_j^{\perp} \oplus \text{span } \{X_0\}$ , that is,

$$T_x M \neq V_i^{\perp} + V_j^{\perp} = (V_i \cap V_j)^{\perp},$$

which is a contradiction by (7). Thus  $V_i^{\perp} \subset \operatorname{Ker} \psi_{ij}$ , for all i, j. By the orthonormality of  $\{w_i\}$  we have  $\psi_{ij} = -\psi_{ji}$ . Therefore,  $T_x M = V_i^{\perp} + V_j^{\perp} \subset \operatorname{Ker} \psi_{ij}$ . Notice that the Ricci equations imply that the  $V_i$ 's are orthogonal. This concludes our proof.

The proof of Theorem 3 can be obtained by combining the diagonalization theorem of [Z] (together with the similar argument of the orthogonality of the special frame) and the proof of the Theorem 1 of [F2]. So we shall omit it here.

#### Final comments

i) Let us explain Theorem 1 a little bit. We have everywhere on  $M^n$  the orthogonal decomposition  $TM = N \oplus V_1 \oplus \cdots \oplus V_p$  of the tangent bundle into distributions. Let  $\widetilde{V}_i$  be the distribution spanned by all vector fields in  $V_i$  and all  $\nabla_{X_1} \cdots \nabla_{X_s} X_{s+1}$ , where all  $X_j \in V_i$ . It is shown in [F2] that  $\widetilde{V}_i \perp \widetilde{V}_j$  whenever  $i \neq j$ , and all  $\widetilde{V}_i$  are parallel distributions (in the neighborhood where they have constant dimensions). Let  $n_i(x)$  be the dimension of  $\widetilde{V}_i$  at x. Each  $n_i$  is a lower semicontinuous integer-valued function. If  $k = n - \sum_{i=1}^p n_i$ , then  $0 \leq k \leq \nu$ . Let  $\mathcal{U}$  be the open dense subset of  $M^n$  which is the disjoint union of open subsets  $\mathcal{U}_j$  in which k(x) takes constant value j. All  $n_i$  are necessarily constant in  $\mathcal{U}_j$ , and we have the desired

local splitting on  $\mathcal{U}_j$ . Observe that, using the Gauss parametrization, it is easy to construct examples of submanifolds with the functions  $n_i$  nonconstant. Therefore, for  $\nu > 0$  we can only obtain the local splitting along an open dense subset. With this is mind, the same argument as in Corollary 2 of [F2] proves the following

**Theorem 5.** Let  $f: M^n \to \mathbb{Q}_c^{2n-r}$ ,  $2 \le r \le n/2$ , be an isometric immersion with flat normal bundle of a connected Riemannian manifold with  $K_M \le c$  and  $Ric_M < c$ . Then c = 0 and f splits locally as a product of r nonpositively curved Euclidean submanifolds, that is,  $f = f_1 \times \cdots \times f_r$  locally, with  $f_i: M_i^{n_i} \to \mathbb{R}^{2n_i-1}$ . The splitting is global provided  $M^n$  is a Hadamard manifold.

Again, the assumption on the Ricci curvature can be replaced by  $\nu = 0$ .

- ii) We believe that the case  $\nu=n-2p>0$  for an isometric immersion  $f:M^n\to\mathbb{Q}^{n+p}_c$ , with  $c\neq 0$ , cannot occur. It would be interesting either to prove its nonexistence or to construct such an example. The complex case should be similar.
- iii) Taking the curvature tensor R as a 4-tensor on  $M^n$ , it is defined the nullity space of  $M^n$  at x as the subspace  $\Gamma(x) = \{X \in T_x M : R(X,Y,Z,W) = 0, \ \forall \ Y,Z,W \in T_xM\}$ . This is an intrinsic subspace, so its dimension  $\mu(x)$  called the nullity index of  $M^n$  is an intrinsic function. For an isometric immersion f of  $M^n$  into Euclidean space we always have that the relative nullity distribution  $\Delta$  of f satisfies  $\Delta \subset \Gamma$ . Thus, our assumption on the relative nullity distribution in Theorem 1 can be replaced by the intrinsic one  $\mu = n 2p$ . The same holds for Corollary 2.
- iv) Now let us consider the more general situation discussed in Theorem 1 of [F2], namely,  $\nu = n p r$ , for some  $2 \le r \le p$ . It is natural to ask if it can be generalized by dropping the flatness of the normal bundle assumption as we did for the case r = p. The answer to this question seems to be negative, since the algebraic decomposition Proposition 4 does not generalizes, even for the case r = p 1, as the following example shows. Take  $A_i$  defined as

The bilinear form  $\alpha=(A_1,A_2,A_3):\mathbb{R}^5\times\mathbb{R}^5\to\mathbb{R}^3$  is nonpositive, has  $\nu=n-p-r=0$  for r=p-1=2 but is not decomposable. It is easy to generalize this example for all p. Thus the analogous result to Proposition 4 is false for  $\nu=n-p-r$  and  $2\leq r\leq p-1$ .

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