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On the dilatation of extremal quasiconformal mappings of polygons

Kurt Strebel

Abstract. A polygon P_N is the unit disk \mathbb{D} with n distinguished boundary points, $4 \leq n \leq N$. An extremal quasiconformal mapping $f_0: \mathbb{D}_z \rightarrow \mathbb{D}_w$ maps each polygon P_N inscribed in \mathbb{D}_z onto a polygon P'_N inscribed in \mathbb{D}_w . Let f_N be the extremal quasiconformal mapping of P_N onto P'_N . Let K_N be its dilatation and let K_0 be the maximal dilatation of f_0 . Then, evidently $\sup K_N \leq K_0$. The problem is, when equality holds. This is completely answered, if f_0 does not have any essential boundary points. For quadrilaterals Q and $Q' = f_0(Q)$ the problem is $\sup(M'/M) = K_0$, with M and M' the moduli of Q and Q' respectively.

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Introduction

1. Let h be a quasisymmetric mapping of the boundary of the unit disk \mathbb{D}_z onto the boundary of \mathbb{D}_w and let f be a quasiconformal extension of h into the disk. It is called extremal and denoted by f_0 if its maximal dilatation K_0 is smallest possible. We always assume $K_0 > 1$. The disk \mathbb{D}_z becomes a quadrilateral Q if we mark four different points z_j , $j = 1, \dots, 4$, in the positive direction on its boundary $\partial\mathbb{D}_z$. The mapping f_0 takes the vertices z_j into points $w_j = f_0(z_j)$ on $\partial\mathbb{D}_w$ and thus the quadrilateral Q into a quadrilateral $Q' = f_0(Q)$ inscribed in \mathbb{D}_w . It follows from the definition of quasiconformality that the conformal moduli M and M' of Q and Q' respectively satisfy (for general properties of quasiconformal mappings, see [3])

$$\frac{1}{K_0} M \leq M' \leq K_0 M. \quad (1)$$

It has been a question for some time, if the bound K_0 is best possible in the inequality (1), in other words, if the maximal dilatation K_0 of the extremal quasiconformal extension f_0 of h can be determined by the ratio of the moduli of

inscribed quadrilaterals,

$$\sup \frac{M'}{M} = K_0. \quad (2)$$

The question has recently been answered in the negative by Anderson and Hinkkanen [1] by laborious computations of a counterexample (horizontal stretching of a parallelogram) and by Reich [4] who reduced it to an approximation problem for holomorphic functions. More counterexamples are given in [9].

2. It is easy to find examples where (2) holds; the above solutions consist therefore in the construction of examples where it does not hold. A type of the first kind is a vertical half strip S and its horizontal stretching by K_0 . Let $z = x + iy$, $S = \{z; 0 < x < a, 0 < y\}$, $w = u + iv$, $S' = \{w; 0 < u < K_0a, 0 < v\}$. We make S to a quadrilateral by marking the vertices $(0, a, a + ib, ib)$ for arbitrary $b > 0$, and similarly S' by marking the image points $(0, K_0a, K_0a + ib, ib)$. Making use of the extremal length definition of the modulus of a quadrilateral ([3], p. 21) as the extremal distance of the vertical sides we easily find the estimates

$$M \leq a/b, \quad M' \geq K_0a/(b + K_0a) \quad (3)$$

and thus

$$K_0 \geq \frac{M'}{M} \geq \frac{K_0a}{b + K_0a} \cdot \frac{b}{a}, \quad (4)$$

which gives

$$\lim_{b \rightarrow \infty} \frac{M'}{M} = K_0. \quad (5)$$

3. The problem with the moduli of quadrilaterals has a different interpretation. We look at the extremal quasiconformal mapping f of Q onto Q' . This is a mapping of \mathbb{D}_z onto \mathbb{D}_w which takes the vertices of Q into those of Q' . Its dilatation is $K = M'/M$, and the question is now what happens with K if we vary the vertices of Q in all possible ways? Of course we always have $K \leq K_0$, but will we have $\sup K = K_0$? In this formulation the problem has a natural generalization to polygons, i.e. disks with an arbitrary finite number $n \geq 4$ of vertices. The basic extremal qc mapping f_0 assigns a polygon P'_n inscribed in \mathbb{D}_w to each polygon P_n inscribed in \mathbb{D}_z . The extremal qc mapping f_n of P_n onto P'_n (i.e. of course of \mathbb{D}_z onto \mathbb{D}_w , but with the only requirement that the vertices of P_n go into the vertices of P'_n) is a Teichmüller mapping with a complex dilatation $\varkappa_n = k_n(\overline{\varphi_n}/|\varphi_n|)$, $k_n = (K_n - 1)/(K_n + 1)$. The quadratic differential φ_n is rational, with at most first order poles at the vertices of P_n . Moreover, $\varphi_n(z) dz^2$ is real along the sides of P_n . Since f_0 also maps the vertices of P_n onto those of P'_n and f_n is extremal with this property, we have $K_n \leq K_0$. The question arises if, by varying the polygon P_n in all possible ways, we have

$$\sup K_n = K_0. \quad (6)$$

4. It follows from general principles of qc mappings (we refer to [3] for the general theory) that this is in fact true if we allow the number n of vertices to become arbitrarily large (for a proof see [5], p. 385, bottom). But how is it, if this number is bounded, $n \leq N$ say? With a certain natural restriction we will characterize the extremal mappings f_0 for which this happens. The proof is an application of the “polygon inequality” ([5], p. 384) and a theorem of R. Fehlmann ([2], p. 567).

The polygon inequality

5. Let f_0 be an extremal qc mapping of \mathbb{D}_z onto \mathbb{D}_w with $f_0|_{\partial\mathbb{D}_z} = h$. Let \varkappa_0 with $\|\varkappa_0\|_{\infty} = k_0$ be its complex dilatation and $K_0 = (1 + k_0)/(1 - k_0)$ its maximal dilatation. Mark n points z_j , $j = 1, \dots, n$, on $\partial\mathbb{D}_z$, $4 \leq n \leq N$. The disk \mathbb{D}_z with the marked boundary points z_j is called a polygon P_n . The image of P_n by f_0 is the polygon P'_n , inscribed in \mathbb{D}_w , with vertices $w_j = f_0(z_j)$. Let f_n be the extremal qc mapping of P_n onto P'_n , $f_n(z_j) = w_j$, and let φ_n , $\|\varphi_n\| = 1$, denote the associated quadratic differential. The complex dilatation of f_n is $k_n(\overline{\varphi_n}/|\varphi_n|)$. Then, the *Polygon Inequality* holds:

$$\operatorname{Re} \iint_{|z|<1} \frac{\varkappa_0(z)\varphi_n(z)}{1-|\varkappa_0(z)|^2} dx dy \geq \frac{k_n}{1-k_n} - \iint_{|z|<1} |\varphi_n(z)| \frac{|\varkappa_0(z)|^2}{1-|\varkappa_0(z)|^2} dx dy. \quad (7)$$

For the proof I refer to ([5], p. 384). In that paper, the inequality was used to prove that the “polygon differentials” φ_n form a Hamilton sequence for \varkappa_0 if the number of vertices tends to infinity and the sides of the polygons P_n become arbitrarily short. This led to a proof of the necessity of the Hamilton–Krushkal condition for extremality. Now, on the contrary, we restrict the number of vertices by a fixed number N , and we denote a polygon with $n \leq N$ vertices generically by P_N .

6.

Theorem 1. *Let $f_0: \mathbb{D}_z \rightarrow \mathbb{D}_w$ with complex dilatation \varkappa_0 , $\|\varkappa_0\|_{\infty} = k_0$, be extremal for its boundary values h . Assume that for a fixed number N the polygon mappings $f_N: P_N \rightarrow P'_N = f_0(P_N)$ with complex dilatation $k_N(\overline{\varphi_N}/|\varphi_N|)$ satisfy*

$$\sup k_N = k_0. \quad (8)$$

(This is of course equivalent to $\sup K_N = K_0$.) Then, there is a sequence of polygon mappings $f_N^{(i)}$ the quadratic differentials $\varphi_N^{(i)}$ of which, $\|\varphi_N^{(i)}\| = 1$, form a Hamilton sequence for \varkappa_0 , i.e.

$$\operatorname{Re} \iint \varkappa_0(z)\varphi_N^{(i)}(z) dx dy \rightarrow k_0, \quad i \rightarrow \infty. \quad (9)$$

Proof. Assume first that f_0 has constant dilatation $|\varkappa_0(z)| = k_0$ a.e. Then, the polygon inequality yields

$$\frac{1}{1-k_0^2} \operatorname{Re} \iint \varkappa_0(z) \varphi_N(z) dx dy \geq \frac{k_N}{1-k_N} - \frac{k_0^2}{1-k_0^2} \quad (10)$$

for all polygons P_N . Let $P_N^{(i)}$ be a sequence of polygons the extremal mappings $f_N^{(i)}$ of which satisfy $k_N^{(i)} \rightarrow k_0$. Then

$$\varliminf_{i \rightarrow \infty} \operatorname{Re} \iint \varkappa_0(z) \varphi_N^{(i)}(z) dx dy \geq \frac{k_0}{1-k_0^2} (1-k_0^2) - k_0^2 = k_0. \quad (11)$$

On the other hand

$$\operatorname{Re} \iint \varkappa_0(z) \varphi_N^{(i)}(z) dx dy \leq \left| \iint \varkappa_0(z) \varphi_N^{(i)}(z) dx dy \right| \leq k_0. \quad (12)$$

This gives the result (9) in the case where $|\varkappa_0(z)| = k_0$ a.e. If $|\varkappa_0(z)|$ is not constant a.e. we proceed as in ([5], p. 386 and p. 382). However, in our present work we only need the case of constant $|\varkappa_0(z)|$. \square

Since the number of vertices of the polygons $P_N^{(i)}$ is smaller or equal to N , we can assume, by passing to a further subsequence, that they converge to a finite number $\leq N$ of points on $\partial\mathbb{D}_z$. We write $P_N^{(i)} \rightarrow P_N$.

The vertical half strip in the introduction is an example where the given quadrilaterals give rise to a Hamilton sequence for the horizontal stretching (which is uniquely extremal).

Extremal mappings without essential boundary point

7. Let f_0 with complex dilatation \varkappa_0 , $\|\varkappa_0\|_\infty = k_0$, be extremal for its boundary values h . A boundary point z of \mathbb{D}_z is called essential, if the following is true: For every neighborhood U of z and every qc mapping g of $U \cap \mathbb{D}_z$ which is equal to h on $U \cap \partial\mathbb{D}_z$ the maximal dilatation of g is at least equal to $K_0 = (1+k_0)/(1-k_0)$.

A theorem of R. Fehlmann ([2]), p. 567) says: If the complex dilatation \varkappa_0 has a degenerating Hamilton sequence (i.e. which tends to zero locally uniformly in the domain), then f_0 has an essential boundary point.

Combining this result with the considerations in ([7], p. 466) we can say: If f_0 does not have an essential boundary point, then, every Hamilton sequence for \varkappa_0 converges in norm to a holomorphic quadratic differential φ_0 , $\|\varphi_0\| = 1$, and $\varkappa_0 = k_0(\overline{\varphi_0}/|\varphi_0|)$ is the complex dilatation of f_0 .

8. Let us apply this to our case. Every polygon differential $\varphi_N^{(i)}$ can be continued across the boundary $\partial\mathbb{D}_z$ by reflection to a rational differential in the whole plane, of norm two. Therefore the limit φ_0 can be reflected. Since its norm is finite, it has at most first order poles at the $n \leq N$ limits of the vertices of the $P_N^{(i)}$, and $\varphi_0(z) dz^2$ is real along the subintervals of $\partial\mathbb{D}_z$ between these limits. Our main result is

Theorem 2. *Let $f_0: \mathbb{D}_z \rightarrow \mathbb{D}_w$ be a qc mapping which is extremal for its boundary values, and assume that it does not have an essential boundary point. For fixed $N \geq 4$ denote the polygons with $4 \leq n \leq N$ vertices inscribed in \mathbb{D}_z generically by P_N . To every P_N the mapping f_0 determines a polygon P'_N inscribed in \mathbb{D}_w , simply by mapping the vertices of P_N onto those of P'_N . Assume that the extremal mappings $f_N: P_N \rightarrow P'_N$ satisfy $\sup k_N = k_0$. Then, there is a convergent sequence $f_N^{(i)}$ of polygon mappings with $\varphi_N^{(i)} \rightarrow \varphi_0$ in norm, where $\varkappa_0 = k_0(\overline{\varphi_0}/|\varphi_0|)$ is the complex dilatation of f_0 . f_0 itself is the extremal qc mapping of a polygon with $n \leq N$ vertices, and every maximizing sequence $f_N^{(i)}$, $k_N^{(i)} \rightarrow k_0$, tends to f_0 uniformly, $\varphi_N^{(i)} \rightarrow \varphi_0$ in norm.*

9. In order to see that the theorem is not empty, let $f: \mathbb{D}_z \rightarrow \mathbb{D}_w$ be an extremal polygon mapping and let φ be the associated rational quadratic differential, $\varkappa = k(\overline{\varphi}/|\varphi|)$ the complex dilatation. The vertices z_j are either first order poles or regular points (i.e. $\varphi(z_j) \neq 0$) or zeroes of φ of any order. Along the sides we have $\varphi(z) dz^2$ real, and thus the sides are composed of trajectories and orthogonal trajectories.

The first order poles and the zeroes are clearly the only candidates for an essential boundary point of f . In order to find the local maximal dilatation H_z at such a point z we first apply the mapping $\Phi = f \sqrt{\varphi}$ and then the horizontal stretching by K . The integral Φ maps an interior half neighborhood of z onto an angle with a horizontal and a vertical side. It is a right angle in the case of a first order pole and an angle which is a multiple of $\frac{1}{2}\pi$ in the case of a zero, possibly many sheeted. In the image \mathbb{D}_w we have the same situation, with a quadratic differential ψ and an integral $\Psi = f \sqrt{\psi}$. The horizontal side of the angle is stretched by K while the vertical side is mapped identically. It is known (and easy to see, using logarithms on both sides, see [6], p. 323) that the local extremal mapping with the given boundary values has dilatation $< K$. Since f itself is extremal with dilatation K , it does not have any essential boundary point, thus satisfying our requirement.

10. Let now $f_0: P_N \rightarrow P'_N$ with complex dilatation $\varkappa_0 = k_0(\overline{\varphi_0}/|\varphi_0|)$ be an extremal polygon mapping. We can clearly take $f_N = f_0$ itself and get $\sup k_N = k_0$. Actually we only need to consider the substantial boundary points of f_0 (= poles of φ_0), since the extremal mapping of the restricted polygon \tilde{P}_N onto \tilde{P}'_N is

the same as f_0 .

Let \tilde{N} be the number of substantial boundary points of f_0 . If, however, we only admit polygons with at most $N' \leq \tilde{N} - 1$ vertices, we find $\sup k_{N'} < k_0$. For, if $\sup k_{N'} = k_0$ we would again arrive, by the same considerations as before, at an extremal polygon mapping $f_{N'}$ with a quadratic differential $\varphi_{N'}$ with at most N' first order poles, whereas φ_0 has \tilde{N} first order poles. Therefore $\varphi_{N'} \neq \varphi_0$, a contradiction.

11. We started with the following question. Let f_0 with complex dilatation \varkappa_0 , $\|\varkappa_0\| = k_0$, be a qc mapping of \mathbb{D}_z onto \mathbb{D}_w which is extremal for its boundary values and which does not have an essential boundary point. Inscribe quadrilaterals Q into \mathbb{D}_z and denote their images by f_0 in \mathbb{D}_w by Q' . The image Q' has, as its vertices, the images by f_0 of the vertices of Q . Let M and M' be the moduli of Q and Q' respectively. The question is, if (2) can hold.

Let f with dilatation K be the extremal mapping of Q onto Q' . The equation (2) is equivalent with

$$\sup K = K_0 \quad (13)$$

where the sup is taken over all quadrilaterals Q . This is the special case of (8) for $N = 4$. We find

Theorem 3. *The extremal mapping f_0 satisfies (13) for the inscribed quadrilaterals Q if and only if it is the extremal mapping of a quadrilateral itself.*

This means that in all other cases we have inequality in (13). The example of Anderson and Hinkkanen is the horizontal stretching of a parallelogram. This mapping f_0 has no essential boundary point and is, in their situation, not the mapping of quadrilaterals. Therefore $\sup(M'/M) < K_0$.

The example of Reich has analytic boundary values. Therefore we have again $\sup(M'/M) < K_0$.

Clearly, in both examples, we still have inequality in (13) even if we allow any inscribed polygons with an arbitrary fixed bound N for the number of vertices.

Added in Proof. After the completion of this paper I have become aware of two papers with related results: Shanshuang Yang, On dilatations and substantial boundary points of homeomorphisms of Jordan curves, *Results Math.* **31** (1979), 180–188, and Qi Yi, A problem in extremal quasiconformal extensions, *Sci. China Ser. A* **41**:11 (1998), 1135–1141.

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