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Autor:	Eckmann, Beno
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Approximating ℓ_2 -Betti numbers of an amenable covering by ordinary Betti numbers

Beno Eckmann

Abstract. It is shown that the ℓ_2 -Betti numbers of an amenable covering of a finite cell-complex can be approximated by ordinary Betti numbers of the finite Følner subcomplexes. This is a new proof, using simple homological arguments, of a recent result of *Dodziuk* and *Mathai*.

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0. Introduction

Let Y be an infinite amenable covering of a finite cell-complex X with covering transformation group G . Then the ℓ_2 -Betti numbers $\overline{\beta}_p(Y)$ can be approximated by the average ordinary Betti numbers of the finite subcomplexes Y_j of a Følner exhaustion of Y . This has been proved by *Dodziuk* and *Mathai* [D-M]. The purpose of the present paper is to give a simple “homological” proof of that result. It consists in examining the ℓ_2 -homology map $H_p(Y_j) \rightarrow H_p(Y)$ induced by the inclusion $Y_j \rightarrow Y$.

1. Følner sequence

1.1. We consider a discrete infinite amenable group G and a free cocompact G -space Y . By this we mean a cell complex Y on which G operates freely by permutation of the cells, with finite orbit complex $X = Y/G$. Then Y is a covering of X with covering transformation group G . Since G is a factor group of the fundamental group of X , and X is a finite complex, G is necessarily finitely generated. In short Y is called an infinite amenable covering of X .

1.2. It is known (Cheeger-Gromov [C-G], see also [E] or [D-M]) that in such a situation there exists in Y a Følner sequence (or Følner exhaustion) Y_j , $j = 1, 2, 3, \dots$.

Here is its description in the form we will need later.

For each closed p -cell σ_p in X we choose an arbitrary lift $\hat{\sigma}_p$ in the corresponding G -orbit. The union of all $\hat{\sigma}_p$, $p \geq 0$, together with its topological closure (i.e. adding if necessary boundary cells of the $\hat{\sigma}_p$) is a closed fundamental domain D for the G -action in Y . The Y_j form an increasing sequence of finite subcomplexes of Y with union Y ; each Y_j is a union of N_j distinct translates $x_\nu D$, $\nu = 1, 2, \dots, N_j$, $x_\nu \in G$, of D . Let further \dot{Y}_j be the topological boundary of Y_j and \dot{N}_j the number of translates of D which meet \dot{Y}_j . From the combinatorial Følner criterion [F] for amenability it follows easily that the sequence Y_j can be chosen such that $\dot{N}_j/N_j \rightarrow 0$ for $j \rightarrow \infty$.

2. ℓ_2 -chains, restricted trace

2.1. The cellular p -chains of Y with \mathbb{R} -coefficients constitute a free $\mathbb{R}G$ -module $C_p(Y)$; as basis we can take the lifts (see 1.2) $\hat{\sigma}_p^i$ of the p -cells σ_p^i of X , $i = 1, 2, \dots, \alpha_p$, where α_p is the number of p -cells of X . Each p -cell of Y can be uniquely written as $x\hat{\sigma}_p^i$, $x \in G$, $i = 1, \dots, \alpha_p$, and in each orbit the G -action is by left translation.

2.2. As Y is an infinite complex, one considers besides the ordinary p -chains also ℓ_2 -chains, i.e. square-summable real linear combinations of the cells of Y . They constitute a Hilbert space $C_p^{(2)}(Y)$ where all the cells $x\hat{\sigma}_p^i$ as above form an orthonormal basis. We sometimes omit Y and simply write $C_p^{(2)}$. The induced action of G on $C_p^{(2)}$ is isometric.

2.3. For any Hilbert subspace H of $C_p^{(2)}$, not necessarily G -invariant, there is the orthogonal projection

$$\Phi : C_p^{(2)} \longrightarrow C_p^{(2)}$$

with image H . We consider the following "restricted trace" of Φ referring to a finite subcomplex Y_j of Y consisting of N_j translates of the fundamental domain D . Here amenability is not required; it is in 3.4 only that Y_j will refer to a Følner sequence in Y .

Let Π_j be the projection $C_p^{(2)} \longrightarrow C_p^{(2)}$ with image $C_p^{(2)}(Y_j)$. Since Y_j is a finite complex, we have $C_p^{(2)}(Y_j) = C_p(Y_j)$; thus Π_j is projection on a finite dimensional \mathbb{R} -subspace of $C_p^{(2)}$ whose basis consists of all cells $x_\nu \hat{\sigma}_p^i$ with $\nu \leq N_j$. One can form the \mathbb{R} -trace

$$d_j(H) = \text{trace}_{\mathbb{R}} \Pi_j \Phi$$

It will be examined for some special subspaces H . Note that it can be expressed

by scalar products in $C_p^{(2)}$ as

$$d_j(H) = \sum_{i=1}^{\alpha_p} \sum_{\nu=1}^{N_j} \langle \Phi(x_\nu \hat{\sigma}_p^i), x_\nu \hat{\sigma}_p^i \rangle + \sum_{\tau_p} \langle \Phi(\tau_p), \tau_p \rangle .$$

where the τ_p are cells in \dot{Y}_j not of the form $x_\nu \hat{\sigma}_p^i$.

2.4. Properties of d_j :

1) Since Φ is idempotent and self-adjoint, the scalar products above are equal to $\langle \Phi(x_\nu \hat{\sigma}_p^i), \Phi(x_\nu \hat{\sigma}_p^i) \rangle$ and $\langle \Phi(\tau_p), \Phi(\tau_p) \rangle$ respectively and thus ≥ 0 : The restricted trace $d_j(H)$ is *non-negative*.

2) Note that one always has

$$d_j(H) \leq \dim_{\mathbb{R}} \Pi_j(H)$$

since

$$\text{tr}_{\mathbb{R}}(\Pi_j \Phi) \leq \|\Pi_j \Phi\| \dim_{\mathbb{R}} \text{im}(\Pi_j \Phi) \leq \dim_{\mathbb{R}} \Pi_j(H).$$

If in particular H is a subspace of $C_p(Y_j)$ then d_j is the same as the trace of the projection of $C_p(Y_j)$ to H . Since these are finite-dimensional vector spaces, the trace is $= \dim_{\mathbb{R}} H$.

3) If H decomposes orthogonally into $H_1 + H_2$ then $d_j(H) = d_j(H_1) + d_j(H_2)$. Just note that then $\Phi = \phi_1 + \phi_2$ where ϕ_i is the projection onto H_i , $i = 1, 2$ and replace Φ in the scalar products above.

4) In case H is G -invariant the projection Φ is G -equivariant and $\langle \Phi(x_\nu \hat{\sigma}_p^i), x_\nu \hat{\sigma}_p^i \rangle$ is equal to $\langle \Phi(\hat{\sigma}_p^i), \hat{\sigma}_p^i \rangle$. But $\sum_{i=1}^{\alpha_p} \langle \Phi(\hat{\sigma}_p^i), \hat{\sigma}_p^i \rangle$ is just the *von Neumann dimension* $\dim_G H$ (see e.g. [L] or [E2]). Thus in that case

$$d_j(H) = N_j \dim_G H$$

plus an "error term" T_j coming from the boundary cells τ_p which is $\leq \dim_{\mathbb{R}} C_p(\dot{Y}_j)$.

3. Mapping $H_p(Y_j)$ into $H_p(Y)$

3.1. In the following, homology H_p is to be understood as "reduced" ℓ_2 -homology (cycles modulo the closure of boundaries). It can be represented by the orthogonal complement of the space of boundaries in the p -cycle space, i.e. by *harmonic chains* (boundary $\partial = 0$ and coboundary $\delta = 0$). In this sense we will consider $H_p(Y)$ as a Hilbert subspace of $C_p^{(2)}(Y)$ and $H_p(Y_j)$ as a subspace of $C_p(Y_j)$.

3.2. Since the boundary operator ∂ in $C_p^{(2)}$ commutes with the G -action, the homology group $H_p(Y)$ considered as a subspace of $C_p^{(2)}$ is G -invariant. According to **2.4, 4** we have

$$d_j(H_p(Y)) = N_j \dim_G H_p(Y) + T_j = N_j \overline{\beta}_p(Y \text{rel. } G) + T_j,$$

where $\bar{\beta}_p$ denotes the ℓ_2 -Betti number and T_j is the error term from 2.4,4).

As for $H_p(Y_j)$, we have by 2.4, 2)

$$d_j(H_p(Y_j)) = \dim_{\mathbb{R}} H_p(Y_j) = \beta_p(Y_j),$$

the ordinary p -th Betti number of Y_j .

3.3. The inclusion of Y_j in Y induces a bounded linear map $\phi : H_p(Y_j) \rightarrow H_p(Y)$. Let K_p be the kernel of ϕ , and K'_p its orthogonal complement in $H_p(Y_j)$; and I_p the image of ϕ , and I'_p its orthogonal complement in $H_p(Y)$.

We will look closer at these harmonic subspaces of $C_p(Y_j)$ and $C_p^{(2)}(Y)$ respectively in order to get estimates for the values of d_j . We recall that ∂ commutes with the inclusion of Y_j in Y but in general not with the the restriction of Y to Y_j , and that for δ things are the other way around. In particular a harmonic chain in Y_j need not be harmonic in Y , but can be made harmonic by adding a well-defined element of the closure of boundaries.

3.4. We decompose the p -chains $c \in C_p^{(2)}$ as $c = \dot{c} + c'$ where all p -cells of \dot{c} intersect the topological boundary \dot{Y}_j and c' does not contain any such cell. This yields an orthogonal decomposition of $C_p^{(2)}$ into \dot{C}_p and C'_p . We now use the amenability of the covering and assume that Y_j is a term of the Følner sequence. Then $\dim_{\mathbb{R}} \dot{C}_p \leq \dot{N}_j \alpha_p$.

1) If $c \in K_p$, with $\partial c = \delta c = 0$ in Y_j , then $c \in \overline{\partial C_{p+1}^{(2)}(Y)}$. If we assume $\dot{c} = 0$, $c = c' \in C'_p$, then δ commutes with the inclusion, i.e. $\delta c = 0$ in Y . But since cocycles are orthogonal to the closure of the space of boundaries, it follows that $c = 0$. Thus $K_p \cap C'_p = 0$, and K_p is isomorphic to a subspace of \dot{C}_p . Therefore

$$d_j(K_p) = \dim_{\mathbb{R}} K_p \leq \dim_{\mathbb{R}} \dot{C}_p \leq \dot{N}_j \alpha_p.$$

2) As for $d_j(I'_p)$ it does not exceed $\dim_{\mathbb{R}} R_p$ where $R_p = \text{res}_j I'_p$ and res_j is the restriction from Y to Y_j . The chains $c \in I'_p$ fulfill $\partial c = \delta c = 0$. Moreover $\langle c, z \rangle = 0$ for all p -cycles z in Y_j since $\phi(z) = z + b$, with $b \in \overline{\partial C_{p+1}^{(2)}(Y)}$. For $r \in R_p$ the same holds except possibly for $\partial r = 0$. But if $r = \dot{c} + c'$ as above, and if we assume $\dot{c} = 0$ then $\partial r = 0$. From $\langle r, z \rangle = 0$ for all p -cycles z in Y_j it follows that r is a coboundary in Y_j , $r = \delta s$. Thus $\langle r, r \rangle = \langle r, \delta s \rangle = \langle \partial r, s \rangle = 0$, whence $r = 0$ and $R_p \cap C'_p = 0$. As before this implies $\dim_{\mathbb{R}} R_p \leq \dot{N}_j \alpha_p$ and we get

$$d_j(I'_p) \leq \dim_{\mathbb{R}} R_p \leq \dot{N}_j \alpha_p.$$

3.5. K'_p is isomorphic as a (finite-dimensional) vector space to I_p . Their d_j need not be equal, but we show that their difference fulfills an inequality similar to

those above. The isomorphism is given by adding to each $c \in K'_p$ a well defined element $b(c) \in \overline{\partial C_{p+1}^{(2)}(Y)}$, in order to get a harmonic chain in Y . If, in particular, $c \in K'_p \cap C'_p$ then $\delta c = 0$ in Y , whence $c \in I_p$. Thus $K'_p \cap C'_p$ is a subspace of I_p which remains unchanged under Π_j . This implies that $d_j(I_p) \geq d_j(K'_p \cap C'_p) = \dim_{\mathbb{R}} K'_p \cap C'_p$ and

$$\dim_{\mathbb{R}} K'_p - d_j(I_p) \leq \dim_{\mathbb{R}} K'_p / K'_p \cap C'_p.$$

But $K'_p / K'_p \cap C'_p$ is isomorphic to $(K'_p + C'_p) / C'_p$ which is contained in $C_p^{(2)} / C'_p$ isomorphic to \dot{C}_p . Thus its dimension is $\leq \dot{N}_j \alpha_p$ whence

$$d_j(K'_p) - d_j(I_p) \leq \dot{N}_j \alpha_p.$$

3.6. Finally we have

$$\beta_p(Y_j) - N_j \overline{\beta_p}(Y \text{rel.} G) = d_j(H_p(Y_j)) - d_j(H_p(Y)) + T_j$$

$$= d_j(K_p) - d_j(I'_p) + (d_j(K'_p) - d_j(I_p)) + T_j$$

where T_j is the error term in **2.4**. By **3.4** and **3.5** and since $T_j \leq \dot{N}_j \alpha_p$ this yields

$$|\frac{1}{N_j} \beta_p(Y_j) - \overline{\beta_p}(Y \text{rel.} G)| \leq 4\alpha_p \frac{\dot{N}_j}{N_j}$$

which goes to 0 with $j \rightarrow \infty$. Thus

$$\lim_{j \rightarrow \infty} \frac{1}{N_j} \beta_p(Y_j) = \overline{\beta_p}(Y \text{rel.} G).$$

This is the approximation statement mentioned in the introduction.

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Beno Eckmann
Forschungsinstitut für Mathematik
Eidg. Technische Hochschule
8092 Zürich
Switzerland

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