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## On the dilatation of extremal quasiconformal mappings of polygons

Kurt Strebel

**Abstract.** A polygon  $P_N$  is the unit disk  $\mathbb{D}$  with  $n$  distinguished boundary points,  $4 \leq n \leq N$ . An extremal quasiconformal mapping  $f_0: \mathbb{D}_z \rightarrow \mathbb{D}_w$  maps each polygon  $P_N$  inscribed in  $\mathbb{D}_z$  onto a polygon  $P'_N$  inscribed in  $\mathbb{D}_w$ . Let  $f_N$  be the extremal quasiconformal mapping of  $P_N$  onto  $P'_N$ . Let  $K_N$  be its dilatation and let  $K_0$  be the maximal dilatation of  $f_0$ . Then, evidently  $\sup K_N \leq K_0$ . The problem is, when equality holds. This is completely answered, if  $f_0$  does not have any essential boundary points. For quadrilaterals  $Q$  and  $Q' = f_0(Q)$  the problem is  $\sup(M'/M) = K_0$ , with  $M$  and  $M'$  the moduli of  $Q$  and  $Q'$  respectively.

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### Introduction

1. Let  $h$  be a quasimetric mapping of the boundary of the unit disk  $\mathbb{D}_z$  onto the boundary of  $\mathbb{D}_w$  and let  $f$  be a quasiconformal extension of  $h$  into the disk. It is called extremal and denoted by  $f_0$  if its maximal dilatation  $K_0$  is smallest possible. We always assume  $K_0 > 1$ . The disk  $\mathbb{D}_z$  becomes a quadrilateral  $Q$  if we mark four different points  $z_j$ ,  $j = 1, \dots, 4$ , in the positive direction on its boundary  $\partial\mathbb{D}_z$ . The mapping  $f_0$  takes the vertices  $z_j$  into points  $w_j = f_0(z_j)$  on  $\partial\mathbb{D}_w$  and thus the quadrilateral  $Q$  into a quadrilateral  $Q' = f_0(Q)$  inscribed in  $\mathbb{D}_w$ . It follows from the definition of quasiconformality that the conformal moduli  $M$  and  $M'$  of  $Q$  and  $Q'$  respectively satisfy (for general properties of quasiconformal mappings, see [3])

$$\frac{1}{K_0}M \leq M' \leq K_0M. \quad (1)$$

It has been a question for some time, if the bound  $K_0$  is best possible in the inequality (1), in other words, if the maximal dilatation  $K_0$  of the extremal quasiconformal extension  $f_0$  of  $h$  can be determined by the ratio of the moduli of

inscribed quadrilaterals,

$$\sup \frac{M'}{M} = K_0. \quad (2)$$

The question has recently been answered in the negative by Anderson and Hinkkanen [1] by laborious computations of a counterexample (horizontal stretching of a parallelogram) and by Reich [4] who reduced it to an approximation problem for holomorphic functions. More counterexamples are given in [9].

**2.** It is easy to find examples where (2) holds; the above solutions consist therefore in the construction of examples where it does not hold. A type of the first kind is a vertical half strip  $S$  and its horizontal stretching by  $K_0$ . Let  $z = x + iy$ ,  $S = \{z; 0 < x < a, 0 < y\}$ ,  $w = u + iv$ ,  $S' = \{w; 0 < u < K_0 a, 0 < v\}$ . We make  $S$  to a quadrilateral by marking the vertices  $(0, a, a + ib, ib)$  for arbitrary  $b > 0$ , and similarly  $S'$  by marking the image points  $(0, K_0 a, K_0 a + ib, ib)$ . Making use of the extremal length definition of the modulus of a quadrilateral ([3], p. 21) as the extremal distance of the vertical sides we easily find the estimates

$$M \leq a/b, \quad M' \geq K_0 a / (b + K_0 a) \quad (3)$$

and thus

$$K_0 \geq \frac{M'}{M} \geq \frac{K_0 a}{b + K_0 a} \cdot \frac{b}{a}, \quad (4)$$

which gives

$$\lim_{b \rightarrow \infty} \frac{M'}{M} = K_0. \quad (5)$$

**3.** The problem with the moduli of quadrilaterals has a different interpretation. We look at the extremal quasiconformal mapping  $f$  of  $Q$  onto  $Q'$ . This is a mapping of  $\mathbb{D}_z$  onto  $\mathbb{D}_w$  which takes the vertices of  $Q$  into those of  $Q'$ . Its dilatation is  $K = M'/M$ , and the question is now what happens with  $K$  if we vary the vertices of  $Q$  in all possible ways? Of course we always have  $K \leq K_0$ , but will we have  $\sup K = K_0$ ? In this formulation the problem has a natural generalization to polygons, i.e. disks with an arbitrary finite number  $n \geq 4$  of vertices. The basic extremal qc mapping  $f_0$  assigns a polygon  $P'_n$  inscribed in  $\mathbb{D}_w$  to each polygon  $P_n$  inscribed in  $\mathbb{D}_z$ . The extremal qc mapping  $f_n$  of  $P_n$  onto  $P'_n$  (i.e. of course of  $\mathbb{D}_z$  onto  $\mathbb{D}_w$ , but with the only requirement that the vertices of  $P_n$  go into the vertices of  $P'_n$ ) is a Teichmüller mapping with a complex dilatation  $\kappa_n = k_n(\overline{\varphi_n}/|\varphi_n|)$ ,  $k_n = (K_n - 1)/(K_n + 1)$ . The quadratic differential  $\varphi_n$  is rational, with at most first order poles at the vertices of  $P_n$ . Moreover,  $\varphi_n(z) dz^2$  is real along the sides of  $P_n$ . Since  $f_0$  also maps the vertices of  $P_n$  onto those of  $P'_n$  and  $f_n$  is extremal with this property, we have  $K_n \leq K_0$ . The question arises if, by varying the polygon  $P_n$  in all possible ways, we have

$$\sup K_n = K_0. \quad (6)$$

4. It follows from general principles of qc mappings (we refer to [3] for the general theory) that this is in fact true if we allow the number  $n$  of vertices to become arbitrarily large (for a proof see [5], p. 385, bottom). But how is it, if this number is bounded,  $n \leq N$  say? With a certain natural restriction we will characterize the extremal mappings  $f_0$  for which this happens. The proof is an application of the “polygon inequality” ([5], p. 384) and a theorem of R. Fehlmann ([2], p. 567).

### The polygon inequality

5. Let  $f_0$  be an extremal qc mapping of  $\mathbb{D}_z$  onto  $\mathbb{D}_w$  with  $f_0|_{\partial\mathbb{D}_z} = h$ . Let  $\kappa_0$  with  $\|\kappa_0\|_\infty = k_0$  be its complex dilatation and  $K_0 = (1 + k_0)/(1 - k_0)$  its maximal dilatation. Mark  $n$  points  $z_j$ ,  $j = 1, \dots, n$ , on  $\partial\mathbb{D}_z$ ,  $4 \leq n \leq N$ . The disk  $\mathbb{D}_z$  with the marked boundary points  $z_j$  is called a polygon  $P_n$ . The image of  $P_n$  by  $f_0$  is the polygon  $P'_n$ , inscribed in  $\mathbb{D}_w$ , with vertices  $w_j = f_0(z_j)$ . Let  $f_n$  be the extremal qc mapping of  $P_n$  onto  $P'_n$ ,  $f_n(z_j) = w_j$ , and let  $\varphi_n$ ,  $\|\varphi_n\| = 1$ , denote the associated quadratic differential. The complex dilatation of  $f_n$  is  $k_n(\overline{\varphi_n}/|\varphi_n|)$ . Then, the *Polygon Inequality* holds:

$$\operatorname{Re} \iint_{|z|<1} \frac{\kappa_0(z)\varphi_n(z)}{1 - |\kappa_0(z)|^2} dx dy \geq \frac{k_n}{1 - k_n} - \iint_{|z|<1} |\varphi_n(z)| \frac{|\kappa_0(z)|^2}{1 - |\kappa_0(z)|^2} dx dy. \quad (7)$$

For the proof I refer to ([5], p. 384). In that paper, the inequality was used to prove that the “polygon differentials”  $\varphi_n$  form a Hamilton sequence for  $\kappa_0$  if the number of vertices tends to infinity and the sides of the polygons  $P_n$  become arbitrarily short. This led to a proof of the necessity of the Hamilton–Krushkal condition for extremality. Now, on the contrary, we restrict the number of vertices by a fixed number  $N$ , and we denote a polygon with  $n \leq N$  vertices generically by  $P_N$ .

### 6.

**Theorem 1.** *Let  $f_0: \mathbb{D}_z \rightarrow \mathbb{D}_w$  with complex dilatation  $\kappa_0$ ,  $\|\kappa_0\|_\infty = k_0$ , be extremal for its boundary values  $h$ . Assume that for a fixed number  $N$  the polygon mappings  $f_N: P_N \rightarrow P'_N = f_0(P_N)$  with complex dilatation  $k_N(\overline{\varphi_N}/|\varphi_N|)$  satisfy*

$$\sup k_N = k_0. \quad (8)$$

(This is of course equivalent to  $\sup K_N = K_0$ .) Then, there is a sequence of polygon mappings  $f_N^{(i)}$  the quadratic differentials  $\varphi_N^{(i)}$  of which,  $\|\varphi_N^{(i)}\| = 1$ , form a Hamilton sequence for  $\kappa_0$ , i.e.

$$\operatorname{Re} \iint \kappa_0(z) \varphi_N^{(i)}(z) dx dy \rightarrow k_0, \quad i \rightarrow \infty. \quad (9)$$

*Proof.* Assume first that  $f_0$  has constant dilatation  $|\kappa_0(z)| = k_0$  a.e. Then, the polygon inequality yields

$$\frac{1}{1 - k_0^2} \operatorname{Re} \iint \kappa_0(z) \varphi_N(z) dx dy \geq \frac{k_N}{1 - k_N} - \frac{k_0^2}{1 - k_0^2} \quad (10)$$

for all polygons  $P_N$ . Let  $P_N^{(i)}$  be a sequence of polygons the extremal mappings  $f_N^{(i)}$  of which satisfy  $k_N^{(i)} \rightarrow k_0$ . Then

$$\lim_{i \rightarrow \infty} \operatorname{Re} \iint \kappa_0(z) \varphi_N^{(i)}(z) dx dy \geq \frac{k_0}{1 - k_0} (1 - k_0^2) - k_0^2 = k_0. \quad (11)$$

On the other hand

$$\operatorname{Re} \iint \kappa_0(z) \varphi_N^{(i)}(z) dx dy \leq \left| \iint \kappa_0(z) \varphi_N^{(i)}(z) dx dy \right| \leq k_0. \quad (12)$$

This gives the result (9) in the case where  $|\kappa_0(z)| = k_0$  a.e. If  $|\kappa_0(z)|$  is not constant a.e. we proceed as in ([5], p. 386 and p. 382). However, in our present work we only need the case of constant  $|\kappa_0(z)|$ .  $\square$

Since the number of vertices of the polygons  $P_N^{(i)}$  is smaller or equal to  $N$ , we can assume, by passing to a further subsequence, that they converge to a finite number  $\leq N$  of points on  $\partial \mathbb{D}_z$ . We write  $P_N^{(i)} \rightarrow P_N$ .

The vertical half strip in the introduction is an example where the given quadrilaterals give rise to a Hamilton sequence for the horizontal stretching (which is uniquely extremal).

## Extremal mappings without essential boundary point

**7.** Let  $f_0$  with complex dilatation  $\kappa_0$ ,  $\|\kappa_0\|_\infty = k_0$ , be extremal for its boundary values  $h$ . A boundary point  $z$  of  $\mathbb{D}_z$  is called essential, if the following is true: For every neighborhood  $U$  of  $z$  and every qc mapping  $g$  of  $U \cap \mathbb{D}_z$  which is equal to  $h$  on  $U \cap \partial \mathbb{D}_z$  the maximal dilatation of  $g$  is at least equal to  $K_0 = (1 + k_0)/(1 - k_0)$ .

A theorem of R. Fehlmann ([2], p. 567) says: If the complex dilatation  $\kappa_0$  has a degenerating Hamilton sequence (i.e. which tends to zero locally uniformly in the domain), then  $f_0$  has an essential boundary point.

Combining this result with the considerations in ([7], p. 466) we can say: If  $f_0$  does not have an essential boundary point, then, every Hamilton sequence for  $\kappa_0$  converges in norm to a holomorphic quadratic differential  $\varphi_0$ ,  $\|\varphi_0\| = 1$ , and  $\kappa_0 = k_0(\overline{\varphi_0}/|\varphi_0|)$  is the complex dilatation of  $f_0$ .

8. Let us apply this to our case. Every polygon differential  $\varphi_N^{(i)}$  can be continued across the boundary  $\partial\mathbb{D}_z$  by reflection to a rational differential in the whole plane, of norm two. Therefore the limit  $\varphi_0$  can be reflected. Since its norm is finite, it has at most first order poles at the  $n \leq N$  limits of the vertices of the  $P_N^{(i)}$ , and  $\varphi_0(z)dz^2$  is real along the subintervals of  $\partial\mathbb{D}_z$  between these limits. Our main result is

**Theorem 2.** *Let  $f_0: \mathbb{D}_z \rightarrow \mathbb{D}_w$  be a qc mapping which is extremal for its boundary values, and assume that it does not have an essential boundary point. For fixed  $N \geq 4$  denote the polygons with  $4 \leq n \leq N$  vertices inscribed in  $\mathbb{D}_z$  generically by  $P_N$ . To every  $P_N$  the mapping  $f_0$  determines a polygon  $P'_N$  inscribed in  $\mathbb{D}_w$ , simply by mapping the vertices of  $P_N$  onto those of  $P'_N$ . Assume that the extremal mappings  $f_N: P_N \rightarrow P'_N$  satisfy  $\sup k_N = k_0$ . Then, there is a convergent sequence  $f_N^{(i)}$  of polygon mappings with  $\varphi_N^{(i)} \rightarrow \varphi_0$  in norm, where  $\varkappa_0 = k_0(\overline{\varphi_0}/|\varphi_0|)$  is the complex dilatation of  $f_0$ .  $f_0$  itself is the extremal qc mapping of a polygon with  $n \leq N$  vertices, and every maximizing sequence  $f_N^{(i)}$ ,  $k_N^{(i)} \rightarrow k_0$ , tends to  $f_0$  uniformly,  $\varphi_N^{(i)} \rightarrow \varphi_0$  in norm.*

9. In order to see that the theorem is not empty, let  $f: \mathbb{D}_z \rightarrow \mathbb{D}_w$  be an extremal polygon mapping and let  $\varphi$  be the associated rational quadratic differential,  $\varkappa = k(\overline{\varphi}/|\varphi|)$  the complex dilatation. The vertices  $z_j$  are either first order poles or regular points (i.e.  $\varphi(z_j) \neq 0$ ) or zeroes of  $\varphi$  of any order. Along the sides we have  $\varphi(z)dz^2$  real, and thus the sides are composed of trajectories and orthogonal trajectories.

The first order poles and the zeroes are clearly the only candidates for an essential boundary point of  $f$ . In order to find the local maximal dilatation  $H_z$  at such a point  $z$  we first apply the mapping  $\Phi = \int \sqrt{\varphi}$  and then the horizontal stretching by  $K$ . The integral  $\Phi$  maps an interior half neighborhood of  $z$  onto an angle with a horizontal and a vertical side. It is a right angle in the case of a first order pole and an angle which is a multiple of  $\frac{1}{2}\pi$  in the case of a zero, possibly many sheeted. In the image  $\mathbb{D}_w$  we have the same situation, with a quadratic differential  $\psi$  and an integral  $\Psi = \int \sqrt{\psi}$ . The horizontal side of the angle is stretched by  $K$  while the vertical side is mapped identically. It is known (and easy to see, using logarithms on both sides, see [6], p. 323) that the local extremal mapping with the given boundary values has dilatation  $< K$ . Since  $f$  itself is extremal with dilatation  $K$ , it does not have any essential boundary point, thus satisfying our requirement.

10. Let now  $f_0: P_N \rightarrow P'_N$  with complex dilatation  $\varkappa_0 = k_0(\overline{\varphi_0}/|\varphi_0|)$  be an extremal polygon mapping. We can clearly take  $f_N = f_0$  itself and get  $\sup k_N = k_0$ . Actually we only need to consider the substantial boundary points of  $f_0$  (= poles of  $\varphi_0$ ), since the extremal mapping of the restricted polygon  $\tilde{P}_N$  onto  $\tilde{P}'_N$  is

the same as  $f_0$ .

Let  $\tilde{N}$  be the number of substantial boundary points of  $f_0$ . If, however, we only admit polygons with at most  $N' \leq \tilde{N} - 1$  vertices, we find  $\sup k_{N'} < k_0$ . For, if  $\sup k_{N'} = k_0$  we would again arrive, by the same considerations as before, at an extremal polygon mapping  $f_{N'}$  with a quadratic differential  $\varphi_{N'}$  with at most  $N'$  first order poles, whereas  $\varphi_0$  has  $\tilde{N}$  first order poles. Therefore  $\varphi_{N'} \neq \varphi_0$ , a contradiction.

**11.** We started with the following question. Let  $f_0$  with complex dilatation  $\varkappa_0$ ,  $\|\varkappa_0\| = k_0$ , be a qc mapping of  $\mathbb{D}_z$  onto  $\mathbb{D}_w$  which is extremal for its boundary values and which does not have an essential boundary point. Inscribe quadrilaterals  $Q$  into  $\mathbb{D}_z$  and denote their images by  $f_0$  in  $\mathbb{D}_w$  by  $Q'$ . The image  $Q'$  has, as its vertices, the images by  $f_0$  of the vertices of  $Q$ . Let  $M$  and  $M'$  be the moduli of  $Q$  and  $Q'$  respectively. The question is, if (2) can hold.

Let  $f$  with dilatation  $K$  be the extremal mapping of  $Q$  onto  $Q'$ . The equation (2) is equivalent with

$$\sup K = K_0 \quad (13)$$

where the sup is taken over all quadrilaterals  $Q$ . This is the special case of (8) for  $N = 4$ . We find

**Theorem 3.** *The extremal mapping  $f_0$  satisfies (13) for the inscribed quadrilaterals  $Q$  if and only if it is the extremal mapping of a quadrilateral itself.*

This means that in all other cases we have inequality in (13). The example of Anderson and Hinkkanen is the horizontal stretching of a parallelogram. This mapping  $f_0$  has no essential boundary point and is, in their situation, not the mapping of quadrilaterals. Therefore  $\sup(M'/M) < K_0$ .

The example of Reich has analytic boundary values. Therefore we have again  $\sup(M'/M) < K_0$ .

Clearly, in both examples, we still have inequality in (13) even if we allow any inscribed polygons with an arbitrary fixed bound  $N$  for the number of vertices.

*Added in Proof.* After the completion of this paper I have become aware of two papers with related results: Shanshuang Yang, On dilatations and substantial boundary points of homeomorphisms of Jordan curves, *Results Math.* **31** (1979), 180–188, and Qi Yi, A problem in extremal quasiconformal extensions, *Sci. China Ser. A* **41:11** (1998), 1135–1141.

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