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Autor: Florit, Luis A. / Zheng, Fangyang
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On nonpositively curved Euclidean submanifolds: splitting results

Luis A. Florit¹ and Fangyang Zheng²

Abstract. In this article, we prove that a n -dimensional, non-positively curved Euclidean submanifold with codimension p and with minimal index of relative nullity $\nu = n - 2p$ is (in an open dense subset) locally the product of p hypersurfaces.

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Let $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$ be an isometric immersion from a Riemannian manifold into a complete simply connected Riemannian manifold of constant sectional curvature c (superscripts will always denote dimensions). Denote by ν the *index of relative nullity* of f ,

$$\nu(x) = \dim\{X \in T_x M : \alpha_f(X, Y) = 0, \forall Y \in T_x M\},$$

where α_f stands for the vector valued second fundamental form of f . It is well known that having $\nu > 0$ imposes strong restrictions on the manifold M^n and on its isometric immersion f . In [F1], the first author proved the inequality $\nu \geq n - 2p$ when the sectional curvature of M^n satisfies $K_M \leq c$ and gave several applications of this result. First let us show that this estimate is sharp.

Example. For each $i = 1, \dots, p$, let $S_i \subseteq \mathbb{R}^3$ be a negatively curved surface. Then the product $M^{2p} = S_1 \times \dots \times S_p \subseteq \mathbb{R}^{3p}$ satisfies the equality $\nu = n - 2p = 0$.

More generically, let $M_i^{n_i} \subseteq \mathbb{R}^{n_i+1}$ be nowhere flat nonpositively curved hypersurfaces, $i = 1, \dots, p$. The Gauss equation tells us that the relative nullity ν_i of $M_i^{n_i}$ is $\nu_i = n_i - 2$. Then, the product manifold $M^n = M_1^{n_1} \times \dots \times M_p^{n_p} \subseteq \mathbb{R}^{n+p}$ also have $\nu = n - 2p$.

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The first author proved in [F2] a general splitting theorem for Euclidean submanifolds f of nonpositive sectional curvature, under the additional assumption that the normal bundle of f is flat. The main purpose of this paper is to drop that assumption in the borderline case $\nu = n - 2p$ to prove that the above example is essentially the unique one with minimal relative nullity index.

Theorem 1. *Let $f : M^n \rightarrow \mathbb{R}^{n+p}$ be an isometric immersion into Euclidean space of a Riemannian manifold with nonpositive sectional curvature. Assume that $\nu = n - 2p$ everywhere. Then there exists an open dense subset $\mathcal{U} \subset M^n$ such that $f|_{\mathcal{U}}$ splits locally as a product of p Euclidean hypersurfaces, that is, for any $x \in \mathcal{U}$, there exist a neighborhood $x \in \mathcal{V} \subseteq \mathcal{U}$ and p nowhere flat Euclidean hypersurfaces $f_i : M_i^{n_i} \rightarrow \mathbb{R}^{n_i+1}$ of nonpositive sectional curvature, such that*

$$\mathcal{V} = M_1 \times \cdots \times M_p \quad \text{and} \quad f|_{\mathcal{V}} = f_1 \times \cdots \times f_p$$

splits.

First of all, note that when f is analytic, the splitting occurs on the entire M . In the general case, each n_i is constant in a connected components of \mathcal{U} , in fact, the universal covering space of any component of \mathcal{U} is the product of p Euclidean hypersurfaces. However, there are examples in which the n_i 's are not constant in the entire \mathcal{U} . Secondly, it is interesting to observe that, from Theorem 1 of [M] we have that $f|_{\mathcal{V}}$ in the above is isometrically rigid if and only if each factor is rigid.

Corollary 2. *Let $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$, $2p \leq n$, be an isometric immersion of a connected Riemannian manifold M^n with $K_M \leq c$ and Ricci curvature $\text{Ric}_M < c$. Then $c = 0$, $n = 2p$ and f splits locally as a product of p negatively curved surfaces of \mathbb{R}^3 . Moreover, the splitting is global provided that M^n is a Hadamard manifold.*

The assumption on the Ricci curvature in the above can be replaced by the weaker one $\nu = 0$. Also, the Hadamard condition can probably be relaxed a bit. Combining our results and [Z], we can state the complex analogue of the above:

Theorem 3. *Let X^n be an immersed complex submanifold of $\mathbb{C}\mathbb{Q}_c^{n+p}$, the complex space form of constant holomorphic sectional curvature c . Assume that X^n has nonpositive extrinsic sectional curvature. Then the index of relative nullity of X^n satisfies $\nu \geq n - p$ and:*

- (1) *when $\nu = n - p = 0$, we must have $c = 0$;*
- (2) *when $c = 0$ and $\nu = n - p$, X^n is locally holomorphically isometric to a product*

$$\mathbb{C}^k \times X^{n_1} \times \cdots \times X^{n_p} \subseteq \mathbb{X}^{n+p}, \quad n = k + \sum_{i=1}^p n_i,$$

for some $0 \leq k \leq \nu$, where each $X^{n_i} \subseteq \mathbb{C}^{n_i+1}$ is a nowhere flat nonpositively curved hypersurface.

Moreover, if X^n is complete, then its universal covering is holomorphically isometric to the product $\mathbb{C}^\nu \times \Sigma_1 \times \cdots \times \Sigma_p$, where each $\Sigma_i \hookrightarrow \mathbb{C}^2$ is a complete immersion of the unit disc. All dimensions here are the complex ones.

Notice that the real analyticity of X^n prevented k from jumping around. The last part of Theorem 3 is because, by a theorem of Abe in [A], any complete immersed complex submanifold of \mathbb{C}^m with one dimensional Gauss image must be a cylinder.

Remark. Any Euclidean hypersurface $g : H^m \rightarrow \mathbb{R}^{m+1}$ of nonpositive sectional curvature without flat points can be described locally by means of the Gauss parametrization in the following way (see [DG] for details). Take a surface $\xi : V^2 \rightarrow \mathbb{S}^m$ in the Euclidean unit sphere and a smooth function γ on V^2 . The map $\Psi : T_\xi^\perp V \rightarrow \mathbb{R}^{m+1}$ given by

$$\Psi(v) = \gamma\xi + \text{grad } \gamma + v$$

parametrizes g over the normal bundle of ξ , in the open set of normal vectors v which satisfies $\det(\gamma\text{Id} + \text{Hess}_\gamma - B_v) < 0$. Here, B_v denotes the second fundamental operator of ξ in the direction v . In this parametrization, ξ is the Gauss map of g and $\gamma = \langle g, \xi \rangle$ its support function. For a discussion on the isometric deformations of those hypersurfaces see [DFT]. Observe that any isometric immersion f as in Theorem 1 can now be explicitly parametrized locally along \mathcal{U} using the Gauss parametrization for each factor.

The flatness of the normal bundle

Let $\alpha : V^n \times V^n \rightarrow W^p$ be a symmetric bilinear map, where V and W are real vector spaces of dimension n and p , respectively, and W is equipped with an inner product $\langle \cdot, \cdot \rangle$. Assume α is *nonpositive* as defined in [F1], i.e.,

$$K_\alpha(X, Y) = \langle \alpha(X, X), \alpha(Y, Y) \rangle - \|\alpha(X, Y)\|^2 \leq 0,$$

for all $X, Y \in V$. Denote by ν the dimension of the null space N of α :

$$N = \{X \in V \mid \alpha(X, Y) = 0, \forall Y \in V\}.$$

Recall that a subspace $T \subseteq V$ is said to be *asymptotic*, if $\alpha(X, Y) = 0$ for all $X, Y \in T$. We know from [F1] that, for the above α , $\nu \geq n - 2p$. The main technical part of this article is the following diagonalization result for the borderline case $\nu = n - 2p$.

Proposition 4. *Let $\alpha : V^n \times V^n \rightarrow W^p$ be a symmetric, nonpositive bilinear map. If $\nu = n - 2p$, then there exist a basis $\{e_1, \dots, e_n\}$ of V and an orthonormal*

basis $\{w_1, \dots, w_p\}$ of W such that $\{e_{2p+1}, \dots, e_n\}$ is a basis of the null space N , and for each $i, j \leq 2p$,

$$\alpha(e_i, e_j) = \delta_{ij}(-1)^i w_{\lfloor \frac{i+1}{2} \rfloor}.$$

Proof. We will carry out the induction on p . When $p = 1$, α is just a symmetric bilinear form, so it can always be diagonalized. The nonpositivity condition will force the rank of α to be less or equal than 2, and when it equals 2, the two nonzero eigenvalues must be of opposite sign. Now assume that the result holds when $\dim W < p$, and consider the case $\dim W = p$.

By restricting α to a subspace \tilde{V}^{2p} such that $V = N \oplus \tilde{V}$, we may assume that $n = 2p$ and $\nu = 0$. Denote by α_X the endomorphism $\alpha_X(Y) = \alpha(X, Y)$. By Proposition 6 of [F1] we know that there exists an asymptotic subspace $T^p \subseteq \tilde{V}^{2p}$ of α . Set

$$r = \min\{\text{rank } \alpha_X : 0 \neq X \in T\} > 0.$$

Fix a vector $X \in T$ with $\text{rank } \alpha_X = r$ and let $V' = \text{Ker}(\alpha_X) \supseteq T$. Thus, by the first claim in the proof of Proposition 6 of [F1], we know that the image $\alpha(V' \times V')$ is perpendicular to the image subspace $\text{Im}(\alpha_X)$, that is, we have the restriction map

$$\alpha|_{V' \times V'}: V' \times V' \rightarrow \text{Im}(\alpha_X)^\perp.$$

Let N' be its null space. If there is $Y \in N' \setminus T$, then $\text{span}(T \cup \{Y\})$ would be an asymptotic subspace of α of dimension $p + 1$. By Proposition 8 of [F1], we get $\nu \geq 1$, a contradiction to our assumption. Therefore, $N' \subseteq T$.

For each $Y \in N' \subseteq T$, we have $\text{Ker}(\alpha_Y) \supseteq V' = \text{Ker}(\alpha_X)$, so $\text{rank } \alpha_Y = r$. Therefore,

$$V' = \text{Ker}(\alpha_Y), \quad \forall 0 \neq Y \in N'. \quad (1)$$

Put $W_0 = \text{span}\{\text{Im}(\alpha_Y) : Y \in N'\}$ which has dimension $r + s$, for some $s \geq 0$. Again from the proof of Proposition 6 of [F1], we know that $\alpha(V' \times V')$ is perpendicular to W_0 , that is,

$$\beta = \alpha|_{V' \times V'}: V' \times V' \rightarrow W_0^\perp$$

is itself a symmetric, nonpositive bilinear map, with $\dim V' = 2p - r$, $\dim W_0^\perp = p - r - s$. Write $q = \dim N'$. Then by Proposition 9 of [F1] we have

$$q \geq (2p - r) - 2(p - r - s) = r + 2s. \quad (2)$$

On the other hand, if $\{Y_1, \dots, Y_q\}$ is a basis of N' and $Z \in V \setminus V'$, from (1) we obtain that the set of vectors $\{\alpha(Y_1, Z), \dots, \alpha(Y_q, Z)\}$ in W_0 must be linearly independent. Thus

$$q \leq r + s. \quad (3)$$

We conclude from (2) and (3) that $s = 0$ and $q = r$. So we can apply the induction hypothesis on β . However, we want to show first that $r = 1$.

Assume the contrary, that is, $q > 1$. Take a subspace V_1^r such that $V_1 \oplus V' = V$. Choose any $Y \in N'$ not collinear with X . Since $s = 0$, (the restriction of) both α_X and α_Y give isomorphisms between V_1 and W_0^\perp . Fix an orthonormal basis $\{w_1, \dots, w_r\}$ of W_0^\perp . Let $\{v_1, \dots, v_r\}$ be the basis of V_1 such that $\alpha_X(v_i) = w_i$ and write $\alpha_Y(v_i) = \sum_{j=1}^r B_{ij}w_j$. That is, we identify V_1 and W_0^\perp through α_X , and use the matrix B to represent α_Y .

If B has a real eigenvalue λ , then $\alpha_{Y-\lambda X}$ would have rank less than r , which contradicts (1). So the matrix B has no real eigenvalues. By considering a complex eigenvector which corresponds to a complex eigenvalue of B , we obtain two 2-planes $P \subseteq V_1$, $Q \subseteq W_0^\perp$, such that both α_X and α_Y give isomorphisms between P and Q .

Now let us fix an orthonormal basis $\{w_1, w_2\}$ of Q , and let $\{e_3, e_4\}$ be the basis of P such that $\alpha_X(e_3) = w_1$, $\alpha_X(e_4) = w_2$. Write

$$\alpha_Y(e_3) = aw_1 + bw_2, \quad \alpha_Y(e_4) = cw_1 + dw_2.$$

Replacing Y by $Y - dX$, we may assume that

$$d = 0.$$

We know that the 2×2 real matrix with entries $a, b, c, 0$ can not have any real eigenvalue, or equivalently,

$$4bc + a^2 < 0.$$

Set $e_1 = X$, $e_2 = Y$. For arbitrary real constants x and y , let us consider the vectors $Z = xe_1 + ye_2 + xe_3 - e_4$ and $Z' = ye_2 + e_3$. We have

$$Z \wedge Z' = xye_1 \wedge e_2 + xe_1 \wedge e_3 + ye_2 \wedge e_4 + e_3 \wedge e_4.$$

Define the symmetric bilinear form R on $\Lambda^2 V$, the curvature of α , as

$$R(Z_1 \wedge Z_2, Z_3 \wedge Z_4) = \langle \alpha(Z_1, Z_3), \alpha(Z_2, Z_4) \rangle - \langle \alpha(Z_1, Z_4), \alpha(Z_2, Z_3) \rangle. \quad (4)$$

Hence, the matrix of R under the partial basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$ is

$$R = \begin{bmatrix} 0 & 0 & 0 & c-b \\ 0 & -1 & -b & -f \\ 0 & -b & -c^2 & -g \\ c-b & -f & -g & -h \end{bmatrix}.$$

Therefore $-R(Z \wedge Z', Z \wedge Z') = x^2 + c^2y^2 + h + 2(2b-c)xy + 2fx + 2gy$. Thus, the nonpositivity of α gives us

$$c^2y^2 + 2((2b-c)x + g)y + (x^2 + 2fx + h) \geq 0.$$

Hence, the discriminant with respect to y must be nonpositive, that is,

$$0 \leq c^2(x^2 + 2fx + h) - ((2b - c)x + g)^2 = (4bc - 4b^2)x^2 + 2(c^2f + cg - 2bg)x + (c^2h - g^2).$$

Since $a^2 + 4bc < 0$, the leading coefficient is negative, which is a contradiction for x sufficiently large. This completes the proof of the claim that $q = r = 1$.

Now applying the induction hypothesis on the restriction map β , we obtain an orthonormal basis $\{w_1, \dots, w_p\}$ of W and a basis $\{e'_1, e_2, e'_2, \dots, e_p, e'_p\}$ of $V' = \text{Ker}(\alpha_X)$ such that $X = e'_1$, $\text{Im}(\alpha_X) = \text{span}\{w_1\}$,

$$\alpha(e_i, e_j) = \delta_{ij}w_i, \quad \alpha(e'_i, e'_j) = -\delta_{ij}w_i, \quad \alpha(e_i, e'_j) = 0, \quad \forall 2 \leq i, j \leq p,$$

and of course $\alpha(e'_1, e'_1) = \alpha(e'_1, e_i) = \alpha(e'_1, e'_i) = 0$, for all $2 \leq i \leq p$.

Choose a vector $e_1 \in V \setminus V'$ such that $\alpha(e_1, e'_1) = w_1$. Write $\alpha = (A^1, \dots, A^p)$, where each $A_{ab}^k = \langle \alpha(e_a, e_b), w_k \rangle$ is a symmetric $2p \times 2p$ matrix. Here for convenience we adopt the notations $e'_i = e_{p+i}$ and $i' = i + p$, for $i \leq p$. Under the basis $\{e_a \wedge e_b; 1 \leq a < b \leq 2p\}$ of $\Lambda^2 V$, the coordinate matrix of the bilinear form R becomes

$$R_{ab,cd} = \sum_{k=1}^p (A_{ac}^k A_{bd}^k - A_{ad}^k A_{bc}^k).$$

The nonpositivity of α simply says that $R(Z_1 \wedge Z_2, Z_1 \wedge Z_2) \leq 0$. For any three vectors Z_i , $i = 1, 2, 3$, by considering the nonpositivity at $Z_1 \wedge (Z_2 + xZ_3)$ for arbitrary x , we have

$$R(Z_1 \wedge Z_2, Z_1 \wedge Z_2) \cdot R(Z_1 \wedge Z_3, Z_1 \wedge Z_3) \geq (R(Z_1 \wedge Z_2, Z_1 \wedge Z_3))^2. \quad (5)$$

For all $2 \leq i \leq p$ and $2 \leq a \neq i, i'$, from the above and $R_{ia,ia} = 0$ we have $R_{1i,ia} = -A_{1a}^i = 0$. That is, $A_{1j}^i = A_{1j'}^i = 0$, for all $2 \leq i \neq j \leq p$. Replacing e_1 by $e_1 - \sum_{i=2}^p (A_{1i}^i e_i - A_{1i'}^i e'_i)$, we may assume that

$$A_{1j}^i \equiv 0, \quad \forall i, j \geq 2. \quad (6)$$

For $2 \leq i \leq p$, set

$$b_i = A_{11}^i, \quad a_i = A_{1i}^1, \quad c_i = A_{1i'}^1.$$

Thus,

$$\begin{aligned} R_{11',11'} &= -1, \\ R_{1i,1i} &= b_i - a_i^2, \quad R_{11',1i} = -a_i, \\ R_{1i',1i'} &= b_i - c_i^2, \quad R_{11',1i'} = -c_i, \end{aligned}$$

since $A_{11'}^1 = 1$. From (5) and $R_{11',11'} R_{1i,1i} \geq (R_{11',1i})^2$ we get $b_i \leq 0$. Similarly, replacing i by i' , we have $b_i \geq 0$. Therefore, all $b_i = 0$.

Now we take any nonsingular 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} A_{11}^1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and set

$$\tilde{e}_1 = ae_1 + ce'_1, \quad \tilde{e}'_1 = be_1 + de'_1, \quad \tilde{e}_i = e_i - a_ie'_1, \quad \tilde{e}'_i = e'_i - c_ie'_1, \quad 2 \leq i \leq p.$$

Then under the new basis $\{\tilde{e}_a\}$ of V , we have $\alpha(\tilde{e}_a, \tilde{e}_b) = 0$, if $a \neq b, b'$, and

$$\alpha(\tilde{e}_i, \tilde{e}_i) = w_i, \quad \alpha(\tilde{e}'_i, \tilde{e}'_i) = -w_i, \quad \forall 1 \leq i \leq p.$$

This completes the proof of Proposition 4. \square

Let us examine the diagonalizing frame $\{w_i\}$ of Proposition 4. Set

$$\mathcal{D} = \{X \in V : \text{rank}(\alpha_X) \leq 1\}.$$

This set of course depends only on α . By Proposition 4, we know that \mathcal{D} is the union of p subspaces of dimension $\nu + 2$, denoted by \mathcal{D}_i , $i = 1, \dots, p$, with $\mathcal{D}_i \cap \mathcal{D}_j = N$ for all $i \neq j$. If we choose a plane $V_i \subseteq \mathcal{D}_i$ which has trivial intersection with N , then V is the direct sum

$$V = N \oplus V_1 \oplus \dots \oplus V_p$$

and $\alpha(\mathcal{D}_i \times \mathcal{D}_j) = 0$ if $i \neq j$, while all $\alpha(\mathcal{D}_i \times \mathcal{D}_i)$ are one dimensional and mutually perpendicular. So the orthonormal frame $\{w_i\}$ is uniquely determined up to permutations.

It is interesting to note that $K \leq 0$ does not implies in general that the symmetric curvature operator R is negative semidefinite. However, it is easy to see using Proposition 4 that, in our case, we really have $R \leq 0$. In fact, $\{e_i \wedge e_{i+p} : 1 \leq i \leq p\}$ is a basis of the orthogonal complement F of the nullity space of R in $\Lambda^2 V$ formed by the unique (up to scaling) decomposable elements in F . Indeed, $e_i \wedge e_{i+p}$ is eigenvector of R of eigenvalue $K(e_i, e_{i+p}) \neq 0$.

We are now in position to give the remaining proofs.

Proofs of Theorem 1 and Corollary 2. For each $x \in M^n$, consider $\alpha_f(x)$ the vector valued second fundamental form of f at x . Since $K_M \leq 0$, the Gauss equation tells us that $\alpha_f(x)$ is nonpositive. Thus, we apply Proposition 4 to it to obtain the

special (smooth) orthonormal frame $\{w_i, 1 \leq i \leq p\}$. By Theorem 1 and Corollary 2 of [F2], we only need to prove that the normal bundle of f is flat. We will show indeed that this frame is normal parallel.

For each $1 \leq i \leq p$, consider the shape tensor A_{w_i} on M^n defined by $\langle A_{w_i}X, Y \rangle = \langle \alpha_f(X, Y), w_i \rangle$. By Proposition 4, $V_i = \text{Im } A_{w_i}$ are two dimensional distributions on M^n such that

$$V_1 \oplus \cdots \oplus V_p = \Delta^\perp, \quad (7)$$

where Δ stands for the relative nullity distribution of f . Let ψ_{ij} be the 1-forms defined by $\psi_{ij}(X) = \langle \nabla_X^\perp w_i, w_j \rangle$. We only need to show that $\psi_{ij} = 0$, for all i, j .

Recall that the Codazzi equation for A_{w_i} is

$$\nabla_X(A_{w_i}Y) - A_{w_i}\nabla_XY - A_{\nabla_X^\perp w_i}Y = \nabla_Y(A_{w_i}X) - A_{w_i}\nabla_YX - A_{\nabla_Y^\perp w_i}X. \quad (8)$$

Taking in (8) $X, Y \in V_i^\perp = \text{Ker } A_{w_i}$ we easily obtain using (7) that

$$A_{w_j}(\psi_{ij}(X)Y - \psi_{ij}(Y)X) = 0, \quad \forall X, Y \in V_i^\perp, \quad 1 \leq j \leq p.$$

Suppose that there is $X_0 \in V_i^\perp$, and $j \neq i$ such that $\psi_{ij}(X_0) \neq 0$. The above equation implies that $V_i^\perp \subset V_j^\perp \oplus \text{span } \{X_0\}$, that is,

$$T_x M \neq V_i^\perp + V_j^\perp = (V_i \cap V_j)^\perp,$$

which is a contradiction by (7). Thus $V_i^\perp \subset \text{Ker } \psi_{ij}$, for all i, j . By the orthonormality of $\{w_i\}$ we have $\psi_{ij} = -\psi_{ji}$. Therefore, $T_x M = V_i^\perp + V_j^\perp \subset \text{Ker } \psi_{ij}$. Notice that the Ricci equations imply that the V_i 's are orthogonal. This concludes our proof. \square

The proof of Theorem 3 can be obtained by combining the diagonalization theorem of [Z] (together with the similar argument of the orthogonality of the special frame) and the proof of the Theorem 1 of [F2]. So we shall omit it here.

Final comments

i) Let us explain Theorem 1 a little bit. We have everywhere on M^n the orthogonal decomposition $TM = N \oplus V_1 \oplus \cdots \oplus V_p$ of the tangent bundle into distributions. Let \tilde{V}_i be the distribution spanned by all vector fields in V_i and all $\nabla_{X_1} \cdots \nabla_{X_s} X_{s+1}$, where all $X_j \in V_i$. It is shown in [F2] that $\tilde{V}_i \perp \tilde{V}_j$ whenever $i \neq j$, and all \tilde{V}_i are parallel distributions (in the neighborhood where they have constant dimensions). Let $n_i(x)$ be the dimension of \tilde{V}_i at x . Each n_i is a lower semicontinuous integer-valued function. If $k = n - \sum_{i=1}^p n_i$, then $0 \leq k \leq \nu$. Let \mathcal{U} be the open dense subset of M^n which is the disjoint union of open subsets \mathcal{U}_j in which $k(x)$ takes constant value j . All n_i are necessarily constant in \mathcal{U}_j , and we have the desired

local splitting on \mathcal{U}_j . Observe that, using the Gauss parametrization, it is easy to construct examples of submanifolds with the functions n_i nonconstant. Therefore, for $\nu > 0$ we can only obtain the local splitting along an open dense subset. With this in mind, the same argument as in Corollary 2 of [F2] proves the following

Theorem 5. *Let $f : M^n \rightarrow \mathbb{Q}_c^{2n-r}$, $2 \leq r \leq n/2$, be an isometric immersion with flat normal bundle of a connected Riemannian manifold with $K_M \leq c$ and $\text{Ric}_M < c$. Then $c = 0$ and f splits locally as a product of r nonpositively curved Euclidean submanifolds, that is, $f = f_1 \times \cdots \times f_r$ locally, with $f_i : M_i^{n_i} \rightarrow \mathbb{R}^{2n_i-1}$. The splitting is global provided M^n is a Hadamard manifold.*

Again, the assumption on the Ricci curvature can be replaced by $\nu = 0$.

ii) We believe that the case $\nu = n - 2p > 0$ for an isometric immersion $f : M^n \rightarrow \mathbb{Q}_c^{n+p}$, with $c \neq 0$, cannot occur. It would be interesting either to prove its nonexistence or to construct such an example. The complex case should be similar.

iii) Taking the curvature tensor R as a 4-tensor on M^n , it is defined the nullity space of M^n at x as the subspace $\Gamma(x) = \{X \in T_x M : R(X, Y, Z, W) = 0, \forall Y, Z, W \in T_x M\}$. This is an intrinsic subspace, so its dimension $\mu(x)$ called the nullity index of M^n is an intrinsic function. For an isometric immersion f of M^n into Euclidean space we always have that the relative nullity distribution Δ of f satisfies $\Delta \subset \Gamma$. Thus, our assumption on the relative nullity distribution in Theorem 1 can be replaced by the intrinsic one $\mu = n - 2p$. The same holds for Corollary 2.

iv) Now let us consider the more general situation discussed in Theorem 1 of [F2], namely, $\nu = n - p - r$, for some $2 \leq r \leq p$. It is natural to ask if it can be generalized by dropping the flatness of the normal bundle assumption as we did for the case $r = p$. The answer to this question seems to be negative, since the algebraic decomposition Proposition 4 does not generalize, even for the case $r = p - 1$, as the following example shows. Take A_i defined as

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The bilinear form $\alpha = (A_1, A_2, A_3) : \mathbb{R}^5 \times \mathbb{R}^5 \rightarrow \mathbb{R}^3$ is nonpositive, has $\nu = n - p - r = 0$ for $r = p - 1 = 2$ but is not decomposable. It is easy to generalize this example for all p . Thus the analogous result to Proposition 4 is false for $\nu = n - p - r$ and $2 \leq r \leq p - 1$.

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Luis A. Florit
IMPA
Estrada Dona Castorina, 110
22460–320 Rio de Janeiro – Brazil
e-mail: luis@impa.br

Fangyang Zheng
The Ohio State University
Columbus, OH 43210-1174 – USA
e-mail: zheng@math.ohio-state.edu

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