

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 74 (1999)

Artikel: Approximating l_2 -Betti numbers of an amenable covering by ordinary Betti numbers
Autor: Eckmann, Beno
DOI: <https://doi.org/10.5169/seals-55779>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 20.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Approximating ℓ_2 -Betti numbers of an amenable covering by ordinary Betti numbers

Beno Eckmann

Abstract. It is shown that the ℓ_2 -Betti numbers of an amenable covering of a finite cell-complex can be approximated by ordinary Betti numbers of the finite Følner subcomplexes. This is a new proof, using simple homological arguments, of a recent result of *Dodziuk* and *Mathai*.

Mathematics Subject Classification (1991). 20, 55, 57.

Keywords. Amenable groups, covering spaces, ℓ_2 -homology, Betti numbers.

0. Introduction

Let Y be an infinite amenable covering of a finite cell-complex X with covering transformation group G . Then the ℓ_2 -Betti numbers $\overline{\beta}_p(Y)$ can be approximated by the average ordinary Betti numbers of the finite subcomplexes Y_j of a Følner exhaustion of Y . This has been proved by *Dodziuk* and *Mathai* [D-M]. The purpose of the present paper is to give a simple “homological” proof of that result. It consists in examining the ℓ_2 -homology map $H_p(Y_j) \rightarrow H_p(Y)$ induced by the inclusion $Y_j \rightarrow Y$.

1. Følner sequence

1.1. We consider a discrete infinite amenable group G and a free cocompact G -space Y . By this we mean a cell complex Y on which G operates freely by permutation of the cells, with finite orbit complex $X = Y/G$. Then Y is a covering of X with covering transformation group G . Since G is a factor group of the fundamental group of X , and X is a finite complex, G is necessarily finitely generated. In short Y is called an infinite amenable covering of X .

1.2. It is known (Cheeger-Gromov [C-G], see also [E] or [D-M]) that in such a situation there exists in Y a Følner sequence (or Følner exhaustion) Y_j , $j = 1, 2, 3, \dots$

Here is its description in the form we will need later.

For each closed p -cell σ_p in X we choose an arbitrary lift $\hat{\sigma}_p$ in the corresponding G -orbit. The union of all $\hat{\sigma}_p$, $p \geq 0$, together with its topological closure (i.e. adding if necessary boundary cells of the $\hat{\sigma}_p$) is a closed fundamental domain D for the G -action in Y . The Y_j form an increasing sequence of finite subcomplexes of Y with union Y ; each Y_j is a union of N_j distinct translates $x_\nu D$, $\nu = 1, 2, \dots, N_j$, $x_\nu \in G$, of D . Let further \dot{Y}_j be the topological boundary of Y_j and \dot{N}_j the number of translates of D which meet \dot{Y}_j . From the combinatorial Følner criterion [F] for amenability it follows easily that the sequence Y_j can be chosen such that $\dot{N}_j/N_j \rightarrow 0$ for $j \rightarrow \infty$.

2. ℓ_2 -chains, restricted trace

2.1. The cellular p -chains of Y with \mathbb{R} -coefficients constitute a free $\mathbb{R}G$ -module $C_p(Y)$; as basis we can take the lifts (see **1.2**) $\hat{\sigma}_p^i$ of the p -cells σ_p^i of X , $i = 1, 2, \dots, \alpha_p$, where α_p is the number of p -cells of X . Each p -cell of Y can be uniquely written as $x\hat{\sigma}_p^i$, $x \in G$, $i = 1, \dots, \alpha_p$, and in each orbit the G -action is by left translation.

2.2. As Y is an infinite complex, one considers besides the ordinary p -chains also ℓ_2 -chains, i.e. square-summable real linear combinations of the cells of Y . They constitute a Hilbert space $C_p^{(2)}(Y)$ where all the cells $x\hat{\sigma}_p^i$ as above form an orthonormal basis. We sometimes omit Y and simply write $C_p^{(2)}$. The induced action of G on $C_p^{(2)}$ is isometric.

2.3. For any Hilbert subspace H of $C_p^{(2)}$, not necessarily G -invariant, there is the orthogonal projection

$$\Phi : C_p^{(2)} \longrightarrow C_p^{(2)}$$

with image H . We consider the following "restricted trace" of Φ referring to a finite subcomplex Y_j of Y consisting of N_j translates of the fundamental domain D . Here amenability is not required; it is in **3.4** only that Y_j will refer to a Følner sequence in Y .

Let Π_j be the projection $C_p^{(2)} \rightarrow C_p^{(2)}$ with image $C_p^{(2)}(Y_j)$. Since Y_j is a finite complex, we have $C_p^{(2)}(Y_j) = C_p(Y_j)$; thus Π_j is projection on a finite dimensional \mathbb{R} -subspace of $C_p^{(2)}$ whose basis consists of all cells $x_\nu \hat{\sigma}_p^i$ with $\nu \leq N_j$. One can form the \mathbb{R} -trace

$$d_j(H) = \text{trace}_{\mathbb{R}} \Pi_j \Phi$$

It will be examined for some special subspaces H . Note that it can be expressed

by scalar products in $C_p^{(2)}$ as

$$d_j(H) = \sum_{i=1}^{\alpha_p} \sum_{\nu=1}^{N_j} \langle \Phi(x_\nu \hat{\sigma}_p^i), x_\nu \hat{\sigma}_p^i \rangle + \sum_{\tau_p} \langle \Phi(\tau_p), \tau_p \rangle .$$

where the τ_p are cells in \dot{Y}_j not of the form $x_\nu \hat{\sigma}_p^i$.

2.4. Properties of d_j :

1) Since Φ is idempotent and self-adjoint, the scalar products above are equal to $\langle \Phi(x_\nu \hat{\sigma}_p^i), \Phi(x_\nu \hat{\sigma}_p^i) \rangle$ and $\langle \Phi(\tau_p), \Phi(\tau_p) \rangle$ respectively and thus ≥ 0 : The restricted trace $d_j(H)$ is *non-negative*.

2) Note that one always has

$$d_j(H) \leq \dim_{\mathbb{R}} \Pi_j(H)$$

since

$$\mathrm{tr}_{\mathbb{R}}(\Pi_j \Phi) \leq \|\Pi_j \Phi\| \dim_{\mathbb{R}} \mathrm{im}(\Pi_j \Phi) \leq \dim_{\mathbb{R}} \Pi_j(H).$$

If in particular H is a subspace of $C_p(Y_j)$ then d_j is the same as the trace of the projection of $C_p(Y_j)$ to H . Since these are finite-dimensional vector spaces, the trace is $= \dim_{\mathbb{R}} H$.

3) If H decomposes orthogonally into $H_1 + H_2$ then $d_j(H) = d_j(H_1) + d_j(H_2)$. Just note that then $\Phi = \phi_1 + \phi_2$ where ϕ_i is the projection onto H_i , $i = 1, 2$ and replace Φ in the scalar products above.

4) In case H is G -invariant the projection Φ is G -equivariant and $\langle \Phi(x_\nu \hat{\sigma}_p^i), x_\nu \hat{\sigma}_p^i \rangle$ is equal to $\langle \Phi(\hat{\sigma}_p^i), \hat{\sigma}_p^i \rangle$. But $\sum_{i=1}^{\alpha_p} \langle \Phi(\hat{\sigma}_p^i), \hat{\sigma}_p^i \rangle$ is just the *von Neumann dimension* $\dim_G H$ (see e.g. [L] or [E2]). Thus in that case

$$d_j(H) = N_j \dim_G H$$

plus an "error term" T_j coming from the boundary cells τ_p which is $\leq \dim_{\mathbb{R}} C_p(\dot{Y}_j)$.

3. Mapping $H_p(Y_j)$ into $H_p(Y)$

3.1. In the following, homology H_p is to be understood as "reduced" ℓ_2 -homology (cycles modulo the closure of boundaries). It can be represented by the orthogonal complement of the space of boundaries in the p -cycle space, i.e. by *harmonic* chains (boundary $\partial = 0$ and coboundary $\delta = 0$). In this sense we will consider $H_p(Y)$ as a Hilbert subspace of $C_p^{(2)}(Y)$ and $H_p(Y_j)$ as a subspace of $C_p(Y_j)$.

3.2. Since the boundary operator ∂ in $C_p^{(2)}$ commutes with the G -action, the homology group $H_p(Y)$ considered as a subspace of $C_p^{(2)}$ is G -invariant. According to 2.4, 4) we have

$$d_j(H_p(Y)) = N_j \dim_G H_p(Y) + T_j = N_j \overline{\beta}_p(Y \mathrm{rel}. G) + T_j,$$

where $\overline{\beta}_p$ denotes the ℓ_2 -Betti number and T_j is the error term from 2.4,4).

As for $H_p(Y_j)$, we have by 2.4, 2)

$$d_j(H_p(Y_j)) = \dim_{\mathbb{R}} H_p(Y_j) = \beta_p(Y_j),$$

the ordinary p -th Betti number of Y_j .

3.3. The inclusion of Y_j in Y induces a bounded linear map $\phi : H_p(Y_j) \rightarrow H_p(Y)$. Let K_p be the kernel of ϕ , and K'_p its orthogonal complement in $H_p(Y_j)$; and I_p the image of ϕ , and I'_p its orthogonal complement in $H_p(Y)$.

We will look closer at these harmonic subspaces of $C_p(Y_j)$ and $C_p^{(2)}(Y)$ respectively in order to get estimates for the values of d_j . We recall that ∂ commutes with the inclusion of Y_j in Y but in general not with the restriction of Y to Y_j , and that for δ things are the other way around. In particular a harmonic chain in Y_j need not be harmonic in Y , but can be made harmonic by adding a well-defined element of the closure of boundaries.

3.4. We decompose the p -chains $c \in C_p^{(2)}$ as $c = \dot{c} + c'$ where all p -cells of \dot{c} intersect the topological boundary \dot{Y}_j and c' does not contain any such cell. This yields an orthogonal decomposition of $C_p^{(2)}$ into \dot{C}_p and C'_p . We now use the amenability of the covering and assume that Y_j is a term of the Følner sequence. Then $\dim_{\mathbb{R}} \dot{C}_p \leq \dot{N}_j \alpha_p$.

1) If $c \in K_p$, with $\partial c = \delta c = 0$ in Y_j , then $c \in \overline{\partial C_{p+1}^{(2)}(Y)}$. If we assume $\dot{c} = 0$, $c = c' \in C'_p$, then δ commutes with the inclusion, i.e. $\delta c = 0$ in Y . But since cocycles are orthogonal to the closure of the space of boundaries, it follows that $c = 0$. Thus $K_p \cap C'_p = 0$, and K_p is isomorphic to a subspace of \dot{C}_p . Therefore

$$d_j(K_p) = \dim_{\mathbb{R}} K_p \leq \dim_{\mathbb{R}} \dot{C}_p \leq \dot{N}_j \alpha_p.$$

2) As for $d_j(I'_p)$ it does not exceed $\dim_{\mathbb{R}} R_p$ where $R_p = \text{res}_j I'_p$ and res_j is the restriction from Y to Y_j . The chains $c \in I'_p$ fulfill $\partial c = \delta c = 0$. Moreover $\langle c, z \rangle = 0$ for all p -cycles z in Y_j since $\phi(z) = z + b$, with $b \in \overline{\partial C_{p+1}^{(2)}}$. For $r \in R_p$ the same holds except possibly for $\partial r = 0$. But if $r = \dot{c} + c'$ as above, and if we assume $\dot{c} = 0$ then $\partial r = 0$. From $\langle r, z \rangle = 0$ for all p -cycles z in Y_j it follows that r is a coboundary in Y_j , $r = \delta s$. Thus $\langle r, r \rangle = \langle r, \delta s \rangle = \langle \partial r, s \rangle = 0$, whence $r = 0$ and $R_p \cap C'_p = 0$. As before this implies $\dim_{\mathbb{R}} R_p \leq \dot{N}_j \alpha_p$ and we get

$$d_j(I'_p) \leq \dim_{\mathbb{R}} R_p \leq \dot{N}_j \alpha_p.$$

3.5. K'_p is isomorphic as a (finite-dimensional) vector space to I_p . Their d_j need not be equal, but we show that their difference fulfills an inequality similar to

those above. The isomorphism is given by adding to each $c \in K'_p$ a well defined element $b(c) \in \overline{\partial C_{p+1}^{(2)}(Y)}$, in order to get a harmonic chain in Y . If, in particular, $c \in K'_p \cap C'_p$ then $\delta c = 0$ in Y , whence $c \in I_p$. Thus $K'_p \cap C'_p$ is a subspace of I_p which remains unchanged under Π_j . This implies that $d_j(I_p) \geq d_j(K'_p \cap C'_p) = \dim_{\mathbb{R}} K'_p \cap C'_p$ and

$$\dim_{\mathbb{R}} K'_p - d_j(I_p) \leq \dim_{\mathbb{R}} K'_p / K'_p \cap C'_p .$$

But $K'_p / K'_p \cap C'_p$ is isomorphic to $(K'_p + C'_p) / C'_p$ which is contained in $C_p^{(2)} / C'_p$ isomorphic to \dot{C}_p . Thus its dimension is $\leq \dot{N}_j \alpha_p$ whence

$$d_j(K'_p) - d_j(I_p) \leq \dot{N}_j \alpha_p .$$

3.6. Finally we have

$$\begin{aligned} \beta_p(Y_j) - N_j \overline{\beta}_p(Y \text{ rel. } G) &= d_j(H_p(Y_j)) - d_j(H_p(Y)) + T_j \\ &= d_j(K_p) - d_j(I'_p) + (d_j(K'_p) - d_j(I_p)) + T_j \end{aligned}$$

where T_j is the error term in **2.4**. By **3.4** and **3.5** and since $T_j \leq \dot{N}_j \alpha_p$ this yields

$$\left| \frac{1}{N_j} \beta_p(Y_j) - \overline{\beta}_p(Y \text{ rel. } G) \right| \leq 4\alpha_p \frac{\dot{N}_j}{N_j}$$

which goes to 0 with $j \rightarrow \infty$. Thus

$$\lim_{j \rightarrow \infty} \frac{1}{N_j} \beta_p(Y_j) = \overline{\beta}_p(Y \text{ rel. } G).$$

This is the approximation statement mentioned in the introduction.

References

- [C-G] J. Cheeger and M. Gromov, L_2 -cohomology and group cohomology, *Topology* **25** (1986), 189-215.
- [D-M] Jozef Dodziuk and Varghese Mathai, Approximating L^2 -invariants of amenable covering spaces: a combinatorial approach, Preprint.
- [E] B. Eckmann, Amenable groups and Euler characteristic, *Comment. Math. Helv.* **67** (1992), 383-393.
- [E2] B. Eckmann, Projective and Hilbert modules over group algebras, and finitely dominated spaces, *Comment. Math. Helv.* **71** (1996), 453-462.
- [F] E. Følner, On groups with full Banach mean value, *Math. Scand.* **3** (1995), 336-334.
- [L] W. Lück, Approximating L^2 -invariants by their finite-dimensional analogues, *GAF* **4** (1994), 455-481.

Beno Eckmann
Forschungsinstitut für Mathematik
Eidg. Technische Hochschule
8092 Zürich
Switzerland

(Received: February 2, 1998)