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# Intersection homology of toric varieties and a conjecture of Kalai 

Tom Braden and Robert MacPherson


#### Abstract

We prove an inequality, conjectured by Kalai, relating the $g$-polynomials of a polytope $P$, a face $F$, and the quotient polytope $P / F$, in the case where $P$ is rational. We introduce a new family of polynomials $g(P, F)$, which measures the complexity of the part of $P$ "far away" from the face $F$; Kalai's conjecture follows from the nonnegativity of these polynomials. This nonnegativity comes from showing that the restriction of the intersection cohomology sheaf on a toric variety to the closure of an orbit is a direct sum of intersection homology sheaves.

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Suppose that a $d$-dimensional convex polytope $P \subset \mathbb{R}^{d}$ is rational, i.e. its vertices have all coordinates rational. Then $P$ gives rise to a polynomial $g(P)=$ $1+g_{1}(P) q+g_{2}(P) q^{2}+\cdots$ with non-negative coefficients as follows. Let $X_{P}$ be the associated toric variety (see $\S 6-$ our variety $X_{P}$ is $d+1$-dimensional and affine). The coefficient $g_{i}(P)$ is the rank of the $2 i$-th intersection cohomology group of $X_{P}$.

The polynomial $g(P)$ turns out to depend only on the face lattice of $P$, (see $\S 1$ ). It can be thought of as a measure of the complexity of $P$; for example, $g(P)=1$ if and only if $P$ is a simplex.

Suppose that $F \subset P$ is a face of dimension $k<d$. We construct an associated polytope $P / F$ as follows: choose an $(d-k-1)$-plane $L$ whose intersection with $P$ is a single point $p$ of the interior of $F$. Let $L^{\prime}$ be a small parallel displacement of $L$ that intersects the interior of $P$. The quotient $P / F$ is the intersection of $P$ with $L^{\prime}$; it is only well-defined up to a projective transformation, but its combinatorial type is well-defined (Formally we put $P / P$ to be the empty polytope). Faces of $P / F$ are in one-to-one correspondence with faces of $P$ which contain $F$.

In Corollary 6, we show that

$$
g(P) \geq g(F) g(P / F)
$$

holds, coefficient by coefficient. This was conjectured by Kalai in [11], where some of its applications were discussed. The special case of the linear and quadratic

[^0]terms was proved in [12]. Roughly, this inequality means that the complexity of $P$ is bounded from below by the complexity of the face $F$ and the normal complexity $g(P / F)$ to the face $F$.

The principal idea is to introduce relative $g$-polynomials $g(P, F)$ for any face $F$ of $P(\S 2)$. These generalize the ordinary $g$-polynomials since $g(P, P)=g(P)$. They are also combinatorially determined by the face lattice. They measure the complexity of $P$ relative to the complexity of $F$. For example, if $P$ is the join of $F$ with another polytope, then $g(P, F)=1$ (the converse, however, does not hold).

Our main result gives an interpretation of the coefficients $g_{i}(P, F)$ of the relative $g$-polynomials as dimensions of vector spaces arising from the topology of the toric variety $X_{P}$. This shows that the coefficients are positive. Kalai's conjecture is a corollary.

The combinatorial definition of the relative $g$-polynomials $g(P, F)$ makes sense whether or not the polytope $P$ is rational. We conjecture that $g(P, F) \geq 0$ for any polytope $P$; this would imply Kalai's conjecture for general polytopes.

This paper is organized as follows: The first three sections are entirely about the combinatorics of polyhedra. They develop the properties of relative $g$-polynomials as combinatorial objects, with the application to Kalai's conjecture. The last three sections concern algebraic geometry. A separate guide to their contents is included in the introduction to $\S \S 4-6$.

## 1. $g$-numbers of polytopes

Let $P \subset \mathbb{R}^{d}$ be a $d$-dimensional convex polytope, i.e. the convex hull of a finite collection of points affinely spanning $\mathbb{R}^{d}$. The set of faces of $P$, ordered by inclusion, forms a poset which we will denote by $\mathcal{F}(P)$. We include the empty face $\emptyset=\emptyset_{P}$ and $P$ itself as members of $\mathcal{F}(P)$. It is a graded poset, with the grading given by the dimension of faces. By convention we set $\operatorname{dim} \emptyset=-1$. Faces of $P$ of dimension 0,1 , and $d-1$ will be referred to as vertices, edges, and facets, respectively.

Given a face $F$ of $P$, the poset $\mathcal{F}(F)$ is isomorphic to the interval $[\emptyset, F] \subset$ $\mathcal{F}(P)$. The interval $[F, P]$ is the face poset of the polytope $P / F$ defined in the introduction.

Given the polytope $P$, there are associated polynomials (first introduced in [14]) $g(P)=\sum g_{i}(P) q^{i}$ and $h(P)=\sum h_{i}(P) q^{i}$, defined recursively as follows:

- $g(\emptyset)=1$
- $h(P)=\Sigma_{\emptyset \leq F<P}(q-1)^{\operatorname{dim} P-\operatorname{dim} F-1} g(F)$, and
- $g_{0}(P)=h_{0}(P), g_{i}(P)=h_{i}(P)-h_{i-1}(P)$ for $0<i \leq \operatorname{dim} P / 2$, and $g_{i}(P)=0$ for all other $i$.
The coefficients of these polynomials will be referred to as the $g$-numbers and $h$-numbers of $P$, respectively. We do not discuss the $h$-polynomial further in this paper.

These numbers depend only on the poset $\mathcal{F}(P)$. In fact, as Bayer and Billera
[1] showed, they depend only on the flag numbers of $P$ : given a sequence of integers $I=\left(i_{1}, \ldots, i_{n}\right)$ with $0 \leq i_{1}<i_{2}<\cdots<i_{n} \leq d$, an $I$-flag is an $n$-tuple $F_{1}<F_{2}<\cdots<F_{n}$ of faces of $P$ with $\operatorname{dim} F_{k}=i_{k}$ for all $k$. The $I$-th flag number $f_{I}(P)$ is the number of $I$-flags. Letting $P$ vary over all polytopes of a given dimension $d$, the numbers $g_{i}(P)$ and $h_{i}(P)$ can be expressed as a $\mathbb{Z}$-linear combination of the $f_{I}(P)$.

Conjecturally all the $g_{i}(P)$ should be nonnegative for all $P$. This is known to be true for $i=1,2[10]$. For higher values of $i$, it can be proved for rational polytopes using the interpretation of $g_{i}(P)$ as an intersection cohomology Betti number of an associated toric variety.

Proposition 1. If $P$ is a rational polytope, then $g_{i}(P) \geq 0$ for all $i$.

## 2. Relative $g$-polynomials

The following proposition defines a relative version of the classical $g$-polynomials.
Proposition 2. There is a unique family of polynomials $g(P, F)$ associated to a polytope $P$ and a face $F$ of $P$, satisfying the following relation: for all $P, F$, we have

$$
\begin{equation*}
\sum_{F \leq E \leq P} g(E, F) g(P / E)=g(P) . \tag{1}
\end{equation*}
$$

Proof. The equation (1) can be used inductively to compute $g(P, F)$, since the left hand side gives $g(P, F) \cdot 1$ plus terms involving $g(E, F)$ where $\operatorname{dim} E<\operatorname{dim} P$. The induction starts when $P=F$, which gives $g(F, F)=g(F)$.

As an example, if $F$ is a facet of $P$, then $g(P, F)=g(P)-g(F)$. Just as before we will denote the coefficient of $q^{i}$ in $g(P, F)$ by $g_{i}(P, F)$.

We have the following notion of relative flag numbers. Let $P$ be a $d$-polytope, and $F$ a face of dimension $e$. Given a sequence of integers $I=\left(i_{1}, \ldots, i_{n}\right)$ with $0 \leq i_{1}<i_{2}<\cdots<i_{n} \leq d$ and a number $1 \leq k \leq n$ with $i_{k} \geq e$, define the relative flag number $f_{I, k}(P, F)$ to be the number of $I$-flags $\left(F_{1}, \ldots, F_{n}\right)$ with $F \leq F_{k}$. Note that letting $k=n$ and $i_{n}=d$ gives the ordinary flag numbers of $P$ as a special case. Also note that the numbers $f_{I, k}$ where $i_{k}=e$ give products of the form $f_{J}(F) f_{J^{\prime}}(P / F)$, and all such products can be expressed this way.

Proposition 3. Fixing $\operatorname{dim} P$ and $\operatorname{dim} F$, the relative $g$-number $g_{i}(P, F)$ is a $\mathbb{Z}$ linear combination of the $f_{I, k}(P, F)$.

Proof. Use induction on $\operatorname{dim} P / F$. If $P=F$, then we have $g(P, P)=g(P)$ and the result is just the corresponding result for the ordinary flag numbers. If $P \neq F$,
the equation (1) gives

$$
g(P, F)=g(P)-\sum_{e=\operatorname{dim}}^{\operatorname{dim} P-1} \sum_{\substack{\operatorname{dim} E=e \\ F \leq E<P}} g(E, F) g(P / E) .
$$

For every $e$ the coefficients of the inner summation on the right hand side are $\mathbb{Z}$-linear combinations of the $f_{I, k}(P, F)$, using the inductive hypothesis.

The following theorem is the main result of this paper. It will be a consequence of Theorem 11.

Theorem 4. If $P$ is a rational polytope and $F$ is any face, then $g_{i}(P, F) \geq 0$ for all $i$.

Corollary 5. (Kalai's conjecture) If $P$ is a rational polytope and $F$ is any face, then

$$
g(P) \geq g(F) g(P / F),
$$

where the inequality is taken coefficient by coefficient.
Proof. For any face $E$ of $P$ the polytope $P / E$ is rational, so we have $g(P)=$ $g(F, F) g(P / F)+$ other nonnegative terms.

## 3. Some examples and formulas

This section contains further combinatorial results on the relative $g$-polynomials. They are not used in the remainder of the paper.

First, we give an interpretation of $g_{1}(P, F)$ and $g_{2}(P, F)$ analogous to the ones Kalai gave for the usual $g_{1}$ and $g_{2}$ in [10]. We begin by recalling those results from [10].

Given a finite set of points $V \subset \mathbb{R}^{d}$ define the space $\mathcal{A} f f(V)$ of affine dependencies of $V$ to be

$$
\left\{a \in \mathbb{R}^{V} \mid \Sigma_{v \in V} a_{v}=0, \Sigma_{v \in V} a_{v} \cdot v=0\right\}
$$

If $V_{P}$ is the set of vertices of a polytope $P \subset \mathbb{R}^{d}$, then $\mathcal{A} f f\left(V_{P}\right)$ is a vector space of dimension $g_{1}(P)$.

To describe $g_{2}(P)$ we need the notion of stress on a framework. A framework $\Phi=(V, E)$ is a finite collection $V$ of points in $\mathbb{R}^{d}$ together with a finite collection $E$ of straight line segments (edges) joining them. Given a finite set $S$, we denote the standard basis elements of $\mathbb{R}^{S}$ by $1_{s}, s \in S$. The space of stresses $\mathcal{S}(\Phi)$ is the kernel of the linear map

$$
\alpha: \mathbb{R}^{E} \rightarrow \mathbb{R}^{V} \otimes \mathbb{R}^{d}
$$

defined by

$$
\alpha\left(1_{e}\right)=1_{v_{1}} \otimes\left(v_{1}-v_{2}\right)+1_{v_{2}} \otimes\left(v_{2}-v_{1}\right),
$$

where $v_{1}$ and $v_{2}$ are the endpoints of the edge $e$. A stress can be described physically as an assignment of a contracting or expanding force to each edge, such that the total force resulting at each vertex is zero.

To a polytope $P$ we can associate a framework $\Phi_{P}$ by taking as vertices the vertices of $P$, and as edges the edges of $P$ together with enough extra edges to triangulate all the 2 -faces of $P$. Then $g_{2}(P)$ is the dimension of $\mathcal{S}\left(\Phi_{P}\right)$.

Given a polytope $P$ and a face $F$, define the closed union of faces $N(P, F)$ to be the union of all facets of $P$ containing $F$. Note that $N(P, \emptyset)=\partial P$, and $N(P, P)=\emptyset$. Let $V_{N}$ be the set of vertices of $P$ in $N(P, F)$, and define a framework $\Phi_{N}$ by taking all edges and vertices of $\Phi_{P}$ contained in $N(P, F)$.

Theorem 6. We have

$$
\begin{gathered}
g_{1}(P, F)=\operatorname{dim}_{\mathbb{R}} \mathcal{A} f f\left(V_{P}\right) / \mathcal{A} f f\left(V_{N}\right), \text { and } \\
g_{2}(P, F)=\operatorname{dim}_{\mathbb{R}} \mathcal{S}\left(\Phi_{P}\right) / \mathcal{S}\left(\Phi_{N}\right),
\end{gathered}
$$

using the obvious inclusions of $\mathcal{A} f f\left(V_{N}\right)$ in $\mathcal{A f f}\left(V_{P}\right)$ and $\mathcal{S}\left(\Phi_{N}\right)$ in $\mathcal{S}\left(\Phi_{P}\right)$.
The proof for $g_{1}$ is an easy exercise; the proof for $g_{2}$ will appear in a forthcoming paper [3].

Next, we have a formula which shows that $g(P, F)$ can be decomposed in the same way $g(P)$ was in Proposition 2. Given two faces $E, F$ of a polytope $P$, let $E \vee F$ be the unique smallest face containing both $E$ and $F$.

Proposition 7. For any polytope $P$ and faces $F^{\prime} \leq F$ of $P$, we have

$$
g(P, F)=\sum_{F^{\prime} \leq E} g\left(E, F^{\prime}\right) g(P / E,(E \vee F) / E) .
$$

Proof. Again, we show that this formula for $g(P, F)$ satisfies the defining relation of Proposition 2. Fix $F^{\prime} \leq F$, and define $\hat{g}(P, F)$ to be the above sum. Then we have

$$
\begin{aligned}
\sum_{F \leq D} \hat{g}(D, F) g(P / D) & =\sum_{\substack{F^{\prime} \leq E \\
F \vee E \leq D}} g(P / D) g\left(E, F^{\prime}\right) g(D / E,(E \vee F) / E) \\
& =\sum_{F^{\prime} \leq E} g\left(E, F^{\prime}\right) g(P / E) \\
& =g(P)
\end{aligned}
$$

Since the computation of $g(P, F)$ from Proposition 2 only involves computation of $g(E, F)$ for other faces $E$ of $P$, this proves that $\hat{g}(P, F)=g(P, F)$, as required.

Finally, we can carry out the inversion implicit in Proposition 2 explicitly. First we need the notion of polar polytopes. Given a polytope $P \subset \mathbb{R}^{d}$, we can assume that the origin lies in the interior of $P$ by moving $P$ by an affine motion. The polar polytope $P^{*}$ is defined by

$$
P^{*}=\left\{x \in\left(\mathbb{R}^{*}\right)^{d} \mid\langle x, y\rangle \leq 1 \text { for all } y \in P\right\} .
$$

The face poset $\mathcal{F}\left(P^{*}\right)$ is canonically the opposite poset to $\mathcal{F}(P)$. Define $\bar{g}(P)=$ $g\left(P^{*}\right)$.

Proposition 8. We have

$$
\begin{equation*}
g(P, F)=\sum_{F \leq F^{\prime} \leq P}(-1)^{\operatorname{dim} P-\operatorname{dim} F^{\prime}} g\left(F^{\prime}\right) \bar{g}\left(P / F^{\prime}\right) . \tag{2}
\end{equation*}
$$

Proof. We use the following formula, due to Stanley [15]: For any polytope $P \neq \emptyset$, we have

$$
\begin{equation*}
\sum_{\emptyset \leq F \leq P}(-1)^{\operatorname{dim} F} \bar{g}(F) g(P / F)=0 . \tag{3}
\end{equation*}
$$

Now define $\hat{g}(P, F)$ to be the right hand side of (2). We will show that the defining property (1) of Proposition 2 holds.

Pick a face $F$ of $P$. We have, using (3),

$$
\begin{aligned}
& \sum_{F \leq E \leq P} \hat{g}(E, F) g(P / E)=\sum_{F \leq F^{\prime} \leq E \leq P}(-1)^{\operatorname{dim} E-\operatorname{dim} F^{\prime} g\left(F^{\prime}\right) \bar{g}\left(E / F^{\prime}\right) g(P / E)} \\
& \quad=\sum_{F \leq F^{\prime} \leq P} g\left(F^{\prime}\right) \sum_{F^{\prime} \leq E \leq P}(-1)^{\operatorname{dim} E-\operatorname{dim} F^{\prime}} \bar{g}\left(E / F^{\prime}\right) g(P / E) \\
& \quad=g(P),
\end{aligned}
$$

as required.

## Introduction to $\S \S 4-6$

The remainder of the paper uses the topology of toric varieties to describe the polynomial $g(P, F)$ when $P$ is rational. Given $P$, there is an associated affine toric variety $X_{P}$, and $g(P)$ gives the local intersection cohomology betti numbers of $X_{P}$ at the unique torus fixed point $p$.

The main topological result is the following (Theorem 10). Let $Y \subset X_{P}$ be the closure of one of the torus orbits. Then the restriction of the intersection cohomology sheaf IC $(X)$ to $Y$ is a direct sum of intersection cohomology sheaves, with shifts, supported on subvarieties of $Y$ (a related result is given by Victor Ginzburg in [8], Lemma 3.5). The polynomial $g_{i}(P, F)$ measures the number of copies of the intersection cohomology sheaf IC $(\{p\})$ that appear with shift $2 i$ in the restriction of the intersection cohomology sheaf of $X_{P}$ to $Y_{F}$, where $Y_{F}$ is the closure of the orbit corresponding to the face $F$.

To prove Theorem 10 we construct a certain resolution (the Seifert resolution, §5) $p: \widetilde{X} \rightarrow X$ of $X$. Its key property is that the inclusion of $\widetilde{Y}=p^{-1}(Y)$ in $\widetilde{X}$ is "Q-homology normally nonsingular" - the restriction of the intersection cohomology sheaf of $\widetilde{X}$ to $\widetilde{Y}$ is an intersection cohomology sheaf (Proposition 14).

This construction, and hence Theorem 10, work in situations other than toric varieties; essentially any variety $X$ with a $\mathbb{C}^{*}$ action contracting $X$ onto the fixed point set $Y$ will satisfy Theorem 10 . The proof we give, while easier than the general result, only works for toric varieties.

## 4. Toric varieties

We will only sketch the properties of toric varieties that we will need. For a more complete presentation, see [7]. Throughout this section let $P$ be a $d$-dimensional rational polytope in $\mathbb{R}^{d}$.

Define a toric variety $X_{P}$ as follows. Embed $\mathbb{R}^{d}$ into $\mathbb{R}^{d+1}$ by

$$
\left(x_{1}, \ldots, x_{d}\right) \mapsto\left(x_{1}, \ldots, x_{d}, 1\right)
$$

and let $\sigma=\sigma_{P}$ be the cone over the image of $P$ with apex at the origin in $\mathbb{R}^{d+1}$. It is a rational polyhedral cone with respect to the standard lattice $N=\mathbb{Z}^{d+1}$. More generally, if $F$ is a face of $P$, let $\sigma_{F}$ be the cone over the image of $F$; set $\sigma_{\emptyset}=\{0\}$.

Define $X=X_{P}$ to be the affine toric variety $X_{\sigma}$ corresponding to $\sigma$. It is the variety Spec $\mathbb{C}\left[M \cap \sigma^{\vee}\right]$, where

$$
\sigma^{\vee}=\left\{\mathbf{x} \in\left(\mathbb{R}^{d+1}\right)^{*} \mid\langle\mathbf{x}, \mathbf{y}\rangle \geq 0 \quad \text { for all } \mathbf{y} \in \sigma\right\}
$$

is the dual cone to $\sigma, M$ is the dual lattice to $N$, and $\mathbb{C}\left[M \cap \sigma^{\vee}\right]$ is the semigroup algebra of $M \cap \sigma^{\vee}$. It is a $(d+1)$-dimensional normal affine algebraic variety, on
which the torus $T=\operatorname{Hom}\left(M, \mathbb{C}^{*}\right)$ acts. Let $f_{\mathrm{v}}: X_{P} \rightarrow \mathbb{C}$ be the regular function corresponding to the point $\mathbf{v} \in M \cap \sigma^{\vee}$.

The orbits of the action of $T$ on $X$ are parametrized by the faces of $P$. Let $F$ be any face of $P$, including the empty face, and let

$$
\sigma_{F}^{\perp}=\left\{\mathbf{x} \in \sigma^{\vee} \mid\langle\mathbf{x}, \mathbf{y}\rangle=0 \quad \text { for all } \quad \mathbf{y} \in \sigma_{F}\right\}
$$

be the face of $\sigma^{\vee}$ dual to $\sigma_{F}$. Then the variety

$$
O_{F}:=\left\{x \in X \mid f_{\mathbf{v}}(x) \neq 0 \Longleftrightarrow \mathbf{v} \in M \cap \sigma_{F}^{\perp}\right\}
$$

is a $T$-orbit, isomorphic to the torus $\left(\mathbb{C}^{*}\right)^{d-e}$, where $e=\operatorname{dim} F$. Furthermore, all $T$-orbits arise this way. In particular, $X_{P}$ has a unique $T$-fixed point $\{p\}=O_{P}$.

Given a face $E$, the union

$$
U_{E}=\bigcup_{F \leq E} O_{F}
$$

is a $T$-invariant open neighborhood of $O_{E}$. There is a non-canonical isomorphism $U_{E} \cong O_{E} \times X_{E}$ where $X_{E}$ is the affine toric variety defined by the cone $\sigma_{E}$, considered as a subset of the affine it spans, with the lattice given by restricting $N$. If $O_{F}^{E}$ denotes the orbit of $X_{E}$ corresponding to a face $F \leq E$, then $O_{F}$ sits in $U_{E} \cong O_{E} \times X_{E}$ as $O_{E} \times O_{F}^{E}$.

The closure of the orbit $O_{E}$ is given by

$$
\overline{O_{E}}=\bigcup_{F \geq E} O_{F} ;
$$

it is isomorphic to the affine toric variety $X_{P / E}$. More precisely, it is the affine toric variety corresponding to the cone $\tau=\sigma / \sigma_{E}$, the image of $\sigma$ projected into $\mathbb{R}^{d+1} / \operatorname{span} \sigma_{E}$, with the lattice given by the projection of $N ; \tau$ is a cone over a polytope projectively equivalent to $P / E$.

The connection between toric varieties and $g$-numbers of polytopes is given by the following result. Proofs appear in [5, 6]. We consider the intersection cohomology sheaf IC $(X)$ of a variety $X$ as an object in the bounded derived category $D^{b}(X)$ of sheaves of $\mathbb{Q}$-vector spaces on $X$. We will take the convention that IC $(X)$ restricts to a constant local system placed in degree zero on an smooth open subset of $X$.

Proposition 9. The local intersection cohomology groups of $X_{P}$ are described as follows. Take $x \in O_{F}$, and let $j_{x}$ be the inclusion. Then

$$
\operatorname{dim} \mathbb{H}^{2 i} j_{x}^{*} \mathbf{I C}\left(X_{P}\right)=g_{i}(F),
$$

and $\mathbb{H}^{k} j_{x}^{*} \mathbf{I C}\left(X_{P}\right)$ vanishes for odd $k$.

Definition. Call an object $\mathbf{A}$ in $D^{b}(X)$ pure if it is a direct sum of shifted intersection cohomology sheaves

$$
\begin{equation*}
\bigoplus_{\alpha} \mathbf{I C}\left(Z_{\alpha} ; \mathcal{L}_{\alpha}\right)\left[n_{\alpha}\right], \tag{4}
\end{equation*}
$$

where each $Z_{\alpha}$ is an irreducible subvariety of $X, \mathcal{L}_{\alpha}$ is a simple local system on a Zariski open subset $U_{\alpha}$ of the smooth locus of $Z_{\alpha}$, and $n_{\alpha}$ is an integer.

Now fix a face $F$ of $P$. The following theorem is the main result of this paper. It will be proved in the following two sections.

Theorem 10. Let $j: \overline{O_{F}} \rightarrow X_{P}$ be the inclusion. Then the pullback $\mathbf{A}=j^{*} \mathbf{I C}\left(X_{P}\right)$ of the intersection cohomology sheaf on $X_{P}$ is pure.

As a result, since the local intersection cohomology exists only in even degrees and gives trivial local systems on the orbits $O_{Y}$, we get

$$
\begin{equation*}
\mathbf{A}=\bigoplus_{E \geq F} \bigoplus_{i \geq 0} \mathbf{I C}\left(\overline{O_{E}}\right)[-2 i] \otimes V_{E}^{i} \tag{5}
\end{equation*}
$$

for some finite dimensional $\mathbb{Q}$-vector spaces $V_{E}^{i}$.
Now we can give an interpretation of the combinatorially defined polynomials $g(P, F)$ for rational polytopes which implies nonnegativity, and hence Theorem 4. Let $\{p\}=O_{P}$ be the unique $T$-fixed point of $X_{P}$.

Theorem 11. The relative $g$-number $g_{i}(P, F)$ is given by

$$
g_{i}(P, F)=\operatorname{dim}_{\mathbb{Q}} V_{P}^{i}
$$

Proof. Taking this for the moment as a definition of $g(P, F)$, we will show that the defining relation of Proposition 2 holds. It will be enough to show that $\operatorname{dim}_{\mathbb{Q}} V_{E}^{i}=$ $g(E, F)$ for a face $F \leq E \neq P$, since then taking the dimensions of the stalk cohomology groups on both sides of (5) gives exactly the desired relation (1).

Consider the commutative diagram of inclusions

where $k$ maps $X_{E} \cong\{x\} \times X_{E}$ into $O_{E} \times X_{E} \cong U_{E} \subset X_{P}$, and $k^{\prime}$ is the restriction of $k$.

Then $k$ is a normally nonsingular inclusion, so we have

$$
\begin{gathered}
\left(j^{\prime}\right)^{*} k^{*} \mathbf{I C}\left(X_{P}\right)=\left(j^{\prime}\right)^{*} \mathbf{I C}\left(X_{E}\right)= \\
\bigoplus_{F \leq F^{\prime} \leq E} \bigoplus_{i \geq 0} \mathbf{I C}\left(\overline{O_{F^{\prime}}^{E}}\right)[-2 i] \otimes W_{F^{\prime}}^{i}
\end{gathered}
$$

for some vector spaces $W_{F^{\prime}}^{i}$. On the other hand, since $k^{\prime}$ is a normally nonsingular inclusion, it is also equal to

$$
\left(k^{\prime}\right)^{*} j^{*} \mathbf{I C}\left(X_{P}\right)=\bigoplus_{F \leq F^{\prime} \leq E} \bigoplus_{i \geq 0} \mathbf{I C}\left(\overline{O_{F^{\prime}}^{E}}\right)[-2 i] \otimes V_{F^{\prime}}^{i}
$$

Where the $V_{F^{\prime}}^{i}$ are as in (5).
Comparing terms, we see that $W_{E}^{i} \cong V_{E}^{i}$, so we have

$$
\operatorname{dim}_{\mathbb{Q}} V_{E}^{i}=\operatorname{dim}_{\mathbb{Q}} W_{E}^{i}=g_{i}(E, F)
$$

as required.

## 5. The Seifert resolution

Fix a face $F$ of the polytope $P$, and let $\tau=\sigma_{F}, X=X_{P}, Y=Y_{F}$. Our proof of Theorem 10 involves constructing a certain resolution $\widetilde{X}$ of $X$, which we call a Seifert resolution of the pair $(X, Y)$. First we need to choose an action of $\mathbb{C}^{*}$ on $X$ for which $Y$ is the fixed-point set.

Let a be any lattice point in the relative interior of the cone $\tau$. Define the rank-one subtorus $T_{\mathbf{a}} \subset T \cong \operatorname{Hom}\left(M, \mathbb{C}^{*}\right)$ to be the kernel of the restriction

$$
\operatorname{Hom}\left(M, \mathbb{C}^{*}\right) \rightarrow \operatorname{Hom}\left(M \cap \mathbf{a}^{\perp}, \mathbb{C}^{*}\right)
$$

The map $M \rightarrow \mathbb{Z}$ given by pairing with a defines a homomorphism $\mathbb{C}^{*}=\operatorname{Hom}\left(\mathbb{Z}, \mathbb{C}^{*}\right)$ $\rightarrow T=\operatorname{Hom}\left(M, \mathbb{C}^{*}\right)$ with image contained in $T_{\mathbf{a}}$, thus defining an action of $\mathbb{C}^{*}$ on $X$.

Proposition 12. $Y$ is the fixed-point set of this action, and for any $x \in X$ we have

$$
\lim _{t \rightarrow 0} t \cdot x \in Y
$$

We say that $Y$ is an attractor for the $\mathbb{C}^{*}$ action
Let $X^{\circ}=X \backslash Y$. By the proposition above, the map $X^{\circ} \times \mathbb{C}^{*} \rightarrow X^{\circ}$ defined by our $\mathbb{C}^{*}$ action extends to a map $X^{\circ} \times \mathbb{C} \rightarrow X$. Let $\widetilde{X}$ be the quotient $X^{\circ} \times \mathbb{C} / \sim$, where the equivalence relation is given by

$$
(x, s) \sim\left(t \cdot x, t^{-1} s\right)
$$

for $t \in \mathbb{C}^{*}$. There is an induced map $p: \widetilde{X} \rightarrow X$. We can let $T$ act on $X^{\circ} \times \mathbb{C}$ by acting on the first factor; this action passes to $\widetilde{X}$, and $p$ is an equivariant map. Let $\widetilde{Y}=p^{-1}(Y), \widetilde{X}^{\circ}=p^{-1}\left(X^{\circ}\right)$.

Proposition 13. The map p is proper, and restricts to an isomorphism $\widetilde{X}^{\circ} \cong X^{\circ}$. Furthermore, $\widetilde{Y} \cong\left(X^{\circ} \times\{0\}\right) / \mathbb{C}^{*}$ is a divisor in $\widetilde{X}$, and is an attractor for the the $\mathbb{C}^{*}$ action on $\widetilde{X}$ defined by the lattice point $a$.

We call the pair ( $\tilde{X}, \widetilde{Y}$ ) a Seifert resolution of $(X, Y)$. The action of $T$ makes $\tilde{X}$ into a toric variety. An explicit description of its fan will be useful. Take a fan consisting of all cones of the form $\rho$ and $\rho_{\mathbf{a}}=\rho+\mathbb{R}_{>0} \mathbf{a}$, where $\rho$ runs over all faces of $\sigma$ which do not contain $\tau$. Then $\widetilde{X}$ is the toric variety defined by this fan, and $\widetilde{Y}$ is the union of the orbits corresponding to the cones $\rho_{\mathbf{a}}$.

The inclusion $\tilde{J}: \widetilde{Y} \rightarrow \widetilde{X}$ looks almost like the inclusion of the zero section of a line bundle; for instance, if $X$ is conical, $Y=\{p\}$ is the cone point and $a$ is chosen to give the conical $\mathbb{C}^{*}$ action, then $\tilde{X}$ is just the blow-up of $X$ along $Y$.

Proposition 14. There is an isomorphism

$$
\tilde{\jmath}^{*} \mathbf{I C}(\widetilde{X}) \cong \mathbf{I C}(\widetilde{Y}) .
$$

We will prove this in the next section; first, we show how it implies Theorem 10. Consider the fiber square

where $q=\left.p\right|_{\widetilde{Y}}$. Because $p$ and $q$ are proper we have

$$
R q_{*} \tilde{\jmath}^{*} \mathbf{I C}(\tilde{X}) \cong j^{*} R p_{*} \mathbf{I C}(\tilde{X})
$$

The left hand side is $R q_{*} \mathbf{I C}(\tilde{Y})$ by Proposition 14, which is pure by the decomposition theorem [2]. The decomposition theorem also implies that $\mathbf{A}=R p_{*} \mathbf{I C}(\widetilde{X})$ is pure, and because $\widetilde{X} \rightarrow X$ is an isomorphism on a Zariski dense subset, the intersection cohomology sheaf of $X$ must occur in $\mathbf{A}$ with zero shift. Thus the right hand side becomes

$$
j^{*}(\mathbf{I C}(X)) \oplus j^{*} \mathbf{A}^{\prime}
$$

where $\mathbf{A}^{\prime}$ is pure. Theorem 10 now follows from the following lemma.
Lemma 15. If $\mathbf{A}, \mathbf{B}$ are objects in $D^{b}(X)$ and $\mathbf{A} \oplus \mathbf{B}$ is pure, then so is $\mathbf{A}$.

Proof. Denote $\mathbf{A} \oplus \mathbf{B}$ by $\mathbf{C}$. Since $\mathbf{C}$ is pure, it is isomorphic to the direct sum

$$
\bigoplus_{i \in \mathbb{Z}}^{p} H^{i}(\mathbf{C})[-i]
$$

of its perverse homology sheaves. Each ${ }^{p} H^{i}(\mathbf{C})={ }^{p} H^{i}(\mathbf{A}) \oplus{ }^{p} H^{i}(\mathbf{B})$ is a pure perverse sheaf, and since the category of perverse sheaves is abelian, ${ }^{p} H^{i}(\mathbf{A})$ is pure. Then the composition

$$
\bigoplus^{p} H^{i}(\mathbf{A})[-i] \rightarrow \bigoplus^{p} H^{i}(\mathbf{C})[-i] \cong \mathbf{C} \rightarrow \mathbf{A}
$$

induces an isomorphism on all the perverse homology sheaves, and hence is an isomorphism (see [2], §1.3).

## 6. Proof of Proposition 14

Let $\mathbf{A}=\tilde{\jmath}^{*} \mathbf{I C}(\widetilde{X})$. We will show that $\mathbf{A}$ satisfies the vanishing conditions for intersection cohomology on the stalk and costalk cohomology groups [9], and thus must be isomorphic to IC $(\tilde{Y})$.

If $\widetilde{X}$ is a line bundle over $\widetilde{Y}$, the result is immediate. In general, we can take a quotient by a finite group which acts trivially on $\widetilde{Y}$ and get a line bundle. This works for more general varieties than toric varieties, but for our purposes a combinatorial proof will suffice.

We continue the notation of the previous section. For each face $\rho$ not containing $\tau$, let $n_{\rho}$ be the index of the lattice $(N \cap \operatorname{span}(\rho))+\mathbf{a} \mathbb{Z}$ in $N$. If $n=\operatorname{lcm} n_{\rho}$, then we can define a lattice $N^{\prime}=N+(\mathbf{a} / n) \mathbb{Z}$ containing $N$. We get a corresponding map of tori $T \rightarrow T^{\prime}$; the kernel $G$ is a finite cyclic group inside $T_{\mathbf{a}}$.

Proposition 16. The quotient $\tilde{X} / G$ is a line bundle over $\tilde{Y} / G \cong \tilde{Y}$.
Using this, we prove Proposition 14. We can retract $\widetilde{X}$ onto $\widetilde{Y}$ using the $\mathbb{C}^{*}$ action; we get an isomorphism

$$
\mathbf{A} \cong R \pi_{*} \mathbf{I C}(\widetilde{X}),
$$

where $\pi: \widetilde{X} \rightarrow \widetilde{Y}$ is the projection defined by the action.
For a point $y \in \widetilde{Y}$, we can find a neighborhood $N \subset \tilde{Y}$ of $y$ so that the stalk and costalk cohomology groups of $\mathbf{A}$ are given by

$$
\begin{aligned}
& \mathbb{H}^{i} i_{y}^{*} \mathbf{A}=I H_{n-i}\left(\pi^{-1}(N), \pi^{-1}(\partial N)\right), \\
& \mathbb{H}^{i} i_{y}^{!} \mathbf{A}=I H_{n-i}\left(\pi^{-1}(N)\right) .
\end{aligned}
$$

Since $G \subset T_{\mathbf{a}}$, elements of $G$ preserve the fibers of $\pi$ and act by transformations which are isotopic to the identity. Thus $G$ acts trivially on the stalks and costalks of $\mathbf{A}$. The following lemma then shows that they are isomorphic to $I H_{n-i}\left(\pi^{-1}(N) / G, \pi^{-1}(\partial N) / G\right)$ and $I H_{n-i}\left(\pi^{-1}(N) / G\right)$, respectively, and hence to $I H_{n-i}(N, \partial N)$ and $I H_{n-i}(N)$, since $\widetilde{X} / G$ is a line bundle over $\widetilde{Y}$. The required vanishing follows immediately.

Lemma 17. Let $X$ be a pseudomanifold, acted on by a finite group $G$, and let $Y$ be a $G$-invariant subspace. Then there is an isomorphism

$$
I H_{*}(X / G, Y / G ; \mathbb{Q}) \cong I H_{*}(X, Y ; \mathbb{Q})^{G}
$$

between the intersection homology of the pair $(X / G, Y / G)$ and the $G$-stable part of the intersection homology of $(X, Y)$.

Proof. Give $X$ a $G$-invariant triangulation. Then the intersection homology of $X$ can be expressed by means of simplicial chains of the barycentric subdivision, see [13, Appendix]. Now the standard argument in [4, p. 120] can be applied.

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