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Commentarii Mathematici Helvetici

Degenerations for representations of extended Dynkin quivers

Grzegorz Zwara

Abstract. Let A be the path algebra of a quiver of extended Dynkin type $\tilde{\mathbb{A}}_n$, $\tilde{\mathbb{D}}_n$, $\tilde{\mathbb{E}}_6$, $\tilde{\mathbb{E}}_7$ or $\tilde{\mathbb{E}}_8$. We show that a finite dimensional A-module M degenerates to another A-module N if and only if there are short exact sequences $0 \to U_i \to M_i \to V_i \to 0$ of A-modules such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$ for $1 \leq i \leq s$ and $N = M_{s+1}$ are true for some natural number s.

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 ${\bf Keywords.}$ Modules, degenerations, extended Dynkin diagrams.

1. Introduction and main results

Let A be a finite dimensional associative K-algebra with an identity over an algebraically closed field K of arbitrary characteristic. If $a_1 = 1, \ldots, a_n$ is a basis of A over K, we have the constant structures a_{ijk} defined by $a_i a_j = \sum a_{ijk} a_k$. The affine variety mod $_A(d)$ of d-dimensional unital left A-modules consists of n-tuples $m = (m_1, \ldots, m_n)$ of $d \times d$ -matrices with coefficients in K such that m_1 is the identity matrix and $m_i m_j = \sum a_{ijk} m_k$ holds for all indices i and j. The general linear group $\operatorname{Gl}_d(K)$ acts on mod $_A(d)$ by conjugation, and the orbits correspond to the isomorphism classes of d-dimensional modules (see [11]). We shall agree to identify a d-dimensional A-module M with the point of mod $_A(d)$. Then one says that a module N in mod $_A(d)$ is a degeneration of a module M in mod $_A(d)$, and we denote this fact by $M \leq_{\operatorname{deg}} N$. Thus $\leq_{\operatorname{deg}}$ is a partial order on the set of isomorphism classes of a given dimension. It is not clear how to characterize $\leq_{\operatorname{deg}}$ in terms of representation theory.

There has been a work by S. Abeasis and A. del Fra [1], K. Bongartz [7], [10], [9], Ch. Riedtmann [13], and A. Skowroński and the author [15], [16], [17] connecting \leq_{deg} with other partial orders \leq_{ext} and \leq on the isomorphism classes in mod_A(d). They are defined in terms of representation theory as follows:

- $M \leq_{\text{ext}} N$: \Leftrightarrow there are modules M_i , U_i , V_i and short exact sequences $0 \to U_i \to M_i \to V_i \to 0$ in mod A such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$, $1 \leq i \leq s$, and $N = M_{s+1}$ for some natural number s.
- $M \leq N$: $\Leftrightarrow [X, M] \leq [X, N]$ holds for all modules X.

Here and later on we abbreviate $\dim_K \operatorname{Hom}_A(X, Y)$ by [X, Y], and furthermore $\dim_K \operatorname{Ext}_A^i(X, Y)$ by $[X, Y]^i$. Then for modules M and N in mod $_A(d)$ the following implications hold:

$$M \leq_{\text{ext}} N \Longrightarrow M \leq_{\text{deg}} N \Longrightarrow M \leq N$$

(see [10], [13]). Unfortunately the reverse implications are not true in general, and it would be interesting to find out when they are. K. Bongartz proved in [10] (see also [8]) that it is the case for all representations of Dynkin quivers and the double arrow. Recently, the author proved in [17] that \leq and \leq_{ext} are also equivalent for all modules over representation-finite blocks of group algebras. Moreover, in [9] K. Bongartz proved that \leq_{deg} and \leq coincide for all representations of extended Dynkin quivers, and conjectured that possibly \leq_{ext} and \leq_{deg} also coincide. The main aim of this paper is to prove the following theorem.

Theorem. The partial orders \leq and \leq_{ext} coincide for modules over all tame concealed algebras.

In particular we get the positive answer to the above question.

Corollary. The partial orders \leq , \leq_{deg} and \leq_{ext} are equivalent for all representations of extended Dynkin quivers.

We mention that K. Bongartz described in [8, Theorem 4] the set-theoretic structure of minimal degenerations of modules provided the partial orders \leq_{ext} and \leq coincide. In a forthcoming paper we shall describe the minimal singularities for representations of extended Dynkin quivers.

The paper is organized as follows. In Section 2 we fix the notation, recall the relevant definitions and facts, and prove some preliminary results on modules which we apply in our investigations. In Section 3 we recall several known facts on tame concealed algebras. In particular we describe some properties of the additive categories of standard stable tubes. Section 4 is devoted to the proof of the Theorem.

For basic background on the topics considered here we refer to [5], [10], [9], [11] and [14]. The results presented in this paper form a part of the author's doctoral dissertation written under supervision of professor A. Skowroński. The author gratefully acknowledges support from the Polish Scientific Grant KBN No. 2 PO3A 020 08.

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2. Preliminary results

2.1. Throughout the paper A denotes a fixed finite dimensional associative K-algebra with an identity over an algebraically closed field K. We denote by mod A the category of finite dimensional left A-modules, by ind A the full subcategory of mod A formed by indecomposable modules, and by rad(mod A) the Jacobson radical of mod A. By an A-module is meant an object from mod A. Further, we denote by Γ_A the Auslander-Reiten quiver of A and by $\tau = \tau_A$ and $\tau^- = \tau_A^-$ the Auslander-Reiten translations DTr and Tr D, respectively. We shall agree to identify the vertices of Γ_A with the corresponding indecomposable modules. For a module M we denote by [M] the image of M in the Grothendieck group $K_0(A)$ of A. Thus [M] = [N] if and only if M and N have the same simple composition factors including the multiplicities. Finally, for a family \mathcal{F} of A-modules, we denote by add(\mathcal{F}) the additive category given by \mathcal{F} , that is, the full subcategory of mod A formed by all modules isomorphic to the direct summands of direct sums of modules from \mathcal{F} .

2.2. Following [13], for M, N from mod A, we set $M \leq N$ if and only if $[X, M] \leq [X, N]$ for all A-modules X. The fact that \leq is a partial order on the isomorphism classes of A-modules follows from a result by M. Auslander [3] (see also [7]). Observe that, if M and N have the same dimension and $M \leq N$, then [M] = [N]. Moreover, M. Auslander and I. Reiten have shown in [4] that, if M and N are A-modules with [M] = [N], then for all nonprojective indecomposable A-modules X and all noninjective indecomposable modules Y the following formulas hold (see [12]):

$$[X, M] - [M, \tau X] = [X, N] - [N, \tau X]$$
$$[M, Y] - [\tau^{-}Y, M] = [N, Y] - [\tau^{-}Y, N]$$

Hence, if [M] = [N], then $M \leq N$ if and only if $[M, X] \leq [N, X]$ for all A-modules X.

2.3. Let M and N be A-modules with [M] = [N] and

$$\Sigma: 0 \to D \to E \to F \to 0$$

an exact sequence in mod A. Following [13] we define the additive functions $\delta_{M,N}$, $\delta'_{M,N}$ and δ_{Σ} on A-modules X as follows

$$\begin{split} \delta_{M,N}(X) &= [N, X] - [M, X] \\ \delta'_{M,N}(X) &= [X, N] - [X, M] \\ \delta_{\Sigma}(X) &= \delta_{E, D \oplus F}(X) = [D \oplus F, X] - [E, X]. \end{split}$$

From the Auslander-Reiten formulas (2.2) we get the following very useful equalities

$$\delta_{M,N}(X) = \delta'_{M,N}(\tau^- X), \qquad \delta_{M,N}(\tau X) = \delta'_{M,N}(X)$$

for all A-modules X. Observe also that $\delta_{M,N}(I) = 0$ for any injective A-module I, and $\delta'_{M,N}(P) = 0$ for any projective A-module P. In particular, the following conditions are equivalent:

 $\begin{array}{ll} (1) & M \leq N, \\ (2) & \delta_{M,N}(X) \geq 0 \text{ for all } X \in \Gamma_A, \\ (3) & \delta'_{M,N}(X) \geq 0 \text{ for all } X \in \Gamma_A. \end{array}$

2.4. For an A-module M and an indecomposable A-module Z, we denote by $\mu(M, Z)$ the multiplicity of Z as a direct summand of M. For a nonprojective indecomposable A-module U, we denote by $\Sigma(U)$ an Auslander-Reiten sequence

$$\Sigma(U): 0 \to \tau U \to E(U) \to U \to 0,$$

and, for an injective indecomposable A-module I, we set E(I) = I/soc(I), $\tau^{-}I = 0$.

We shall need the following lemma.

Lemma 2.5. Let M, N be A-modules with [M] = [N] and U an indecomposable A-module. Then

$$\mu(N,U) - \mu(M,U) = \delta_{M,N}(U) - \delta_{M,N}(E(U)) + \delta_{M,N}(\tau U).$$

Proof. If U is nonprojective, then the Auslander-Reiten sequence $\Sigma(U)$ induces an exact sequence

$$0 \to \operatorname{Hom}_A(M, \tau U) \to \operatorname{Hom}_A(M, E(U)) \to \operatorname{rad}(M, U) \to 0,$$

and hence we get

$$[M, \tau U \oplus U] - [M, E(U)] = [M, U] - \dim_K \operatorname{rad}(M, U) = \mu(M, U).$$

Similarly, we have

$$[N, \tau U \oplus U] - [N, E(U)] = \mu(N, U).$$

Then we obtain the equalities

$$\mu(N,U) - \mu(M,U) = ([N,\tau U \oplus U] - [M,\tau U \oplus U]) - (N,[E(U)] - [M,E(U)]) = \delta_{M,N}(\tau U) + \delta_{M,N}(U) - \delta_{M,N}(E(U)).$$

Assume now that U is projective. Then $\operatorname{Hom}_A(M, \operatorname{rad} U) \simeq \operatorname{rad}(M, U)$, and so

$$[M,U] - [M, \operatorname{rad} U] = \mu(M,U).$$

Similarly, we have

$$[N, U] - [N, \operatorname{rad} U] = \mu(N, U).$$

Therefore, we get

$$\mu(N, U) - \mu(M, U) = ([N, U] - [M, U]) - ([N, \operatorname{rad} U] - [M, \operatorname{rad} U])$$

= $\delta_{M,N}(U) - \delta_{M,N}(\operatorname{rad} U)$
= $\delta_{M,N}(U) - \delta_{M,N}(E(U)) + \delta_{M,N}(\tau U).$

2.6. A component Γ of Γ_A , without oriented cycles and such that any τ -orbit contains a projective module is called *preprojective*. Also any module $X \in add(\Gamma)$ is called *preprojective*. There is a partial order \preceq on the set of vertices of a preprojective component Γ with $U \preceq V$ if there exists a path in Γ leading from U to V. Preinjective components and preinjective modules are defined dually. **2.7.** Let M and N be A-modules with M < N. A short nonsplittable exact

sequence

$$\Sigma: 0 \to L_1 \to M' \to L_2 \to 0$$

is said to be *admissible for* (M, N) if $M = M' \oplus V$ for some A-module V and $[L_1 \oplus L_2 \oplus V, X] \leq [N, X]$ for any A-module X (equivalently, $\delta_{\Sigma} \leq \delta_{M,N}$ or $\delta'_{\Sigma} \leq \delta'_{M,N}$). We shall need the following fact.

Proposition. Let M and N be A-modules with [M] = [N], and assume that M is preprojective and M < N holds. Then there exists an admissible sequence $0 \rightarrow L_1 \rightarrow M \rightarrow L_2 \rightarrow 0$ for (M, N).

Proof. We can repeat the proof of Theorem 4.1 in [10], since Bongartz has used the fact that N is preprojective only to prove that M is preprojective.

3. Some properties of modules over tame concealed algebras

Here and later on A denotes a fixed tame concealed algebra [14]. **3.1.** We recall those aspects of the representation theory of tame concealed algebras that we will need later (see [14], [10]). We have a decomposition of Γ_A into the preprojective part \mathcal{P} , the preinjective part \mathcal{I} and the regular one \mathcal{R} , where \mathcal{R} is a sum of stable tubes \mathcal{T}_{μ} of ranks $r_{\mu} \geq 1$, for $\mu \in \mathbb{P}^1(K) = K \cup \{\infty\}$. For any A-module X we can write $X = X_P \oplus X_R \oplus X_I$, where $X_P \in \text{add}(\mathcal{P}), X_I \in \text{add}(\mathcal{I})$ and $X_R = \bigoplus_{\mu \in \mathbb{P}^1(K)} X_\mu$ with $X_\mu \in \text{add}(\mathcal{T}_\mu)$. All connected components of Γ_A are standard (see [14] for definition). A tube of rank 1 is called *homogeneous* and \mathcal{T}_{μ} is not homogeneous for at most three $\mu \in \mathbb{P}^1(K)$. For any $X, Y \in \Gamma_A$, if [X, Y] > 0

and X and Y do not belong to the same connected component of Γ_A , then X is preprojective or Y is preinjective. The abelian category $\operatorname{add}(\mathcal{T}_{\mu})$ is serial and closed under extensions, so we may speak about simple regular modules, composition series in $\operatorname{add}(\mathcal{T}_{\mu})$, and so on. A tube \mathcal{T}_{μ} has r_{μ} simple regular modules, which are conjugate under τ . If a tube T_{μ} is homogeneous $(r_{\mu} = 1)$, then we denote a unique simple regular module in \mathcal{T}_{μ} by E_{μ} . For any simple regular module E in \mathcal{T}_{μ} we denote by

$$\cdots \to \varphi^3 E \to \varphi^2 E \to \varphi E \to \varphi^0 E = E$$

a unique infinite sectional path in \mathcal{T}_{μ} of epimorphisms and by

$$E = \psi^0 E o \psi E o \psi^2 E o \psi^3 E o \cdots$$

a unique infinite sectional path in \mathcal{T}_{μ} of monomorphisms. Then every indecomposable module in \mathcal{T}_{μ} is of the form $\varphi^{j} E$ and $\psi^{j} E'$ for some $j \geq 0$ and simple regular modules E, E' in \mathcal{T}_{μ} . In an obvious way we define functions

$$\varphi^k, \psi^k : \mathcal{T}_\mu \to \mathcal{T}_\mu \cup \{0\}$$

for any integer k, such that for any simple regular module E in \mathcal{T}_{μ} and $l \geq 0$ we have:

- $\varphi^k(\varphi^l E) = \varphi^{k+l}E$ if $k+l \ge 0$, and $\varphi^k(\varphi^l E) = 0$ otherwise; $\psi^k(\psi^l E) = \psi^{k+l}E$ if $k+l \ge 0$, and $\psi^k(\psi^l E) = 0$ otherwise.

Observe that for any integer k and $X \in \mathcal{T}_{\mu}$ we have $\tau X = \psi^{-}\varphi X$, $\tau^{-}X = \varphi^{-}\psi X$ and $\varphi^{kr}X = \psi^{kr}X$, where $r = r_{\mu}$.

There is a positive, sincere vector \underline{h} in $K_0(A)$, such that

$$[\varphi^{kr_{\mu}-1}E] = [\psi^{kr_{\mu}-1}E] = k \cdot \underline{h}$$

for any simple regular module E in \mathcal{T}_{μ} and $k \geq 1$.

3.2 The global dimension of A is at most 2. All preprojective and regular modules have projective dimension at most 1, and dually all preinjective and regular modules have injective dimension at most 1. The bilinear form on $K_0(A) = \mathbb{Z}^n$ which extends the equality

$$< [M], [N] >= [M, N] - [M, N]^1 + [M, N]^2$$

and the associated quadratic form $\chi: K_0(A) \to \mathbb{Z}, \, \chi(\underline{y}) = \langle \underline{y}, \underline{y} \rangle$, will play an important role. If M has no non-zero preinjective direct summand or N has no non-zero preprojective direct summand, then

$$< [M], [N] >= [M, N] - [M, N]^{1}.$$

The quadratic form χ is positive semidefinite and controls the category mod A (see [14]). This means that the following conditions are satisfied:

- (1) For any $X \in \Gamma_A$, $\chi([X]) \in \{0, 1\}$.
- (2) For any connected, positive vector \underline{y} with $\chi(\underline{y}) = 1$, there is precisely one $X \in \Gamma_A$ with [X] = y.
- (3) For any connected, positive vector \underline{y} with $\chi(\underline{y}) = 0$, there is an infinite family of pairwise nonisomorphic modules $X \in \Gamma_A$ with [X] = y.

Moreover, $\chi(\underline{h}) = 0$ and $\langle \underline{h}, \underline{y} \rangle = - \langle \underline{y}, \underline{h} \rangle$ for any $\underline{y} \in K_0(A)$. Finally, we define a linear function $\partial : K_0(A) \to \mathbb{Z}$, called the *defect*, as follows

$$\partial \underline{y} = \langle \underline{h}, \underline{y} \rangle = - \langle \underline{y}, \underline{h} \rangle.$$

The main property of ∂ is that the value $\partial[X]$ is negative for any $X \in \mathcal{P}$, positive for any $X \in \mathcal{I}$, and zero for any $X \in \mathcal{R}$.

Lemma 3.3. If $M \leq N$, then $\partial[M_P] - \partial[N_P] = \partial[N_I] - \partial[M_I] \geq 0$.

Proof. Since [M] = [N], then

$$\partial[M_P] + \partial[M_R] + \partial[M_I] = \partial[N_P] + \partial[N_R] + \partial[N_I].$$

The equalities $\partial[M_R] = \partial[N_R] = 0$ imply $\partial[M_P] - \partial[N_P] = \partial[N_I] - \partial[M_I]$. Take a homogeneous tube \mathcal{T}_{μ} with $(M \oplus N)_{\mu} = 0$. Then

$$0 \leq [N, E_{\mu}] - [M, E_{\mu}] = [N_P, E_{\mu}] - [M_P, E_{\mu}] = \langle [N_P], [E_{\mu}] \rangle - \langle [M_P], [E_{\mu}] \rangle = \langle [N_P], \underline{h} \rangle - \langle [M_P], \underline{h} \rangle = \partial [M_P] - \partial [N_P].$$

3.4. Fix a tube \mathcal{T}_{μ} , $\mu \in \mathbb{P}^{1}(K)$, and a module $X \in \operatorname{add}(\mathcal{T}_{\mu})$. Let $H(X) \geq 0$ be the minimal number such that for any indecomposable direct summand $\varphi^{j}E$ of X, where E is a simple regular module in \mathcal{T}_{μ} , we have j < H(X) (so H(X) is the maximal quasi-length of an indecomposable direct summand of X). For any simple regular module E in \mathcal{T}_{μ} we denote by $\ell_{E}(X)$ the multiplicity of E as a composition factor of a composition series of X in the category $\operatorname{add}(\mathcal{T}_{\mu})$. If E_{1}, \ldots, E_{r} $(r = r_{\mu})$ denote all simple regular modules in \mathcal{T}_{μ} , then

$$[X] = \ell_{E_1}(X)[E_1] + \ell_{E_2}(X)[E_2] + \dots + \ell_{E_r}(X)[E_r].$$

Moreover, the following lemma holds (see Lemma 5.1 in [15]).

Lemma 3.5. Let X be a module in $\operatorname{add}(\mathcal{T}_{\mu})$ and E be any simple regular module in \mathcal{T}_{μ} . Then for any $k \geq H(X) - 1$ we have

$$[X, \psi^k E] = \ell_E(X) = [\varphi^k E, X].$$

As a consequence of the above lemma we obtain

Lemma 3.6. Let i, j be integers with $j \ge 0$ and E be any simple regular module in T_{μ} . Then

- (i) [φ^sψ^tE, ψ^{r-1}E] = 1 for all s ≥ 0, 0 ≤ t < r, and [X, ψ^{r-1}E] = 0 for the remaining indecomposable modules X ∈ T_μ.
 (ii) [φ^sψ^tE, ψ^{r-1}φ^jE] [φ^sψ^tE, ψ⁻φ^jE] = 1 for all s ≥ j, 0 ≤ t < r, and [X, ψ^{r-1}φ^jE] [X, ψ⁻φ^jE] = 0 for the remaining indecomposable modules X ∈ T_μ. (ii) $X \in \mathcal{T}_{\mu}$.
- (iii) If $j \ge r$, then $[\psi^j E, \psi^j E] > 1$. (iv) $[E, \psi^j E] = 1$ and $[E', \psi^j E] = 0$ for all simple regular modules $E' \ne E$ in T_{μ} .

Applying Lemmas 4.3 and 4.6 in [15], we obtain the following result (see also Corollary 2.2 in [2]).

Lemma 3.7. Let $X \in T_{\mu}$, $s,t \geq 0$ be integers, and M, N be A-modules with [M] = [N]. Then

(i) There exists a nonsplittable exact sequence

$$\Sigma: 0 \to \varphi^s X \to \varphi^s \psi^{t+1} X \oplus \varphi^- X \to \varphi^- \psi^{t+1} X \to 0.$$

Moreover, if s < r or t < r, then $\delta_{\Sigma}(\varphi^i \psi^j X) = 1$ for all $0 \le i \le s$, $0 \leq j \leq t$, and $\delta_{\Sigma}(Y) = 0$ for the remaining indecomposable A-modules. (ii)

$$\sum_{0 \le i \le s} \sum_{0 \le j \le t} \mu(N, \varphi^{-i} \psi^j X) - \mu(M, \varphi^{-i} \psi^j X)$$
$$= \delta_{M,N}(\psi^- \varphi^{s+1} X) - \delta_{M,N}(\psi^- X) - \delta_{M,N}(\varphi^{s+1} \psi^t X) + \delta_{M,N}(\psi^t X).$$

Lemma 3.8. Let M, N be A-modules with $M \leq N$ and $\partial[M_P] = \partial[N_P]$. Then

- (i) $[M_P] \ge [N_P]$.
- (ii) For any indecomposable simple regular module E in a tube T_{μ} we have

$$\ell_E(M_\mu) \le \ell_E(N_\mu).$$

(iii) For any tube T_{μ} , $[M_{\mu}] \leq [N_{\mu}]$ holds.

Proof. (i) Let I be any indecomposable injective A-module. We shall show that $[M_P, I] \ge [N_P, I]$. For all but finitely many k > 0, the vector $k \cdot \underline{h} - [I]$ is positive

and connected. Moreover,

$$\chi(k \cdot \underline{h} - [I]) = < k \cdot \underline{h} - [I], k \cdot \underline{h} - [I] > = < [I], [I] > = \chi([I]) = 1.$$

Thus for all but finitely many k > 0 there is an indecomposable A-module X_k with $[X_k] = k \cdot \underline{h} - [I]$. Of course

$$\partial[X_k] = <\underline{h}, k \cdot \underline{h} - [I] > = - <\underline{h}, [I] > = -\partial[I] < 0,$$

which implies that X_k is preprojective. Take k > 0 such that there exists a preprojective A-module X_k with $[X_k] = k\underline{h} - [I]$ and $[M_P \oplus N_P, X_k]^1 = 0$. Then

$$\begin{split} [M_P, I] = &< [M_P], [I] > = -k\partial[M_P] - < [M_P], [X_k] > = -k\partial[M_P] - [M_P, X_k] \\ \geq -k\partial[N_P] - [N_P, X_k] = -k\partial[N_P] - < [N_P], [X_k] > = < [N_P], [I] > \\ = [N_P, I]. \end{split}$$

Hence, $[M_P] \ge [N_P]$.

(ii) Let $r = r_{\mu}$ and s be a natural number such that $sr \ge H(M_{\mu} \oplus N_{\mu})$. Then

$$0 \leq [N, \psi^{sr-1}E] - [M, \psi^{sr-1}E] = [N_P, \psi^{sr-1}E] - [M_P, \psi^{sr-1}E] + [N_\mu, \psi^{sr-1}E] - [M_\mu, \psi^{sr-1}E] = < [N_P], s \cdot \underline{h} > - < [M_P], s \cdot \underline{h} > +\ell_E(N_\mu) - \ell_E(M_\mu) = -s(\partial[N_P] - \partial[M_P]) + \ell_E(N_\mu) - \ell_E(M_\mu) = \ell_E(N_\mu) - \ell_E(M_\mu),$$

by Lemma 3.5.

(iii) follows from (ii), since for any $X \in \text{add}(\mathcal{T}_{\mu})$ we have

$$[X] = \ell_{E_1}(X)[E_1] + \ldots + \ell_{E_r}(X)[E_r],$$

where $r = r_{\mu}$ and E_1, \ldots, E_r denote all simple regular modules in \mathcal{T}_{μ} .

Lemma 3.9. Let Γ' be a disjoint union of some tubes in Γ_A and $\Gamma'' = \Gamma_A \setminus \Gamma'$. Then for any $X \in \operatorname{add}(\Gamma'')$ and $R_1, R_2 \in \operatorname{add}(\Gamma')$ with $[R_1] = [R_2]$ we have

$$[X, R_1] = [X, R_2]$$
 and $[R_1, X] = [R_2, X].$

Proof. By duality, it is enough to prove the first equality. We may assume that X is indecomposable and preprojective, because $[X, R_1] = [X, R_2] = 0$ for any regular or preinjective A-module $X \in \text{add}(\Gamma'')$. Hence, we get

$$[X, R_1] - [X, R_1]^1 = \langle [X], [R_1] \rangle = \langle [X], [R_2] \rangle = [X, R_2] - [X, R_2]^1$$

Since $[X, R_1]^1 = [X, R_2]^1 = 0$ for any preprojective A-module X, we obtain the required equality $[X, R_1] = [X, R_2]$.

4. Proof of the Theorem

We shall divide our proof of the Theorem into several steps. We use the notations introduced in Sections 2 and 3.

Proposition 4.1. Let M and $N = N_0 \oplus N_1$ be A-modules without any common indecomposable direct summands. Assume that M < N and N_0 is a preprojective indecomposable A-module with $[N_0, N] = [N_0, M]$. If there is no admissible sequence of the form $0 \to N_0 \to M \to C \to 0$ for (M, N), then there exist a homogeneous tube \mathcal{T}_{ν} in Γ_A , for which $(M \oplus N)_{\nu} = 0$, and a nonsplittable exact sequence

$$0 \to L \to M \to E_{\nu} \to 0,$$

such that $[L \oplus E_{\nu}, X] \leq [N, X]$ for any indecomposable A-module $X \notin \mathcal{T}_{\nu}$.

Proof. By Theorem 2.4 in [10] N_0 embeds into M and the closure $\overline{\mathcal{Q}}$ of the quotients of M by N_0 contains N_1 . Let $t = \dim_K M + 1$ and $\Gamma' \cup \mathcal{T}_{\mu_1} \cup \cdots \cup \mathcal{T}_{\mu_t}$ be the disjoin union of all homogeneous tubes which do not contain any indecomposable direct summand of $M \oplus N$. We set $\Gamma'' = \Gamma_A \setminus \Gamma'$. Then Γ'' is the disjoint union of finitely many connected components of Γ_A , and for any natural number d, there is only a finite number of isomorphism classes of d-dimensional modules from $\operatorname{add}(\Gamma'')$. We decompose the set \mathcal{Q} into a finite union of pairwise disjoint subsets $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_r$ such that two modules $U_1 \oplus U_2$ and $V_1 \oplus V_2$ from \mathcal{Q} with $U_1, V_1 \in \operatorname{add}(\Gamma''), U_2, V_2 \in \operatorname{add}(\Gamma')$, belong to the same $\mathcal{D}_i, 1 \leq i \leq r$, if and only if $U_1 \simeq V_1$. Since $\overline{\mathcal{Q}} = \overline{\mathcal{D}}_1 \cup \overline{\mathcal{D}}_2 \cup \cdots \cup \overline{\mathcal{D}}_r$, the module N_1 belongs to $\overline{\mathcal{D}}_i$ for some $1 \leq i \leq r$. Take any $V \oplus R \in \mathcal{D}_i$ with $V \in \operatorname{add}(\Gamma'')$ and $R \in \operatorname{add}(\Gamma')$. Then any module from \mathcal{D}_i is, up to isomorphism, of the form $V \oplus R'$ for some $R' \in \operatorname{add}(\Gamma'')$ with [R'] = [R]. Consequently, for any indecomposable module $X \in \operatorname{add}(\Gamma'')$ we have [R', X] = [R, X], by Lemma 3.9. Applying upper semicontinuity of the function $(Z \to \dim_K \operatorname{Hom}_A(Z, X))$, we conclude that the set

$$\mathcal{S}_X = \{ Z \in \overline{\mathcal{D}_i}; \ [Z, X] \ge [V \oplus R, X] = [V \oplus R', X] \}$$

is closed (see [11],[13]), for any $X \in \operatorname{add}(\Gamma'')$. Since \mathcal{D}_i is a subset of \mathcal{S}_X , we obtain that $[N_1, X] \ge [V \oplus R, X]$ for any $X \in \operatorname{add}(\Gamma'')$. Take a tube $\mathcal{T}_{\mu_c} \subset \Gamma''$, for some $1 \le c \le t$, such that any direct summand of $V \oplus N_1$ does not belong to \mathcal{T}_{μ_c} . It is possible, because $\dim_K V < t$.

Assume that R = 0. Then by Lemma 3.9, for any $\mathcal{T}_{\lambda} \subset \Gamma'$ and $j \ge 0$, we have

$$[N_1, \varphi^j E_{\lambda}] = [N_1, \varphi^j E_{\mu_c}] \ge [V, \varphi^j E_{\mu_c}] = [V, \varphi^j E_{\lambda}].$$

This leads to a contradiction, since the sequence $0 \to N_0 \to M \to V \to 0$ is admissible for (M, N). So, there is a tube $\mathcal{T}_{\nu} \subset \Gamma'$ such that $V \oplus R = I \oplus \varphi^j E_{\nu}$ for

some A-module I and $j \ge 0$. Then, for an epimorphism $p: \varphi^j E_\nu \to E_\nu$ we obtain the following commutative diagram with exact rows and columns

Hence, for any $\mathcal{T}_{\lambda} \subset (\Gamma' \setminus \mathcal{T}_{\nu})$ and $k \geq 0$, applying Lemma 3.9, we get

$$[N, \varphi^k E_{\lambda}] = [N, \varphi^k E_{\mu_c}] \ge [N_0 \oplus V \oplus R, \varphi^k E_{\mu_c}]$$
$$= [N_0 \oplus I \oplus \varphi^j E_{\nu}, \varphi^k E_{\mu_c}]$$
$$= [N_0 \oplus I \oplus \varphi^{j-1} E_{\nu} \oplus E_{\nu}, \varphi^k E_{\mu_c}]$$
$$\ge [L \oplus E_{\nu}, \varphi^k E_{\mu_c}] = [L \oplus E_{\nu}, \varphi^k E_{\lambda}].$$

This leads to $[L \oplus E_{\nu}, X] \leq [N, X]$ for any $X \in \Gamma_A \setminus \mathcal{T}_{\nu}$.

Proposition 4.2. Let M and N be A-modules without any common indecomposable direct summand and such that M < N and $M_P \oplus N_P$ is nonzero. Let $r = r_{\mu}$ and E be any simple regular module in \mathcal{T}_{μ} for some $\mu \in \mathbb{P}^1(K)$. If there is no admissible sequence for (M, N), then

- $\begin{array}{ll} (i) & \partial [M_P] = \partial [N_P]. \\ (ii) & \delta'_{M,N}(\varphi^s \psi^t E) = 0 \mbox{ holds for some } s \geq 0 \mbox{ and } 0 \leq t < r. \end{array}$
- (iii) For any $j \ge 1$ such that $\psi^- \varphi^j E$ is a direct summand of M, the equality $\begin{array}{l} \delta'_{M,N}(\varphi^s\psi^t E) = 0 \ \ holds \ for \ some \ s \geq j \ \ and \ 0 \leq t < r. \\ (iv) \ \ There \ are \ infinitely \ many \ modules \ X \ \ in \ \ T_{\mu} \ \ with \ \delta'_{M,N}(X) = 0. \\ (v) \ \ There \ are \ infinitely \ many \ modules \ X \ \ in \ \ T_{\mu} \ \ with \ \delta_{M,N}(X) = 0. \end{array}$

Proof. (i) If $\delta_{M,N}(X) = 0$ for all indecomposable preprojective A-modules, then, by Lemma 2.5, $\mu(M_P, X) = \mu(N_P, X)$ for any indecomposable preprojective Amodule, and consequently $M_P = N_P = 0$, which gives a contradiction. Let N_0 be a minimal, with respect to \preceq , indecomposable preprojective A-module with $\delta_{M,N}(N_0) > 0$. Then by Lemma 2.5 we get

$$\mu(N, N_0) - \mu(M, N_0) = \delta_{M,N}(N_0) > 0,$$

because $X \prec N_0$ for any indecomposable direct summand X of $E(N_0) \oplus \tau N_0$. This implies that $N = N_0 \oplus N_1$ for some A-module N_1 . Of course, $\delta'_{M,N}(N_0) = \delta_{M,N}(\tau N_0) = 0$ and consequently $[N_0, N] = [N_0, M]$. By Proposition 4.1, there is a nonsplittable exact sequence

$$0 \to L \to M \to E_{\nu} \to 0$$

such that \mathcal{T}_{ν} is a homogeneous tube for which $(M \oplus N)_{\nu} = 0$ and $[L \oplus E_{\nu}, X] \leq [N, X]$ for any indecomposable A-module $X \notin \mathcal{T}_{\nu}$. Observe that $L_R \oplus L_I = M_R \oplus M_I$. Then we get a nonsplittable exact sequence

$$\Sigma: 0 \to L_P \to M_P \to E_\nu \to 0$$

such that $\delta_{\Sigma}(X) \leq \delta_{M,N}(X)$ for any indecomposable A-module $X \notin \mathcal{T}_{\nu}$. Thus there is $t \geq 0$ such that $\delta_{\Sigma}(\varphi^t E_{\nu}) > \delta_{M,N}(\varphi^t E_{\nu})$, because Σ is not admissible for (M, N). We set $F = E_{\nu}$. Since $\tau^- \varphi^t F = \varphi^t F$, we get

$$\delta_{\Sigma}(\varphi^t F) = \delta'_{\Sigma}(\varphi^t F) = [\varphi^t F, L_P \oplus F] - [\varphi^t F, M_P] = [\varphi^t F, F] = 1$$

and

$$\begin{split} \delta_{M,N}(\varphi^t F) =& [N,\varphi^t F] - [M,\varphi^t F] = [N_P,\varphi^t F] - [M_P,\varphi^t F] = <[N_P], [\varphi^t F] > \\ & - <[M_P], [\varphi^t F] > = <[N_P], (t+1) \cdot \underline{h} > - <[M_P], (t+1) \cdot \underline{h} > \\ & = (t+1)(\partial[M_P] - \partial[N_P]). \end{split}$$

This leads to $\partial[M_P] - \partial[N_P] < 1$ and, by Lemma 3.3, we have $\partial[M_P] = \partial[N_P]$.

(ii) Since $M_P \leq_{\text{ext}} L_P \oplus E_{\nu}$, then $M_P \leq L_P \oplus E_{\nu}$. Let X be any indecomposable A-module. If $X \notin \mathcal{P} \cup \mathcal{T}_{\mu}$, then $[X, M_P] = [X, L_P \oplus \psi^{r-1}E] = 0$. If $X \in \mathcal{T}_{\mu}$, then $0 = [X, M_P] \leq [X, L_P \oplus \psi^{r-1}E]$. Since $[E_{\nu}] = \underline{h} = [\psi^{r-1}E]$, applying Lemma 3.9 for any preprojective module X, we obtain

$$0 \le [X, L_P \oplus \psi^{r-1}E] - [X, M_P] = [X, L_P \oplus E_{\nu}] - [X, M_P]$$

= [X, L \overline E_{\nu}] - [X, M] \le [X, N] - [X, M].

Thus $M_P \leq L_P \oplus \psi^{r-1}E$ and

$$[X, L_P \oplus \psi^{r-1}E] - [X, M_P] \le [X, N] - [X, M]$$

for any indecomposable A-module $X \notin \mathcal{T}_{\mu}$. By Proposition 2.7, there is an admissible sequence

$$\Sigma_0: 0 \to L_1 \to M_P \to L_2 \to 0$$

for $(M_P, L_P \oplus \psi^{r-1}E)$. Hence, $[X, L_1 \oplus L_2] \leq [X, L_P \oplus \psi^{r-1}E] = 0$ for any indecomposable module $X \notin \mathcal{P} \cup \mathcal{T}_{\mu}$. This implies that $L_1 \oplus L_2 \in \operatorname{add}(\mathcal{P} \cup \mathcal{T}_{\mu})$. Since the sequence Σ_0 is not admissible for (M, N), we get

$$[X, \psi^{r-1}E] = [X, L_P \oplus \psi^{r-1}E] - [X, M_P] > [X, N] - [X, M]$$

for some indecomposable module $X \in \mathcal{T}_{\mu}$. By Lemma 3.6(i), $[\varphi^{s}\psi^{t}E, \psi^{r-1}E] = 1$ for all $s \geq 0, 0 \leq t < r$ and $[X, \psi^{r-1}E] = 0$ for the remaining modules $X \in \mathcal{T}_{\mu}$. Hence, $\delta'_{M,N}(X) = [X, N] - [X, M] = 0$ for some $X = \varphi^{s}\psi^{t}E, s \geq 0$ and $0 \leq t < r$.

(iii) Assume that $\psi^- \varphi^j E$ is a direct summand of M for some $j \ge 1$. Take the admissible sequence

$$\Sigma_0: 0 \to L_1 \to M_P \to L_2 \to 0$$

for $(M_P, L_P \oplus \psi^{r-1}E)$, considered in (ii). We can write $L_2 = L'_2 \oplus Y$ such that $L_1 \oplus L'_2$ is preprojective and $Y \in \operatorname{add}(\mathcal{T}_{\mu})$. If Y = 0, then $[X, L_1 \oplus L_2] - [X, M_P] = 0$ for any $X \in \mathcal{T}_{\mu}$, and moreover Σ_0 is an admissible sequence for (M, N). Hence $Y \neq 0$, and consequently

$$[X,Y] = [X,L_1 \oplus L'_2 \oplus Y] - [X,M_P] \le [X,L_P \oplus \psi^{r-1}E] - [X,M_P] = [X,\psi^{r-1}E]$$

for any X in \mathcal{T}_{μ} . Applying Lemma 3.6(iv) we get $[E, Y] \leq [E, \psi^{r-1}E] = 1$ and $[E', Y] \leq [E', \psi^{r-1}E] = 0$, for all simple regular modules $E' \neq E$ in \mathcal{T}_{μ} , and consequently Y is indecomposable and $Y = \psi^k E$ for some $k \geq 0$. Since $[Y, Y] \leq [Y, \psi^{r-1}E] \leq 1$, we obtain k < r, by Lemma 3.6. Let

$$e: L'_2 \oplus \varphi^j \psi^k E o L'_2 \oplus \psi^k E = L_2$$

be a natural epimorphism. Then the pull back of Σ_0 under e is a sequence of the form

$$\Sigma_j: 0 \to L_1 \to M_P \oplus \psi^- \varphi^j E \to L_2' \oplus \varphi^j \psi^k E \to 0,$$

because ker e is isomorphic to $\psi^- \varphi^j E$ and $Ext^1(M_P, \psi^- \varphi^j E) = 0$. Observe that $M_P \oplus \psi^- \varphi^j E$ is a direct summand of M and $\delta'_{\Sigma_j} \leq \delta'_{\Sigma_0}$. This implies that $\delta'_{\Sigma_j}(X) \leq \delta'_{M,N}(X)$ for any indecomposable A-module $X \notin \mathcal{T}_{\mu}$. Since the sequence Σ_j is not admissible for (M, N), we get $\delta'_{\Sigma_j}(X) > \delta'_{M,N}(X)$ for some $X \in \mathcal{T}_{\mu}$. Then

$$\delta_{\Sigma_j}'(X) = [X,\varphi^j\psi^k E] - [X,\psi^-\varphi^j E] \leq [X,\varphi^j\psi^{r-1}E] - [X,\psi^-\varphi^j E],$$

because $\varphi^j \psi^k E$ may be treated as a submodule of $\varphi^j \psi^{r-1} E$. Applying Lemma 3.6(ii) we get that $[\varphi^s \psi^t E, \varphi^j \psi^{r-1} E] - [\varphi^s \psi^t E, \psi^- \varphi^j E] = 1$ for all $s \ge j, 0 \le t < r$, and $[Y, \varphi^j \psi^{r-1} E] - [Y, \psi^- \varphi^j E] = 0$ for the remaining indecomposable modules $Y \in \mathcal{T}_{\mu}$. Thus, $X = \varphi^s \psi^t E$ and $\delta'_{M,N}(X) = 0$ for some $s \ge j$ and $0 \le t < r$.

(iv) Suppose that the required claim is not true. Take a maximal $s \ge 0$ and a simple regular module E' in \mathcal{T}_{μ} such that $\delta'_{M,N}(\varphi^s E') = 0$. Applying (ii) for the simple regular module $\tau^- E'$, we infer that there are numbers $s' \ge 0$ and $0 \le t' < r$ with $\delta'_{M,N}(\varphi^{s'}\psi^{t'}\tau^- E') = \delta'_{M,N}(\varphi^{s'-1}\psi^{t'+1}E') = 0$. Take a pair (s',t') with maximal number s'. Since $\delta'_{M,N}(\varphi^{s'}\psi^{t'}\tau^- E') = \varphi^{s'+t'}(\tau^{-t'-1}E')$, then $s' \le s' + t' \le s$, by maximality of s. Thus, $\delta'_{M,N}(\varphi^k\psi^{t}\tau^- E') > 0$ for all

k > s' and $0 \le l < r$. Applying Lemma 3.7(ii), we get

$$\begin{split} \sum_{s' \leq i \leq s} \sum_{0 \leq j \leq t'} \mu(N, \varphi^{i} \psi^{j} E') - \mu(M, \varphi^{i} \psi^{j} E') &= \delta_{M,N}(\psi^{-} \varphi^{s+1} E') \\ &- \delta_{M,N}(\psi^{-} \varphi^{s'} E') - \delta_{M,N}(\varphi^{s+1} \psi^{t'} E') + \delta_{M,N}(\varphi^{s'} \psi^{t'} E') \\ &\leq \delta'_{M,N}(\varphi^{s} E') - \delta'_{M,N}(\varphi^{s+1} \psi^{t'} \tau^{-} E') + \delta'_{M,N}(\varphi^{s'-1} \psi^{t'+1} E') \\ &= - \delta'_{M,N}(\varphi^{s+1} \psi^{t'} \tau^{-} E') < 0, \end{split}$$

because s+1>s' and $0\leq t'< r.$ Thus $\varphi^i\psi^jE'$ is a direct summand of M for some $s'\leq i\leq s$ and $0\leq j< r.$ Let $E=\tau^{-j-1}E'.$ Then $\psi^-\varphi^{i+j+1}E$ is a direct summand of M, and applying (iii), we get numbers $p\geq i+j+1$ and $0\leq q< r$ with $\delta'_{M,N}(\varphi^p\psi^qE)=0.$ Observe that $\varphi^p\psi^qE=\varphi^{p-j}\psi^{q+j}\tau^-E'$ and $0\leq q+j<2r.$ If q+j< r, then $\delta'_{M,N}(\varphi^{p-j}\psi^{q+j}\tau^-E')=0$, because $p-j\geq i+1>s'.$ This leads to $q+j\geq r,$ and $\varphi^{p-j}\psi^{q+j}\tau^-E'=\varphi^{p-j+r}\psi^{q+j-r}\tau^-E'.$ But then $\delta'_{M,N}(\varphi^{p-j+r}\psi^{q+j-r}\tau^-E')=0$, because p-j+r>s' and $0\leq q+j-r< r,$ which is a contradiction.

(v) follows from (iv) and the formula $\delta_{M,N}(X) = \delta'_{M,N}(\tau^- X)$.

Proposition 4.3. Let M and N be A-modules with M < N. Assume that there is a tube \mathcal{T}_{μ} in Γ_A such that $\delta_{M,N}(\psi^j E) = 0$ and $\delta_{M,N}(\psi^{j-1}E) > 0$ for some simple regular module E in \mathcal{T}_{μ} and $j \geq H(M_{\mu} \oplus N_{\mu}) + r$, where $r = r_{\mu}$. Then there exists an admissible sequence for (M, N).

Proof. Applying Lemma 3.5 we get

$$\begin{split} \delta_{M,N}(\psi^{j}E) = & [N,\psi^{j}E] - [M,\psi^{j}E] = [N_{P} \oplus N_{\mu},\psi^{j}E] - [M_{P} \oplus M_{\mu},\psi^{j}E] \\ &= < [N_{P}], [\psi^{j}E] > - < [M_{P}], [\psi^{j}E] > + \ell_{E}(N_{\mu}) - \ell_{E}(M_{\mu}), \end{split}$$

and similarly

$$\delta_{M,N}(\psi^{j-r}E) = \langle [N_P], [\psi^{j-r}E] \rangle - \langle [M_P], [\psi^{j-r}E] \rangle + \ell_E(N_\mu) - \ell_E(M_\mu).$$

This leads to

$$\delta_{M,N}(\psi^{j-r}E) = \langle [N_P], [\psi^{j-r}E] - [\psi^j E] \rangle - \langle [M_P], [\psi^{j-r}E] - [\psi^j E] \rangle \\ = \langle [N_P], -\underline{h} \rangle - \langle [M_P], -\underline{h} \rangle = \partial [N_P] - \partial [M_P] = 0.$$

Take a maximal number k such that $j - r \leq k \leq j - 2$ and $\delta_{M,N}(\psi^k E) = 0$. Then we have $\delta_{M,N}(\psi^t E) > 0$ for any k < t < j. If $\delta_{M,N}(\varphi^c \psi^d E) > 0$ for all $-k - 1 \leq c \leq 0$ and k < d < j, then we set Y = 0, p = -k - 2 and q = k + 1. Assume now that this is not the case. Take a maximal number c and a number d

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such that $-k - 1 \leq c \leq 0$, k < d < j and $\delta_{M,N}(\varphi^c \psi^d E) = 0$. Of course, c < 0. Applying Lemma 3.7(ii), we get

$$\sum_{c \le p < 0} \sum_{k < q \le d} \mu(N, \varphi^c \psi^d E) - \mu(M, \varphi^c \psi^d E) = \delta_{M,N}(\psi^k E) + \delta_{M,N}(\varphi^c \psi^d E) - \delta_{M,N}(\psi^d E) - \delta_{M,N}(\varphi^c \psi^k E) \le -\delta_{M,N}(\psi^d E) < 0,$$

because k < d < j. Hence, $Y = \varphi^p \psi^q E$ is a direct summand of M for some $c \leq p < 0$ and $k < q \leq d$.

We set $V = \psi^q E$ and $W = \varphi^p \psi^j E$. Applying Lemma 3.7(i) for $X = \varphi^{p+1} \psi^q E$, s = -p - 1, t = j - q - 1, we get a short exact sequence

$$\Omega: 0 \to V \xrightarrow{\begin{pmatrix} i \\ f \end{pmatrix}} \psi^{j} E \oplus Y \xrightarrow{(f_{1}, f_{2})} W \to 0,$$

where $i: V \to \psi^j E$ is a monomorphism. Further, $\delta_{\Omega}(X) = 1$ for any $X \in \mathcal{Y} = \{\varphi^v \psi^w E; \ p < v \leq 0, q \leq w < j\}$ and $\delta_{\Omega}(X) = 0$ for the remaining indecomposable *A*-modules *X*, because t < r. Thus, $\delta_{\Omega} \leq \delta_{M,N}$, and so $M \oplus V \oplus W \leq N \oplus Y \oplus \psi^j E$. Moreover,

$$0 \leq [N \oplus Y \oplus \psi^j E, \psi^j E] - [M \oplus V \oplus W, \psi^j E)] \leq [N, \psi^j E] - [M, \psi^j E] = 0$$

and $M \oplus V \oplus W \leq_{\text{deg}} N \oplus Y \oplus \psi^j E$, by Proposition 3 in [9]. Observe that the set of isomorphism classes of kernels of epimorphisms $M \oplus (V \oplus W) \to \psi^j E$ is finite. Therefore, there is a nonsplittable short exact sequence

$$\Theta: 0 \to L \to M \oplus V \oplus W \xrightarrow{g} \psi^j E \to 0$$

such that $L \leq_{\text{deg}} N \oplus Y$, by Theorem 2.4 in [10]. Of course, $M = M' \oplus Y$ for some A-module M'. We may consider the module V as a submodule of $\psi^j E$.

We claim that for any $g' \in \operatorname{Hom}_A(Y \oplus V \oplus W, \psi^j E)$ we have im $g' \subseteq V$. Indeed, since

$$E \subset \psi E \subset \cdots \subset V = \psi^q E \subset \cdots \subset \psi^j E$$

is the unique composition series of $\psi^j E$ in $\operatorname{add}(\mathcal{T}_\mu)$, we get im $g' = \psi^{j'} E$ for some $0 \leq j' \leq j$. On the other hand, the equality im $g' = \psi^{j'} E$ implies that there is an indecomposable direct summand $\varphi^k \psi^{j'} E$ of $(Y \oplus V \oplus W)$, for some $k \geq 0$. This leads to $j' \leq q$, which proves our claim.

Then the epimorphism g is of the form

$$g = (g_1, \iota g_2) : M' \oplus (Y \oplus V \oplus W) \to \psi^j E,$$

for some $g_1: M' \to \psi^j E$ and $g_2: Y \oplus V \oplus W \to V$. Consider the pull back of the sequence

$$0 \to L \to M' \oplus (Y \oplus V \oplus W) \oplus Y \xrightarrow{\begin{pmatrix} g_1 & ig_2 & 0\\ 0 & 0 & 1_Y \end{pmatrix}} \psi^j E \oplus Y \to 0$$

under the monomorphism $\binom{i}{f}: V \to \psi^j E \oplus Y$. Then we obtain the following commutative diagram with exact rows and columns

Hence we get an exact sequence

$$0 \to Z \to M' \oplus (Y \oplus V \oplus W) \oplus Y \xrightarrow{(f_1g_1, f_1ig_2, f_2)} W \to 0.$$

We may consider the module Z as a submodule of $M' \oplus (Y \oplus V \oplus W) \oplus Y$. Since $f_1 \iota g_2 = -f_2 f g_2$, we obtain a submodule $Z' = \{(0, m, f g_2(m)); m \in Y \oplus V \oplus W\}$ of Z. It is easy to see that $Z' \simeq Y \oplus V \oplus W, Z = Z' \oplus Z_1$ for some A-module Z_1 , and there exists an exact sequence of the form

$$\Psi: 0 \to Z_1 \to M' \oplus Y = M \to W \to 0$$

Observe that, for any A-module X, we have

$$\begin{split} \delta_{\Psi}(X) = & [Z_1 \oplus W, X] - [M, X] = [Z_1 \oplus W \oplus Y \oplus V, X] - [M \oplus Y \oplus V, X] \\ = & [Z, X] - [M \oplus Y \oplus V, X] \leq [L \oplus V, X] - [M \oplus Y \oplus V, X] \\ = & [L, X] - [M \oplus Y, X] \leq [N \oplus Y, X] - [M \oplus Y, X] = \delta_{M,N}(X), \end{split}$$

because $Z \leq_{\text{ext}} L \oplus V$ and $L \leq_{\text{deg}} N \oplus Y$. Thus the sequence Ψ is admissible for (M, N), and this finishes the proof.

4.4. Proof of Theorem. Let M and N be two A-modules such that M < N. We shall show that $M <_{\text{ext}} N$. By Lemma 1.2 in [10], we may assume that the relation M < N is minimal.

We claim that there is an admissible exact sequence for (M, N). Suppose that this is not the case. We may assume that M and N have no common indecomposable direct summand. If $M_P = N_P = M_I = N_I = 0$, then by Theorem 1 in [15], or

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Section 3 in [9], $M = M_R <_{\text{ext}} N_R = N$. Then by definition of the relation \leq_{ext} , there is an admissible sequence for (M, N), and we get a contradiction. Hence, up to duality, we may assume that $M_P \oplus N_P$ is nonzero. Then by Proposition 4.2(i), $\partial[M_P] = \partial[N_P]$ and applying Lemma 3.8(i) and its dual we obtain

$$[M_P] \ge [N_P]$$
 and $[M_I] \ge [N_I]$.

Assume that $[M_P] = [N_P]$ and let V be any indecomposable A-module. If V is preprojective, then

$$\delta_{M_P,N_P}(V) = [N_P,V] - [M_P,V] = [N,V] - [M,V] \ge 0,$$

otherwise

$$\delta_{M_P,N_P}(V) = \delta'_{M_P,N_P}(\tau^- V) = [\tau^- V, N_P] - [\tau^- V, M_P] = 0 - 0 = 0.$$

This implies that $M_P < N_P$ and by Corollary 4.2 in [10], $M_P <_{\text{ext}} N_P$. Then, by definition of the relation \leq_{ext} , there is an admissible sequence for (M_P, N_P) . Since $\delta_{M_P,N_P} \leq \delta_{M,N}$, this sequence is admissible for (M, N), again a contradiction.

Hence, $[M_P] > [N_P]$, and consequently $\sum [M_{\mu}] < \sum [N_{\mu}]$, where the summation runs through all $\mu \in \mathbb{P}^1(K)$. Applying Lemma 3.8(iii), we conclude that there is $\mu \in \mathbb{P}^1(K)$ such that $[M_{\mu}] < [N_{\mu}]$. We set $r = r_{\mu}$ and let E_1, \ldots, E_r be all simple regular modules in \mathcal{T}_{μ} . Then by Lemma 3.8(ii) there is a simple regular module E in \mathcal{T}_{μ} with $\ell_E(M_{\mu}) < \ell_E(N_{\mu})$, because $[X] = \ell_{E_1}(X)[E_1] + \cdots + \ell_{E_r}(X)[E_r]$ for any $X \in \operatorname{add}(\mathcal{T}_{\mu})$. Applying Lemma 3.5, we get

$$\begin{split} \delta_{M,N}(\psi^{sr-1}E) = & [N,\psi^{sr-1}E] - [M,\psi^{sr-1}E] = [N_P,\psi^{sr-1}E] \\ & - [M_P,\psi^{sr-1}E] + [N_\mu,\psi^{sr-1}E] - [M_\mu,\psi^{sr-1}E] \\ & = < [N_P], [\psi^{sr-1}E] > - < [M_P], [\psi^{sr-1}E] > + \ell_E(N_\mu) - \ell_E(M_\mu) \\ & > < [N_P], s \cdot \underline{h} > - < [M_P], s \cdot \underline{h} > = -s\partial[N_P] + s\partial[M_P] = 0, \end{split}$$

for any integer s satisfying $sr \geq H(M_{\mu} \oplus N_{\mu})$. Hence $\delta_{M,N}(X) > 0$ for infinitely many X in \mathcal{T}_{μ} .

Applying Proposition 4.2(v), we infer that there are a simple regular module F in \mathcal{T}_{μ} and a number $j > H(M_{\mu} \oplus N_{\mu}) + r$ such that $\delta_{M,N}(\psi^{j}F) = 0$ and either $\delta_{M,N}(\psi^{j-1}F) > 0$ or $\delta_{M,N}(\varphi^{-}\psi^{j}F) > 0$. Let $F' = \tau^{-j-1}F$. Then either $\delta_{M,N}(\psi^{j}F) = 0 < \delta_{M,N}(\psi^{j-1}F)$ or $\delta'_{M,N}(\varphi^{j}F') = 0 < \delta'_{M,N}(\varphi^{j-1}F')$. Then by Proposition 4.3 or its dual there exists an admissible exact sequence for (M, N). This proves our claim.

Take an admissible sequence $0 \to L_1 \to M' \to L_2 \to 0$ for (M, N). This implies that $M = M' \oplus V$ for some A-module V and we obtain $M <_{\text{ext}} L_1 \oplus L_2 \oplus V \leq N$. Since the relation M < N is minimal, then $N = L_1 \oplus L_2 \oplus V$. This leads to $M <_{\text{ext}} N$, and completes the proof.

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