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## Approximating $\ell_2$ -Betti numbers of an amenable covering by ordinary Betti numbers

Beno Eckmann

**Abstract.** It is shown that the  $\ell_2$ -Betti numbers of an amenable covering of a finite cell-complex can be approximated by ordinary Betti numbers of the finite Følner subcomplexes. This is a new proof, using simple homological arguments, of a recent result of *Dodziuk* and *Mathai*.

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### 0. Introduction

Let  $Y$  be an infinite amenable covering of a finite cell-complex  $X$  with covering transformation group  $G$ . Then the  $\ell_2$ -Betti numbers  $\overline{\beta}_p(Y)$  can be approximated by the average ordinary Betti numbers of the finite subcomplexes  $Y_j$  of a Følner exhaustion of  $Y$ . This has been proved by *Dodziuk* and *Mathai* [D-M]. The purpose of the present paper is to give a simple “homological” proof of that result. It consists in examining the  $\ell_2$ -homology map  $H_p(Y_j) \rightarrow H_p(Y)$  induced by the inclusion  $Y_j \rightarrow Y$ .

### 1. Følner sequence

**1.1.** We consider a discrete infinite amenable group  $G$  and a free cocompact  $G$ -space  $Y$ . By this we mean a cell complex  $Y$  on which  $G$  operates freely by permutation of the cells, with finite orbit complex  $X = Y/G$ . Then  $Y$  is a covering of  $X$  with covering transformation group  $G$ . Since  $G$  is a factor group of the fundamental group of  $X$ , and  $X$  is a finite complex,  $G$  is necessarily finitely generated. In short  $Y$  is called an infinite amenable covering of  $X$ .

**1.2.** It is known (Cheeger-Gromov [C-G], see also [E] or [D-M]) that in such a situation there exists in  $Y$  a Følner sequence (or Følner exhaustion)  $Y_j$ ,  $j = 1, 2, 3, \dots$

Here is its description in the form we will need later.

For each closed  $p$ -cell  $\sigma_p$  in  $X$  we choose an arbitrary lift  $\hat{\sigma}_p$  in the corresponding  $G$ -orbit. The union of all  $\hat{\sigma}_p$ ,  $p \geq 0$ , together with its topological closure (i.e. adding if necessary boundary cells of the  $\hat{\sigma}_p$ ) is a closed fundamental domain  $D$  for the  $G$ -action in  $Y$ . The  $Y_j$  form an increasing sequence of finite subcomplexes of  $Y$  with union  $Y$ ; each  $Y_j$  is a union of  $N_j$  distinct translates  $x_\nu D$ ,  $\nu = 1, 2, \dots, N_j$ ,  $x_\nu \in G$ , of  $D$ . Let further  $\dot{Y}_j$  be the topological boundary of  $Y_j$  and  $\dot{N}_j$  the number of translates of  $D$  which meet  $\dot{Y}_j$ . From the combinatorial Følner criterion [F] for amenability it follows easily that the sequence  $Y_j$  can be chosen such that  $\dot{N}_j/N_j \rightarrow 0$  for  $j \rightarrow \infty$ .

## 2. $\ell_2$ -chains, restricted trace

**2.1.** The cellular  $p$ -chains of  $Y$  with  $\mathbb{R}$ -coefficients constitute a free  $\mathbb{R}G$ -module  $C_p(Y)$ ; as basis we can take the lifts (see **1.2**)  $\hat{\sigma}_p^i$  of the  $p$ -cells  $\sigma_p^i$  of  $X$ ,  $i = 1, 2, \dots, \alpha_p$ , where  $\alpha_p$  is the number of  $p$ -cells of  $X$ . Each  $p$ -cell of  $Y$  can be uniquely written as  $x\hat{\sigma}_p^i$ ,  $x \in G$ ,  $i = 1, \dots, \alpha_p$ , and in each orbit the  $G$ -action is by left translation.

**2.2.** As  $Y$  is an infinite complex, one considers besides the ordinary  $p$ -chains also  $\ell_2$ -chains, i.e. square-summable real linear combinations of the cells of  $Y$ . They constitute a Hilbert space  $C_p^{(2)}(Y)$  where all the cells  $x\hat{\sigma}_p^i$  as above form an orthonormal basis. We sometimes omit  $Y$  and simply write  $C_p^{(2)}$ . The induced action of  $G$  on  $C_p^{(2)}$  is isometric.

**2.3.** For any Hilbert subspace  $H$  of  $C_p^{(2)}$ , not necessarily  $G$ -invariant, there is the orthogonal projection

$$\Phi : C_p^{(2)} \longrightarrow C_p^{(2)}$$

with image  $H$ . We consider the following "restricted trace" of  $\Phi$  referring to a finite subcomplex  $Y_j$  of  $Y$  consisting of  $N_j$  translates of the fundamental domain  $D$ . Here amenability is not required; it is in **3.4** only that  $Y_j$  will refer to a Følner sequence in  $Y$ .

Let  $\Pi_j$  be the projection  $C_p^{(2)} \longrightarrow C_p^{(2)}$  with image  $C_p^{(2)}(Y_j)$ . Since  $Y_j$  is a finite complex, we have  $C_p^{(2)}(Y_j) = C_p(Y_j)$ ; thus  $\Pi_j$  is projection on a finite dimensional  $\mathbb{R}$ -subspace of  $C_p^{(2)}$  whose basis consists of all cells  $x_\nu \hat{\sigma}_p^i$  with  $\nu \leq N_j$ . One can form the  $\mathbb{R}$ -trace

$$d_j(H) = \text{trace}_{\mathbb{R}} \Pi_j \Phi$$

It will be examined for some special subspaces  $H$ . Note that it can be expressed

by scalar products in  $C_p^{(2)}$  as

$$d_j(H) = \sum_{i=1}^{\alpha_p} \sum_{\nu=1}^{N_j} \langle \Phi(x_\nu \hat{\sigma}_p^i), x_\nu \hat{\sigma}_p^i \rangle + \sum_{\tau_p} \langle \Phi(\tau_p), \tau_p \rangle .$$

where the  $\tau_p$  are cells in  $\dot{Y}_j$  not of the form  $x_\nu \hat{\sigma}_p^i$ .

#### 2.4. Properties of $d_j$ :

1) Since  $\Phi$  is idempotent and self-adjoint, the scalar products above are equal to  $\langle \Phi(x_\nu \hat{\sigma}_p^i), \Phi(x_\nu \hat{\sigma}_p^i) \rangle$  and  $\langle \Phi(\tau_p), \Phi(\tau_p) \rangle$  respectively and thus  $\geq 0$ : The restricted trace  $d_j(H)$  is *non-negative*.

2) Note that one always has

$$d_j(H) \leq \dim_{\mathbb{R}} \Pi_j(H)$$

since

$$\mathrm{tr}_{\mathbb{R}}(\Pi_j \Phi) \leq \|\Pi_j \Phi\| \dim_{\mathbb{R}} \mathrm{im}(\Pi_j \Phi) \leq \dim_{\mathbb{R}} \Pi_j(H).$$

If in particular  $H$  is a subspace of  $C_p(Y_j)$  then  $d_j$  is the same as the trace of the projection of  $C_p(Y_j)$  to  $H$ . Since these are finite-dimensional vector spaces, the trace is  $= \dim_{\mathbb{R}} H$ .

3) If  $H$  decomposes orthogonally into  $H_1 + H_2$  then  $d_j(H) = d_j(H_1) + d_j(H_2)$ . Just note that then  $\Phi = \phi_1 + \phi_2$  where  $\phi_i$  is the projection onto  $H_i$ ,  $i = 1, 2$  and replace  $\Phi$  in the scalar products above.

4) In case  $H$  is  $G$ -invariant the projection  $\Phi$  is  $G$ -equivariant and  $\langle \Phi(x_\nu \hat{\sigma}_p^i), x_\nu \hat{\sigma}_p^i \rangle$  is equal to  $\langle \Phi(\hat{\sigma}_p^i), \hat{\sigma}_p^i \rangle$ . But  $\sum_{i=1}^{\alpha_p} \langle \Phi(\hat{\sigma}_p^i), \hat{\sigma}_p^i \rangle$  is just the *von Neumann dimension*  $\dim_G H$  (see e.g. [L] or [E2]). Thus in that case

$$d_j(H) = N_j \dim_G H$$

plus an "error term"  $T_j$  coming from the boundary cells  $\tau_p$  which is  $\leq \dim_{\mathbb{R}} C_p(\dot{Y}_j)$ .

### 3. Mapping $H_p(Y_j)$ into $H_p(Y)$

**3.1.** In the following, homology  $H_p$  is to be understood as "reduced"  $\ell_2$ -homology (cycles modulo the closure of boundaries). It can be represented by the orthogonal complement of the space of boundaries in the  $p$ -cycle space, i.e. by *harmonic* chains (boundary  $\partial = 0$  and coboundary  $\delta = 0$ ). In this sense we will consider  $H_p(Y)$  as a Hilbert subspace of  $C_p^{(2)}(Y)$  and  $H_p(Y_j)$  as a subspace of  $C_p(Y_j)$ .

**3.2.** Since the boundary operator  $\partial$  in  $C_p^{(2)}$  commutes with the  $G$ -action, the homology group  $H_p(Y)$  considered as a subspace of  $C_p^{(2)}$  is  $G$ -invariant. According to 2.4, 4) we have

$$d_j(H_p(Y)) = N_j \dim_G H_p(Y) + T_j = N_j \overline{\beta}_p(Y \mathrm{rel}. G) + T_j,$$

where  $\overline{\beta}_p$  denotes the  $\ell_2$ -Betti number and  $T_j$  is the error term from 2.4,4).

As for  $H_p(Y_j)$ , we have by 2.4, 2)

$$d_j(H_p(Y_j)) = \dim_{\mathbb{R}} H_p(Y_j) = \beta_p(Y_j),$$

the ordinary  $p$ -th Betti number of  $Y_j$ .

**3.3.** The inclusion of  $Y_j$  in  $Y$  induces a bounded linear map  $\phi : H_p(Y_j) \rightarrow H_p(Y)$ . Let  $K_p$  be the kernel of  $\phi$ , and  $K'_p$  its orthogonal complement in  $H_p(Y_j)$ ; and  $I_p$  the image of  $\phi$ , and  $I'_p$  its orthogonal complement in  $H_p(Y)$ .

We will look closer at these harmonic subspaces of  $C_p(Y_j)$  and  $C_p^{(2)}(Y)$  respectively in order to get estimates for the values of  $d_j$ . We recall that  $\partial$  commutes with the inclusion of  $Y_j$  in  $Y$  but in general not with the restriction of  $Y$  to  $Y_j$ , and that for  $\delta$  things are the other way around. In particular a harmonic chain in  $Y_j$  need not be harmonic in  $Y$ , but can be made harmonic by adding a well-defined element of the closure of boundaries.

**3.4.** We decompose the  $p$ -chains  $c \in C_p^{(2)}$  as  $c = \dot{c} + c'$  where all  $p$ -cells of  $\dot{c}$  intersect the topological boundary  $\dot{Y}_j$  and  $c'$  does not contain any such cell. This yields an orthogonal decomposition of  $C_p^{(2)}$  into  $\dot{C}_p$  and  $C'_p$ . We now use the amenability of the covering and assume that  $Y_j$  is a term of the Følner sequence. Then  $\dim_{\mathbb{R}} \dot{C}_p \leq \dot{N}_j \alpha_p$ .

1) If  $c \in K_p$ , with  $\partial c = \delta c = 0$  in  $Y_j$ , then  $c \in \overline{\partial C_{p+1}^{(2)}(Y)}$ . If we assume  $\dot{c} = 0$ ,  $c = c' \in C'_p$ , then  $\delta$  commutes with the inclusion, i.e.  $\delta c = 0$  in  $Y$ . But since cocycles are orthogonal to the closure of the space of boundaries, it follows that  $c = 0$ . Thus  $K_p \cap C'_p = 0$ , and  $K_p$  is isomorphic to a subspace of  $\dot{C}_p$ . Therefore

$$d_j(K_p) = \dim_{\mathbb{R}} K_p \leq \dim_{\mathbb{R}} \dot{C}_p \leq \dot{N}_j \alpha_p.$$

2) As for  $d_j(I'_p)$  it does not exceed  $\dim_{\mathbb{R}} R_p$  where  $R_p = \text{res}_j I'_p$  and  $\text{res}_j$  is the restriction from  $Y$  to  $Y_j$ . The chains  $c \in I'_p$  fulfill  $\partial c = \delta c = 0$ . Moreover  $\langle c, z \rangle = 0$  for all  $p$ -cycles  $z$  in  $Y_j$  since  $\phi(z) = z + b$ , with  $b \in \overline{\partial C_{p+1}^{(2)}}$ . For  $r \in R_p$  the same holds except possibly for  $\partial r = 0$ . But if  $r = \dot{c} + c'$  as above, and if we assume  $\dot{c} = 0$  then  $\partial r = 0$ . From  $\langle r, z \rangle = 0$  for all  $p$ -cycles  $z$  in  $Y_j$  it follows that  $r$  is a coboundary in  $Y_j$ ,  $r = \delta s$ . Thus  $\langle r, r \rangle = \langle r, \delta s \rangle = \langle \partial r, s \rangle = 0$ , whence  $r = 0$  and  $R_p \cap C'_p = 0$ . As before this implies  $\dim_{\mathbb{R}} R_p \leq \dot{N}_j \alpha_p$  and we get

$$d_j(I'_p) \leq \dim_{\mathbb{R}} R_p \leq \dot{N}_j \alpha_p.$$

**3.5.**  $K'_p$  is isomorphic as a (finite-dimensional) vector space to  $I_p$ . Their  $d_j$  need not be equal, but we show that their difference fulfills an inequality similar to

those above. The isomorphism is given by adding to each  $c \in K'_p$  a well defined element  $b(c) \in \overline{\partial C_{p+1}^{(2)}(Y)}$ , in order to get a harmonic chain in  $Y$ . If, in particular,  $c \in K'_p \cap C'_p$  then  $\delta c = 0$  in  $Y$ , whence  $c \in I_p$ . Thus  $K'_p \cap C'_p$  is a subspace of  $I_p$  which remains unchanged under  $\Pi_j$ . This implies that  $d_j(I_p) \geq d_j(K'_p \cap C'_p) = \dim_{\mathbb{R}} K'_p \cap C'_p$  and

$$\dim_{\mathbb{R}} K'_p - d_j(I_p) \leq \dim_{\mathbb{R}} K'_p / K'_p \cap C'_p .$$

But  $K'_p / K'_p \cap C'_p$  is isomorphic to  $(K'_p + C'_p) / C'_p$  which is contained in  $C_p^{(2)} / C'_p$  isomorphic to  $\dot{C}_p$ . Thus its dimension is  $\leq \dot{N}_j \alpha_p$  whence

$$d_j(K'_p) - d_j(I_p) \leq \dot{N}_j \alpha_p .$$

**3.6.** Finally we have

$$\begin{aligned} \beta_p(Y_j) - N_j \overline{\beta}_p(Y \text{ rel. } G) &= d_j(H_p(Y_j)) - d_j(H_p(Y)) + T_j \\ &= d_j(K_p) - d_j(I'_p) + (d_j(K'_p) - d_j(I_p)) + T_j \end{aligned}$$

where  $T_j$  is the error term in **2.4**. By **3.4** and **3.5** and since  $T_j \leq \dot{N}_j \alpha_p$  this yields

$$\left| \frac{1}{N_j} \beta_p(Y_j) - \overline{\beta}_p(Y \text{ rel. } G) \right| \leq 4\alpha_p \frac{\dot{N}_j}{N_j}$$

which goes to 0 with  $j \rightarrow \infty$ . Thus

$$\lim_{j \rightarrow \infty} \frac{1}{N_j} \beta_p(Y_j) = \overline{\beta}_p(Y \text{ rel. } G).$$

This is the approximation statement mentioned in the introduction.

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