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Schubert polynomials and Bott-Samelson varieties

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Abstract. Schubert polynomials generalize Schur polynomials, but it is not clear how to generalize several classical formulas: the Weyl character formula, the Demazure character formula, and the generating series of semistandard tableaux. We produce these missing formulas and obtain several surprising expressions for Schubert polynomials.

The above results arise naturally from a new geometric model of Schubert polynomials in terms of Bott-Samelson varieties. Our analysis includes a new, explicit construction for a Bott-Samelson variety Z as the closure of a B-orbit in a product of flag varieties. This construction works for an arbitrary reductive group G, and for $G = GL(n)$ it realizes Z as the representations of a certain partially ordered set.

This poset unifies several well-known combinatorial structures: generalized Young diagrams with their associated Schur modules; reduced decompositions of permutations; and the chamber sets of Berenstein-Fomin-Zelevinsky, which are crucial in the combinatorics of canonical bases and matrix factorizations. On the other hand, our embedding of Z gives an elementary construction of its coordinate ring, and allows us to specify a basis indexed by tableaux.

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Introduction

The classical Schur polynomials appear in many contexts: 1) as characters of the irreducible representations of $GL(n, \mathbf{C})$ (the Schur modules); 2) as an algebraic model for the cohomology ring of a Grassmannian (product of Schur polynomials $\leftrightarrow$ intersection of Schubert classes); 3) as an orthogonal basis for the symmetric functions in a polynomial ring; and 4) as generating functions enumerating semistandard Young tableaux. (See [10] for a unified account of this theory.)

In recent decades many generalizations of Schur polynomials have appeared, among the most interesting being the Schubert polynomials first defined by Lascoux and Schutzenberger [15]. These are known to generalize each of the above interpretations. They are: 1) characters of representations of the group B of upper

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triangular matrices [12]; 2) an algebraic model of the cohomology ring of a flag variety [4], [7]; 3) an orthogonal basis for a polynomial ring [15]; and 4) generating functions for certain mysterious tableaux defined by compatiblity conditions in the plactic monoid [16].

Nevertheless, many of the rich properties of Schur polynomials have no known analogs for Schubert polynomials. In this paper we supply several such missing analogs, mainly concerning interpretations 1) and 4): analogs of the Weyl and Demazure character formulas; and a straightforward construction for the mysterious tableaux of Lascoux and Schutzenberger, showing how they "quantize" our Demazure formula. These results also hold for a broad class of Schur-like polynomials associated to generalized Young diagrams, such as skew Schur polynomials [1], [23], [24], [25], [21].

These results are purely combinatorial, but we obtain them by generalizing a powerful tool of representation theory, the Borel-Weil Theorem, which states that Schur modules (whose characters are Schur polynomials) are graded pieces in the coordinate ring of a flag variety (c.f. [10]). The theory of Schur polynomials can be developed from this point of view, and this is what we do for Schubert polynomials and their associated B-modules. Instead of flag varieties, however, we must use the more general varieties defined by Bott and Samelson, which are a well-known tool in geometric representation theory. (They are indexed by reduced decompositions of permutations into simple transpositions.)

This method follows our paper [21], but we must do extra geometric work here, giving a precise connection between our B-modules and the Bott-Samelson varieties Z. As a by-product of our analysis, we obtain a new construction of the Bott-Samelson varieties for an arbitrary reductive group G. In our case $G = GL(n)$, the new construction realizes Z as the variety of representations of a partially ordered set. This poset is equivalent to two well-known but previously unconnected combinatorial pictures, and our approach reveals deep relations between them: first, generalized Young diagrams, which are used to construct generalized Schur modules; and second, reduced decompositions of permutations, which are pictured via the wiring diagrams and chamber sets of Berenstein, Fomin, and Zelevinsky [2], [18], crucial in the combinatorics of matrix factorizations, total positivity, and canonical bases.

The paper is organized into three parts, which may be read independently and have separate introductions. The first part (§1) introduces Bott-Samelson varieties for a general reductive group, and shows the isomorphism between our new construction and the classical one. This lays the groundwork for our papers [13], [14] with V. Lakshmibai, giving a Standard Monomial Theory for Bott-Samelson varieties.

The second part (§§2-3) makes this construction explicit for $GL(n)$, discusses the combinatorial models and their connections, defines generalized Schur modules and Schur polynomials, and proves the Demazure character formula for them.

The last part (§4) states all the applications to Schubert polynomials in elementary combinatorial language.

1. Bott-Samelson varieties

Let G be a complex reductive Lie group (or more generally a reductive algebraic group over an infinite field of arbitrary characteristic or over $\mathbf{Z}$), and let B be a Borel subgroup.

The Bott-Samelson varieties are an important tool in the representation theory of G and the geometry of the flag variety G/B. First defined in [5] as a desingularization of the Schubert varieties in G/B, they were exploited by Demazure [7] to analyze the singular cohomology or Chow ring $H^{\cdot}(G/B, \mathbf{C})$ (the Schubert calculus), and the projective coordinate ring $\mathbf{C}[G/B]$. Since the irreducible representations of G are embedded in the coordinate ring, Demazure was able to obtain an iterative character formula [8] for these representations.

Bott-Samelson varieties are so useful because they "factor" the flag variety into a "product" of projective lines. More precisely, they are iterated $\mathbf{P}^1$-fibrations and they each have a natural, birational map to G/B. The Schubert subvarieties themselves lift birationally to iterated $\mathbf{P}^1$-fibrations under this map (hence the desingularization). The combinatorics of Weyl groups enters because a given G/B can be "factored" in many ways, indexed by sequences $\mathbf{i} = (i_1, i_2, \ldots, i_l)$ such that $w_0 = s_{i_1} s_{i_2} \cdots s_{i_l}$ is a reduced decompostion of the longest Weyl group element w_0 into simple reflections.

The Bott-Samelson variety $Z_{\mathbf{i}}$ is usually defined as a product of l minimal parabolic subgroups modulo an action of B^l, but we give a new, dual construction of $Z_{\mathbf{i}}$ as a subvariety rather than a quotient. It is the closure of an orbit of the Borel subgroup B inside a product of flag varieties:

$$Z_{\mathbf{i}} \cong \overline{B \cdot (s_{i_1}B, s_{i_1}s_{i_2}B, \ldots, w_0 B)} \subset (G/B)^l,$$

where B acts diagonally on $(G/B)^l$. (We give several variations of this definition below.)

This embedding of $Z_{\mathbf{i}}$ allows us to apply the tools of Standard Monomial Theory, producing a standard monomial basis for the space of sections of an effective line bundle (a graded piece of $\mathbf{C}[Z_{\mathbf{i}}]$). We pursue this in our papers [13], [14] with V. Lakshmibai.

In §1.3, we give another definition of the Bott-Samelson variety in terms of incidence conditions; and in §1.4, we show that the map $Z_{\mathbf{i}} \to G/B$ compactifies the matrix factorizations of Berenstein-Fomin-Zelevinsky [2], [3].

1.1. Three constructions

Let W be the Weyl group generated by simple reflections $s_1, \ldots, s_r$, where r is the rank of G. For $w \in W$, $\ell(w)$ denotes the length l of a reduced (i. e. minimal) decompostion $w = s_{i_1} \ldots s_{i_l}$, and w_0 denotes the element of maximal length.

We let B be a Borel subgroup, $T \subset B$ a maximal torus (Cartan subgroup). Let $P_k \supset B$ be the *minimal* parabolic associated to the simple reflection s_k, so that

$P_i/B \cong \mathbf{P}^1$, the projective line. Also, take $\widehat{P}_k \supset B$ to be the *maximal* parabolic associated to the reflections $s_1, \ldots, \widehat{s_k}, \ldots, s_r$. Finally, we have the Schubert variety as a B-orbit closure inside the flag variety:

$$X_w = \overline{BwB} \subset G/B$$

For what follows, we fix a reduced decompostion of some $w \in W$,

$$w = s_{i_1} \ldots s_{i_l},$$

and we denote $\mathbf{i} = (i_1, \ldots, i_l)$.

Now let $P \supset B$ be any parabolic subgroup of G, and X any space with B-action. Then the *induced P-space* is the quotient

$$P \overset{B}{\times} X \overset{\text{def}}{=} (P \times X)/B$$

where the quotient is by the free action of B on $P \times X$ given by $(p, x) \cdot b = (pb, b^{-1}x)$. (Thus $(pb, x) = (p, bx)$ in the quotient.) The key property of this construction is that

$$\begin{array}{ccc} X & \rightarrow & P \overset{B}{\times} X \\ & & \downarrow \\ & & P/B \end{array}$$

is a fiber bundle with fiber X and base P/B. We can iterate this construction for a sequence of parabolics $P, P', \ldots,$

$$P \overset{B}{\times} P' \overset{B}{\times} \cdots \overset{\text{def}}{=} P \overset{B}{\times} (P' \overset{B}{\times} (\cdots)).$$

Then the **quotient Bott-Samelson variety** of the reduced word $\mathbf{i}$ is

$$Z_{\mathbf{i}}^{\text{quo}} \overset{\text{def}}{=} P_{i_1} \overset{B}{\times} \cdots \overset{B}{\times} P_{i_l}/B.$$

Because of the fiber-bundle property of induction, $Z_{\mathbf{i}}^{\text{quo}}$ is clearly a smooth, irreducible variety of dimension l. It is a subvariety of

$$X_l \overset{\text{def}}{=} \underbrace{G \overset{B}{\times} \cdots \overset{B}{\times} G}_{l \text{ factors}}/B.$$

B acts on these spaces by multiplying the first coordinate:

$$b \cdot (p_1, p_2, \ldots, p_l) \overset{\text{def}}{=} (bp_1, p_2, \ldots, p_l).$$

The original purpose of the Bott-Samelson variety was to desingularize the Schubert variety X_w via the multiplication map:

$$\begin{aligned} Z_{\mathbf{i}}^{\text{quo}} &\to X_w \subset G/B \\ (p_1, \dots, p_l) &\mapsto p_1 p_2 \cdots p_l B, \end{aligned}$$

a birational morphism.

Next, consider the *fiber product*

$$G/B \underset{G/P}{\times} G/B \overset{\text{def}}{=} \{(g_1, g_2) \in (G/B)^2 \mid g_1 P = g_2 P\}.$$

We may define the **fiber product Bott-Samelson variety**

$$Z_{\mathbf{i}}^{\text{fib}} \overset{\text{def}}{=} eB \underset{G/P_{i_1}}{\times} G/B \underset{G/P_{i_2}}{\times} \cdots \underset{G/P_{i_l}}{\times} G/B \subset (G/B)^{l+1}.$$

We let B act diagonally on $(G/B)^{l+1}$; that is, simultaneously on each factor:

$$b \cdot (g_0 B, g_1 B, \dots, g_l B) \overset{\text{def}}{=} (b g_0 B, b g_1 B, \dots, b g_l B).$$

This action restricts to $Z_{\mathbf{i}}^{\text{fib}}$. The natural map to the flag variety is the projection to the last coordinate:

$$\begin{aligned} Z_{\mathbf{i}}^{\text{fib}} &\to G/B \\ (eB, g_1 B, \dots, g_l B) &\mapsto g_l B \end{aligned}$$

This construction is related to the correspondences of Fulton [10], Ch. 10.3.

Finally, let us define the B**-orbit Bott-Samelson variety** as the closure (in either the Zariski or analytic topologies) of the orbit of a point $z_{\mathbf{i}}$:

$$Z_{\mathbf{i}}^{\text{orb}} \overset{\text{def}}{=} \overline{B \cdot z_{\mathbf{i}}} \subset G/\widehat{P}_{i_1} \times \cdots \times G/\widehat{P}_{i_l},$$

where

$$z_{\mathbf{i}} = (s_{i_1} \widehat{P}_{i_1},\ s_{i_1} s_{i_2} \widehat{P}_{i_2},\ \dots,\ s_{i_1} \cdots s_{i_l} \widehat{P}_{i_l})$$

Again, B acts diagonally. In this case the map to G/B is more difficult to describe, but see the Examples in §2.3.

1.2. Isomorphism theorem

The three types of Bott-Samelson variety are isomorphic.

Theorem 1. *(i) Let*

$$\begin{array}{rccc} \phi: & X_l & \to & (G/B)^{l+1} \\ & (g_1, g_2, \ldots, g_l) & \mapsto & (\overline{e}, \overline{g_1}, \overline{g_1 g_2}, \ldots, \overline{g_1 g_2 \cdots g_l}), \end{array}$$

where $\overline{g}$ means the coset of g. Then ϕ restricts to an isomorphism of B-varieties

$$\phi : Z_{\mathbf{i}}^{\mathrm{quo}} \xrightarrow{\sim} Z_{\mathbf{i}}^{\mathrm{fib}}.$$

(ii) Let

$$\begin{array}{rcccccccc} \psi: & X_l & \to & G/\widehat{P}_{i_1} & \times & G/\widehat{P}_{i_2} & \times \cdots \times & G/\widehat{P}_{i_l} \\ & (g_0, g_1, \ldots, g_l) & \mapsto & (\overline{g_1} & , & \overline{g_1 g_2} & , \ \cdots \ , & \overline{g_1 g_2 \cdots g_l}), \end{array}$$

where $\overline{g}$ means the coset of g. Then ψ restricts to an isomorphism of B-varieties

$$\psi : Z_{\mathbf{i}}^{\mathrm{quo}} \xrightarrow{\sim} Z_{\mathbf{i}}^{\mathrm{orb}}.$$

Proof. (i) It is trivial to verify that ϕ is a B-equivariant isomorphism from X_l to $eB \times (G/B)^l$ and that $\phi(Z_{\mathbf{i}}^{\mathrm{quo}}) \subset Z_{\mathbf{i}}^{\mathrm{fib}}$, so it suffices to show the reverse inclusion. Suppose

$$z_f = (eB, g_1 B, \ldots, g_l B) \in Z_{\mathbf{i}}^{\mathrm{fib}}.$$

Then

$$z_q = \phi^{-1}(z_f) = (g_1, g_1^{-1} g_2, g_2^{-1} g_3, \ldots) \in X_l.$$

By definition, $eP_{i_1} = g_1 P_{i_1}$, so $g_1 \in P_{i_1}$. Also $g_1 P_{i_2} = g_2 P_{i_2}$, so $g_1^{-1} g_2 \in P_{i_2}$, and similarly $g_{k-1}^{-1} g_k \in P_{i_k}$. Hence $z_q \in Z_{\mathbf{i}}^{\mathrm{quo}}$, and $\phi(z_q) = z_f$.
(ii) First let us show that ψ is injective on $Z_{\mathbf{i}}^{\mathrm{quo}}$. Suppose $\psi(p_1, \ldots, p_l) = \psi(q_1, \ldots, q_l)$ for $p_k, q_k \in P_{i_k}$. Then $p_1 \widehat{P}_{i_1} = q_1 \widehat{P}_{i_1}$, so that $p_1^{-1} q_1 \in \widehat{P}_{i_1} \cap P_{i_1} = B$. Thus $q_1 = p_1 b_1$ for $b_1 \in B$. Next, we have

$$p_1 p_2 \widehat{P}_{i_2} = q_1 q_2 \widehat{P}_{i_2} = p_1 b_1 q_2 \widehat{P}_{i_2},$$

so that $p_2^{-1} b_1 q_2 \in \widehat{P}_{i_2} \cap P_{i_2} = B$, and $q_2 = b_1^{-1} p_2 b_2$ for $b_2 \in B$. Continuing in this way, we find that

$$\begin{aligned} (q_1, q_2, \ldots, q_l) &= (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{l-1}^{-1} p_l b_l) \\ &= (p_1, p_2, \ldots, p_l) \in X_l \end{aligned}$$

Thus ψ is injective on $Z_{\mathbf{i}}^{\text{quo}}$.

Since we are working with algebraic morphisms, we must also check that ψ is injective on tangent vectors of $Z_{\mathbf{i}}^{\text{quo}}$. Now, the degeneracy locus

$$\{z \in Z_{\mathbf{i}}^{\text{quo}} \mid \text{Ker } d\psi_z \neq 0\}$$

is a B-invariant, closed subvariety of $Z_{\mathbf{i}}^{\text{quo}}$, and by Borel's Fixed Point Theorem it must contain a B-fixed point. But it is easily seen that the degenerate point

$$z_0 = (e, \ldots, e) \in X_l$$

is the only fixed point of $Z_{\mathbf{i}}^{\text{quo}}$. Thus if $d\psi$ is injective at z_0, then the degeneracy locus is empty, and $d\psi$ is injective on each tangent space. The injectivity at z_0 is easily shown by an argument completely analogous to that for global injectivity given above, but written additively in terms of Lie algebras instead of multiplicatively with Lie groups.

Thus it remains to show surjectivity: that ψ takes $Z_{\mathbf{i}}^{\text{quo}}$ onto $Z_{\mathbf{i}}^{\text{orb}}$. Consider

$$z_{\mathbf{i}}^{\text{quo}} = (s_{i_1}, \ldots, s_{i_l}) \in X_l,$$

a well-defined point in $Z_{\mathbf{i}}^{\text{quo}}$. Then

$$\psi(z_{\mathbf{i}}^{\text{quo}}) = z_{\mathbf{i}} = (s_{i_1}\widehat{P}_{i_1}, s_{i_1}s_{i_2}\widehat{P}_{i_2}, \ldots),$$

and ψ is B-equivariant, so that $\psi(Z_{\mathbf{i}}^{\text{quo}}) \supset \psi(B \cdot z_{\mathbf{i}}^{\text{quo}}) = B \cdot z_{\mathbf{i}}$. However $Z_{\mathbf{i}}^{\text{quo}}$ is a projective variety, so its image under the regular map ψ is closed. Hence $\psi(Z_{\mathbf{i}}^{\text{quo}}) \supset \overline{B \cdot z_{\mathbf{i}}} = Z_{\mathbf{i}}^{\text{orb}}$. □

1.3. Incidence relations

We give another characterization of the Bott-Samelson variety as a subvariety $Z_{\mathbf{i}}^{\text{orb}} \subset G/\widehat{P}_{i_1} \times \cdots \times G/\widehat{P}_{i_l}$ in terms of certain *incidence conditions*, which can easily be translated into algebraic equations defining $Z_{\mathbf{i}}^{\text{orb}}$ as a variety.

Given two parabolic subgroups $P, Q \supset B$, we say the cosets gP and $g'Q$ are *incident* (written $gP \sim g'Q$) if any of the following equivalent conditions holds:

(i) $(gP, g'Q)$ lies in the image of the diagonal map $G/(P \cap Q) \to G/P \times G/Q$;
(ii) $gP \cap g'Q \neq \emptyset$;
(iii) $g^{-1}g' \in PQ$;
(iv) $g^{-1}g'B \in X_w$, the Schubert variety of G/B associated to the unique longest element w in the set $W_P W_Q \subset W$, the product of the subgroups of W corresponding to P and Q.

For $G = GL(n)$ and P, Q maximal parabolics, the spaces $G/P, G/Q$ are Grassmannians, and our definition of incidence reduces to the inclusion relation between subspaces. (See §2.3.)

The incidence relation $\sim$ is reflexive and symmetric, but only partially transitive. One substitute for transitivity is the following property. Suppose g_1Q_1, g_2Q_2, g_3Q_3 are cosets of any parabolics with $g_iQ_i \sim g_jQ_j$ for all i, j. Then there exists g_0 with $g_1Q_1 \sim g_0(Q_1 \cap Q_2) \sim g_2Q_2$, and since $(Q_1 \cap Q_2)Q_3 = Q_1Q_3 \cap Q_2Q_3$ (by [6], Ch. 4, Ex. 1), we conclude that $g_0(Q_1 \cap Q_2) \sim g_3Q_3$. An immediate consequence of this property is:

Lemma 2. *Consider any parabolics $Q_1, Q_2, \ldots \supset B$. Then the point $(g_1Q_1, g_2Q_2, \ldots)$ lies in the image of the diagonal map $G/(\cap_i Q_i) \to \prod_i G/Q_i$ if and only if $g_iQ_i \sim g_jQ_j$ for all i, j.*

This lemma generalizes the description of $GL(n)/B$ as the variety of flags of subspaces.

The incidence relation has another transitivity property. Suppose s, s', s'' are simple reflections of W such that s' is between s and s'' in the Coxeter graph of W: that is, if $s^{(1)}, s^{(2)}, \ldots, s^{(N)}$ is any sequence of simple reflections such that $s = s^{(1)}$, $s'' = s^{(N)}$ and $s^{(j)}s^{(j+1)} \neq s^{(j+1)}s^{(j)}$ for all j, then $s' = s^{(j)}$ for some j. Let $\widehat{P}, \widehat{P}', \widehat{P}''$ be the maximal parabolic subgroups of G corresponding to s, s', s''. Then we may easily show that $\widehat{P}\widehat{P}'\widehat{P}'' = \widehat{P}\widehat{P}''$, so that

$$g\widehat{P} \sim g'\widehat{P}' \text{ and } g'\widehat{P}' \sim g''\widehat{P}'' \quad \Rightarrow \quad g\widehat{P} \sim g''\widehat{P}'' .$$

From this and the previous Lemma, we obtain:

Lemma 3. *Let $\widehat{P}_1, \ldots, \widehat{P}_r \supset B$ be all the maximal parabolic subgroups of G. Then the point $(g_1\widehat{P}_1, \ldots, g_r\widehat{P}_r)$ lies in the image of the diagonal embedding $G/B \to \prod_{i=1}^r G/\widehat{P}_i$ if and only if $g_i\widehat{P}_i \sim g_j\widehat{P}_j$ for all i, j with $s_is_j \neq s_js_i$.*

To our word $\mathbf{i} = (i_1, \ldots, i_l)$ we now associate a graph $\Gamma_\mathbf{i}$ whose vertices are the symbols $1^*, 2^*, \ldots r^*$ and $1, 2, \ldots, l$. (Recall that $r = \operatorname{rank} G$.) The edges of $\Gamma_\mathbf{i}$ are all pairs of vertices of the forms:

$$\begin{array}{rl} (i^*, k) & \text{with } i \neq i_p \text{ for } 1 \leq p \leq k \text{ and } s_i s_{i_k} \neq s_{i_k} s_i, \\ (j, k) & \text{with } i_j \neq i_p \text{ for } j < p \leq k \text{ and } s_{i_j} s_{i_k} \neq s_{i_k} s_{i_j}. \end{array}$$

The graph $\Gamma_\mathbf{i}$ is closely related to the wiring diagrams and chamber weights of Berenstein, Fomin, and Zelevinsky [2], [3].

Now, it follows from Theorem 1 that $Z_\mathbf{i}^{\text{orb}}$ is the image of $Z_\mathbf{i}^{\text{fib}}$ under the natural projection

$$\begin{array}{ccc} (G/B)^{l+1} & \to & \prod_{j=1}^l G/\widehat{P}_{i_j} \\ (g_0B, g_1B, \ldots, g_lB) & \mapsto & (g_1\widehat{P}_{i_1}, \ldots, g_l\widehat{P}_{i_l}). \end{array}$$

Translating this into incidence conditions using the above Lemmas, we obtain:

Theorem 4.

$$Z_{\mathbf{i}}^{\rm orb} = \left\{ (g_1\widehat{P}_{i_1}, \ldots, g_l\widehat{P}_{i_l}) \;\middle|\; \begin{array}{rl} e\widehat{P}_i \sim g_k\widehat{P}_{i_k} \text{ for all } & (i^*, k) \in \Gamma_{\mathbf{i}} \\ g_j\widehat{P}_{i_j} \sim g_k\widehat{P}_{i_k} \text{ for all } & (j, k) \in \Gamma_{\mathbf{i}} \end{array} \right\}$$

See §2.3 below for examples in the case of $G = GL(n)$.

1.4. Open cells and matrix factorizations

In view of Theorem 1, we will let $Z_{\mathbf{i}}$ denote the abstract Bott-Samelson variety defined by any of our three versions. It contains the degenerate B-fixed point z_0 defined by:

$$\begin{aligned} z_0 &= (e, e, \ldots) \in Z_{\mathbf{i}}^{\rm quo} \\ &= (eB, eB, \ldots) \in Z_{\mathbf{i}}^{\rm fib} \\ &= (e\widehat{P}_{i_1}, e\widehat{P}_{i_2}, \ldots) \in Z_{\mathbf{i}}^{\rm orb} \end{aligned}$$

as well as the generating T-fixed point whose B-orbit is dense in $Z_{\mathbf{i}}$:

$$\begin{aligned} z_{\mathbf{i}} &= (s_{i_1}, s_{i_2}, s_{i_3}, \ldots) \in Z_{\mathbf{i}}^{\rm quo} \\ &= (eB, s_{i_1}B, s_{i_1}s_{i_2}B, \ldots) \in Z_{\mathbf{i}}^{\rm fib} \\ &= (s_{i_1}\widehat{P}_{i_1}, s_{i_1}s_{i_2}\widehat{P}_{i_2}, \ldots) \in Z_{\mathbf{i}}^{\rm orb} \end{aligned}$$

Big cell. We may parametrize the dense orbit $B \cdot z_{\mathbf{i}} \subset Z_{\mathbf{i}}$ by an affine cell. Consider the normal ordering of the positive roots associated to the reduced word $\mathbf{i}$. That is, let

$$\beta_1 = \alpha_{i_1},\ \beta_2 = s_{i_1}(\alpha_{i_2}),\ \beta_3 = s_{i_1}s_{i_2}(\alpha_{i_3}),\ \cdots$$

Let U_{β_k} be the one-dimensional unipotent subgroup of B corresponding to the positive root β_k. Then we have a direct product:

$$B = U_{\beta_1} \cdots U_{\beta_l} \cdot (B \cap wBw^{-1}),$$

so that the multiplication map

$$\begin{array}{ccc} U_{\beta_1} \times \cdots \times U_{\beta_l} & \to & B \cdot z_{\mathbf{i}} \\ (u_1, \ldots, u_l) & \mapsto & u_1 \cdots u_l \cdot z_{\mathbf{i}} \end{array}$$

is injective, and an isomorphism of varieties. The left-hand side is isomorphic to an affine space $\mathbf{C}^l$ (or $\mathbf{A}^l$ for G over a general field).

Opposite big cell. $Z_{\mathbf{i}}$ also contains an opposite big cell centered at z_0 which is not the orbit of a group. Let $U_{-\alpha_i}$ be the one-dimensional unipotent subgroup of $w_0 B w_0$ corresponding to the negative simple root $-\alpha_i$. The map

$$\begin{array}{rcl} \mathbf{C}^l \cong U_{-\alpha_{i_1}} \times \cdots \times U_{-\alpha_{i_l}} & \to & Z_{\mathbf{i}}^{\mathrm{quo}} \\ (u_1, \ldots, u_l) & \mapsto & (u_1, \ldots, u_l) \end{array}$$

is an open embedding.

In the case of $G = GL(n)$, $B =$ upper triangular matrices, we may write an element of $U_{-\alpha_{i_k}}$ as $u_k = I + t_k e_k$, where I is the identity matrix, e_k is the sub-diagonal coordinate matrix $e_{(i_k+1, i_k)}$, and $t_k \in \mathbf{C}$. If we further map $Z_{\mathbf{i}}^{\mathrm{quo}}$ to G/B via the natural multiplication map, we get

$$\begin{array}{ccc} (t_1, \ldots, t_l) & \mapsto & (I + t_1 e_1) \cdots (I + t_l e_l) \\ \mathbf{C}^l & \to & N_- \\ \cap & & \cap \\ Z_{\mathbf{i}}^{\mathrm{quo}} & \to & G/B \\ (p_1, \ldots, p_l) & \mapsto & p_1 \cdots p_l B \end{array}$$

where N_- denotes the unipotent lower triangular matrices (mod B). Thus the natural map in the bottom row compactifies the matrix factorization map in the top row, which has been studied by Berenstein, Fomin, and Zelevinsky [2]; and the corresponding statement holds in the general case of [3].

2. Bott-Samelson varieties for $GL(n)$

We begin again, restating our results in explicit combinatorial form for the general linear group $G = GL(n, \mathbf{C})$. We define the Bott-Samelson variety in an explicit and elementary way, which will easily show that its coordinate ring consists of generalized Schur modules. That is, a generalized Schur module bears the same relation to a Bott-Samelson variety as an ordinary (irreducible) Schur module bears to a flag variety according to the Borel-Weil Theorem. Therefore the characters, generalized Schur polyomials, can be computed by powerful Riemann-Roch type theorems just like ordinary Schur polynomials.

Our purpose in this section is to get enough combinatorial control over the Bott-Samelson varieties to make such theorems explicit. For a reduced decomposition $\mathbf{i}$, the Bott-Samelson variety $Z_{\mathbf{i}}$ is the space of flagged representations of a certain partially ordered set $D_{\mathbf{i}}^+$: that is, the variety of all embeddings of the poset $D_{\mathbf{i}}^+$ into the poset of subspaces of $\mathbf{C}^n$. (Such an embedding is *flagged* if a certain chain in $D_{\mathbf{i}}^+$ maps to the standard flag $\mathbf{C}^1 \subset \mathbf{C}^2 \subset \cdots \mathbf{C}^n$.)

The posets $D_{\mathbf{i}}^+$ can be specified by several equivalent combinatorial devices. They can be naturally embedded into the Boolean lattice of subsets of $[n] = \{1, 2, \ldots, n\}$. The image of such an embedding is a *chamber family*, associated to a reduced decomposition via its *wiring diagram*. This is easily translated into the language of *generalized Young diagrams* in the plane: the columns of a diagram correspond to the elements of a chamber family. It is remarkable that these different combinatorial pictures come together to describe our varieties.

In the rest of this paper, $G = GL(n)$. To make our statements more elementary, we will use $\mathbf{C}$ for our base field, but everything goes through without change over an infinite field of arbitrary characteristic or over $\mathbf{Z}$. We let B be the group of invertible upper triangular matrices, T the group of invertible diagonal matrices, and $\mathrm{Gr}(k, \mathbf{C}^n)$ the Grassmannian of k-dimensional subspaces of complex n-space. Also $W =$ permutation matrices, $\ell(w) =$ the number of inversions of a permutation w, $s_i =$ the transposition $(i, i+1)$, and the longest permutation is $w_0 = n \dots 321$. We will frequently use the notation

$$[k] = \{1, 2, 3, \dots, k\}.$$

2.1. Chamber families

Define a *subset family* to be a collection $D = \{C_1, C_2, \dots\}$ of subsets $C_k \subset [n]$. The order of the subsets is irrelevant in the family, and we do not allow subsets to be repeated.

Now suppose the list of indices $\mathbf{i} = (i_1, i_2, \dots, i_l)$ encodes a reduced decomposition $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ of a permutation into a minimal number of simple transpositions. We associate a subset family, the *chamber family*

$$D_{\mathbf{i}} \stackrel{\text{def}}{=} \{s_{i_1}[i_1], s_{i_1} s_{i_2}[i_2], \dots, w[i_l]\}.$$

Here $w[j] = \{w(1), w(2), \dots, w(j)\}$. Further, define the *full chamber family*

$$D_{\mathbf{i}}^{+} \stackrel{\text{def}}{=} \{[1], [2], \dots, [n]\} \cup D_{\mathbf{i}}.$$

We tentatively connect these structures with geometry. Let $\mathbf{C}^n$ have the standard basis $e_1, \dots, e_n$. For any subset $C = \{j_1, \dots, j_k\} \subset [n]$, the coordinate subspace

$$E_C = \mathrm{Span}_{\mathbf{C}}\{e_{j_1}, \dots, e_{j_k}\} \in \mathrm{Gr}(k) = \mathrm{Gr}(k, \mathbf{C}^n)$$

is a T-fixed point in a Grassmannian. A subset family corresponds to a T-fixed point in a product of Grassmannians

$$z_D = (E_{C_1}, E_{C_2}, \dots) \in \mathrm{Gr}(D) \stackrel{\text{def}}{=} \mathrm{Gr}(\,|C_1|\,) \times \mathrm{Gr}(\,|C_2|\,) \times \dots .$$

We will define Bott-Samelson varieties as orbit closures of such points (see §2.3).

Examples. For $n = 3$, $G = GL(3)$, $\mathbf{i} = 121$, we have the reduced chamber family

$$\begin{aligned} D_{121} &= \{\, s_1[1],\ s_1s_2[2],\ s_1s_2s_1[1]\,\} \\ &= \{\,\{2\}, \{2,3\}, \{3\}\,\} \\ &= \{2, 23, 3\}. \end{aligned}$$

The full chamber family is $D^+_{121} = \{1, 12, 123, 2, 23, 3\}$. The chamber family of the other reduced word $\mathbf{i} = 212$ is $D_{212} = \{13, 3, 23\}$, $D^+_{212} = \{1, 12, 123, 13, 3, 23\}$.

For $n = 4$, the subset family $D = \{12, 123, 2, 3\}$ is associated to the T-fixed point

$$z_D = (E_{12}, E_{123}, E_2, E_3) \in \mathrm{Gr}(D) = \mathrm{Gr}(2) \times \mathrm{Gr}(3) \times \mathrm{Gr}(1) \times \mathrm{Gr}(1). \qquad \square$$

Chamber families have a rich structure. (See [18], [25].) Given a full chamber family $D^+_{\mathbf{i}}$, we may omit some of its elements to get a subfamily $D \subset D^+_{\mathbf{i}}$. The resulting *chamber subfamilies* can be characterized as follows.

For two sets $S, S' \subset [n]$, we say S *is elementwise less than* S', $S \overset{elt}{<} S'$, if $s < s'$ for all $s \in S$, $s' \in S'$. Now, a pair of subsets $C, C' \subset [n]$ is *strongly separated* if

$$(C \setminus C') \overset{elt}{<} (C' \setminus C) \quad \text{or} \quad (C' \setminus C) \overset{elt}{<} (C \setminus C')\ ,$$

where $C \setminus C'$ denotes the complement of C' in C. A family of subsets is called strongly separated if each pair of subsets in it is strongly separated.

Proposition 5. (Leclerc-Zelevinsky [18]) *A family D of subsets of $[n]$ is a chamber subfamily, $D \subset D^+_{\mathbf{i}}$ for some* $\mathbf{i}$*, if and only if D is strongly separated.*

Remarks. (a) Reiner and Shimozono [25] give an equivalent description of strongly separated families. Place the subsets of the family into lexicographic order. Then $D = (C_1 \overset{lex}{\leq} C_2 \overset{lex}{\leq} \cdots)$ is strongly separated if and only if it is "%-avoiding": that is, if $i_1 \in C_{j_1}$, $i_2 \in C_{j_2}$ with $i_1 > i_2$, $j_1 < j_2$, then $i_1 \in C_{j_2}$ or $i_2 \in C_{j_1}$.
(b) If $\mathbf{i} = (i_1, \ldots, i_l)$ is an initial subword of $\mathbf{i}' = (i_1, \ldots, i_l, \ldots, i_N)$, then $D_{\mathbf{i}} \subset D_{\mathbf{i}'}$. Thus the chamber families associated to decompositions of the longest permutation w_0 are the maximal strongly separated families.
(c) In §4.3 below, we describe the "orthodontia" algorithm to determine a chamber family $D^+_{\mathbf{i}}$ which contains to a given strongly separated family D. See also [25].

Examples. (a) For $n = 3$, the chamber families $D^+_{121} = \{1, 12, 123, 2, 23, 3\}$ and $D^+_{212} = \{1, 12, 123, 13, 3, 23\}$ are the only maximal strongly separated families. The sets 13 and 2 are the only pair not strongly separated from each other.
(b) For $n = 4$, the strongly separted family $D = \{24, 34, 4\}$ is contained in the chamber sets of the reduced words $\mathbf{i} = 312132$ and $\mathbf{i} = 123212$.

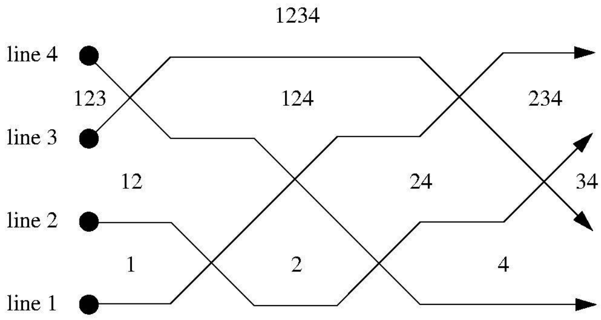

Figure 1.

2.2. Pictures of chamber families

Wiring diagrams. Chamber families can be represented pictorially in several ways, the most natural being due to Berenstein, Fomin, and Zelevinsky [2]. The *wiring diagram* or *braid diagram* of the reduced word $\mathbf{i}$ is best defined via an example.

Let $G = GL(4)$, $w = w_0$ (the longest permutation), and $\mathbf{i} = 312132$. On the right and left ends of the wiring diagram are the points 1,2,3,4 in two columns, with 1 at the bottom and 4 at the top. Each point i on the left is connected to the point $w^{-1}(i)$ on the right by a curve which is horizontal and disjoint from the other curves, except for certain X-shaped crossings. The crossings, read left to right, correspond to the entries of $\mathbf{i}$. The first entry $i_1 = 3$ corresponds to a crossing of the curve on level 3 with the one on level 4. The curves on other levels continue horizontally. The second entry $i_2 = 1$ indicates a crossing of the curves on levels 1 and 2, the others continuing horizontally, and so on.

If we add crossings only up to the l^{th} step, we obtain the wiring diagram of the truncated word $s_{i_1} s_{i_2} \cdots s_{i_l}$.

Now we may construct the chamber family

$$D_{\mathbf{i}}^{+} = (1, 12, 123, 1234, 124, 2, 24, 4, 234, 34)$$

as follows. Label each of the curves of the wiring diagram by its point of origin on the left. Into each of the connected regions between the curves, write the numbers of those curves which pass below the region. Then the sets of numbers inscribed in these chambers are the members of the family $D_{\mathbf{i}}^{+}$. If we list the chambers from left to right, we recover the natural order in which these subsets appear in $D_{\mathbf{i}}^{+}$. (Warning: In BFZ's terminology, our $D_{\mathbf{i}}^{+}$ would be the chamber family of the *reverse word* of $\mathbf{i}$, a reduced decomposition of w^{-1}.)

Young diagrams. Another way to picture a chamber family, or any subset family, is as follows. We may consider a subset $C = \{j_1, j_2, \ldots, j_c\} \subset [n]$ as a column of c

squares in the rows $j_1, j_2, \ldots$. For each subset C_k in the chamber family, form the column associated to it, and place these columns next to each other. The result is an array of squares in the plane called a *generalized Young diagram.*

For our word $\mathbf{i} = 312132$, we draw the (reduced) chamber family as:

$$D_{\mathbf{i}} \;=\; \begin{array}{c|cccccc} 1 & \square & & & & & \\ 2 & \square & \square & \square & & \square & \\ 3 & & & & & \square & \square \\ 4 & \square & & \square & \square & \square & \square \end{array}$$

where we indicate the row numbers on the left of the diagram.

2.3. Varieties

To any subset family D we have associated a T-fixed point in a product of Grassmannians, $z_D \in \mathrm{Gr}(D)$, and we may define the *configuration variety* of D to be the closure of the G-orbit of z_D:

$$\mathcal{F}_D = \overline{G \cdot z_D} \subset \mathrm{Gr}(D);$$

and the *flagged configuration variety* to be the closure of its B-orbit:

$$\mathcal{F}_D^B = \overline{B \cdot z_D} \subset \mathrm{Gr}(D).$$

Furthermore, if $D = D_{\mathbf{i}}$, a chamber family, then the *Bott-Samelson variety* is the flagged configuration variety of $D_{\mathbf{i}}$:

$$Z_{\mathbf{i}} = Z_{\mathbf{i}}^{\mathrm{orb}} = \mathcal{F}_D^B \,.$$

(We could also use the full chamber family $D_{\mathbf{i}}^+$, since the extra coordinates correspond to the standard flag fixed under the B-action.)

Thus $\mathcal{F}_D$, $\mathcal{F}_D^B$, and $Z_{\mathbf{i}}$ can be considered as varieties of configurations of subspaces in $\mathbf{C}^n$, like the flag and Schubert varieties. We will give defining equations for the Bott-Samelson varieties analogous to those for Schubert varieties.

For a subset family D with partial order given by inclusion, define the *variety of flagged representations* of D

$$\mathcal{R}_D^B = \left\{ (V_C)_{C \in D} \in \mathrm{Gr}(D) \;\middle|\; \begin{array}{c} \forall\, C, C' \in D,\;\; C \subset C' \Rightarrow V_C \subset V_{C'} \\ \text{and } \;\forall\, [i] \in D,\;\; V_{[i]} = \mathbf{C}^i \end{array} \right\}.$$

("Flagged" refers to the condition that a space $V_{[i]}$ corresponding to an initial interval $[i] \in D$ is fixed to be an element of the standard flag $\mathbf{C}^1 \subset \mathbf{C}^2 \subset \cdots$.) Let B act diagonally on $\mathcal{R}_D^B$.

The following proposition is a special case of Prop. 4 of §1.3.

Proposition 6. *For every reduced word* $\mathbf{i}$*, we have* $Z_{\mathbf{i}} \cong \mathcal{R}_{D^+}^B$.

2.4. Examples of varieties

Example. For $n = 4$, $\mathbf{i} = 312132$, we may use the picture in the above example to write the Bott-Samelson variety $Z_{\mathbf{i}} = \mathcal{R}^B_{D^+}$ as the set of all 6-tuples of subspaces of $\mathbf{C}^4$, $(V_{124}, V_2, V_{24}, V_4, V_{234}, V_{34})$ with $\dim(V_C) = |C|$ and satisfying the following inclusions:

$$\begin{array}{ccccccc}
 & & & \mathbf{C}^4 = V_{1234} & & & \\
 & & \nearrow & \uparrow & \nwarrow & & \\
\mathbf{C}^3 = V_{123} & & & V_{124} & & V_{234} & \\
\uparrow & & \nearrow & & \nwarrow & \uparrow & \nwarrow \\
\mathbf{C}^2 = V_{12} & & & & & V_{24} & & V_{34} \\
\uparrow & & \nwarrow & & \nearrow & \uparrow & \nearrow \\
\mathbf{C}^1 = V_1 & & & V_2 & & V_4 & \\
 & & \nwarrow & \uparrow & \nearrow & & \\
 & & & 0 & & &
\end{array}$$

where the arrows indicate inclusion of subspaces. The natural map onto the flag variety projects $(V_{124}, \ldots, V_{34})$ to the flag at the right edge of the picture: $(0 \subset V_4 \subset V_{34} \subset V_{234} \subset \mathbf{C}^4)$.

Example. Desingularizing a Schubert variety. Let $n = 7$, and consider the family D comprising the single subset $C = 12457$. Its configuration variety is the Grassmannian $\mathcal{F}_D = \mathrm{Gr}(5, \mathbf{C}^7)$, and its flagged configuration variety is a Schubert variety X_λ in this Grassmannian:

$$\mathcal{F}^B_D = X_\lambda = \{V \in \mathrm{Gr}(5, \mathbf{C}^7) \mid \mathbf{C}^2 \subset V,\ \dim(\mathbf{C}^5 \cap V) \geq 4\}.$$

Here the indexing partition $\lambda = (0, 0, 1, 1, 2)$ is obtained from the subset $C = 12457$ by subtracting $1, 2, \ldots$ from its elements: $0 = 1 - 1$, $0 = 2 - 2$, $1 = 4 - 3$, $1 = 5 - 4, 2 = 7 - 5$.) Now, we know by Proposition 5 that any strongly separated family is part of some chamber family $D_{\mathbf{i}}$. In fact, we may take $\mathbf{i}$ so that the projection map $Z_{\mathbf{i}} = \mathcal{F}^B_{\tilde{D}} \to \mathcal{F}^B_D$ is birational. The orthodontia algorithm of §4.3 below produces such an $\mathbf{i}$.

By orthodontia, we find that our variety is desingularized by the reduced word $\mathbf{i} = 3465$, for which $D_{\mathbf{i}} = \{124, 1245, 123457, 12457\}$ and

$$Z_{\mathbf{i}} = \left\{ \begin{array}{c} (V_{124}, V_{1245}, V_{123457}, V_{12457}) \in \mathrm{Gr}(3) \times \mathrm{Gr}(4) \times \mathrm{Gr}(6) \times \mathrm{Gr}(5) \\ \text{such that} \quad \mathbf{C}^2 \subset V_{124} \subset \mathbf{C}^4 \subset V_{123457} \ \ , \ \ V_{1245} \subset \mathbf{C}^5 \ , \\ V_{124} \subset V_{1245} \subset V_{12457} \subset V_{123457} \end{array} \right\}.$$

The desingularization map is the projection

$$\begin{array}{cccc} \pi : & Z_{\mathbf{i}} & \to & \mathcal{F}^B_D = X_\lambda \\ & (V_{124}, V_{1245}, V_{123457}, V_{12457}) & \mapsto & V_{12457}. \end{array}$$

In our paper [21] and Zelevinsky's work [26], there are given other desingularizations of Schubert varieties, all of them expressible as flagged configuration varieties.

Conjecture 7. *For any subset family* D*, a configuration* $(V_C)_{C\in D} \in \mathrm{Gr}(D)$ *lies in* $\mathcal{F}_D$ *exactly if, for every subfamily* $D' \subset D$*,*

$$\dim(\bigcap_{C\in D'} V_C) \geq |\cap_{C\in D'} C|$$

$$\dim(\sum_{C\in D'} V_C) \leq |\cup_{C\in D'} C|$$

Remarks. (a) If $D = D_{\mathbf{i}}$ is a chamber family, the conjecture reduces to the previous theorem.
(b) The conjecture is known if D satisfies the "northwest condition" (see [21]): that is, the elements of D can be arranged in an order $C_1, C_2, \ldots$ such that if $i_1 \in C_{j_1}$, $i_2 \in C_{j_2}$, then $\min(i_1, i_2) \in C_{\min(j_1,j_2)}$. In fact, it suffices in this case to consider only the intersection conditions of the conjecture.
(c) Note that a configuration $(V_1, \ldots, V_l) \in \mathrm{Gr}(D)$ lies in the flagged configuration variety $\mathcal{F}_D^B$ if and only if $(\mathbf{C}^1, \ldots, \mathbf{C}^n, V_1, \ldots, V_l)$ lies in the unflagged variety $\mathcal{F}_{D^+}$ of the augmented diagram $D^+ \stackrel{\mathrm{def}}{=} \{[1], [2], \ldots [n]\} \cup D$. Hence the conjecture gives conditions defining flagged configuration varieties as well as unflagged.
(d) It would be interesting to know whether the determinantal equations implied by the conditions of the conjecture (and the previous theorem) define $\mathcal{F}_D \subset \mathrm{Gr}(D)$ scheme-theoretically.

3. Schur and Weyl modules

The most familiar construction of Schur modules is in terms of Young symmetrizers acting on a large tensor power of $\mathbf{C}^n$. This construction is limited to characteristic zero, however, so we use an alternative construction in the spirit of DeRuyts [10], Desarmenien-Kung-Rota [9], and Carter-Lusztig. This construction is universally valid and is more natural geometrically. (We sketch the connection with the symmetrizer picture at the end of §3.1.) Using the same arguments as in [21], our Borel-Weil Theorem is immediate, and we work out a version of Demazure's character formula to get a new expression for generalized Schur polynomials.

3.1. Definitions

We have associated to any subset family $D = \{C_1, \ldots, C_k\}$ a configuration variety $\mathcal{F}_D$ with G-action, and a flagged configuration variety $\mathcal{F}_D^B$ with B-action. Now, assign an integer multiplicity $\mathbf{m}(C) \geq 0$ to each subset $C \in D$. For each pair

$(D, \mathbf{m})$, we define a G-module and a B-module, which will turn out to sections of a line bundle on $\mathcal{F}_D$ and $\mathcal{F}_D^B$. We construct these "Weyl modules" $M_{D,\mathbf{m}}$ inside the coordinate ring of $n \times n$ matrices, and their flagged versions $M_{D,\mathbf{m}}^B$ inside the coordinate ring of upper-triangular matrices.

Let $\mathbf{C}[x_{ij}]$ (resp. $\mathbf{C}[x_{ij}]_{i\leq j}$) denote the polynomial functions in the variables x_{ij} with $i,j \in [n]$ (resp. x_{ij} with $1 \leq i \leq j \leq n$). For $R, C \subset [n]$ with $|R| = |C|$, let

$$\Delta_C^R = \det(x_{ij})_{(i\in R, j\in C)} \in \mathbf{C}[x_{ij}]$$

be the minor of the matrix $x = (x_{ij})$ on the rows R and the columns C. Further, let

$$\widetilde{\Delta}_C^R = \Delta_C^R|_{x_{ij}=0,\ \forall\, i>j} \in \mathbf{C}[x_{ij}]_{i\leq j}$$

be the same minor evaluated on an upper triangular matrix of variables.

Now, for a subset family $D = \{C_1, \dots, C_l\}$, $\mathbf{m} = (m_1, \dots, m_l)$, define the *Weyl module*

$$M_{D,\mathbf{m}} = \mathrm{Span}_{\mathbf{C}} \left\{ \Delta_{C_1}^{R_{11}} \cdots \Delta_{C_1}^{R_{1m_1}} \Delta_{C_2}^{R_{21}} \cdots \Delta_{C_l}^{R_{lm_l}} \;\middle|\; \begin{matrix} \forall\, k,m \quad R_{km} \subset [n] \\ \text{and} \quad |R_{km}| = |C_k| \end{matrix} \right\}.$$

That is, a spanning vector is a product of minors with column indices equal to the elements of D and row indices taken arbitrarily.

For two sets $R = \{i_1 < \cdots < i_c\}$, $C = \{j_1 < \cdots < j_c\}$ we say $R \overset{comp}{\leq} C$ (componentwise inequality) if $i_1 \leq j_1$, $i_2 \leq j_2$, $\dots$. Define the *flagged Weyl module*

$$M_{D,\mathbf{m}}^B = \mathrm{Span}_{\mathbf{C}} \left\{ \widetilde{\Delta}_{C_1}^{R_{11}} \cdots \widetilde{\Delta}_{C_1}^{R_{1m_1}} \widetilde{\Delta}_{C_2}^{R_{21}} \cdots \widetilde{\Delta}_{C_l}^{R_{lm_l}} \;\middle|\; \begin{matrix} \forall\, k,m \quad R_{km} \subset [n] \\ |R_{km}| = |C_k|,\ R_{km} \overset{comp}{\leq} C_k \end{matrix} \right\}.$$

(In fact, the condition $R_{km} \overset{comp}{\leq} C_k$ is superfluous, since $\widetilde{\Delta}_C^R = 0$ unless $R_{km} \overset{comp}{\leq} C_k$.)

For $f(x) \in \mathbf{C}[x_{ij}]$, a matrix $g \in G$ acts by left translation, $(g \cdot f)(x) = f(g^{-1}x)$. It is easily seen that this restricts to a G-action on $M_{D,\mathbf{m}}$ and similarly we get a B-action on $M_{D,\mathbf{m}}^B$.

We clearly have the diagram of B-modules:

$$\begin{array}{ccc} M_{D,\mathbf{m}} & \subset & \mathbf{C}[x_{ij}] \\ \downarrow & & \downarrow \\ M_{D,\mathbf{m}}^B & \subset & \mathbf{C}[x_{ij}]_{i\leq j} \end{array}$$

where the vertical maps ($x_{ij} \mapsto 0$ for $i > j$) are surjective. That is, $M_{D,\mathbf{m}}^B$ is a quotient of $M_{D,\mathbf{m}}$.

The *Schur modules* are defined to be the duals

$$S_{D,\mathbf{m}} \stackrel{\text{def}}{=} (M_{D,\mathbf{m}})^* \qquad S^B_{D,\mathbf{m}} \stackrel{\text{def}}{=} (M^B_{D,\mathbf{m}})^*.$$

We will deal mostly with the Weyl modules, but everything we say will of course have a dual version applying to Schur modules.

Example. Let $n = 4$, $D = \{234, 34, 4\}$, $\mathbf{m} = (2, 0, 3)$. (That is, $m(234) = 2$, $m(34) = 0$, $m(4) = 3$.) We picture this as a generalized Young diagram by writing each column repeatedly, according to its multiplicity. Zero multiplicity means we omit the column. Thus

$$(D,\mathbf{m}) = \begin{array}{cccccc} 1 & & & & & \\ 2 & \square & \square & & & \\ 3 & \square & \square & & & \\ 4 & \square & \square & \square & \square & \square \end{array} \qquad \tau = \begin{array}{cccccc} 1 & & & & & \\ 2 & 1 & 1 & & & \\ 3 & 3 & 2 & & & \\ 4 & 4 & 3 & 2 & 4 & 3 \end{array}$$

The spanning vectors for $M_{D,\mathbf{m}}$ correspond to all column-strict fillings of this diagram by indices in $[n]$. For example, the filling τ above corresponds to

$$\Delta^{134}_{234}\,\Delta^{123}_{234}\,\Delta^{2}_{4}\,\Delta^{4}_{4}\,\Delta^{3}_{4}$$

$$= \begin{vmatrix} x_{12} & x_{13} & x_{14} \\ x_{32} & x_{33} & x_{34} \\ x_{42} & x_{43} & x_{44} \end{vmatrix} \cdot \begin{vmatrix} x_{12} & x_{13} & x_{14} \\ x_{22} & x_{23} & x_{24} \\ x_{32} & x_{33} & x_{34} \end{vmatrix} \cdot x_{24} \cdot x_{44} \cdot x_{34}$$

$$= \left(\begin{array}{ccccc|ccccc} 1 & 1 & & & & 2 & 2 & & & \\ 3 & 2 & & & & 3 & 3 & & & \\ 4 & 3 & 2 & 4 & 3 & 4 & 4 & 4 & 4 & 4 \end{array} \right)$$

The last expression is in the letter-place notation of Rota et al [9].

A basis may be extracted from this spanning set by considering only the row-decreasing fillings (a normalization of the semi-standard tableaux), and in fact the Weyl module is the dual of the classical Schur module S_λ associated to the shape D considered as the Young diagram $\lambda = (0, 2, 2, 5)$.

The spanning elements of the flagged Weyl module $M^B_{D,\mathbf{m}}$ correspond to the "flagged" fillings of the diagram: those for which the number i does not appear above the i^{th} level. For the diagram above, all the column-strict fillings are flagged, and $M_{D,\mathbf{m}} \cong M^B_{D,\mathbf{m}}$.

However, for

$$(D',\mathbf{m}) = \begin{array}{cccccc} 1 & & & & & \\ 2 & \square & \square & & & \\ 3 & \square & \square & \square & \square & \square \\ 4 & \square & \square & & & \end{array}$$

$$\tau_1 = \begin{array}{cccccc} 1 & & & & & \\ 2 & 2 & 1 & & & \\ 3 & 3 & 2 & 4 & 3 & 4 \\ 4 & 4 & 3 & & & \end{array} \qquad \tau_2 = \begin{array}{cccccc} 1 & & & & & \\ 2 & 2 & 1 & & & \\ 3 & 3 & 2 & 3 & 2 & 3 \\ 4 & 4 & 4 & & & \end{array}$$

the filling τ_1 is *not* flagged, since 4 appears on the 3rd level, but τ_2 *is* flagged, and corresponds to the spanning element

$$\widetilde{\Delta}^{234}_{234}\,\widetilde{\Delta}^{124}_{234}\,\widetilde{\Delta}^{3}_{3}\,\widetilde{\Delta}^{2}_{3}\,\widetilde{\Delta}^{3}_{3} = \begin{vmatrix} x_{22} & x_{23} & x_{24} \\ 0 & x_{33} & x_{34} \\ 0 & 0 & x_{44} \end{vmatrix} \cdot \begin{vmatrix} x_{12} & x_{13} & x_{14} \\ x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{44} \end{vmatrix} \cdot x_{33} \cdot x_{23} \cdot x_{33}.$$

We have $M_{D,\mathbf{m}} \cong M_{D',\mathbf{m}} \cong M^B_{D,\mathbf{m}} \cong S^*_{(0,2,2,5)}$, the dual of a classical (irreducible) Schur module for $GL(4)$, and $M^B_{D',\mathbf{m}} \cong S^*_{(0,2,5,2)}$, the dual of the Demazure module with lowest weight $(0,2,5,2)$ and highest weight $(5,2,2,0)$. Cf. [23].

Remarks. (a) In [13], [14] and §4.4 below, we make a general definition of "standard tableaux" giving bases of the Weyl modules for strongly separated families.
(b) We briefly indicate the equivalence between our definition of the Weyl modules and the tensor product definition given in [1], [23], [21].

Let $Y = Y_{D,\mathbf{m}} \subset \mathbf{N} \times \mathbf{N}$ be the generalized Young diagram of squares in the plane associated to $(D, \mathbf{m})$ as in the above examples, and let $U = (\mathbf{C}^n)^*$. One defines $M^{\text{tensor}}_Y = U^{\otimes Y}\gamma_Y$, where γ_Y is a generalized Young symmetrizer. The spanning vectors Δ_τ of $M_{D,\mathbf{m}}$ correspond to the fillings $\tau : Y \to [n]$. Then the map

$$\begin{array}{ccc} M_{D,\mathbf{m}} & \to & M^{\text{tensor}}_Y \\ \Delta_\tau & \mapsto & \left(\bigotimes_{(i,j)\in Y} e^*_{\tau(i,j)}\right)\gamma_Y \end{array}$$

is a well-defined isomorphism of G-modules, and similarly for the flagged versions. This is easily seen from the definitions, and also follows from the Borel-Weil theorems proved below and in [21].

3.2. Borel-Weil theory

A configuration variety $\mathcal{F}_D \subset \mathrm{Gr}(D)$ has a natural family of line bundles defined by restricting the determinant or Plucker bundles on the factors of $\mathrm{Gr}(D)$. For $D = (C_1, C_2, \ldots)$, and multiplicities $\mathbf{m} = (m_1, m_2, \ldots)$, we define

$$\begin{array}{ccc} \mathcal{L}_{\mathbf{m}} & \subset & \mathcal{O}(m_1, m_2, \ldots) \\ \downarrow & & \downarrow \\ \mathcal{F}_D & \subset & \mathrm{Gr}(D) = \mathrm{Gr}(|C_1|) \times \mathrm{Gr}(|C_2|) \\ \times \cdots & & \end{array}$$

We denote by the same symbol $\mathcal{L}_{\mathbf{m}}$ this line bundle restricted to $\mathcal{F}_D^B$. Note that in the case of a Bott-Samelson variety $\mathcal{F}_D = Z_{\mathbf{i}}$, this is the well-known line bundle

$$\mathcal{L}_{\mathbf{m}} \cong \frac{P_{i_1} \times \cdots \times P_{i_l} \times \mathbf{C}}{B^l}$$

$$(p_1, \dots, p_l, v) \cdot (b_1, \dots, b_l) \stackrel{\text{def}}{=} (p_1 b_1, \dots, b_{l-1}^{-1} p_l b_l,\ \varpi_{i_1}(b_1^{-1})^{m_1} \cdots \varpi_{i_l}(b_l^{-1})^{m_l}\, v),$$

ϖ_i denoting the fundamental weight $\varpi_i(\mathrm{diag}(x_1, \dots, x_n)) = x_1 x_2 \cdots x_i$.

Note that if $m_k \geq 0$ for all k (resp. $m_k > 0$ for all k) then $\mathcal{L}_{\mathbf{m}}$ is effective (resp. very ample). However, $\mathcal{L}_{\mathbf{m}}$ may be effective even if some $m_k < 0$.

Proposition 8. *Let $(D, \mathbf{m})$ be a strongly separated subset family with multiplicity. Then we have*
(i) $M_{D,\mathbf{m}} \cong H^0(\mathcal{F}_D, \mathcal{L}_{\mathbf{m}})$ and $H^i(\mathcal{F}_D, \mathcal{L}_{\mathbf{m}}) = 0$ for $i > 0$.
(ii) $M_{D,\mathbf{m}}^B \cong H^0(\mathcal{F}_D^B, \mathcal{L}_{\mathbf{m}})$ and $H^i(\mathcal{F}_D^B, \mathcal{L}_{\mathbf{m}}) = 0$ for $i > 0$.
(iii) $\mathcal{F}_D$ and $\mathcal{F}_D^B$ are normal varieties, projectively normal with respect to $\mathcal{L}_{\mathbf{m}}$, and have rational singularities.

Proof. First, recall that we can identify the sections of a bundle over a single Grassmannian, $\mathcal{O}(1) \to \mathrm{Gr}(i)$, with linear combinations of $i \times i$ minors $\Delta^R(x)$ in the homogeneous Stiefel coordinates

$$x = \begin{pmatrix} x_{11} & \cdots & x_{1i} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{ni} \end{pmatrix} \in \mathrm{Gr}(i),$$

where R denotes any set of row indices $R \subset [n]$, $|R| = i$. Thus, a typical spanning element of $H^0(\mathrm{Gr}(D), \mathcal{O}(\mathbf{m}))$ is the section

$$\Delta^{R_{11}}(x^{(1)}) \cdots \Delta^{R_{11}}(x^{(1)})\ \Delta^{R_{21}}(x^{(2)}) \cdots \Delta^{R_{lm_l}}(x^{(l)}),$$

where $x^{(k)}$ represents the homogeneous coordinates on each factor $\mathrm{Gr}(|C_k|)$ of $\mathrm{Gr}(D)$, and R_{km} are arbitrary subsets with $|R_{km}| = i_k$.

Now, restrict the above section to $\mathcal{F}_D \subset \mathrm{Gr}(D)$ and then further to the dense G-orbit $G \cdot z_D \subset \mathcal{F}_D$. Parametrizing the orbit by $g \mapsto g \cdot z_D$, we pull back the resulting sections of $H^0(\mathcal{F}_D, \mathcal{L}_{\mathbf{m}})$ to certain functions on $G \subset \mathrm{Mat}_{n\times n}(\mathbf{C})$, which are precisely the products of minors defining the spanning set of $M_{D,\mathbf{m}}$. This shows that

$$M_{D,\mathbf{m}} \cong \mathrm{Im}\left[H^0(\mathrm{Gr}(D), \mathcal{O}(\mathbf{m}))\ \stackrel{\text{rest}}{\to}\ H^0(\mathcal{F}_D, \mathcal{L}_{\mathbf{m}})\right].$$

Similarly for B-orbits, we have

$$M_{D,\mathbf{m}}^B \cong \mathrm{Im}\left[H^0(\mathrm{Gr}(D), \mathcal{O}(\mathbf{m}))\ \stackrel{\text{rest}}{\to}\ H^0(\mathcal{F}_D^B, \mathcal{L}_{\mathbf{m}})\right].$$

Given this description of $M_{D,\mathbf{m}}$, our Proposition becomes a restatement of the vanishing results in [21], Props. 25 and 28 (due to W. van der Kallen and S.P. Inamdar, applying the work of O. Mathieu [22], P. Polo, et.al.). The conditions (α) and (β) of these propositions apply to $\mathcal{F}_D$ because D is contained in a chamber family $D_{\mathbf{i}}^{+}$ (Prop. 5 above). Furthermore, the proof of [21], Props. 25 and 28 go through identically with $\mathcal{F}_D^B$ in place of $\mathcal{F}_D$, merely replacing $\mathcal{F}_{w_0;u_1,\ldots,u_r}$ by $\mathcal{F}_{e;u_1,\ldots,u_r}$. □

We recall another result from [21]: For the unflagged case, the following proposition is a restatement of [21], Prop. 28(a). Again, the proof given there goes through almost identically for the flagged case.

Proposition 9. *Suppose $(D,\mathbf{m})$, $(\widetilde{D},\widetilde{\mathbf{m}})$ are strongly separated subset families with $D \subset \widetilde{D}$, $\widetilde{\mathbf{m}}(C) = \mathbf{m}(C)$ for $C \in D$, $\mathbf{m}(C) = 0$ otherwise. Then the natural projection $\pi : \mathrm{Gr}(\widetilde{D}) \to \mathrm{Gr}(D)$ restricts to a surjection $\pi : \mathcal{F}_{\widetilde{D}} \to \mathcal{F}_D$, and induces an isomorphism*

$$\pi^* : H^0(\mathcal{F}_D, \mathcal{L}_{\mathbf{m}}) \xrightarrow{\sim} H^0(\mathcal{F}_{\widetilde{D}}, \mathcal{L}_{\widetilde{\mathbf{m}}}),$$

and similarly for the flagged case.

Remarks. (a) Note that the proposition holds even if $\dim \mathcal{F}_{\widetilde{D}} > \dim \mathcal{F}_D$.
(b) We will use the proposition in the case where D is a strongly separated family which is part of the chamber family $\widetilde{D} = D_{\mathbf{i}}$. The above propositions give:

$$M_{D,\mathbf{m}} \cong H^0(\mathcal{F}_D, \mathcal{L}_{\mathbf{m}}) \cong H^0(\mathcal{F}_D\ , \mathcal{L}_{\widetilde{\mathbf{m}}}) = H^0(Z_{\mathbf{i}}, \mathcal{L}_{\widetilde{\mathbf{m}}}).$$

In the next section, we apply the Demazure formula for Bott-Samelson varieties to compute the character of $M_{D,\mathbf{m}}$.
(c) We may conjecture that all the results of this section hold not only in the strongly separated case, but for all subset families and configuration varieties.

3.3. Demazure character formula

We now examine how the iterative structure of Bott-Samelson varieties helps to understand the associated Weyl modules.

Define Demazure's isobaric divided difference operator $\Lambda_i : \mathbf{C}[x_1, \ldots, x_n] \to \mathbf{C}[x_1, \ldots, x_n]$,

$$\Lambda_i f = \frac{x_i f - x_{i+1} s_i f}{x_i - x_{i+1}}.$$

For example for $f(x_1, x_2, x_3) = x_1^2 x_2^2 x_3$,

$$\begin{aligned}\Lambda_2 f(x_1, x_2, x_3) &= \frac{x_2(x_1^2 x_2^2 x_3) - x_3(x_1^2 x_3^2 x_2)}{x_2 - x_3} \\ &= x_1^2 x_2 x_3(x_2 + x_3).\end{aligned}$$

For any permutation with a reduced decompostion $w = s_{i_1} \dots s_{i_l}$, define

$$\Lambda_w \stackrel{\text{def}}{=} \Lambda_{i_1} \cdots \Lambda_{i_l},$$

which is known to be independent of the reduced decomposition chosen.

By the dual character of a G- or B-module M, we mean

$$\text{char}^* M = \text{tr}(\text{diag}(x_1, \dots, x_n)|M^*) \in \mathbf{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

(The dual character of a Weyl module is the ordinary character of the corresponding Schur module, a polynomial in $x_1, x_2, \dots$.) Let ϖ_i denote the ith fundamental weight, the multiplicative character of B defined by $\varpi_i(\text{diag}(x_1, \dots, x_n)) = x_1 x_2 \cdots x_i$.

Proposition 10. *Suppose $(D, \mathbf{m})$ is strongly separated, and*

$$D \subset D_{\mathbf{i}}^{+} = \{[1], \dots, [n], C_1, \dots, C_l\},$$

for some reduced word $\mathbf{i} = (i_1, \dots, i_l)$. Define $\widetilde{\mathbf{m}} = (k_1, \dots, k_n, m_1, \dots, m_l)$ by $\widetilde{\mathbf{m}}(C) = \mathbf{m}(C)$ for $C \in D$, $\widetilde{\mathbf{m}}(C) = 0$ otherwise. Then

$$\text{char}^* M_{D,\mathbf{m}}^B = \varpi_1^{k_1} \cdots \varpi_n^{k_n} \Lambda_{i_1}(\varpi_{i_1}^{m_1} \dots (\Lambda_{i_l} \varpi_{i_l}^{m_l}) \dots).$$

Furthermore,

$$\text{char}^* M_{D,\mathbf{m}} = \Lambda_{w_0} \, \text{char}^* M_{D,\mathbf{m}}^B,$$

where w_0 denotes the longest permutation.

Remark. We explain in §4.4 below (and in [13], [14]) how one can recursively generate a set of standard tableaux for M_D^B by "quantizing" this character formula.

We devote the rest of this section to proving the Proposition.

For a subset $C = \{j_1, j_2, \dots\} \subset [n]$, and a permutation w, let $wC = \{w(j_1), w(j_2), \dots\}$, and for a subset family $D = \{C_1, C_2, \dots\}$, let $wD = \{wC_1, wC_2, \dots\}$. Now, for $i \in [n-1]$, let

$$\Lambda_i D \stackrel{\text{def}}{=} \{s_i[i]\} \cup s_i D,$$

where $s_i[i] = \{1, 2, \dots, i-1, i+1\}$. We say that D is *i-free* for $i \in [n]$ if for every $C \in D$, we have $C \cap \{i, i+1\} \neq \{i+1\}$.

Lemma 11. *Suppose $(D, \mathbf{m})$ is strongly separated and i-free. Then:*
(i) $\mathcal{F}_{\Lambda_i D}^B \cong P_i \times^B \mathcal{F}_D^B$.
(ii) $\mathcal{F}_{s_i D}^B \cong P_i \cdot \mathcal{F}_D^B \subset \text{Gr}(D)$.
(iii) The projection $\mathcal{F}_{\Lambda_i D}^B \to \mathcal{F}_{s_i D}^B$ is regular, surjective, and birational.

(iv) Let $\widetilde{\mathbf{m}}$ *be the multiplicity on* $\Lambda_i D$ *defined by* $\widetilde{\mathbf{m}}(s_i C) = \mathbf{m}(C)$ *for* $C \in D$, $\widetilde{\mathbf{m}}(s_i[i]) = m_0$*. The bundle* $\mathcal{L}_{\widetilde{\mathbf{m}}} \to \mathcal{F}^B_{\Lambda_i D}$ *is isomorphic to*

$$\mathcal{L}_{\widetilde{\mathbf{m}}} \cong P_i \overset{B}{\times} ((\varpi_i^{m_0})^* \otimes \mathcal{L}_{\mathbf{m}}),$$

where $(\varpi_i^{m_0})^* \otimes \mathcal{L}_{\mathbf{m}}$ *indicates the bundle* $\mathcal{L}_{\mathbf{m}} \to \mathcal{F}^B_D$ *with its* B*-action twisted by the multiplicative character* $(\varpi_i^{m_0})^* = \varpi_i^{-m_0}$.

Proof. (i) Since D is i-free, we have $U_i z_D = z_D$, where U_i is the one-dimensional unipotent subgroup corresponding to the simple root α_i. We may factor B into a direct product of subgroups, $B = U_i B' = B' U_i$. Then

$$\mathcal{F}^B_D = \overline{B \cdot z_D} = \overline{B' \cdot z_D}.$$

Hence the T-fixed point $(s_i, z_D) \in P_i \overset{B}{\times} \mathcal{F}^B_D$ has a dense B-orbit:

$$\begin{aligned} \overline{B \cdot (s_i, z_D)} &= \overline{(U_i B' s_i, z_D)} \\ &= (\overline{U_i s_i}, \overline{B' \cdot z_D}) \\ &= P_i \overset{B}{\times} \mathcal{F}^B_D. \end{aligned}$$

Clearly, the injective map

$$\begin{array}{rccc} \psi: & P_i \overset{B}{\times} \mathrm{Gr}(D) & \to & \mathrm{Gr}(i) \times \mathrm{Gr}(D) \\ & (p, V) & \mapsto & (p\mathbf{C}^i, pV) \end{array}$$

takes $\psi(s_i, z_D) = z_{\Lambda_i D}$, the B-generating point of $\mathcal{F}^B_{\Lambda_i D}$. Thus $\psi : P_i \overset{B}{\times} \mathcal{F}^B_D \to \mathcal{F}^B_{\Lambda_i D}$ is an isomorphism.
(ii+iii) By the above, the projection is a bijection on the open B-orbit, and hence is birational. The image of the projection is $P_i \cdot \mathcal{F}^B_D$, which must be closed since $P_i \times^B \mathcal{F}^B_D$ is a proper (i.e. compact variety).
(iv) Clear from the definitions. □

Lemma 12. *Let* $(D, \mathbf{m})$ *be a strongly separated family and* $i \in [n-1]$*. Let*

$$\begin{aligned} \mathcal{F}' &= P_i \overset{B}{\times} \mathcal{F}^B_D \\ \mathcal{L}' &= P_i \overset{B}{\times} \mathcal{L}_{\mathbf{m}}. \end{aligned}$$

so that $\mathcal{L}' \to \mathcal{F}'$ *is a line bundle. Then*

$$\mathrm{char}^* H^0(\mathcal{F}', \mathcal{L}') = \Lambda_i \, \mathrm{char}^* H^0(\mathcal{F}^B_D, \mathcal{L}_{\mathbf{m}}).$$

Proof. By Demazure's analysis of induction to P_i (see [7], "construction élémentaire") we have

$$\Lambda_i \operatorname{char}^* H^0(\mathcal{F}_D^B, \mathcal{L}_{\mathbf{m}}) = \operatorname{char}^* H^0(\mathcal{F}', \mathcal{L}') - \operatorname{char}^* H^1(P_i/B, H^1(\mathcal{F}_D^B, \mathcal{L}_{\mathbf{m}})).$$

However, we know by [21], Prop. 28(a) that $H^0(\mathcal{F}_D^B, \mathcal{L}_{\mathbf{m}})$ has a good filtration, so that the H^1 term above is zero. □

Corollary 13. *If $(D, \mathbf{m})$ is strongly separated and i-free, and $(\Lambda_i D, \widetilde{\mathbf{m}})$ is a diagram with multiplicities $\widetilde{\mathbf{m}}(s_i C) = m(C)$ for $C \in D$, $\widetilde{\mathbf{m}}(s_i[i]) = m_0$, then*

$$\operatorname{char}^* M^B_{\Lambda_i D, \widetilde{\mathbf{m}}} = \Lambda_i \varpi_i^{m_0} \operatorname{char}^* M^B_{D,\mathbf{m}}.$$

If $m_0 = 0$, then

$$\operatorname{char}^* M^B_{s_i D,\mathbf{m}} = \operatorname{char}^* M^B_{\Lambda_i D, \widetilde{\mathbf{m}}} = \Lambda_i \operatorname{char}^* M^B_{D,\mathbf{m}}$$

This follows immediately from the above Lemmas and Proposition 9.

Proof of Proposition. The first formula of the Proposition now follows from the above Lemmas and Prop. 9. The second statement follows similarly from Demazure's character formula and the vanishing statements of [21], Prop. 28. □

4. Schubert polynomials

We now apply our theory to compute the Schubert polynomials $\mathcal{S}(w)$ of permutations $w \in S_n$, which generalize the Schur polynomials $s_\lambda(x_1, \dots, x_k)$. They were originally considered as representatives of Schubert classes in the Borel picture of the cohomology of the flag variety $GL(n)/B$, though we will give a completely different geometric interpretation in §4.2. As a general reference, see Macdonald [20] or Fulton [10].

Although our results follow from the geometric theory of previous sections, we phrase them in a purely elementary and self-contained way (except in §4.2). Most of our computations in §§4.3–4.5 are valid for the character of the generalized Schur module of any strongly separated family.

We first state the combinatorial definition of Schubert polynomials, and then prove the theorem of Kraskiewicz and Pragacz [12], that Schubert modules are the characters of flagged Schur modules associated to a Rothe diagram. Finally, we give three new, explicit formulas for Schubert polynomials.

4.1. Definitions

The Schubert polynomials $\mathcal{S}(w)$ in variables $x_1 \ldots , x_n$ are constructed combinatorially in terms of the following *divided difference operators.* First, the operator ∂_i is defined by

$$\partial_i f(x_1, \ldots , x_n) = \frac{f(x_1, \ldots , x_i, x_{i+1}, \ldots , x_n) - f(x_1, \ldots , x_{i+1}, x_i, \ldots , x_n)}{x_i - x_{i+1}}.$$

Then for a reduced decomposition of a permutation $u = s_{i_1} s_{i_2} \cdots$, the operator $\partial_u = \partial_{i_1} \partial_{i_2} \cdots$ is independent of the reduced decomposition chosen. Also, take $\partial_e = \mathrm{id}$.

Now we may define the Schubert polynomials as follows. Let w_0 be the longest permutation ($w_0(i) = n + 1 - i$), and take $u = w^{-1}w_0$, so that $wu = w_0$. Then

$$\mathcal{S}(w) \stackrel{\text{def}}{=} \partial_u(x_1^{n-1} x_2^{n-2} \cdots x_{n-2}^2 x_{n-1}).$$

We have deg $\mathcal{S}(w) = \ell(w)$.

To compute any $\mathcal{S}(w)$, we write $w_0 = ws_{i_1} \cdots s_{i_r}$ for some reduced word $s_{i_1} \cdots s_{i_r}$ ($s_i = (i, i + 1)$ denoting a simple transposition in S_n). In particular, we may take i_k to be the *first ascent* of $w_k = ws_{i_1} \cdots s_{i_{k-1}}$; that is, $i_k =$ the smallest i such that $w_k(i + 1) > w_k(i)$.

Examples. (a) For $w \in S_3$, we have $\mathcal{S}(w_0) = x_1^2 x_2$, $\mathcal{S}(s_1 s_2) = x_1 x_2$, $\mathcal{S}(s_2 s_1) = x_1^2$, $\mathcal{S}(s_2) = x_1 + x_2$, $\mathcal{S}(s_1) = x_1$, $\mathcal{S}(e) = 1$.
(b) For the permutation $w = 24153 \in S_5$, by inverting first ascents we get $ws_1 s_3 s_2 s_1 s_4 s_3 = w_0$, so

$$\begin{aligned}\mathcal{S}(w) &= \partial_1 \partial_3 \partial_2 \partial_1 \partial_4 \partial_3 (x_1^4 x_2^3 x_3^2 x_4) \\ &= x_1 x_2 (x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4).\end{aligned}$$

(c) Given $n > k$, the partition $\lambda = (0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \lambda_k \leq n)$ is "strictified" to the subset $C = \{\lambda_1 + 1 < \lambda_2 + 2 < \cdots \lambda_k + k\} \subset [n]$, which is completed to a Grassmannian permutation w by adjoining $[n] \setminus C$. Then the Schubert polynomial of w is equal to the Schur polynomial of λ: $\mathcal{S}(w) = s_\lambda(x_1, \ldots , x_k)$. For instance, for $n = 7, k = 5, C = 12457, \lambda = 00112,$ we have $\mathcal{S}(1245736) = s_{00112}(x_1, \ldots , x_5)$.

Now, a *diagram* (generalized Young diagram) is a subset $D \subset \mathbf{N} \times \mathbf{N}$. The point (i, j) is in row i, column j, and we think of a diagram as a list of columns $C \subset \mathbf{N}$: $D = (C_1, C_2, \ldots)$. Two diagrams are *column equivalent*, $D \cong D'$, if one is obtained from the other by switching the order of columns (and ignoring empty columns). For a column $C \subset \mathbf{N}$, the multiplicity $\mathrm{mult}_D(C)$ is the number of columns of D with content equal to C. An equivalence class of diagrams is another way to express our subset families with multiplicity in §3.1. Sum of diagrams

$$D \oplus D'$$

means placing D horizontally next to D' (concatenating lists of columns), and $D \setminus \{C\}$ means removing one column whose content is equal to C.

The *Rothe diagram* of a permutation $w \in S_n$ is

$$D(w) = \{\, (i,j) \in [n]\times[n] \mid i < w^{-1}(j),\ j < w(i) \,\}.$$

It is easy to see that $D(w)$ is a strongly separated subset family. (In fact, it is northwest. See [23], [24], [21].)

Example. For the same $w = 24153$, we have

$$D = D(w) = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{ccc} \square & & \\ \square & & \square \\ & & \\ & & \square \end{array} \;\cong\; \begin{array}{cc} \square & \\ \square & \square \\ & \\ & \square \end{array} \;=\; (\,\{1,2\},\{2,4\}\,)$$

Recall that $[i]$ denotes the interval $\{1, 2, 3, \ldots, i\}$.

4.2. Theorem of Kraskiewicz and Pragacz

The geometric significance of the Schubert polynomials is as follows. There are two classical computations of the singular cohomology ring $H^{\cdot}(G/B, \mathbf{C})$ of the flag variety. That of Borel identifies the cohomology with a coinvariant algebra

$$c : H^{\cdot}(G/B, \mathbf{C}) \xrightarrow{\sim} \mathbf{C}[x_1, \ldots, x_n]/I_+,$$

where I_+ is the the ideal generated by the non-constant symmetric polynomials. The map c is an isomorphism of graded $\mathbf{C}$-algebras, and the generator x_i represents the Chern class of the i^{th} quotient of the tautological flag bundle, which is not the dual of an effective divisor. The alternative computation of Schubert gives as a linear basis for $H^{\cdot}(G/B, \mathbf{C})$ the Schubert classes $\sigma_w = [X_{w_0 w}]$, the Poincare duals of the Schubert varieties.

The isomorphism between these computations was defined by Bernstein-Gelfand-Gelfand [4] and Demazure [7], and given a precise combinatorial form by Lascoux and Schutzenberger [15]. It states that the Schubert polynomials $\mathcal{S}(w)$ defined above are representatives of the Schubert classes in the cohomology ring.

We now give a completely different geometric interpretation of the polynomials $\mathcal{S}(w)$ in terms of Weyl modules.

Theorem 14. (Kraskiewicz-Pragacz [12])

$$\mathcal{S}(w) = \operatorname{char}^* M^B_{D(w)},$$

where $M^B_{D(w)}$ is the Weyl module of §3.1 associated to $D(w)$ (thought of as a subset family with multiplicity).

Proof. (Magyar-Reiner-Shimozono) Let $\chi(w) = \text{char}^* M^B_{D(w)}$. We must show that $\chi(w)$ satisfies the defining relations of $\mathcal{S}(w)$.

First, $D(w_0) = ([1], \dots, [n-1])$, and

$$M^B_{D(w_0)} = \mathbf{C} \cdot \widetilde{\Delta}^1_1 \widetilde{\Delta}^{12}_{12} \dots \widetilde{\Delta}^{[n-1]}_{[n-1]},$$

a one-dimensional B-module, so $\chi(w_0) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$.

Now, suppose $ws_i < w$, and i is the first ascent of ws_i. Then the $w(i+1)^{\text{th}}$ element of $D(w)$ is $C_{w(i+1)}(w) = [i]$. Letting

$$D'(w) \stackrel{\text{def}}{=} D(w) \setminus \{\, [i] \,\},$$

it is easily seen that:
(i) $D'(w)$ is i-free,
(ii) $D(w) \cong D'(w) \oplus \{\, [i] \,\}$, and
(iii) $D(ws_i) \cong s_i D'(w) \oplus \{\, [i-1] \,\}$ (where $[0] = \emptyset$) .

Hence we obtain trivially:

$$\chi(w) = x_1 \cdots x_i \ \text{char}^* M^B_{D'(w)}$$
$$\chi(ws_i) = x_1 \cdots x_{i-1} \ \text{char}^* M^B_{s_i D'(w)}.$$

Since $D'(w)$ is strongly separated and i-free, Corollary 13 implies that

$$\text{char}^* M^B_{s_i D'(w)} = \Lambda_i \, \text{char}^* M^B_{D'(w)}.$$

Thus we have

$$\begin{aligned} \chi(ws_i) &= (x_1 \cdots x_{i-1}) \ \Lambda_i \, \text{char}^* M^B_{D'(w)} \\ &= \Lambda_i x_i^{-1} (x_1 \cdots x_i) \ \text{char}^* M^B_{D'(w)} \\ &= \Lambda_i x_i^{-1} \, \chi(w) \\ &= \partial_i \, \chi(w). \end{aligned}$$

But now, using the the first-ascent sequence to write $w_0 = ws_{i_1} \cdots s_{i_r}$, we compute

$$\chi(w) = \partial_{i_1} \cdots \partial_{i_r} (x_1^{n-1} x_2^{n-2} \cdots x_{n-1}) = \mathcal{S}(w). \qquad \square$$

4.3. Orthodontia and Demazure character formula

We will use the Demazure character formula (Prop. 10) to compute Schubert polynomials. To make this formula explicit, however, we must embed our Rothe diagram into a chamber family. The algorithm we give below will work for any strongly separated family.

Let $D = (C_1, C_2, \dots)$ be a Rothe diagram. We require a reduced word $\mathbf{i} = (i_1, \dots, i_l)$ and a multiplicity list $\mathbf{m} = (k_1, \dots, k_n, m_1, \dots, m_l)$, $k_i, m_j \geq 0$, which generate D in the following sense. Define a diagram by

$$D_{\mathbf{i},\mathbf{m}} = \bigoplus_{i=1}^{n} k_i \cdot [i] \quad \oplus \quad \bigoplus_{j=1}^{l} m_j \cdot (s_{i_1} s_{i_2} \cdots s_{i_j}[i_j]),$$

where $m \cdot C = C \oplus \cdots \oplus C$ (m copies of C), and $0 \cdot C = \emptyset$, an empty column. Then we require that $D \cong D_{\mathbf{i},\mathbf{m}}$..

As our first step in generating $\mathbf{i}$ and $\mathbf{m}$, let $k_i = \mathrm{mult}_D([i])$, $1 \leq i \leq n$, and remove from D all columns of the form $C = [i]$ to get a new diagram D_-.

Given a column $C \subset [n]$, a *missing tooth* of C is a positive integer i such that $i \notin C$, but $i + 1 \in C$. The only C without any missing teeth are the intervals $[i]$, so we can choose a missing tooth i_1 of the first column of D_-. Now switch rows i_1 and $i_1 + 1$ of $D_- = \{C_1, C_2, \dots\}$ to get a new diagram D' with closer teeth (orthodontia). That is,

$$D' = s_{i_1} D_- = \{s_{i_1} C_1, s_{i_1} C_2, \dots\}.$$

In the second step, repeat the above with D' instead of D. That is, let $m_1 = \mathrm{mult}_{D'}([i_1])$, and remove all columns of the form $C = [i_1]$ from D' to get a new diagram D'_-. Find a missing tooth i_2 of the first column of D'_-, and switch rows to get a new diagram $D'' = s_{i_2} D'_-$.

Iterate this procedure until all columns have been removed. It is easily seen that the sequences $\mathbf{i}$ and $\mathbf{m}$ thus defined have the desired properties.

Example. For $w = 24153$,

$$D = D(w) = \begin{array}{lll} 1 & \square & \\ 2 & \square & \square \\ 3 & & \\ 4 & & \square \end{array} \qquad D_- = \begin{array}{ll} 1 & \circ \\ 2 & \square \\ 3 & \\ 4 & \square \end{array} \qquad D' = D'_- = \begin{array}{ll} 1 & \square \\ 2 & \\ 3 & \circ \\ 4 & \square \end{array}$$

$$D'' = D''_- = \begin{array}{ll} 1 & \square \\ 2 & \circ \\ 3 & \square \end{array} \qquad D''' = \begin{array}{ll} 1 & \square \\ 2 & \square \end{array} \qquad D'''_- = \emptyset$$

so that the sequence of missing teeth (as indicated by $\circ$) gives $\mathbf{i} = (1, 3, 2)$, and $\mathbf{m} = (k_1 = 0, k_2 = 1, k_3 = 0, k_4 = 0, m_1 = 0, m_2 = 0, m_3 = 1)$. Furthermore $D = \{[2]^1,\ (s_1 s_3 s_2[2])^1\} = \{\ \{1, 2\}, \{2, 4\}\ \}$.

Note that $s_{i_1} \cdots s_{i_l} = s_1 s_3 s_2$ is a reduced subword of the first-ascent sequence $s_1 s_3 s_2 s_1 s_4 s_3$ which raises w to the maximal permutation w_0. This is always the case, and we can give an algorithm for extracting this subword.

Note. To apply this algorithm to a general strongly separated family D (with multiplicity), first choose an order $D = \{C_1, C_2, \dots\}$ for the subsets in the family (the columns) such that if $i < j$ then $(C_i \setminus C_j) \overset{dt}{<} (C_j \setminus C_i)$, in the notation of §2.2. For example, the obvious lexicographic order will do.

Now, the definition of $\mathcal{S}(w)$ involves *descending* induction (lowering the degree), but we give the following *ascending* algorithm, which follows immediately from Prop. 10.

Proposition 15. *Given a permutation w, let $(\mathbf{i}, \mathbf{m})$ be a generating sequence such as the orthodontic sequence above. Let $\Lambda_i = \partial_i x_i$ and $\varpi_i = x_1 x_2 \cdots x_i$. Then*

$$\mathcal{S}(w) = \varpi_1^{k_1} \cdots \varpi_n^{k_n} \Lambda_{i_1}(\varpi_{i_1}^{m_1} \dots (\Lambda_{i_l} \varpi_{i_l}^{m_l}) \dots).$$

Example. For our permutation $w = 24153$, we may verify that

$$\mathcal{S}(w) = x_1 x_2 \, \Lambda_1 \Lambda_3 \Lambda_2 (x_1 x_2).$$

Note that this algorithm is more efficient than the usual one if the permutation $w \in S_n$ has small length compared to n.

Remark. The above proposition computes a Schubert polynomial $\mathcal{S}(w)$ in terms of a word $\mathbf{i}$. This word $\mathbf{i}$ is *not* a decomposition of w. We may view the formula of the proposition as computing the character of a space of sections over the Bott-Samelson variety $Z_{\mathbf{i}}$ (cf. §3.3). This variety is *not* the Schubert variety X_w, nor any desingularization of it, since in general $\dim X_w \neq \dim Z_{\mathbf{i}}$. There is no obvious combinatorial relationship between w and $\mathbf{i}$, nor any obvious geometric relationship between X_w and $Z_{\mathbf{i}}$.

4.4. Young tableaux

The work of Lascoux-Schutzenberger [17] and Littlemann [19] allows us to "quantize" our Demazure formula, realizing the terms of the polynomial by certain tableaux endowed with a crystal graph structure. Reiner and Shimozono have shown that our construction gives the same non-commutative Schubert polynomials as those in [16]. Our tableaux are different, however, from the "balanced tableaux" of Fomin, Greene, Reiner, and Shimozono [11]. For proofs see [14], and see also [24], [25].

Recall that a *column-strict filling* (with entries in $\{1,\dots,n\}$) of a diagram D is a map t , mapping the points (i,j) of D to numbers from 1 to n, strictly increasing down each column. The *weight* of a filling t is the monomial $x^t = \prod_{(i,j)\in D} x_{t(i,j)}$, so that the exponent of x_i is the number of times i appears in the filling. We will define a set of fillings $\mathcal{T}$ of the Rothe diagram $D(w)$ which satisfy

$$\mathcal{S}(w) = \sum_{t\in\mathcal{T}} x^t.$$

We will need the *root operators* first defined in [17]. These are operators f_i which take a filling t of a diagram D either to another filling of D or are undefined. To define them we first encode a filling t in terms of its *reading word*: that is, the sequence of its entries starting at the upper left corner, and reading down the columns one after another: $t(1,1), t(2,1), t(3,1), \dots, t(1,2), t(2,2), \dots$.

If it is defined, the lowering operator f_i changes one of the i entries to $i+1$, according to the following rule. First, we ignore all the entries in t except those containing i or $i+1$; if an i is followed by an $i+1$ (ignoring non i or $i+1$ entries in between), then henceforth we ignore that pair of entries; we look again for an i followed (up to ignored entries) by an $i+1$, and henceforth ignore this pair; and iterate until we obtain a subword of the form $i+1, i+1, \dots, i+1, i, i, \dots, i$. If there are *no* i entries in this word, then $f_i(t)$ is undefined. If there are some i entries, then the *leftmost* is changed to $i+1$.

For example, we apply f_2 to the word

$$\begin{array}{rcl}
t & = & 1\ 2\ 2\ 1\ 3\ 2\ 1\ 4\ 2\ 2\ 3\ 3\\
 & & .\ 2\ 2\ .\ 3\ 2\ .\ .\ 2\ 2\ 3\ 3\\
 & & .\ 2\ .\ .\ .\ 2\ .\ .\ 2\ .\ .\ 3\\
 & & .\ 2\ .\ .\ .\ 2\ .\ .\ .\ .\ .\ .\\
f_2(t) & = & 1\ 3\ 2\ 1\ 3\ 2\ 1\ 4\ 2\ 2\ 3\ 3\\
f_2^2(t) & = & 1\ 3\ 2\ 1\ 3\ 3\ 1\ 4\ 2\ 2\ 3\ 3\\
f_2^3(t) & = & \text{undefined}
\end{array}$$

Decoding the image word back into a filling of the same diagram D, we have defined our operators.

Moreover, define the quantized Demazure operator $\widetilde{\Lambda}_i$ taking a tableau t to a set of tableaux:

$$\widetilde{\Lambda}_i(t) = \{t, f_i(t), (f_i)^2(t), \dots\}.$$

Also, for a set $\mathcal{T}$ of tableaux, $\widetilde{\Lambda}(\mathcal{T}) = \bigcup_{t\in\mathcal{T}} \widetilde{\Lambda}(t)$. Note that this means ordinary union of sets, without counting any multiplicities.

Now, consider the column $\phi_i = \{1, 2, \dots, i\}$ and its minimal column-strict filling ϖ_i (jth row maps to j). For a filling t of any diagram $D = (C_1, C_2, \dots)$, define in the obvious way the composite filling $\varpi_i \oplus t$ of the concatenated diagram $\phi_i \oplus D = (\phi_i, C_1, C_2, \dots)$. This corresponds to concatenating the words $(1, 2, \dots, m)$ and t. Similarly, let $[\varpi_i]^m \oplus t$ denote concatenating m copies of ϖ_m before t.

Proposition 16. *For a permutation w, let* **i**, **m** *be a generating sequence as in the previous Proposition. Define the set of tableaux*

$$\mathcal{T} = \varpi_1^{k_1} \oplus \cdots \oplus \varpi_n^{k_n} \oplus \widetilde{\Lambda}_{i_1}(\varpi_{i_1}^{m_1} \oplus \dots (\widetilde{\Lambda}_{i_l} \varpi_{i_l}^{m_l}) \dots).$$

Then the Schubert polynomial $\mathcal{S}(w)$ is the generating function of $\mathcal{T}$:

$$\mathcal{S}(w) = \sum_{t \in \mathcal{T}} x^t.$$

Proof. Follows immediately from the Demazure formula above, and the combinatorial properties of root operators described in [19] Sec. 5.

Example. Continuing the example of the previous section, the set $\mathcal{T}$ of tableaux (words) is built up as follows:

$$\{\varpi_2 = 12\} \overset{\widetilde{\Lambda}_2}{\to} \{12, 13\} \overset{\widetilde{\Lambda}_3}{\to} \{12, 13, 14\} \overset{\widetilde{\Lambda}_1}{\to}= \{12, 13, 14, 23, 24\}$$

$$\overset{\varpi_2 \oplus}{\to} \quad \mathcal{T} = \mathcal{T}_2 = \{1212, 1213, 1214, 1223, 1224\}.$$

This clearly gives us the Schubert polynomial as generating function, and furthermore we see the crystal graph (with vertices the tableaux in $\mathcal{T}$ and edges all pairs of the form $(t, f_i t)$):

$$\begin{array}{ccccc} 1223 & \overset{1}{\leftarrow} & 1213 & & \\ 3\downarrow & & \downarrow 3 & & 1212 \\ 1224 & \overset{1}{\leftarrow} & 1214 & & \end{array}$$

The highest-weight elements in each component are the Yamanouchi words $\mathrm{Yam}(\mathcal{T}) = \{1213, 1212\}$, and by looking at the corresponding lowest elements, we may deduce the expansion of the Schubert polynomial in terms of key polynomials (characters of Demazure modules): $\mathcal{S}(w) = \kappa_{x_1 x_2^2 x_4} + \kappa_{x_1^2 x_2^2} = \kappa_{1201} + \kappa_{2200}$. Lascoux and Schutzenberger [17] have obtained another characterization of such lowest-weight tableaux.

4.5. Weyl character formula

Our final formula reduces to the the Weyl character formula (Jacobi bialternant) in case $\mathcal{S}(w)$ is a Schur polynomial.

Geometrically, the idea is to apply the Atiyah-Bott Fixed Point Theorem to the Bott-Samelson variety to compute the character of its space of sections (the Schubert polynomial). This would be very inefficient, however, since the formula would involve 2^l terms (where l is the length of the $\mathbf{i}$ found by orthodontia). We obtain a much smaller expression from considering a smaller configuration variety $\mathcal{F}_{\widetilde{D}}$ which is smooth and birational to the Bott-Samelson variety, and which desingularizes the configuration variety $\mathcal{F}_{D(w)}$. (See [21] for details.) The formula below applies also to the subset families of northwest type considered in [21], but for a general strongly separated family one has only the inefficient formula coming from the full Bott-Samelson resolution.

Combinatorially, we define a certain extension $\widetilde{D}$ of the Rothe diagram $D = D(w)$. Define the staircase diagram to be the set of columns $\Phi = \{[1], [2], \ldots, [n]\}$. Let the *flagged diagram* $\Phi \oplus D$ be the sum (concatenation) of the two diagrams. Now, given $\Phi \oplus D = (C_1, \ldots, C_r)$, define the *blowup* of the flagged diagram $\widehat{\Phi \oplus D} = (C_1, \ldots, C_r, C'_1, C'_2, \ldots)$, where the extra columns are the intersections $\tilde{C} = C_{i_1} \cap C_{i_2} \cap \cdots \subset \mathbf{N}$, for all lists C_{i_1}, $C_{i_2}, \ldots$ of columns of $\Phi \oplus D$; but if an intersection $\tilde{C}$ is already a column of $\Phi \oplus D$, then we do not append it. Let $\widetilde{D} = \widehat{\Phi \oplus D}$.

Define a *standard tabloid* t of $\widetilde{D}$ to be a column-strict filling such that if C, C' are columns of $\widetilde{D}$ with C horizontally contained in C', then the numbers filling C all appear in the filling of C'. In symbols, $t : \widetilde{D} \to \{1, \ldots, n\}$, $t(i,j) < t(i+1,j)$ for all i, j, and $C \subset C' \Rightarrow t(C) \subset t(C')$.

For $1 \leq i \neq j \leq n$ and a tabloid t of $\widetilde{D}$, we define certain integers: $d_{ij}(t)$ is the number of connected components of the following graph. The vertices are columns C of $\widetilde{D}$ such that $i \in t(C)$, $j \notin t(C)$; the edges are (C, C') such that $C \subset C'$ or $C' \subset C$.

Finally, since there are inclusions of diagrams D, $\Phi \subset \widetilde{D}$, we have the *restrictions* of a tabloid t for $\widetilde{D}$ to D and Φ, which we denote $t|D$ and $t|\Phi$. For t a filling of D, let

$$x^t = \prod_{(i,j) \in D} x_{t(i,j)},$$

the weight of the filling.

Proposition 17.

$$\mathcal{S}(w) = \sum_t \frac{x^{(t|D)}}{\prod_{i<j} (1 - x_i^{-1} x_j)^{d_{ij}(t)-1} \, (1 - x_j^{-1} x_i)^{d_{ji}(t)}},$$

where t runs over the standard tabloids for $\widehat{\Phi \oplus D}$ such that $(t|\Phi)(i,j) = i$ for all $(i,j) \in \Phi$.

Example. For the same $w = 24153$,

$$D = D(w) = \begin{array}{c|cc} 1 & \square & \\ 2 & \square & \square \\ 3 & & \\ 4 & & \square \end{array} \qquad \Phi \oplus D = \begin{array}{cccccc} \square & \square & \square & \square & \square & \\ & \square & \square & \square & \square & \square \\ & & \square & \square & & \\ & & & \square & & \square \end{array}$$

$$\Phi \widehat{\oplus} D = \begin{array}{c|ccccccc} 1 & \square & \square & \square & \square & \square & & \\ 2 & & \square & \square & \square & \square & \square & \square \\ 3 & & & \square & \square & & & \\ 4 & & & & \square & & \square & \end{array} .$$

There are six standard tabloids of the type occurring in the theorem. Their restrictions to the last three columns of $\Phi \widehat{\oplus} D$ are:

$$\begin{array}{ccc} 1 & & \\ 2 & 1 & 1 \\ & & \\ & 2 & \end{array}, \quad \begin{array}{ccc} 1 & & \\ 2 & 1 & 1 \\ & & \\ & 3 & \end{array}, \quad \begin{array}{ccc} 1 & & \\ 2 & 1 & 1 \\ & & \\ & 4 & \end{array}, \quad \begin{array}{ccc} 1 & & \\ 2 & 1 & 2 \\ & & \\ & 2 & \end{array}, \quad \begin{array}{ccc} 1 & & \\ 2 & 2 & 2 \\ & & \\ & 3 & \end{array}, \quad \begin{array}{ccc} 1 & & \\ 2 & 2 & 2 \\ & & \\ & 4 & \end{array}.$$

The integers $d_{ij}(t)$ are 0, 1, or 2, and we obtain

$$\begin{aligned} S(w) = {} & \frac{x_1^2 x_2^2}{(1 - x_1^{-1} x_2)(1 - x_2^{-1} x_3)(1 - x_2^{-1} x_4)} + \frac{x_1^2 x_2 x_3}{(1 - x_1^{-1} x_2)(1 - x_3^{-1} x_4)(1 - x_3^{-1} x_2)} \\ & + \frac{x_1^2 x_2 x_4}{(1 - x_1^{-1} x_2)(1 - x_4^{-1} x_2)(1 - x_4^{-1} x_3)} + \frac{x_1^2 x_2^2}{(1 - x_1^{-1} x_3)(1 - x_1^{-1} x_4)(1 - x_2^{-1} x_1)} \\ & + \frac{x_1 x_2^2 x_3}{(1 - x_2^{-1} x_1)(1 - x_3^{-1} x_4)(1 - x_3^{-1} x_1)} + \frac{x_1 x_2^2 x_4}{(1 - x_2^{-1} x_1)(1 - x_4^{-1} x_1)(1 - x_4^{-1} x_3)} . \end{aligned}$$

Note that it is not clear a priori why this rational function should simplify to a polynomial (with positive integer coefficients).

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