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On products in algebraic K-theory

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Abstract. This paper investigates the product structure in algebraic K -theory of rings. The first objective is to understand the relationships between products and the kernel of the Hurewicz homomorphism relating the algebraic K -theory of any ring to the integral homology of its linear groups. The second part of the paper is devoted to the ring of integers \mathbb{Z} . Using recent results of V. Voevodsky we completely determine the products in $K_*(\mathbb{Z})$ tensored with the ring of 2-adic integers.

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0. Introduction

The purpose of this paper is to study the Loday's product homomorphism

$$\star : K_i(R) \otimes K_k(\mathbb{Z}) \longrightarrow K_{i+k}(R)$$

in the algebraic K -theory of any ring R with identity, for positive integers i and k (see [21]). Our first goal is to exhibit very strong connections between the image of that product and the kernel of the non-stable Hurewicz homomorphisms relating the K -groups of R to the integral homology groups of its linear groups

$$h_i : K_i(R) = \pi_i BGL(R)^+ \longrightarrow H_i BGL(R)^+ \cong H_i GL(R) \quad \text{for } i \geq 1,$$

respectively $h_i : K_i(R) \rightarrow H_i E(R)$ for $i \geq 2$ and $h_i : K_i(R) \rightarrow H_i St(R)$ for $i \geq 3$, where $GL(R)$ is the infinite general linear group (considered as a discrete group) over R , $E(R)$ its subgroup generated by elementary matrices, and $St(R)$ the infinite Steinberg group over R . A universal approximation of the exponent of the kernel and some information on the cokernel of these Hurewicz homomorphisms

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have been obtained in [3], [4] and Section 5 of [5]. Our argument is based on the understanding, from various viewpoints, of the stable Hurewicz homomorphism between the algebraic K -theory and the homology of the K -theory spectrum. We establish in particular the following result (see Theorem 3.2):

For any ring R and any integer $i \geq 2$, the image of $\star : K_i(R) \otimes K_1(\mathbb{Z}) \rightarrow K_{i+1}(R)$ is contained in the kernel of the Hurewicz homomorphisms $h_{i+1} : K_{i+1}(R) \rightarrow H_{i+1}GL(R)$ and $h_{i+1} : K_{i+1}(R) \rightarrow H_{i+1}E(R)$; the same holds for $h_{i+1} : K_{i+1}(R) \rightarrow H_{i+1}St(R)$ if $i \geq 3$.

In low dimensions, we actually prove exactness results for any ring R (see Theorems 4.1 and 4.3).

(a) *There is an exact sequence*

$$K_4(R) \xrightarrow{h_4} H_4E(R) \longrightarrow \Gamma(K_2(R)) \longrightarrow K_3(R) \xrightarrow{h_3} H_3E(R) \longrightarrow 0,$$

where $\Gamma(-)$ is the quadratic functor defined on abelian groups by J.H.C. Whitehead in Section 5 of [37]; moreover, $\ker h_3$ is isomorphic to $K_2(R) \star K_1(\mathbb{Z})$.

(b) *There is an exact sequence*

$$K_5(R) \xrightarrow{h_5} H_5St(R) \longrightarrow K_3(R) \otimes K_1(\mathbb{Z}) \xrightarrow{\star} K_4(R) \xrightarrow{h_4} H_4St(R) \longrightarrow 0$$

and the kernel of h_5 fits into a short exact sequence

$$0 \longrightarrow K_4(R) \star K_1(\mathbb{Z}) \longrightarrow \ker h_5 \longrightarrow Q \longrightarrow 0,$$

where Q is a quotient of the subgroup of elements of order 2 in the group $K_3(R)$.

The second objective of the paper is to compute explicitly products in the algebraic K -theory of the ring of integers \mathbb{Z} . First of all, we determine in low dimensions the products $K_i(\mathbb{Z}) \star K_k(\mathbb{Z})$, the homology groups of $SL(\mathbb{Z})$ and $St(\mathbb{Z})$, and the Hurewicz homomorphism (see Proposition 5.1). Secondly, we consider maps

$$K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) \xrightarrow{\star} K_{i+k}(\mathbb{Z}) \longrightarrow K_{i+k}(\mathbb{Z}) \otimes \hat{\mathbb{Z}}_2$$

for all positive integers i and k , where the second arrow is the tensor product of $K_{i+k}(\mathbb{Z})$ with the inclusion of \mathbb{Z} into the ring of 2-adic integers $\hat{\mathbb{Z}}_2$. We call these maps 2-adic products for $K_\star(\mathbb{Z})$ and continue to denote them by the symbol \star . We deduce from a topological argument based on results by M. Bökstedt [12], V. Voevodsky [33], J. Rognes and C. Weibel [35] and [28] the calculation of all such

2-adic products (see Theorems 5.6, 5.7, Corollary 5.8 and Theorem 5.9):

The 2-adic product $\star : K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) \longrightarrow K_{i+k}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2$ is trivial for all positive integers i and k , except if $i \equiv k \equiv 1 \pmod{8}$ or $i \equiv 1 \pmod{8}$ and $k \equiv 2 \pmod{8}$ (or $i \equiv 2 \pmod{8}$ and $k \equiv 1 \pmod{8}$) where its image is cyclic of order 2.

We also mention in Proposition 5.11 an interesting relationship between products in algebraic K -theory and the kernel of the Dwyer-Friedlander map.

For any odd prime l and any integer $n \geq 2$, the image of the product map

$$\star : K_{2n-1}(\mathbb{Z}) \otimes K_{2n-1}(\mathbb{Z}) \longrightarrow K_{4n-2}(\mathbb{Z})$$

is contained in the kernel of the Dwyer-Friedlander map $K_{4n-2}(\mathbb{Z}) \rightarrow K_{4n-2}^{\text{ét}}(\mathbb{Z}[\frac{1}{l}])$.

Observe that the 2-adic products in the K -theory of \mathbb{Z} have a very small image. On the other hand, we finally prove in Theorem 6.4 that the image of the product

$$\star : K_1(E) \otimes K_{2n-1}(E) \longrightarrow K_{2n}(E)$$

is huge when E is a cyclotomic field and n and odd integer.

The paper is organized as follows. In Section 1, we give a new construction of the Whitehead exact sequence for spectra. Section 2 presents another approach of the study of the Hurewicz homomorphism for spectra using the so-called Postnikov cofibrations. Section 3 is devoted to general results on the relations between products in algebraic K -theory and the kernel of the stable and of the non-stable Hurewicz homomorphism. Section 4 provides the above exact sequences involving the K -groups and the homology groups of the linear groups in dimensions ≤ 5 . In Section 5, we calculate the 2-adic products in the algebraic K -theory of the ring of integers \mathbb{Z} . We finally discuss in Section 6 products in the K -theory of cyclotomic fields.

Throughout the paper, all rings are supposed to have an identity. We consider all ordinary homology groups with (trivial) coefficients in \mathbb{Z} except if explicitly mentioned. If G is an abelian group, G_l denotes the l -torsion subgroup of G (for a prime l), $K(G, s)$ the Eilenberg-MacLane space having all homotopy groups trivial except for G in dimension s and $H(G)$ the Eilenberg-MacLane spectrum having all homotopy groups trivial except for G in dimension 0. If X is any CW-complex or any CW-spectrum and i any integer, we write $\alpha_i : X \rightarrow X[i]$ for its i -th Postnikov section (i.e., $\pi_k X[i] = 0$ for $k > i$ and $(\alpha_i)_* : \pi_k X \xrightarrow{\cong} \pi_k X[i]$ for $k \leq i$) and $\gamma_i : X(i) \rightarrow X$ for the fiber of α_i ; in other words, $X(i)$ is the i -connected cover of X . For $j \geq i+1$, $X(i, j)$ denotes $X(i)[j]$, whose homotopy groups are $\pi_k X(i, j) = 0$ if $k \leq i$ or $k > j$ and $\pi_k X(i, j) \cong \pi_k X$ if $i+1 \leq k \leq j$.

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1. The Whitehead exact sequence for spectra

Let S be the sphere spectrum and $S \xrightarrow{\alpha_0} S[0] = H(\mathbb{Z})$ its 0-th Postnikov section. By taking the smash product of any spectrum X with the cofibration $S(0) \xrightarrow{\gamma_0} S \xrightarrow{\alpha_0} S[0]$, where $S(0)$ is 0-connected, one obtains the cofibration of spectra

$$X \wedge S(0) \xrightarrow{\text{id} \wedge \gamma_0} X \wedge S \simeq X \xrightarrow{\text{id} \wedge \alpha_0} X \wedge H(\mathbb{Z}),$$

whose homotopy exact sequence is the long Whitehead exact sequence

$$\cdots \longrightarrow \pi_i(X \wedge S(0)) \xrightarrow{\bar{\chi}_i} \pi_i X \xrightarrow{\bar{h}_i} H_i X \xrightarrow{\bar{\nu}_i} \pi_{i-1}(X \wedge S(0)) \longrightarrow \cdots$$

of X ; here i is any integer, $\bar{\nu}_i$ is the connecting homomorphism, $\bar{\chi}_i$ is induced by $(\text{id} \wedge \gamma_0)$ and \bar{h}_i by $(\text{id} \wedge \alpha_0)$, i.e., \bar{h}_i is the stable Hurewicz homomorphism. The groups $\pi_i(X \wedge S(0))$ are usually denoted by $\Gamma_i(X)$: that definition coincides actually with the homotopy groups of the fiber of the Dold-Thom map (see [16]) and it was recently proved in [29] that they are isomorphic to the groups introduced in the original paper [37] by J.H.C. Whitehead.

Now, let us assume that the spectrum X is $(r - 1)$ -connected for some integer r . The advantage of the above approach is that one can compute the groups $\Gamma_i(X)$ with the Atiyah-Hirzebruch spectral sequence for the $S(0)$ -homology of X :

$$E_{s,t}^2 \cong H_s(X; \pi_t S(0)) \implies \Gamma_{s+t}(X).$$

Notice that $E_{s,t}^2 = 0$ if $s \leq r - 1$ or $t \leq 0$. This implies in particular that $\Gamma_i(X) = 0$ for $i \leq r$ (Hurewicz theorem) and that $(\rho_1 \rho_2 \cdots \rho_{i-r}) \Gamma_i(X) = 0$ for $i \geq r + 1$, where ρ_k denotes the exponent of the homotopy group $\pi_k S$ for $k \geq 1$ (see also [29] for another proof and [5] for corresponding results for the generalized Hurewicz homomorphisms). The first interesting Gamma group of an $(r - 1)$ -connected spectrum X is

$$\Gamma_{r+1}(X) \cong E_{r,1}^2 \cong \pi_r X \otimes \pi_1 S$$

(this was in fact established a long time ago by J.H.C. Whitehead, see for instance Section 14 of [37]). Our first goal is to understand the homomorphism $\bar{\chi}_{r+1} : \Gamma_{r+1}(X) \cong \pi_r X \otimes \pi_1 S \rightarrow \pi_{r+1} X$. Let us start with the following general result on the external product $\pi_i X \otimes \pi_k S \xrightarrow{\wedge} \pi_{i+k}(X \wedge S) \cong \pi_{i+k} X$ (see [32], p. 270 for the definition of the external product).

Lemma 1.1. *Let X be any spectrum, i and k two integers with $k \geq 1$. Then the image of the external product $\wedge : \pi_i X \otimes \pi_k S \rightarrow \pi_{i+k} X$ is contained in the kernel*

of the stable Hurewicz homomorphism $\bar{h}_{i+k} : \pi_{i+k}X \rightarrow H_{i+k}X$ for all integers i and all positive integers k .

Proof. The commutative diagram

$$\begin{array}{ccc}
 \pi_i X \otimes \pi_k S(0) & \xrightarrow[\cong]{(\text{id})_* \otimes (\gamma_0)_*} & \pi_i X \otimes \pi_k S \\
 \downarrow \wedge & & \downarrow \wedge \\
 \Gamma_{i+k}(X) & \xrightarrow{\bar{\chi}_{i+k}} & \pi_{i+k} X
 \end{array}$$

shows that the image of $\wedge : \pi_i X \otimes \pi_k S \rightarrow \pi_{i+k} X$ is contained in image $\bar{\chi}_{i+k} \cong \ker \bar{h}_{i+k}$. Another proof of this fact is given by Lemma 1 of [6]. \square

In the case where X is $(r-1)$ -connected and $i = r, k = 1$, we have the following exactness result:

Proposition 1.2. *For an $(r-1)$ -connected spectrum X , the homomorphism $\bar{\chi}_{r+1} : \Gamma_{r+1}(X) \rightarrow \pi_{r+1}X$ in the Whitehead exact sequence is exactly the external product $\wedge : \pi_r X \otimes \pi_1 S \rightarrow \pi_{r+1}X$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
 H_r X \otimes H_1 S(0) & \xrightarrow[\cong]{\wedge} & H_{r+1}(X \wedge S(0)) & & \\
 \uparrow \cong & & \uparrow \cong & & \\
 \pi_r X \otimes \pi_1 S(0) & \xrightarrow{\wedge} & \pi_{r+1}(X \wedge S(0)) & = & \Gamma_{r+1}(X) \\
 \cong \downarrow (\text{id})_* \otimes (\gamma_0)_* & & \downarrow \bar{\chi}_{r+1} & & \\
 \pi_r X \otimes \pi_1 S & \xrightarrow{\wedge} & \pi_{r+1}(X \wedge S) & \cong & \pi_{r+1}X.
 \end{array}$$

The top horizontal homomorphism is an isomorphism by Künneth formula and the two top vertical arrows, which are Hurewicz homomorphisms, are isomorphisms since X is $(r-1)$ -connected, $S(0)$ is 0-connected and $X \wedge S(0)$ is r -connected. Consequently, the external product in the middle of the diagram is an isomorphism. The homomorphism $(\text{id})_* \otimes (\gamma_0)_*$ is an isomorphism because $(\gamma_0)_* : \pi_1 S(0) \xrightarrow{\cong} \pi_1 S$. Therefore, $\bar{\chi}_{r+1}$ is exactly the external product $\pi_r X \otimes \pi_1 S \xrightarrow{\wedge} \pi_{r+1}X$. \square

Corollary 1.3. *For any $(r - 1)$ -connected spectrum X , the following sequence is exact:*

$$\begin{aligned} \cdots \longrightarrow \Gamma_{r+2}(X) \xrightarrow{\bar{\chi}_{r+2}} \pi_{r+2}X \xrightarrow{\bar{h}_{r+2}} H_{r+2}X \xrightarrow{\bar{\nu}_{r+2}} \\ \pi_r X \otimes \pi_1 S \xrightarrow{\wedge} \pi_{r+1}X \xrightarrow{\bar{h}_{r+1}} H_{r+1}X \longrightarrow 0. \end{aligned}$$

2. Postnikov cofibrations

The purpose of this section is to present another approach of the study of the Hurewicz homomorphism. For an $(r - 1)$ -connected spectrum X , consider for all integers $i \geq r + 1$ the cofibrations of spectra

$$\Sigma^i H(\pi_i X) \xrightarrow{\gamma_{i-1}} X[i] \xrightarrow{\alpha_{i-1}} X[i - 1],$$

where α_{i-1} is the $(i - 1)$ -st Postnikov section of $X[i]$: let us call them the Postnikov cofibrations of X . The associated homology exact sequences are

$$\begin{aligned} \cdots \longrightarrow H_{i+1}X[i] \xrightarrow{(\alpha_{i-1})_*} H_{i+1}X[i - 1] \xrightarrow{\bar{\partial}} \underbrace{H_i(\Sigma^i H(\pi_i X))}_{\cong \pi_i X} \xrightarrow{(\gamma_{i-1})_*} \\ \underbrace{H_i X[i]}_{\cong H_i X} \xrightarrow{(\alpha_{i-1})_*} H_i X[i - 1] \longrightarrow 0, \end{aligned}$$

and it is easy to check that $(\gamma_{i-1})_*$ is the stable Hurewicz homomorphism \bar{h}_i . Thus, we obtain the following

Proposition 2.1. *Let X be an $(r - 1)$ -connected spectrum and i an integer $\geq r + 1$. There is an exact sequence*

$$\cdots \longrightarrow H_{i+1}X[i] \xrightarrow{(\alpha_{i-1})_*} H_{i+1}X[i - 1] \xrightarrow{\bar{\partial}} \pi_i X \xrightarrow{\bar{h}_i} H_i X \xrightarrow{(\alpha_{i-1})_*} H_i X[i - 1] \longrightarrow 0.$$

Now let us try to understand the homomorphism $\bar{\partial}$ for the cases $i = r + 1$ and $i = r + 2$.

Proposition 2.2. (a) *For any $(r - 1)$ -connected spectrum X , there is an exact sequence*

$$0 \longrightarrow H_{r+2}X[r + 1] \xrightarrow{\bar{\varphi}} \pi_r X \otimes \pi_1 S \xrightarrow{\bar{\partial}} \pi_{r+1}X \xrightarrow{\bar{h}_{r+1}} H_{r+1}X \longrightarrow 0.$$

(b) The homomorphism $\bar{\partial}$ is again exactly the external product $\wedge : \pi_r X \otimes \pi_1 S \rightarrow \pi_{r+1} X$.

Proof. Let us look at the Postnikov cofibration of X for $i = r + 1$,

$$\Sigma^{r+1}H(\pi_{r+1}X) \xrightarrow{\gamma_r} X[r+1] \xrightarrow{\alpha_r} X[r] \simeq \Sigma^r H(\pi_r X),$$

and take its homology exact sequence

$$\begin{array}{c} \underbrace{H_{r+2}(\Sigma^{r+1}H(\pi_{r+1}X))}_{=0} \longrightarrow H_{r+2}X[r+1] \xrightarrow{\bar{\varphi}} H_{r+2}(\Sigma^r H(\pi_r X)) \xrightarrow{\bar{\partial}} \\ \pi_{r+1}X \xrightarrow{\bar{h}_{r+1}} H_{r+1}X \longrightarrow \underbrace{H_{r+1}(\Sigma^r H(\pi_r X))}_{=0} \end{array}$$

where $\bar{\varphi}$ is written for $(\alpha_r)_*$. Since $\Sigma^r H(\pi_r X)$ is an Eilenberg-MacLane spectrum, it is clear that

$$H_{r+2}(\Sigma^r H(\pi_r X)) \cong \Gamma_{r+1}(\Sigma^r H(\pi_r X)) \cong \pi_r X \otimes \pi_1 S,$$

and we get assertion (a). Then, consider the map $\alpha_0 : S \rightarrow H(\mathbb{Z})$ and denote by ζ the composition

$$(\text{id} \wedge \alpha_0) \gamma_r : \Sigma^{r+1}H(\pi_{r+1}X) \longrightarrow X[r+1] \simeq X[r+1] \wedge S \longrightarrow X[r+1] \wedge H(\mathbb{Z}),$$

and by F its fiber. By smashing with $H(\mathbb{Z})$ the cofibration obtained by looping the base spectrum of the cofibration $\Sigma^{r+1}H(\pi_{r+1}X) \xrightarrow{\gamma_r} X[r+1] \xrightarrow{\alpha_r} \Sigma^r H(\pi_r X)$, we get the commutative diagram

$$\begin{array}{ccccc} \Sigma^{r-1}H(\pi_r X) \wedge H(\mathbb{Z}) & \longrightarrow & \Sigma^{r+1}H(\pi_{r+1}X) \wedge H(\mathbb{Z}) & \xrightarrow{\gamma_r \wedge \text{id}} & X[r+1] \wedge H(\mathbb{Z}) \\ \uparrow & & \uparrow \text{id} \wedge \alpha_0 & & \uparrow \text{id} \\ F & \longrightarrow & \Sigma^{r+1}H(\pi_{r+1}X) & \xrightarrow{\zeta} & X[r+1] \wedge H(\mathbb{Z}) \\ \downarrow & & \downarrow \gamma_r & & \downarrow \text{id} \\ X[r+1] \wedge S(0) & \xrightarrow{\text{id} \wedge \gamma_0} & X[r+1] & \xrightarrow{\text{id} \wedge \alpha_0} & X[r+1] \wedge H(\mathbb{Z}) \end{array}$$

in which all rows are cofibrations. Then look at their homotopy exact sequences

$$\begin{array}{ccccccc}
 H_{r+2}X[r+1] & \longrightarrow & \underbrace{H_{r+1}(\Sigma^{r-1}H(\pi_r X))}_{\cong H_{r+2}(\Sigma^r H(\pi_r X))} & \xrightarrow{\bar{\delta}} & \underbrace{H_{r+1}(\Sigma^{r+1}H(\pi_{r+1} X))}_{\cong \pi_{r+1} X} & \xrightarrow[\cong]{(\gamma_r \wedge \text{id})_*} & \underbrace{H_{r+1}X[r+1]}_{\cong H_{r+1} X} \\
 \uparrow = & & \uparrow & & \text{Hurewicz} \uparrow \cong & & \uparrow = \\
 H_{r+2}X[r+1] & \longrightarrow & \pi_{r+1}F & \longrightarrow & \pi_{r+1}X & \xrightarrow{\zeta_*} & H_{r+1}X[r+1] \\
 \downarrow = & & \downarrow & & (\gamma_r)_* \downarrow \cong & & \downarrow = \\
 H_{r+2}X[r+1] & \longrightarrow & \underbrace{\Gamma_{r+1}(X[r+1])}_{\cong \Gamma_{r+1}(X)} & \xrightarrow{\bar{\chi}_{r+1}} & \underbrace{\pi_{r+1}X[r+1]}_{\cong \pi_{r+1} X} & \xrightarrow{\bar{h}_{r+1}} & \underbrace{H_{r+1}X[r+1]}_{\cong H_{r+1} X}.
 \end{array}$$

Observe that the three horizontal arrows on the left of the diagram are injective and conclude by the five lemma that the two vertical arrows starting from $\pi_{r+1}F$ are isomorphisms: assertion (b) can then be deduced from Proposition 1.2. \square

Remark 2.3. It follows from Corollary 1.3 and Proposition 2.2 that the cokernel of $\bar{h}_{r+2} : \pi_{r+2}X \rightarrow H_{r+2}X$ is isomorphic to image $\bar{\nu}_{r+2} = \ker(\wedge : \pi_r X \otimes \pi_1 S \rightarrow \pi_{r+1}X) \cong H_{r+2}X[r+1]$ for any $(r-1)$ -connected spectrum X .

Similarly, we can investigate the stable Hurewicz homomorphism in dimension $r+2$. Consider the Postnikov cofibration of an $(r-1)$ -connected spectrum X for $i = r+2$ and its homology exact sequence

$$\begin{array}{c}
 \dots \longrightarrow H_{r+3}X[r+1] \xrightarrow{\bar{\psi}} \underbrace{H_{r+2}(\Sigma^{r+2}H(\pi_{r+2}X))}_{\cong \pi_{r+2} X} \xrightarrow{\bar{h}_{r+2}} \\
 \underbrace{H_{r+2}X[r+2]}_{\cong H_{r+2} X} \longrightarrow H_{r+2}X[r+1] \longrightarrow 0,
 \end{array}$$

where $\bar{\psi}$ is written for $\bar{\delta}$. The next two lemmas describe the group $H_{r+3}X[r+1]$ and the homomorphism $\bar{\psi}$.

Lemma 2.4. *There is an exact sequence*

$$\dots \longrightarrow \pi_{r+1}X \otimes \pi_1 S \xrightarrow{\bar{\theta}} H_{r+3}X[r+1] \xrightarrow{\bar{\eta}} {}_2(\pi_r X) \longrightarrow 0,$$

where ${}_2(\pi_r X)$ denotes the subgroup of elements of order dividing 2 in the group $\pi_r X$.

Proof. Let us look again at the cofibration

$$\Sigma^{r+1}H(\pi_{r+1}X) \xrightarrow{\gamma_r} X[r+1] \xrightarrow{\alpha_r} \Sigma^r H(\pi_r X)$$

and at its homology exact sequence

$$\cdots \longrightarrow H_{r+3}(\Sigma^{r+1}H(\pi_{r+1}X)) \xrightarrow{\bar{\theta}} H_{r+3}X[r+1] \xrightarrow{\bar{\eta}} H_{r+3}(\Sigma^r H(\pi_r X)) \longrightarrow 0,$$

where $\bar{\theta}$ and $\bar{\eta}$ are the homomorphisms induced by γ_r and α_r respectively. It turns out that

$H_{r+3}(\Sigma^{r+1}H(\pi_{r+1}X)) \cong \Gamma_{r+2}(\Sigma^{r+1}H(\pi_{r+1}X)) \cong \pi_{r+1}X \otimes \pi_1 S$ because of the results of Section 1 and that $H_{r+3}(\Sigma^r H(\pi_r X)) \cong {}_2(\pi_r X)$, according to Théorème 2 of [14]. \square

Lemma 2.5. *The composition $\bar{\psi}\bar{\theta} : \pi_{r+1}X \otimes \pi_1 S \rightarrow \pi_{r+2}X$ is the external product \wedge .*

Proof. The obvious map $X(r) \rightarrow X$ provides the commutative diagram of cofibrations

$$\begin{array}{ccccc} \Sigma^{r+2}H(\pi_{r+2}X) & \longrightarrow & X(r, r+2] & \longrightarrow & X(r, r+1] \simeq \Sigma^{r+1}H(\pi_{r+1}X) \\ \downarrow \text{id} & & \downarrow & & \downarrow \gamma_r \\ \Sigma^{r+2}H(\pi_{r+2}X) & \longrightarrow & X[r+2] & \longrightarrow & X[r+1] \end{array}$$

which induces the commutative square

$$\begin{array}{ccc} \underbrace{H_{r+3}(\Sigma^{r+1}H(\pi_{r+1}X))}_{\cong \pi_{r+1}X \otimes \pi_1 S} & \longrightarrow & \underbrace{H_{r+2}(\Sigma^{r+2}H(\pi_{r+2}X))}_{\cong \pi_{r+2}X} \\ \downarrow \bar{\theta} & & \downarrow = \\ H_{r+3}X[r+1] & \xrightarrow{\bar{\psi}} & \underbrace{H_{r+2}(\Sigma^{r+2}H(\pi_{r+2}X))}_{\cong \pi_{r+2}X} \end{array}$$

Then, the statement of Proposition 2.2 for the r -connected spectrum $X(r)$ shows that the top horizontal arrow is the external product. \square

We may summarize our results on the stable Hurewicz homomorphism \bar{h}_{r+2} as follows.

Proposition 2.6. *Let X be an $(r - 1)$ -connected spectrum.*

(a) *There is an exact sequence*

$$\cdots \longrightarrow H_{r+3}X[r + 1] \xrightarrow{\bar{\psi}} \pi_{r+2}X \xrightarrow{\bar{h}_{r+2}} H_{r+2}X \longrightarrow H_{r+2}X[r + 1] \longrightarrow 0.$$

(b) *The kernel of \bar{h}_{r+2} fits into the short exact sequence*

$$0 \longrightarrow \wedge(\pi_{r+1}X \otimes \pi_1S) \longrightarrow \ker \bar{h}_{r+2} \longrightarrow Q \longrightarrow 0,$$

where Q is a quotient of ${}_2(\pi_rX)$.

Remark 2.7. Since $\pi_iX[r + 1] = 0$ for $i \geq r + 2$,

$$H_{r+3}X[r + 1] \cong \Gamma_{r+2}(X[r + 1]).$$

The Postnikov section $X \rightarrow X[r + 1]$ induces a map f between the Atiyah-Hirzebruch spectral sequences

$$H_s(X; \pi_tS(0)) \implies \Gamma_{s+t}(X) \quad \text{and} \quad H_s(X[r + 1]; \pi_tS(0)) \implies \Gamma_{s+t}(X[r + 1]).$$

The lines $s + t = r + 2$ in these spectral sequences give the following picture:

$$\begin{array}{ccccccc} H_{r+2}(X; \pi_1S) & \xrightarrow{d^2} & \pi_rX \otimes \pi_2S & \longrightarrow & \Gamma_{r+2}(X) & \longrightarrow & H_{r+1}(X; \pi_1S) \longrightarrow 0 \\ \downarrow f_1 & & \cong \downarrow f_2 & & \downarrow f_3 & & \cong \downarrow f_4 \\ H_{r+2}(X[r + 1]; \pi_1S) & \xrightarrow{d^2} & \pi_rX[r + 1] \otimes \pi_2S & \longrightarrow & \Gamma_{r+2}(X[r + 1]) & \longrightarrow & H_{r+1}(X[r + 1]; \pi_1S) \longrightarrow 0. \end{array}$$

By the universal coefficient theorem, one has $H_{r+2}(X; \pi_1S) \cong (H_{r+2}X \otimes \pi_1S) \oplus \text{Tor}(H_{r+1}X, \pi_1S)$ and $H_{r+2}(X[r + 1]; \pi_1S) \cong (H_{r+2}(X[r + 1]) \otimes \pi_1S) \oplus \text{Tor}(H_{r+1}X, \pi_1S)$; thus, one can check that f_1 is surjective because of Whitehead's theorem and deduce from the five lemma that

$$\Gamma_{r+2}(X) \cong \Gamma_{r+2}(X[r + 1]) \cong H_{r+3}X[r + 1].$$

Moreover, one can show with the argument of the proof of Proposition 2.2 (b) that the homomorphism $\bar{\psi}$ of Proposition 2.6 (a) is actually $\bar{\chi}_{r+2} : \Gamma_{r+2}(X) \rightarrow \pi_{r+2}X$ of Corollary 1.3. Consequently, the part

$$H_{r+3}X[r + 1] \xrightarrow{\bar{\psi}} \pi_{r+2}X \xrightarrow{\bar{h}_{r+2}} H_{r+2}X$$

of the sequence given by Proposition 2.6 (a) is a piece of the Whitehead exact sequence.

Remark 2.8. It follows from Lemma 2.4 and the previous remark that the group $\Gamma_{r+2}(X)$ is described by the exact sequence

$$\cdots \longrightarrow \pi_{r+1}X \otimes \pi_1S \xrightarrow{\bar{\theta}} \Gamma_{r+2}(X) \xrightarrow{\bar{\eta}} {}_2(\pi_r X) \longrightarrow 0,$$

and in particular that its exponent divides 4 (this was already known by [11], Section 4).

Remark 2.9. All exact sequences introduced in Sections 1 and 2 are obviously natural in X .

3. Products and Hurewicz homomorphisms in algebraic K -theory

If R is any ring, let us denote by X_R the connective K -theory spectrum of R , i.e., a (-1) -connected Ω -spectrum whose 0-th space is the infinite loop space $BGL(R)^+ \times K_0(R)$. We shall also consider the $(r-1)$ -connected spectra $X_R(r-1)$ for $r \geq 0$, i.e., the fiber of the Postnikov section $X_R \rightarrow X_R[r-1]$, and call γ_{r-1} the obvious map $X_R(r-1) \rightarrow X_R$. Observe that $K_i(R) \cong \pi_i X_R(r-1)$ for $i \geq r$. Remember that the infinite loop spaces corresponding to $X_R(0)$, $X_R(1)$ and $X_R(2)$ are $BGL(R)^+$, $BE(R)^+$ and $BSt(R)^+$ respectively. If R and R' are two rings, there is a pairing $\mu : X_R \wedge X_{R'} \rightarrow X_{R \otimes R'}$ and the product in algebraic K -theory is defined as follows

$$\begin{aligned} \star : K_i(R) \otimes K_k(R') &\cong \pi_i X_R \otimes \pi_k X_{R'} \xrightarrow{\wedge} \pi_{i+k}(X_R \wedge X_{R'}) \\ &\xrightarrow{\mu_*} \pi_{i+k} X_{R \otimes R'} \cong K_{i+k}(R \otimes R') \end{aligned}$$

for any two integers $i \geq 0$ and $k \geq 0$ (see for instance [21], Proposition 2.4.2). We shall actually concentrate our attention to the special case where R' is the ring of integers \mathbb{Z} : the goal of Sections 3 and 4 is to investigate the relationships between the image of the product

$$\star : K_i(R) \otimes K_k(\mathbb{Z}) \longrightarrow K_{i+k}(R \otimes \mathbb{Z}) \cong K_{i+k}(R)$$

and the kernel of the stable and the non-stable Hurewicz homomorphism.

Remember that $X_{\mathbb{Z}}$ is a ring spectrum and let us call $j : S \rightarrow X_{\mathbb{Z}}$ its identity. Notice that j corresponds to the map $B\Sigma_{\infty}^+ \rightarrow BGL(\mathbb{Z})^+$ given by the inclusion of the infinite symmetric group Σ_{∞} into $GL(\mathbb{Z})$. This map j induces an isomorphism $j_* : \pi_1 S \xrightarrow{\cong} \pi_1 X_{\mathbb{Z}} \cong K_1(\mathbb{Z})$ and the image of $j_* : \pi_k S \rightarrow K_k(\mathbb{Z})$ for $k \geq 2$ is described in [22] and [26]. For any ring R , the above pairing μ provides then X_R

with an $X_{\mathbb{Z}}$ -module structure. Let us first translate the results of Sections 1 and 2 in terms of algebraic K-theory.

Proposition 3.1. *Let R be a ring, i and k two integers with $i \geq 0$, $k \geq 1$, and consider an element $x \in K_i(R)$ and an element $y \in K_k(\mathbb{Z})$ belonging to the image of $j_* : \pi_k S \rightarrow K_k(\mathbb{Z})$.*

- (a) *For all $r \leq i$, $x \star y$ is an element of the kernel of the stable Hurewicz homomorphism $\bar{h}_{i+k} : K_{i+k}(R) \rightarrow H_{i+k}X_R(r-1)$.*
- (b) *If $k \leq i-1$, then $x \star y$ is an element of the kernel of the non-stable Hurewicz homomorphisms $h_{i+k} : K_{i+k}(R) \rightarrow H_{i+k}E(R)$ and $h_{i+k} : K_{i+k}(R) \rightarrow H_{i+k}GL(R)$.*
- (c) *If $i \geq 3$ and $k \leq i-1$, then $x \star y$ is an element of the kernel of $h_{i+k} : K_{i+k}(R) \rightarrow H_{i+k}St(R)$.*

Proof. The first assertion is a consequence of Lemma 1.1 and of the commutativity of the diagram

$$\begin{array}{ccc}
 \pi_i X_R(r-1) \otimes \pi_k S & \xrightarrow{\wedge} & \pi_{i+k} X_R(r-1) & \xrightarrow{\bar{h}_{i+k}} & H_{i+k} X_R(r-1) \\
 \cong \downarrow (\gamma_{r-1})_* \otimes \text{id} & & \cong \downarrow (\gamma_{r-1})_* & & \\
 \pi_i X_R \otimes \pi_k S & \xrightarrow{\wedge} & \pi_{i+k} X_R & & \\
 \downarrow \cong \otimes j_* & & \downarrow \cong & & \\
 K_i(R) \otimes K_k(\mathbb{Z}) & \xrightarrow{\star} & K_{i+k}(R), & &
 \end{array}$$

where the bottom square commutes because X_R is an $X_{\mathbb{Z}}$ -module. In order to prove the last two assertions, consider the $(i-1)$ -connected cover $BGL(R)^+(i-1)$ of the CW-complex $BGL(R)^+$, for $i \geq k+1 \geq 2$. The iterated homology suspension $\sigma : H_{i+k}BGL(R)^+(i-1) \rightarrow H_{i+k}X_R(i-1)$, which is an isomorphism since $k \leq i-1$, and the commutative diagram

$$\begin{array}{ccc}
 K_{i+k}(R) & \xrightarrow{h_{i+k}} & H_{i+k}BGL(R)^+(i-1) \\
 \downarrow = & & \sigma \downarrow \cong \\
 K_{i+k}(R) & \xrightarrow{\bar{h}_{i+k}} & H_{i+k}X_R(i-1)
 \end{array}$$

show that $h_{i+k} : K_{i+k}(R) \rightarrow H_{i+k}BGL(R)^+(i-1)$ fulfills $h_{i+k}(x \star y) = 0$ according to (a) for $r = i$. Since $i \geq 2$, assertion (b) then follows from the composition with

the obvious homomorphism

$$H_{i+k}BGL(R)^+(i-1) \longrightarrow H_{i+k}BE(R)^+ \longrightarrow H_{i+k}BGL(R)^+.$$

If $i \geq 3$, this homomorphism factors even through $H_{i+k}BSt(R)^+$ and we get (c). \square

Now, let us consider the case $k = 1$ and $i = r$: the fact that $j_* : \pi_1 S \rightarrow K_1(\mathbb{Z})$ is an isomorphism implies the following result, where $X_R(i-1, i+1]$ is written for $X_R(i-1)[i+1]$.

Theorem 3.2. *Let R be any ring.*

(a) *For any integer $i \geq 0$, there is a natural exact sequence*

$$K_{i+2}(R) \xrightarrow{\bar{h}_{i+2}} H_{i+2}X_R(i-1) \xrightarrow{\bar{v}_{i+2}} K_i(R) \otimes K_1(\mathbb{Z}) \xrightarrow{\star} K_{i+1}(R) \xrightarrow{\bar{h}_{i+1}} H_{i+1}X_R(i-1) \longrightarrow 0.$$

Moreover, $\ker(\star) = \text{image } \bar{v}_{i+2} \cong H_{i+2}X_R(i-1, i+1]$.

- (b) *For any integer $i \geq 2$, the image of $\star : K_i(R) \otimes K_1(\mathbb{Z}) \rightarrow K_{i+1}(R)$ is contained in the kernel of the non-stable Hurewicz homomorphisms $h_{i+1} : K_{i+1}(R) \rightarrow H_{i+1}E(R)$ and $h_{i+1} : K_{i+1}(R) \rightarrow H_{i+1}GL(R)$.*
- (c) *For any integer $i \geq 3$, the image of $\star : K_i(R) \otimes K_1(\mathbb{Z}) \rightarrow K_{i+1}(R)$ is contained in the kernel of $h_{i+1} : K_{i+1}(R) \rightarrow H_{i+1}St(R)$.*

Proof. Assertion (a) follows from Corollary 1.3 and Remark 2.3 for the spectrum $X_R(i-1)$ since the diagram

$$\begin{array}{ccc} K_i(R) \otimes \pi_1 S & \xrightarrow{\wedge} & \pi_{i+1} X_R \\ \cong \downarrow \text{id} \otimes j_* & & \downarrow \cong \\ K_i(R) \otimes K_1(\mathbb{Z}) & \xrightarrow{\star} & K_{i+1}(R) \end{array}$$

commutes again because of the $X_{\mathbb{Z}}$ -module structure of X_R . Assertions (b) and (c) are direct consequences of Proposition 3.1. (b) and (c). \square

It is possible to obtain a similar information on the stable and the non-stable Hurewicz homomorphism in any dimension $i \geq r+1$. Proposition 2.1 provides the exact sequence

$$\begin{aligned} \cdots \longrightarrow H_{i+1}X_R(r-1, i] &\xrightarrow{(\alpha_{i-1})^*} H_{i+1}X_R(r-1, i-1] \xrightarrow{\bar{\delta}} \\ &K_i(R) \xrightarrow{\bar{h}_i} H_i X_R(r-1) \xrightarrow{(\alpha_{i-1})^*} H_i X_R(r-1, i-1] \longrightarrow 0. \end{aligned}$$

Proposition 3.3. *Let R be any ring, i and r positive integers such that $r \leq i \leq 2r - 1$, then the kernel of the non-stable Hurewicz homomorphism*

$$h_i : K_i(R) \rightarrow H_i BGL(R)^+(r - 1)$$

is exactly the image of $\bar{\partial}$.

Proof. Let us consider the homology exact sequence of the Postnikov cofibration

$$\Sigma^i(H(K_i(R))) \xrightarrow{\gamma_{i-1}} X_R(r - 1, i] \xrightarrow{\alpha_{i-1}} X_R(r - 1, i - 1],$$

and the corresponding homology exact sequence obtained from the Serre spectral sequence of the fibration of CW-complexes

$$K(K_i(R), i) \longrightarrow BGL(R)^+(r - 1, i] \longrightarrow BGL(R)^+(r - 1, i - 1].$$

We obtain the commutative diagram

$$\begin{array}{ccccc} H_{i+1}X_R(r - 1, i - 1] & \xrightarrow{\bar{\partial}} & K_i(R) & \xrightarrow{\bar{h}_i} & H_iX_R(r - 1) \\ \uparrow \sigma & & \uparrow = & & \uparrow \\ H_{i+1}BGL(R)^+(r - 1, i - 1] & \xrightarrow{\partial} & K_i(R) & \xrightarrow{h_i} & H_iBGL(R)^+(r - 1), \end{array}$$

where the horizontal sequences are exact and the three vertical arrows are iterated suspensions. The left iterated homology suspension σ is surjective if $i + 1 \leq 2r$ and even an isomorphism if $i + 1 \leq 2r - 1$ (see [36], p. 382); consequently we may conclude that $\text{image } \partial = \text{image } \bar{\partial}$. □

4. Products and the non-stable Hurewicz homomorphism in low dimensions

The purpose of this section is to study the relationships between the algebraic K-theory of a ring R and the integral homology of its linear groups in low dimensions. In dimension 2, the following isomorphisms are known (see [4]):

$$K_2(R) \cong H_2E(R) \quad \text{and} \quad H_2GL(R) \cong K_2(R) \oplus \Lambda^2(K_1(R)).$$

Let us start by looking at dimensions 3 and 4. Let $\Gamma(-)$ be the quadratic functor defined on abelian groups by J.H.C. Whitehead in Section 5 of [37]: if Y

is a simply connected CW-complex, then the group $\Gamma_3(Y)$ in the Whitehead exact sequence of the space Y turns out to be isomorphic to $\Gamma(\pi_2 Y)$.

Theorem 4.1. *For any ring R , there is a natural exact sequence*

$$K_4(R) \xrightarrow{h_4} H_4 E(R) \xrightarrow{\nu_4} \Gamma(K_2(R)) \xrightarrow{\chi_3} K_3(R) \xrightarrow{h_3} H_3 E(R) \longrightarrow 0$$

and $\ker h_3$ is isomorphic to the image of the product homomorphism $\star : K_2(R) \otimes K_1(\mathbb{Z}) \rightarrow K_3(R)$. In particular, $H_3 E(R) \cong K_3(R)/(K_2(R) \star K_1(\mathbb{Z}))$.

Proof. The exact sequence is just the Whitehead exact sequence (see [37]) of the space $BE(R)^+$ since $\Gamma_3(BE(R)^+) = \Gamma(K_2(R))$. In order to determine the image of χ_3 , consider the exact sequence given by Proposition 2.2 for $X = X_R(1)$ and $r = 2$, and also the corresponding exact sequence obtained from the Serre spectral sequence of the fibration of CW-complexes

$$K(K_3(R), 3) \longrightarrow BE(R)^+[3] \longrightarrow K(K_2(R), 2).$$

We get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_4 X_R(1, 3) & \xrightarrow{\bar{\varphi}} & \overbrace{H_4(\Sigma^2 H(K_2(R)))}^{\cong K_2(R) \otimes K_1(\mathbb{Z})} & \xrightarrow{\star} & K_3(R) & \xrightarrow{\bar{h}_3} & H_3 X_R(1) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow \sigma & & \uparrow = & & \uparrow & & \\ 0 & \longrightarrow & H_4 BE(R)^+[3] & \xrightarrow{\varphi} & H_4(K(K_2(R), 2)) & \xrightarrow{\partial} & K_3(R) & \xrightarrow{h_3} & H_3 BE(R)^+ & \longrightarrow & 0, \end{array}$$

where the vertical arrows are iterated suspensions. It turns out that

$$H_4(K(K_2(R), 2)) \cong \Gamma_3(K(K_2(R), 2)) \cong \Gamma(K_2(R))$$

and the argument of the proof of Proposition 2.2 shows again that the homomorphism χ_3 in the Whitehead exact sequence is exactly ∂ . According to the proof of Proposition 3.3, the iterated homology suspension σ is surjective and one gets $\text{image } \chi_3 = \text{image } \partial = K_2(R) \star K_1(\mathbb{Z})$. Notice that this computation of the image of χ_3 can also be deduced from Section 2.2.6 of [21]. \square

Remark 4.2. This extends the result for fields given in [31], Corollary 5.2, to the case of any ring R .

In order to understand the 4-dimensional and 5-dimensional Hurewicz homomorphisms, let us use exactly the same idea (but now for $r = 3$) for the 2-connected CW-complex $BSt(R)^+$, respectively the 2-connected K -theory spectrum $X_R(2)$.

Theorem 4.3. *Let R be any ring.*

(a) *There is a natural exact sequence*

$$H_6X_R(2, 4] \xrightarrow{\bar{\psi}} K_5(R) \xrightarrow{h_5} H_5St(R) \xrightarrow{\nu_5} K_3(R) \otimes K_1(\mathbb{Z}) \xrightarrow{\star} K_4(R) \xrightarrow{h_4} H_4St(R) \longrightarrow 0.$$

In particular, $H_4St(R) \cong K_4(R)/(K_3(R) \star K_1(\mathbb{Z}))$.

(b) *There is a natural exact sequence*

$$K_4(R) \otimes K_1(\mathbb{Z}) \xrightarrow{\bar{\theta}} H_6X_R(2, 4] \xrightarrow{\bar{\eta}} {}_2(K_3(R)) \longrightarrow 0.$$

(c) *The composition $\bar{\psi}\bar{\theta}$ is the product map $\star : K_4(R) \otimes K_1(\mathbb{Z}) \rightarrow K_5(R)$. Consequently, there is a natural short exact sequence*

$$0 \longrightarrow K_4(R) \star K_1(\mathbb{Z}) \longrightarrow \ker h_5 \longrightarrow Q \longrightarrow 0,$$

where Q is a quotient of ${}_2(K_3(R))$.

Proof. As in the previous proof, we use Proposition 2.2, but consider in this case the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_5X_R(2, 4] & \xrightarrow{\bar{\varphi}} & \overbrace{H_5(\Sigma^3 H(K_3(R)))}^{\cong K_3(R) \otimes K_1(\mathbb{Z})} & \xrightarrow{\star} & K_4(R) & \xrightarrow{\bar{h}_4} & H_4X_R(2) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow \sigma & & \uparrow = & & \uparrow & & \\ 0 & \longrightarrow & H_5BSt(R)^+[4] & \xrightarrow{\varphi} & H_5(K(K_3(R), 3)) & \xrightarrow{\partial} & K_4(R) & \xrightarrow{h_4} & H_4BSt(R)^+ & \longrightarrow & 0. \end{array}$$

However, this time, σ is even an isomorphism. Observe that $H_5BSt(R)^+[4]$ is isomorphic to the kernel of $\star : K_3(R) \otimes K_1(\mathbb{Z}) \rightarrow K_4(R)$. The Whitehead exact sequence of $BSt(R)^+$ is

$$\begin{array}{ccccccc} \dots & \longrightarrow & \Gamma_5(BSt(R)^+) & \xrightarrow{\chi_5} & K_5(R) & \xrightarrow{h_5} & H_5St(R) & \xrightarrow{\nu_5} & \Gamma_4(BSt(R)^+) \\ & & & & & & & & \xrightarrow{\chi_4} & K_4(R) & \xrightarrow{h_4} & H_4St(R) & \longrightarrow & 0 \end{array}$$

and it is easy to check that $\Gamma_4(BSt(R)^+) \cong K_3(R) \otimes K_1(\mathbb{Z}) \cong H_5(K(K_3(R), 3))$. In order to understand the kernel of h_5 , let us use the exact sequence given by Proposition 2.6 for $r = 3, i = 5$, and the exact sequence coming from the homology Serre spectral sequence of the fibration of CW-complexes

$$K(K_5(R), 5) \longrightarrow BSt(R)^+[5] \longrightarrow BSt(R)^+[4].$$

We obtain the commutative diagram

$$\begin{array}{ccccccc}
 H_6X_R(2, 4] & \xrightarrow{\bar{\psi}} & K_5(R) & \xrightarrow{\bar{h}_5} & H_5X_R(2) & \longrightarrow & H_5X_R(2, 4] \longrightarrow 0 \\
 \uparrow \sigma & & \uparrow = & & \uparrow & & \uparrow \\
 H_6BSt(R)^+[4] & \xrightarrow{\psi} & K_5(R) & \xrightarrow{h_5} & H_5BSt(R)^+ & \longrightarrow & H_5BSt(R)^+[4] \longrightarrow 0,
 \end{array}$$

where the vertical arrows are iterated suspensions. According to the proof of Proposition 3.3, the iterated homology suspension σ is surjective and therefore image $\psi = \text{image } \bar{\psi}$. On the other hand, the group $H_6X_R(2, 4]$ and the image of $\bar{\psi}$ may be described by Proposition 2.6 and Lemmas 2.4 and 2.5. \square

The following corollary follows from the five lemma and the argument of the proofs of Theorems 4.1 and 4.3.

Corollary 4.4. *For any ring R , the iterated homology suspensions $H_3E(R) \cong H_3BE(R)^+ \rightarrow H_3X_R(1)$ and $H_4St(R) \cong H_4BSt(R)^+ \rightarrow H_4X_R(2)$ are isomorphisms.*

Remark 4.5. Observe that $h_4 : K_4(R) \rightarrow H_4St(R)$ is an isomorphism up to 2-torsion. This produces the following consequence of Proposition 9 of [9]. Let l be an odd prime, ξ_l a primitive l -root of unity of order l . Let $R = \mathbb{Z}[\xi_l + \xi_l^{-1}]$ be the ring of integers of the maximal real subfield of the cyclotomic field $\mathbb{Q}(\xi_l)$. The vanishing of the group $H_4St(R)$ in this case would imply the Kummer-Vandiver conjecture for the prime l .

5. Products in the algebraic K -theory of the ring of integers \mathbb{Z}

This section is devoted to the study of products in the algebraic K -theory of the ring of integers \mathbb{Z} :

$$\star : K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) \longrightarrow K_{i+k}(\mathbb{Z}).$$

Let us start by describing the results on low-dimensional products given by Section 4 in the case where $R = \mathbb{Z}$.

Proposition 5.1.

- (a) *The product homomorphism $\star : K_i(\mathbb{Z}) \otimes K_1(\mathbb{Z}) \rightarrow K_{i+1}(\mathbb{Z})$ is an isomorphism if $i = 1$, injective if $i = 2$, and trivial if $i \not\equiv 1$ or $2 \pmod{8}$.*
- (b) *The product homomorphism $\star : K_i(\mathbb{Z}) \otimes K_2(\mathbb{Z}) \rightarrow K_{i+2}(\mathbb{Z})$ is trivial if $i \not\equiv 1 \pmod{8}$.*

- (c) $H_4SL(\mathbb{Z}) \cong \mathbb{Z}/2$ and $H_4St(\mathbb{Z}) = 0$.
- (d) There is a short exact sequence

$$0 \longrightarrow K_5(\mathbb{Z}) \xrightarrow{h_5} H_5St(\mathbb{Z}) \xrightarrow{\nu_5} \mathbb{Z}/2 \longrightarrow 0.$$

Proof. The assertion (a) is well known for $i = 1$. Theorem 4.1 produces the exact sequence

$$K_4(\mathbb{Z}) \xrightarrow{h_4} H_4SL(\mathbb{Z}) \xrightarrow{\nu_4} \underbrace{\Gamma(\mathbb{Z}/2)}_{\mathbb{Z}/4} \xrightarrow{\chi_3} \underbrace{K_3(\mathbb{Z})}_{\mathbb{Z}/48} \xrightarrow{h_3} \underbrace{H_3SL(\mathbb{Z})}_{\mathbb{Z}/24} \longrightarrow 0$$

(see [2], [19] and [37], Sections 5 and 13) and asserts that the product $\star : K_2(\mathbb{Z}) \otimes K_1(\mathbb{Z}) \rightarrow K_3(\mathbb{Z})$ is injective. Recently, J. Rognes and C. Weibel deduced from the work of V. Voevodsky [33] the complete calculation of the 2-torsion of the algebraic K-theory of \mathbb{Z} (see Table 1 of [35] and Theorem 0.6 of [28]). This, together with another argument of J. Rognes, shows that $K_4(\mathbb{Z}) = 0$ and implies that $H_4SL(\mathbb{Z})$ is cyclic of order 2. Moreover, $K_i(\mathbb{Z})$ is a finite odd torsion group if i is a positive integer $\equiv 0, 4,$ or $6 \pmod{8}$. Therefore, $K_i(\mathbb{Z}) \star K_1(\mathbb{Z}) = 0$ if $i \equiv 0, 4,$ or $6 \pmod{8}$ or if $i + 1 \equiv 0, 4,$ or $6 \pmod{8}$. This gives (a), and (b) follows from (a) since $K_2(\mathbb{Z}) = K_1(\mathbb{Z}) \star K_1(\mathbb{Z})$. Note that the first author proved the triviality of $\star : K_3(\mathbb{Z}) \otimes K_1(\mathbb{Z}) \rightarrow K_4(\mathbb{Z})$ in [6] before Rognes and Weibel’s proof of the vanishing of $K_4(\mathbb{Z})$. The calculation of $K_i(\mathbb{Z}) \star K_1(\mathbb{Z})$ when $i \equiv 1$ or $2 \pmod{8}$ and of $K_i(\mathbb{Z}) \star K_2(\mathbb{Z})$ when $i \equiv 1 \pmod{8}$ will be given by Theorems 5.7 and 5.9 below.

Now, let us apply Theorem 4.3. The map $\bar{\psi}$ is actually the connecting homomorphism of the homology exact sequence of the cofibration

$$\Sigma^5 H(K_5(\mathbb{Z})) \longrightarrow X_{\mathbb{Z}}(2, 5] \longrightarrow X_{\mathbb{Z}}(2, 4].$$

It is of course possible to consider the analogous cofibration for the sphere spectrum S

$$\Sigma^5 H(\pi_5 S) \longrightarrow S(2, 5] \longrightarrow S(2, 4].$$

The identity $j : S \rightarrow X_{\mathbb{Z}}$ of the ring spectrum $X_{\mathbb{Z}}$ induces the commutative diagram

$$\begin{array}{ccc} H_6 S(2, 4] & \longrightarrow & \pi_5 S = 0 \\ \downarrow j_* & & \downarrow \\ H_6 X_{\mathbb{Z}}(2, 4] & \xrightarrow{\bar{\psi}} & K_5(\mathbb{Z}) \end{array}$$

Proof. This follows from the commutativity of the diagram

$$\begin{array}{ccccccc}
 K_3(\mathbb{Z}) \otimes K_1(\mathbb{Z}) & \xrightarrow{\star} & K_4(\mathbb{Z}) & = & 0 & & \\
 \cong \downarrow \ell_* \otimes \text{id} & & \downarrow \ell_* & & & & \\
 K_3(R) \otimes K_1(\mathbb{Z}) & \xrightarrow{\star} & K_4(R) & \xrightarrow{h_4} & H_4St(R) & \longrightarrow & 0,
 \end{array}$$

where the rows are the exact sequences given by Theorem 4.3 (a). □

The argument of the proof of Proposition 5.1 (d) produces also the next two corollaries

Corollary 5.3. *Let R be a ring such that the composition of $j : S \rightarrow X_{\mathbb{Z}}$ with the obvious map $X_{\mathbb{Z}} \rightarrow X_R$ induces an isomorphism ${}_2(\pi_3 S) \xrightarrow{\cong} {}_2(K_3(R))$. Then $K_4(R) \star K_1(\mathbb{Z}) \cong \ker(h_5 : K_5(R) \rightarrow H_5St(R))$.*

Corollary 5.4. *If R is a ring as in Corollary 5.3 (for instance, if $R = \mathbb{Z}$ or any localization of \mathbb{Z}), then the iterated homology suspension $H_5St(R) \cong H_5BSt(R)^+ \rightarrow H_5X_R(2)$ is an isomorphism.*

Proof. By hypothesis, one has actually the exact sequences

$$\begin{array}{ccccccc}
 H_6(\Sigma^4 H(K_4(R))) & \xrightarrow{\bar{\psi} \bar{\theta}} & K_5(R) & \xrightarrow{\bar{h}_5} & H_5X_R(2) & \longrightarrow & H_5X_R(2, 4) \longrightarrow 0 \\
 \uparrow \sigma & & \uparrow = & & \uparrow & & \uparrow \sigma' \\
 H_6K(K_4(R), 4) & \xrightarrow{\psi \theta} & K_5(R) & \xrightarrow{h_5} & H_5BSt(R)^+ & \longrightarrow & H_5BSt(R)^+[4] \longrightarrow 0,
 \end{array}$$

and the iterated homology suspension σ is clearly an isomorphism; the same is true for σ' because of the commutativity of

$$\begin{array}{ccc}
 H_5X_R(2, 4) & \xrightarrow{\cong} & H_5(\Sigma^3 H(K_3(R))) \\
 \uparrow \sigma' & & \uparrow \cong \\
 H_5BSt(R)^+[4] & \xrightarrow{\cong} & H_5K(K_3(R), 3).
 \end{array}$$

The assertion then follows from the five lemma. □

Let us now consider maps

$$K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) \xrightarrow{\star} K_{i+k}(\mathbb{Z}) \longrightarrow K_{i+k}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2$$

for all positive integers i and k , where the second arrow is the tensor product of $K_{i+k}(\mathbb{Z})$ with the inclusion of \mathbb{Z} into the ring of 2-adic integers $\widehat{\mathbb{Z}}_2$. We call these maps 2-adic products for $K_*(\mathbb{Z})$ and continue to denote them by the symbol \star . Again because of Table 1 of [35] (see also Theorem 0.6 of [28]), $K_i(\mathbb{Z})$ is a finite odd torsion group if i is a positive integer $\equiv 0, 4$, or $6 \pmod 8$ and $\cong \mathbb{Z} \oplus$ (finite odd torsion group) if $i \equiv 5 \pmod 8$; thus, the only 2-adic products which can be non trivial are the following:

$$\begin{aligned} K_{8s+1}(\mathbb{Z}) \otimes K_{8t+1}(\mathbb{Z}) &\xrightarrow{\star} K_{8(s+t)+2}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2 \\ K_{8s+1}(\mathbb{Z}) \otimes K_{8t+2}(\mathbb{Z}) &\xrightarrow{\star} K_{8(s+t)+3}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2 \\ K_{8s+2}(\mathbb{Z}) \otimes K_{8t+5}(\mathbb{Z}) &\xrightarrow{\star} K_{8(s+t)+7}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2 \\ K_{8s+2}(\mathbb{Z}) \otimes K_{8t+7}(\mathbb{Z}) &\xrightarrow{\star} K_{8(s+t+1)+1}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2 \\ K_{8s+3}(\mathbb{Z}) \otimes K_{8t+7}(\mathbb{Z}) &\xrightarrow{\star} K_{8(s+t+1)+2}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2 \\ K_{8s+5}(\mathbb{Z}) \otimes K_{8t+5}(\mathbb{Z}) &\xrightarrow{\star} K_{8(s+t+1)+2}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2 \end{aligned}$$

for s and $t \geq 0$. We now want to determine these products.

The inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces a map $\lambda : BGL(\mathbb{Z})^+ \rightarrow BO$ and the induced homomorphism

$$\lambda_* : K_*(\mathbb{Z}) \longrightarrow \pi_* BO$$

is a ring homomorphism since it can be written as the composition $\lambda_* : K_*(\mathbb{Z}) \rightarrow K_*(\mathbb{R}) \rightarrow \pi_* BO$, where both arrows are ring homomorphisms (see [13], p. 50 and Section 3). One can understand the kernel of λ_* at the prime 2 by the following argument. If p is a prime $\equiv 3$ or $5 \pmod 8$, M. Bökstedt introduced in [12] (see also [23] and Section 4 of [18]) a space $J(p)$ which is defined by the pull-back diagram

$$\begin{array}{ccc} J(p) & \xrightarrow{\lambda'} & BO \\ \downarrow f_p & & \downarrow c \\ F\Psi^p & \xrightarrow{b} & BU, \end{array}$$

where $F\Psi^p$ is the fiber of $(\Psi^p - 1) : BU \rightarrow BU$ (recall that $F\Psi^p \simeq BGL(\mathbb{F}_p)^+$ by Theorem 7 of [24]), b the Brauer lifting and c the complexification. The fibers of the horizontal maps are homotopy equivalent to the unitary group $U \simeq SU \times S^1$. More precisely, Bökstedt was interested in the covering space $JK(\mathbb{Z}, p)$ of $J(p)$ corresponding to the cyclic subgroup of order 2 of $\pi_1 J(p) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. After completion at the prime 2, he constructed a map

$$\tilde{\varphi} : (BGL(\mathbb{Z}^+)_2)^\wedge \longrightarrow JK(\mathbb{Z}, p)_2^\wedge$$

which induces a split surjection on all homotopy groups. Let us write $\hat{\lambda}$ and $\hat{\lambda}'$ for the 2-completion of the maps λ and λ' respectively: it turns out that the composition

$$(BGL(\mathbb{Z}^+)_2)^\wedge \xrightarrow{\tilde{\varphi}} JK(\mathbb{Z}, p)_2^\wedge \longrightarrow J(p)_2^\wedge \xrightarrow{\hat{\lambda}'} BO_2^\wedge$$

is exactly $\hat{\lambda}$. Recall that the localization exact sequence in K -theory implies that

$$(BGL(\mathbb{Z}[\frac{1}{2}])^+)_2^\wedge \simeq (BGL(\mathbb{Z}^+)_2)^\wedge \times (S^1)_2^\wedge.$$

Therefore, $\tilde{\varphi}$ provides a map

$$\varphi : (BGL(\mathbb{Z}[\frac{1}{2}])^+)_2^\wedge \longrightarrow J(p)_2^\wedge$$

which also induces a split surjection on all homotopy groups. Since the 2-torsion of $K_*(\mathbb{Z})$ is known by Table 1 of [35] and Theorem 0.6 of [28], it is easy to check that $\tilde{\varphi}$ and φ are actually homotopy equivalences. Consequently, we obtain (see also Corollary 8 of [35]):

Proposition 5.5. *For all primes $p \equiv 3$ or $5 \pmod{8}$, there is a pull-back diagram*

$$\begin{array}{ccc} (BGL(\mathbb{Z}^+)_2)^\wedge \times (S^1)_2^\wedge & \xrightarrow{\hat{\lambda}'} & BO_2^\wedge \\ \downarrow \hat{f}_p & & \downarrow \hat{c} \\ (F\Psi^p)_2^\wedge & \xrightarrow{\hat{b}} & BU_2^\wedge. \end{array}$$

Consequently, there is a fibration

$$SU_2^\wedge \xrightarrow{\eta} (BGL(\mathbb{Z}^+)_2)^\wedge \xrightarrow{\hat{\lambda}} BO_2^\wedge.$$

This fibration induces the long exact sequence

$$\dots \longrightarrow \pi_i SU \otimes \widehat{\mathbb{Z}}_2 \xrightarrow{\eta_*} K_i(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2 \xrightarrow{\widehat{\lambda}_*} \pi_i BO \otimes \widehat{\mathbb{Z}}_2 \longrightarrow \pi_{i-1} SU \otimes \widehat{\mathbb{Z}}_2 \longrightarrow \dots$$

Remember that $\pi_i SU = 0$ if i is even and $\pi_i SU \cong \mathbb{Z}$ if i is odd ≥ 3 .

Theorem 5.6. *The 2-adic products*

$$K_{8s+3}(\mathbb{Z}) \otimes K_{8t+7}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t+1)+2}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2$$

$$K_{8s+5}(\mathbb{Z}) \otimes K_{8t+5}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t+1)+2}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2$$

are trivial for all integers s and $t \geq 0$.

Proof. For any product mentioned in the statement of the theorem, let us consider the commutative diagram

$$\begin{array}{ccccc} K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) & \xrightarrow{\star} & K_{i+k}(\mathbb{Z}) & \longrightarrow & K_{i+k}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2 \\ \downarrow \lambda_* \otimes \lambda_* & & \downarrow \lambda_* & & \downarrow \widehat{\lambda}_* \\ \pi_i BO \otimes \pi_k BO & \longrightarrow & \pi_{i+k} BO & \longrightarrow & \pi_{i+k} BO \otimes \widehat{\mathbb{Z}}_2, \end{array}$$

where the bottom left horizontal arrow is the product map in $\pi_* BO$ and where the right horizontal arrows denote the tensor product with $\widehat{\mathbb{Z}}_2$. Let $x \in K_i(\mathbb{Z})$ and $y \in K_k(\mathbb{Z})$. One has clearly $\lambda_*(y) = 0$ since $\pi_k BO = 0$ for $k = 8t + 5$ or $k = 8t + 7$. Thus, $\lambda_*(x \star y) = \lambda_*(x)\lambda_*(y) = 0$. This shows that the 2-adic product $x \star y \in K_{i+k}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2$ belongs to the kernel of $\widehat{\lambda}_*$, and consequently to the image of $\eta_* : \pi_{i+k} SU \otimes \widehat{\mathbb{Z}}_2 \rightarrow K_{i+k}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2$. Since $i + k$ is even, the group $\pi_{i+k} SU$ is trivial and $x \star y$ vanishes. \square

In the next theorem, we look at the groups $K_{8s+1}(\mathbb{Z})$ for $s \geq 0$. Remember that $K_1(\mathbb{Z}) \cong \mathbb{Z}/2$ and that $K_{8s+1}(\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus$ (finite odd torsion group) for $s \geq 1$ by Table 1 of [35] and Theorem 0.6 of [28]. The fibration given by Proposition 5.5 provides the exact sequence

$$0 \longrightarrow \pi_{8s+1} SU \otimes \widehat{\mathbb{Z}}_2 \cong \widehat{\mathbb{Z}}_2 \xrightarrow{\eta_*} K_{8s+1}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2 \xrightarrow{\widehat{\lambda}_*} \pi_{8s+1} BO \otimes \widehat{\mathbb{Z}}_2 \cong \mathbb{Z}/2 \longrightarrow 0$$

if $s \geq 1$ (if $s = 0$, $\pi_1 SU = 0$). Let us write x_s for the element of order 2 in $K_{8s+1}(\mathbb{Z})$. We denote by y_s , for $s \geq 1$, the generator of the infinite cyclic summand

of $K_{8s+1}(\mathbb{Z})$ whose image under the 2-adic completion $K_{8s+1}(\mathbb{Z}) \rightarrow K_{8s+1}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2$ is exactly the image of a generator of $\pi_{8s+1}SU \otimes \widehat{\mathbb{Z}}_2$ under the homomorphism $\eta_* : \pi_{8s+1}SU \otimes \widehat{\mathbb{Z}}_2 \cong \widehat{\mathbb{Z}}_2 \rightarrow K_{8s+1}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2$.

Theorem 5.7. *Consider the 2-adic product*

$$K_{8s+1}(\mathbb{Z}) \otimes K_{8t+1}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t)+2}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2$$

for any integers $s, t \geq 0$.

(a) For all s and $t \geq 1$, $y_s \star y_t = 0$.

(b) For all $s \geq 0$ and all $t \geq 1$, $x_s \star y_t = 0$.

(c) For all s and $t \geq 0$, $x_s \star x_t$ is the generator of $K_{8(s+t)+2}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2 \cong \mathbb{Z}/2$.

Proof. The commutativity of the square

$$\begin{array}{ccc} K_{8s+1}(\mathbb{Z}) & \longrightarrow & K_{8s+1}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2 \\ \downarrow \lambda_* & & \downarrow \widehat{\lambda}_* \\ \pi_{8s+1}BO & \xrightarrow{\cong} & \pi_{8s+1}BO \otimes \widehat{\mathbb{Z}}_2, \end{array}$$

where the horizontal arrows denote the tensor product with $\widehat{\mathbb{Z}}_2$, and the definition of y_s show that $\lambda_*(y_s) = 0$. We deduce similarly that $\lambda_*(y_t) = 0$. This implies the vanishing of $\widehat{\lambda}_*(y_s \star y_t)$ and $\widehat{\lambda}_*(x_s \star y_t)$. The fact that $\pi_{8(s+t)+2}SU = 0$ then enables us to deduce the triviality of the products $y_s \star y_t$ and $x_s \star y_t$ in $K_{8(s+t)+2}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2$. The image of x_s under λ_* is the generator c_s of $\pi_{8s+1}BO \cong \mathbb{Z}/2$ and it is known that $c_s c_t$ is non trivial in $\pi_{8(s+t)+2}BO$ (see [32], p. 304). Therefore, it follows from the equality $\lambda_*(x_s \star x_t) = c_s c_t$ that the product $x_s \star x_t$ does not vanish. \square

Corollary 5.8. *The 2-adic products*

$$K_{8s+2}(\mathbb{Z}) \otimes K_{8t+5}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t)+7}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2$$

$$K_{8s+2}(\mathbb{Z}) \otimes K_{8t+7}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t+1)+1}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2$$

are trivial for all s and $t \geq 0$.

Proof. According to Theorem 5.7 (c),

$$K_{8s+2}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2 \cong (K_1(\mathbb{Z}) \star K_{8s+1}(\mathbb{Z})) \otimes \widehat{\mathbb{Z}}_2.$$

This implies the assertion because the products

$$K_{8s+1}(\mathbb{Z}) \otimes K_{8t+5}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t)+6}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2 = 0$$

and

$$K_{8s+1}(\mathbb{Z}) \otimes K_{8t+7}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t)+1}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2 = 0$$

are obviously trivial (see Table 1 of [35] and Theorem 0.6 of [28]). □

For the next result, let us call z_t the element of order 2 in $K_{8t+2}(\mathbb{Z}) \cong \mathbb{Z}/2 \oplus$ (finite odd torsion group).

Theorem 5.9. *Consider the 2-adic product*

$$K_{8s+1}(\mathbb{Z}) \otimes K_{8t+2}(\mathbb{Z}) \xrightarrow{\star} K_{8(s+t)+3}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2$$

for any integers $s, t \geq 0$.

(a) For all $s \geq 1$ and all $t \geq 0$, $y_s \star z_t = 0$.

(b) For all s and $t \geq 0$, $x_s \star z_t$ is an element of order 2 in $K_{8(s+t)+3}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2$

Proof. Because of Theorem 5.7 (c),

$$(K_{8s+1}(\mathbb{Z}) \star K_{8t+2}(\mathbb{Z})) \otimes \widehat{\mathbb{Z}}_2 \cong (K_1(\mathbb{Z}) \star K_{8s+1}(\mathbb{Z}) \star K_{8t+1}(\mathbb{Z})) \otimes \widehat{\mathbb{Z}}_2$$

and assertion (a) follows from $y_s \star z_t = x_0 \star y_s \star x_t = 0$ by Theorem 5.7 (b). Similarly, $x_s \star z_t = x_0 \star x_s \star x_t$ is non trivial according to Proposition 12.17 of [1] and Corollary 4.6 of [13]. □

We may summarize our results on the 2-adic products in the K -theory of \mathbb{Z} as follows.

Corollary 5.10. *The 2-adic product*

$$\star : K_i(\mathbb{Z}) \otimes K_k(\mathbb{Z}) \longrightarrow K_{i+k}(\mathbb{Z}) \otimes \widehat{\mathbb{Z}}_2$$

is trivial for all positive integers i and k , except if $i \equiv k \equiv 1 \pmod 8$ or $i \equiv 1 \pmod 8$ and $k \equiv 2 \pmod 8$ (or $i \equiv 2 \pmod 8$ and $k \equiv 1 \pmod 8$) where its image is cyclic of order 2.

Let us conclude this section by the following observation about the relationships between products in algebraic K -theory of the ring of integers \mathbb{Z} and the Dwyer-Friedlander map relating the algebraic K -theory of \mathbb{Z} to its étale K -theory (see Section 4 of [17]).

Proposition 5.11. *For any odd prime l and any integer $n \geq 2$, the image of the product map*

$$\star : K_{2n-1}(\mathbb{Z}) \otimes K_{2n-1}(\mathbb{Z}) \longrightarrow K_{4n-2}(\mathbb{Z})$$

is contained in the kernel of the Dwyer-Friedlander map $K_{4n-2}(\mathbb{Z}) \rightarrow K_{4n-2}^{\text{ét}}(\mathbb{Z}[\frac{1}{l}])$.

Proof. The products in algebraic K-theory and étale K-theory commute with the Dwyer-Friedlander map. Observe that

$$K_{2n-1}^{\text{ét}}(\mathbb{Z}[\frac{1}{l}]) = \begin{cases} \hat{\mathbb{Z}}_l & \text{if } n \text{ is odd,} \\ \mathbb{Z}/|w_n(\mathbb{Q})|_l^{-1} & \text{if } n \text{ is even.} \end{cases}$$

Hence $K_{2n-1}^{\text{ét}}(\mathbb{Z}[\frac{1}{l}]) \cong H_{\text{ét}}^1(\mathbb{Z}[\frac{1}{l}]; \mathbb{Z}_l(n))$ is cyclic. But the product in étale K-theory is just the cup product in étale cohomology. This shows that the product

$$K_{2n-1}^{\text{ét}}(\mathbb{Z}[\frac{1}{l}]) \otimes K_{2n-1}^{\text{ét}}(\mathbb{Z}[\frac{1}{l}]) \longrightarrow K_{4n-2}^{\text{ét}}(\mathbb{Z}[\frac{1}{l}])$$

is zero because the cup product $H_{\text{ét}}^1 \otimes H_{\text{ét}}^1 \rightarrow H_{\text{ét}}^2$ is anticommutative. □

6. Products in the algebraic K-theory of cyclotomic fields

The results of Section 5 indicate that the 2-adic products in $K_*(\mathbb{Z})$ are trivial or have a very small image. In this section, we show that in the case of products in the K-theory of number fields, the image of product maps can be quite big. In the proof, we use the methods of [9] and [10]. Let us consider an odd prime number l , a positive integer m , and the cyclotomic field $E = \mathbb{Q}(\xi_{l^m})$ obtained from \mathbb{Q} by adding a primitive root of unity ξ_{l^m} of order l^m . Our goal is to show that for n odd, the product homomorphism

$$\star : K_1(E) \otimes K_{2n-1}(E)_l \longrightarrow K_{2n}(E)_l$$

has a big image.

If R is a commutative ring, X_R is a ring spectrum with respect to $\mu : X_R \wedge X_R \rightarrow X_{R \otimes R} \rightarrow X_R$, where the first map is the pairing which was also called μ at the beginning of Section 3 (see [21], Proposition 2.4.2) and the second is induced by the multiplication $R \otimes R \rightarrow R$. The product structure of $K_*(R)$, also denoted by \star , is given by the composition

$$\star : K_i(R) \otimes K_k(R) \cong \pi_i X_R \otimes \pi_k X_R \xrightarrow{\wedge} \pi_{i+k}(X_R \wedge X_R) \xrightarrow{\mu_*} \pi_{i+k} X_R \cong K_{i+k}(R).$$

Recall that the K-theory with \mathbb{Z}/l^m -coefficients (for a prime l and a positive integer m , with $m \geq 2$ if $l = 2$) may be defined by $K_k(R; \mathbb{Z}/l^m) = \pi_k(M \wedge X_R)$, where M is the mod l^m Moore spectrum (i.e., such that $H_0 M \cong \mathbb{Z}/l^m$, $H_k M = 0$ for $k \neq 0$). Notice that M is a ring spectrum with identity i_M and product μ_M (see [27], p. 22). We also consider the following products (see also [13]):

$$\star : K_i(R) \otimes K_k(R; \mathbb{Z}/l^m) \cong \pi_i X_R \otimes \pi_k(M \wedge X_R) \xrightarrow{\wedge} \pi_{i+k}(M \wedge X_R \wedge X_R) \xrightarrow{\mu_*}$$

$$\pi_{i+k}(M \wedge X_R) \cong K_{i+k}(R; \mathbb{Z}/l^m)$$

and

$$\begin{aligned} \star : K_i(R; \mathbb{Z}/l^m) \otimes K_k(R; \mathbb{Z}/l^m) &\cong \pi_i(M \wedge X_R) \otimes \pi_k(M \wedge X_R) \xrightarrow{\wedge} \\ \pi_{i+k}(M \wedge M \wedge X_R \wedge X_R) &\xrightarrow{(\mu_M \wedge \mu)^*} \pi_{i+k}(M \wedge X_R) \cong K_{i+k}(R; \mathbb{Z}/l^m). \end{aligned}$$

Remark 6.1. Since M is a ring spectrum, the diagram

$$\begin{array}{ccccc} S \wedge S & \xrightarrow{\text{id} \wedge i_M} & S \wedge M & \xrightarrow{i_M \wedge \text{id}} & M \wedge M \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \mu_M \\ S & \xrightarrow{i_M} & M & \xrightarrow{\text{id}} & M \end{array}$$

commutes and implies the compatibility of the three products, i.e., the commutativity of the diagram

$$\begin{array}{ccccc} K_i(R) \otimes K_k(R) & \xrightarrow{\text{id} \otimes \text{red}} & K_i(R) \otimes K_k(R; \mathbb{Z}/l^m) & \xrightarrow{\text{red} \otimes \text{id}} & K_i(R; \mathbb{Z}/l^m) \otimes K_k(R; \mathbb{Z}/l^m) \\ \downarrow \star & & \downarrow \star & & \downarrow \star \\ K_{i+k}(R) & \xrightarrow{\text{red}} & K_{i+k}(R; \mathbb{Z}/l^m) & \xrightarrow{\text{id}} & K_{i+k}(R; \mathbb{Z}/l^m), \end{array}$$

where red is the map which is induced on K -theory by the reduction of coefficients mod l^m .

For any ring R , there is the following Bockstein long exact sequence

$$\dots \longrightarrow K_k(R) \xrightarrow{l^m} K_k(R) \longrightarrow K_k(R; \mathbb{Z}/l^m) \xrightarrow{\mathfrak{b}} K_{k-1}(R) \longrightarrow \dots,$$

where \mathfrak{b} is the Bockstein homomorphism.

Lemma 6.2. For any $x \in K_i(R)$ and any $y \in K_k(R; \mathbb{Z}/l^m)$, one has $\mathfrak{b}(x \star y) = x \star \mathfrak{b}(y) \in K_{i+k-1}(R)$.

Proof. The Bockstein homomorphism \mathfrak{b} is induced by the obvious map $\varepsilon : M \rightarrow \Sigma^{-1}S$ which fits into the commutative diagram

$$\begin{array}{ccc} S \wedge M & \xrightarrow{\cong} & M \\ \downarrow \text{id} \wedge \varepsilon & & \downarrow \varepsilon \\ S \wedge \Sigma^{-1}S & \xrightarrow{\cong} & \Sigma^{-1}S \end{array}$$

and provides the commutativity of

$$\begin{array}{ccc} K_i(R) \otimes K_k(R; \mathbb{Z}/l^m) & \xrightarrow{\star} & K_{i+k}(R; \mathbb{Z}/l^m) \\ \downarrow \text{id} \otimes \mathfrak{b} & & \downarrow \mathfrak{b} \\ K_i(R) \otimes K_{k-1}(R) & \xrightarrow{\star} & K_{i+k-1}(R) \end{array}$$

and the statement of the lemma. □

This lemma implies the formula

$$\mathfrak{b}(Tr_{E/\mathbb{Q}}(u \star \beta_m^{\star n})) = Tr_{E/\mathbb{Q}}(u \star \mathfrak{b}(\beta_m^{\star n})),$$

where u is any element $\in K_1(E) = E^\times$, $\beta_m = \beta(\xi_{l^m}) \in K_2(E; \mathbb{Z}/l^m)$ is the Bott element (see Definition 2.7.2 of [34]) and $Tr_{E/\mathbb{Q}}$ is the transfer map (see [25], Section 4). Using this equality, we can rewrite the definition of the Stickelberger pseudosplitting homomorphism Λ from [8], Section IV.1, or [10], Definition 3.2, as follows.

Definition 6.3. There is a homomorphism

$$\Lambda : \bigoplus_p K_{2n-1}(\mathbb{F}_p)_l \longrightarrow K_{2n}(\mathbb{Q})_l,$$

where $\Lambda = \prod_p \Lambda_p$ and $\Lambda_p : K_{2n-1}(\mathbb{F}_p)_l \rightarrow K_{2n}(\mathbb{Q})_l$ is given by the formula

$$\Lambda_p(\kappa_p) = \begin{cases} Tr_{E/\mathbb{Q}}(\lambda_{\mathfrak{b}}(\mathfrak{p}) \star \mathfrak{b}(\beta_k^{\star n})^{b^n \gamma_l}) & \text{if } l \text{ does not divide } n, \\ Tr_{E/\mathbb{Q}}(\lambda_{\mathfrak{b}}(\mathfrak{p}) \star \mathfrak{b}(\beta_k^{\star n})^{nb^n \gamma_l}) & \text{if } l \text{ divides } n, \end{cases}$$

where b is a natural number such that $(b, w_{n+1}(\mathbb{Q})) = 1$ and κ_p is a generator of the group $K_{2n-1}(\mathbb{F}_p)_l$. In addition, $\lambda_b(\mathfrak{p}) \in E^\times$ are the twisted Gauss sums (see [9], Definition 3) and $\gamma_l = 1 + l^n + l^{2n} + l^{3n} \dots = \frac{1}{1-l^n} \in \widehat{\mathbb{Z}}_l$.

Theorem 6.4. *Let I be the image of the map*

$$K_1(E) \otimes K_{2n-1}(E)_l \xrightarrow{\star} K_{2n}(E)_l \xrightarrow{Tr_{E/\mathbb{Q}}} K_{2n}(\mathbb{Q})_l,$$

where n is an odd integer. Then the exponent of the group $K_{2n}(\mathbb{Q})_l/I$ divides the number $(\#K_{2n}(\mathbb{Z})_l)^2$.

Proof. Consider the localization sequence

$$0 \longrightarrow K_{2n}(\mathbb{Z})_l \longrightarrow K_{2n}(\mathbb{Q})_l \xrightarrow{\partial} \bigoplus_p K_{2n-1}(\mathbb{F}_p)_l \longrightarrow 0.$$

By Definition 6.3, we see that $\text{image } \Lambda \subseteq I$. On the other hand, by Proposition 2 of [9], the composition $\partial \Lambda$ acts by raising into the power with exponent an integer $|(b^{n+1} - 1)\zeta_{\mathbb{Q}}(-n)|_l^{-1}$ (recall that the Gauss sums used in the construction of Λ depend on b), where $\zeta_{\mathbb{Q}}(s)$ is the Riemann zeta function. Consider now $x \in K_{2n}(\mathbb{Q})_l$ and $z = \partial(x) \in \bigoplus_p K_{2n-1}(\mathbb{F}_p)_l$. The computations above show that for every b ,

$$\frac{\Lambda(z)}{x^{|(b^{n+1} - 1)\zeta_{\mathbb{Q}}(-n)|_l^{-1}}} \in K_{2n}(\mathbb{Z})_l.$$

The greatest common divisor of all $|(b^{n+1} - 1)|_l^{-1}$ over the integers b which are relatively prime to $w_{n+1}(\mathbb{Q})$ equals the number $|w_{n+1}(\mathbb{Q})|_l^{-1}$, by Lemma 2.3 of [15]. Since the integer $|w_{n+1}(\mathbb{Q})\zeta_{\mathbb{Q}}(-n)|_l^{-1}$ divides the number t of elements in the group $K_{2n}(\mathbb{Z})_l$, it follows that $x^{t^2} \in \text{image } \Lambda \subseteq I$. □

Remark 6.5. Observe that in an abelian group the product of a torsion element and a nontorsion element is again nontorsion. Hence, every torsion element in $K_{2n-1}(E)$ can be written as a quotient of two nontorsion elements. Consequently, the subgroup of $K_1(E) \star K_{2n-1}(E)$ generated by the subset $\{x \star y \mid x \in K_1(E), y \in K_{2n-1}(E), y \text{ nontorsion}\}$ contains the subgroup $K_1(E) \star K_{2n-1}(E)_l$ of $K_{2n}(E)$.

Remark 6.6. It follows from Theorem 6.4 and Theorem 3.4 of [10] that the image of $K_1(E) \star K_{2n-1}(E)_l$ under the transfer $Tr_{E/\mathbb{Q}} : K_{2n}(E) \rightarrow K_{2n}(\mathbb{Q})$ contains the group of divisible elements $D(n)_l \cong K_{2n}^{\acute{e}t}(\mathbb{Z}[\frac{1}{l}])$ from Section 5.2 of [9] (see also [8], Section IV.3).

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