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Minimal orbits close to periodic frequencies

Ugo Bessi and Vera Semijopuva

Abstract. Let $\mathcal{L}(Q,\dot{Q})=\frac{1}{2}|\dot{Q}|^2+h(Q,\dot{Q})$ with h analytic of small norm. The problem of Arnold's diffusion consists in finding conditions on h which guarantee the existence of orbits Q of $\mathcal{L}$ with $\dot{Q}$ connecting two arbitrary points of frequency space. Recently, J. N. Mather has found a sufficient condition for Arnold's diffusion; this condition is not read on h itself, but on the set of all action-minimizing orbits of $\mathcal{L}$. In this paper we try to characterize those action-minimizing orbits whose mean frequency is close to periodic.

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Introduction

One of the problems of the theory of Hamiltonian Systems is to understand the dynamics of lagrangians of the following kind

$$\mathcal{L}(Q,\dot{Q})=A(\dot{Q})-h(Q,\dot{Q})\qquad(Q,\dot{Q})\in\mathbf{T}^m\times\mathbf{R}^m \tag{1}$$

where A is real analytic, $\partial^2_{\dot{Q}\dot{Q}}A>0$ and h is a real analytic function of small norm. In particular, a question has been recently much studied: suppose we are given h of small norm and $\delta>0$; we must find for which $\dot{Q}_1,\dot{Q}_2\in\mathbf{R}^m$ there is an orbit Q of $\mathcal{L}$ and $t_1<t_2\in\mathbf{R}$ such that

$$|\dot{Q}(t_i)-\dot{Q}_i|\le\delta\qquad i=1,2. \tag{2}$$

Obviously, if $h=0$, there is no such orbit if $\delta<\frac{1}{2}|\dot{Q}_2-\dot{Q}_1|$ and, if $\delta\ge\frac{1}{2}|\dot{Q}_2-\dot{Q}_1|$, one is trivially found. If $m=2$, $|\dot{Q}_2-\dot{Q}_1|>\sqrt{\|h\|}>2\delta$, there is again no such orbit because of the KAM theorem.

In [11] a theorem is proven which gives a sufficient condition in order to have (2); this theorem holds not only for Lagrangians satisfying (1), but for all lagrangians of class C^2 which are convex and superlinear in $\dot{Q}$ and whose Euler-Lagrange flow (from now on E-L flow) is complete. The sufficient condition is read not on the lagrangian itself, but on the set of all action-minimizing orbits. In this paper we consider the more restricted class of quasi-integrable lagrangians, like (1), and try to characterize some of its action-minimizing orbits (those whose frequency is close to periodic) in terms of h.

In order to explain our results we need to recall some of the terminology of [10] and [11]. Let us denote by $\mathcal{M}$ the set of all probability measures on $\mathbf{T}^m \times \mathbf{R}^m$ with compact support invariant by the Euler-Lagrange (from now on E-L) flow of $\mathcal{L}$. In [10], the following functional is introduced

$$\alpha: H^1(\mathbf{T}^m, \mathbf{R}) \to \mathbf{R}$$

$$-\alpha(c) = \min\{\int_{\mathbf{T}^m \times \mathbf{R}^m} (\mathcal{L} - \eta_c) \mathrm{d}\mu \;:\; \mu \in \mathcal{M}\}$$

where η_c is a closed 1-form in the cohomology class c, considered as a function $\eta_c: \mathbf{T}^m \times \mathbf{R}^m \to \mathbf{R}$. It can be shown that the minimum is achieved, that $\alpha(c)$ does not depend on the choiche of η_c and that the E-L flow of $\mathcal{L} - \eta_c$ is the same as that of $\mathcal{L}$. In the following, we will choose the representative of c with constant components and we will write, with an abuse of notation, $c = \eta_c \in \mathbf{R}^m$. We list below some of the properties of α proven in [10]:

- α is convex and superlinear;
- If $\alpha(c)$ is attained on μ, then it is attained on almost all the measures on the ergodic decomposition of μ.
- If there is a positive-definite KAM torus, then there is a unique c such that $\alpha(c)$ is attained on the ergodic measure on the KAM torus; moreover, $\alpha(c)$ is attained only on that measure.

We denote by A. C.$(\mathbf{R}, \mathbf{T}^m)$ the space of absolutely continuous functions from $\mathbf{R}$ to $\mathbf{T}^m$. Following [11] we say that an orbit $Q \in$ A. C.$(\mathbf{R}, \mathbf{T}^m)$ is a c-minimizer ($c \in \mathbf{R}^m$) if, for any $a < b \in \mathbf{R}$, any $d < e \in \mathbf{R}$ and any $Q_1 \in$ A. C.$([d,e], \mathbf{T}^m)$ such that

$$Q_1(d) = Q(a), \quad Q_1(e) = Q(b)$$

we have

$$\int_a^b [\mathcal{L}(Q, \dot{Q}) - \langle c, \dot{Q}\rangle + \alpha(c)]\mathrm{d}t \leq \int_d^e [\mathcal{L}(Q_1, \dot{Q}_1) - \langle c, \dot{Q}_1\rangle + \alpha(c)]\mathrm{d}t. \tag{3}$$

If we had $b - a = e - d$ then the integral of $\alpha(c)$ would be the same on the right and on the left and we could drop the term $\alpha(c)$ in the integrand; if $Q|_{[a,b]}$ were in the same homotopy class as $Q_1|_{[d,e]}$, then we could also drop $-\langle c, \dot{Q}\rangle$ and recover the usual notion of minimal orbit. In other words the term $-\langle c, \dot{Q}\rangle$ makes the functional sensitive to the homotopy class of the orbit, and $\alpha(c)$ makes it sensitive to the time of travel. In [11], proposition 5.2, it is shown that the orbits in the support of the measures realizing $\alpha(c)$ are c-minimal; however, they are not the only c-minimal ones: for instance, if the c-minimal ergodic measures are evenly distributed along periodic orbits, then some of the homoclinic or heteroclinic connections among them can be c-minimal orbits. The sufficient condition of [11] is read on the set of all c-minimal orbits as c varies in $\mathbf{R}^m$.

Our approach is based on the fact that there is a change of coordinates (the Nekhorocheff normal form near periodic frequencies) which brings the lagrangian to a very simple form, thus easying the problem of finding minimal orbits. Before discussing the properties of the normal form we remark that these new coordinates are defined only in open sets of the type $\mathbf{T}^m \times B(\frac{\bar{k}}{\bar{T}}, \frac{C}{\bar{T}}\|h\|^{\frac{1}{2m}})$, where $\bar{k} \in \mathbf{Z}^m$ and $0 < \bar{T} \le C'\|h\|^{-\frac{1}{2}+\frac{1}{2m}}$. Thus we must check the following things: first, that for $c \in B(\frac{\bar{k}}{\bar{T}}, \frac{1}{2}\frac{C}{\bar{T}}\|h\|^{\frac{1}{2m}})$ all c-minimal orbits live inside $\mathbf{T}^m \times B(\frac{\bar{k}}{\bar{T}}, \frac{C}{\bar{T}}\|h\|^{\frac{1}{2m}})$, where the normal form is defined. Second, that the change of coordinates preserves the minimality of the orbits. Third, that the balls $B(\frac{\bar{k}}{\bar{T}}, \frac{1}{2}\frac{C}{\bar{T}}\|h\|^{\frac{1}{2m}})$ cover frequency space, so that we can study by this method c-minimal orbits for all $c \in \mathbf{R}^m$. All these facts, which are a reformulation of results of Bernstein-Katok and Lochak, are proven in the appendix for completeness' sake. In particular, we refer the reader to [8] for a proof of Nekhorocheff theorem based on periodic orbits, and to [9] for a survey of the problem of Arnold's diffusion.

In the new variables, the perturbation is the sum of two terms: the first one, which we call V, depends only on the components of Q orthogonal to $\frac{\bar{k}}{\bar{T}}$, and if the perturbation h has Fourier development

$$h(Q,\dot{Q}) = \sum_{k\in\mathbf{Z}^m} a_k(\dot{Q})e^{i\langle k,Q\rangle}$$

then we have that

$$V(Q,\dot{Q}) \simeq \sum_{k\perp\bar{k}} a_k(\dot{Q})e^{i\langle k,Q\rangle}. \tag{4}$$

The second term, which we call γf, is exponentially small in $\|h\|$, and has little influence. For the moment, let us restrict ourselves to the very particular case in which V does not depend on $\dot{Q}$, $f = 0$ and the lagrangian in normal form reads $\mathcal{L}(Q,\dot{Q}) = \frac{1}{2}|\dot{Q}|^2 - V(Q)$. Since V does not depend on the $\frac{\bar{k}}{\bar{T}}$ direction, it is easy to see that the $\frac{\bar{k}}{\bar{T}}$-minimal measures are given by the convex combinations of the measures uniformly distributed along $Q^i(t) = \frac{\bar{k}}{\bar{T}}t + a_i$, with $\{a_i\}$ the set of the maxima of V. Indeed, it is only on these orbits that the integrand $\mathcal{L}(Q,\dot{Q}) - \langle\frac{\bar{k}}{\bar{T}},\dot{Q}\rangle$ reaches its minimum value. Moreover, if the maxima of V are nondegenerate in the direction orthogonal to $\frac{\bar{k}}{\bar{T}}$, the orbit Q^i will be hyperbolic and thus will survive the second, exponentially small term of the Nekhorocheff normal form. We will call Q^i_γ the periodic orbit close to Q^i surviving the perturbation. In theorem 1.3 we show that, under suitable hypotheses on V, the measures evenly distributed along the Q^i_γ are the ergodic $\frac{\bar{k}}{\bar{T}}$-minimal measures. We remark that, by (4), the nondegeneracy of the maxima of V can be read directly on h, without the need of actually performing the change of variables.

Thanks to the particular form of V, which allows us to treat separately the motion in the $\frac{\bar{k}}{T}$ direction (the fast variable) and the one in the orthogonal directions (the slow variables) we are able to study which are the c-minimal orbits for c close to $\frac{\bar{k}}{T}$ and γf small. In theorem 2.3 we study Λ, the minimal supporting domain of α containing $\frac{\bar{k}}{T}$. Under the same hypotheses on V as in theorem 1.3 we show that Λ is contained in the affine hyperplane $\frac{\bar{k}}{T}+\left(\frac{\bar{k}}{T}\right)^{\perp}$ and has nonempty interior relative to it; if c belongs to this interior, then the only c-minimal orbits are the Q^i_γ and the heteroclinic connections between couples of them. In theorem 2.4 we show that, if we neglect the exponentially small term γf, we still get a good approximation of Λ.

If $m=2$ (the twist map case) we re-read in this framework some well-known results. In particular, in proposition 2.6 we show that, if c is just outside Λ, then the c-minimal orbits are close to sequences of heteroclinic or homoclinic connections between the Q^i_γ. The study of orbits close to a sequence of homoclinics or heteroclinics has been initiated in [14] (see also [6]); if the methods of these papers could be applied to this situation they could provide another way to prove the existence of Arnold's diffusion.

We remark that in [10] Mather considers time-dependent Lagrangians, with period τ in time, $\mathcal{L}\colon \mathbf{T}^m\times\mathbf{R}^m\times\mathbf{T}^1\to\mathbf{R}$, together with their extended E-L flow, i. e. the flow on $\mathbf{T}^m\times\mathbf{R}^m\times\mathbf{T}^1$. Moreover, he compactifies this space and shows that the extended E-L flow on $\mathbf{T}^m\times\mathbf{R}^m\times\mathbf{T}^1\cup\infty$ is continuous. He defines $\mathcal{M}$ to be the space of invariant probability measures on $\mathbf{T}^m\times\mathbf{R}^m\times\mathbf{T}^1\cup\infty$. We don't compactify since in [10] it is shown that c-minimal measures have compact support (obviously, in [10] one has to compactify, otherwise one is not certain that $\mathcal{M}$ is not empty!) Thus to follow [10] we should consider the space of compactly supported invariant probability measures on $\mathbf{T}^m\times\mathbf{R}^m\times\mathbf{T}^1$; it is easy to show that, in the autonomous case, it makes no difference to consider this space or our $\mathcal{M}$. Moreover, in [11] there is also a slightly different definition of c-minimal orbit; indeed, in this paper the numbers a,b,c,d of (3) are restricted to be in $\tau\mathbf{Z}$. Since our lagrangian is autonomous, we can take τ any element of $\mathbf{R}^+$, and thus our definition amounts to the same of [11].

Section 1

We will denote by d the metric induced on $\mathbf{T}^m$ by the Euclidean distance on $\mathbf{R}^m$ and by $\langle\cdot,\cdot\rangle$ the standard scalar product of $\mathbf{R}^m$.

By the arguments of the appendix we can restrict our study to lagrangians already in normal form. Thus we will consider a lagrangian

$$\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q})=\frac{1}{2}|\dot{Q}|^2-\epsilon\left[V(Q,\dot{Q})+b(\dot{Q})\right]-\gamma f(Q,\dot{Q})$$

and $(\bar{k},\bar{T}) \in \mathbf{Z}^m \times [1,\infty)$ satisfying the following conditions

$$G1) \qquad \epsilon,\gamma > 0, \qquad \|b\|_{C^3}, \|V\|_{C^3}, \|f\|_{C^3} \le 1$$

$$G2) \qquad V(Q+\frac{\bar{k}}{\bar{T}}s,\dot{Q}) = V(Q,\dot{Q}) \qquad \forall (Q,\dot{Q},s) \in \mathbf{T}^m \times \mathbf{R}^m \times \mathbf{R}$$

$$G3) \qquad \forall \dot{Q} \in \mathbf{R}^m \qquad \max_Q V(Q,\dot{Q}) = 0$$

and there are p functions of class C^2, $a_1(\dot{Q}),\dots,a_p(\dot{Q})$ such that

$$\forall \dot{Q} \in \mathbf{R}^n \qquad V(Q,\dot{Q}) = 0 \Longleftrightarrow Q \in \bigcup_{i=1}^{p} \{a_i(\dot{Q}) + \frac{\bar{k}}{\bar{T}}s\}_{s\in\mathbf{R}}.$$

Moreover, there is $A > 0$ such that

$$\forall \dot{Q} \in \mathbf{R}^m \qquad \mathcal{A}(\dot{Q}) := -\frac{\partial^2}{\partial Q^2} V|_{(\frac{\bar{k}}{\bar{T}})^\perp}(a_i(\dot{Q}),\dot{Q}) \ge A \qquad i = 1,\dots,p$$

$$G4) \qquad |\frac{\bar{k}}{\bar{T}}| = 1.$$

We take the integrable part to be $\frac{1}{2}|\dot{Q}|^2$ because the fact that this fuction is the Legendre transform of itself will allow simpler formulas. We don't make any analyticity assumption on V and f but we remark that, if $\mathcal{L}_{\epsilon,\gamma}$ is the normal form of a lagrangian $\mathcal{L}$ like the ones considered in the appendix, then by (A.13) if $\bar{T}$ is not too big we can read G1-4) directly on $\mathcal{L}$, without actually performing the change of variables.

It is easy to check that, for ϵ and γ small enough, $\mathcal{L}_{\epsilon,\gamma}$ satisfies the hypotheses of [10], i.e

$$i) \qquad \frac{\partial^2}{\partial \dot{Q}^2}\mathcal{L} > 0$$

$$ii) \qquad \lim_{|\dot{Q}|\to\infty} \frac{\mathcal{L}(Q,\dot{Q})}{|\dot{Q}|} = +\infty \quad \text{uniformly in } Q$$

iii) the E-L flow of $\mathcal{L}$ is complete.

In our case, point *iii)* is a consequence of the fact that the energy surfaces are compact.

We define

$$-\alpha_\gamma(c) = \min\{\int_{\mathbf{T}^m\times\mathbf{R}^m} (\mathcal{L}_{\epsilon,\gamma} - c)\mathrm{d}\mu \ : \ \mu \in \mathcal{M}\}$$

where $\mathcal{M}$ is the space of probability measures on $\mathbf{T}^m \times \mathbf{R}^m$ which are compactly supported and invariant by the flow of $\mathcal{L}_{\epsilon,\gamma}$. The aim of this section is to find the measures which realize $\alpha_\gamma(\frac{\bar{k}}{\bar{T}})$.

Let us consider the periodic orbits of $\mathcal{L}_{\epsilon,0}$ given by

$$Q(i, \frac{1}{2}r^2)\colon t \to a_i(r\frac{\bar{k}}{\bar{T}}) + r\frac{\bar{k}}{\bar{T}}t \qquad i = 1, \dots, p.$$

If V does not depend on $\dot{Q}$ and $b \equiv 0$, it is easy to see that $Q(i, \frac{1}{2}r^2)$ is hyperbolic with stable and unstable manifolds projecting diffeomorphically onto

$$M^i_\delta = \{x \in \mathbf{T}^m \ : \ \inf_{t\in\mathbf{R}} d(x, a_i(r\frac{\bar{k}}{\bar{T}}) + t\cdot\frac{\bar{k}}{\bar{T}}) \le \delta\}.$$

We will suppose that this situation is true also for the V and b we consider. Since hyperbolic periodic orbits are stable under small perturbations of the flow, for γ small enough we can find a solution of $\mathcal{L}_{\epsilon,\gamma}$, $Q(i, h, \gamma)$ which depends smoothly on γ, having energy h and such that $Q(i, h, 0) = Q(i, h)$. This leads us to an additional hypothesis.

G5) We suppose that γ is so small that all the $Q(i, h, \gamma)$ depend C^2 on (h, γ). Moreover, the $Q(i, h, \gamma)$ are hyperbolic with stable and unstable manifolds projecting diffeomorphically onto M^i_δ. We require that, if $T(i, h, \gamma)$ is the period of $Q(i, h, \gamma)$, then

$$g^i(h) = \frac{1}{T(i,h,\gamma)}\int_0^{T(i,h,\gamma)} [\mathcal{L}_{\epsilon,\gamma}(Q(i,h,\gamma), \dot{Q}(i,h,\gamma)) - \langle\frac{\bar{k}}{\bar{T}}, \dot{Q}(i,h,\gamma)\rangle]\mathrm{d}t$$

has a unique minimum close to $h = \frac{1}{2}\left(\frac{\bar{k}}{\bar{T}}\right)^2 = \frac{1}{2}$, which we call h_i. Moreover, $\frac{1}{2}\bar{T} \le T(i, h_i, \gamma) \le 2\bar{T}$.

We set

$$Q^i_\gamma(t) = Q(i, h_i, \gamma), \qquad T^i_\gamma = T(i, h_i, \gamma)$$

and define

$$\bar{G} = \min_{i=1,\dots,p} g^i(h_i).$$

We define the set $I \subset \{1, \dots, p\}$ by

$$i \in I \quad \text{iff} \quad \bar{G} = g^i(h_i).$$

Lemma 1.1. *Let G1-5) hold. Let μ be an ergodic $\frac{\bar{k}}{T}$-minimal measure whose support is contained in $M^i_\delta \times \mathbf{R}^m$, $i \in (1, \dots, p)$. Then $i \in I$ and μ is the pull-back of the Lebesgue measure by $(Q^i_\gamma, \dot{Q}^i_\gamma)$.*

Proof. We will follow the argument of [4] and [10], based on the Weierstrass method (an exposition is in the appendix of [10] or in [5], chapters 3 and 12). We note that the stable and unstable manifolds of Q^i_γ are lagrangian submanifolds for the canonical 2-form, invariant by the E-L flow; moreover, the local stable and unstable manifolds of Q^i_γ project diffeomorphically on M^i_δ by G5). We define $\Phi \colon M^i_\delta \times \mathbf{R}^+ \to \mathbf{T}^m \times \mathbf{R}^m$ in the following way: $\Phi(x, t)$ is the evolution of the orbit on the stable manifold of Q^i_γ such that the projection of $\Phi(x, 0)$ on $\mathbf{T}^n$ is x. Let us call $\tilde{M}^i_\delta$ the universal cover of M^i_δ. Under these hypotheses, the references quoted above ensure that we can find a function $S \colon \tilde{M}^i_\delta \times \mathbf{R}^+ \to \mathbf{R}$ such that, if we define

$$\mathcal{L}^*(x, \dot{x}, t) = \mathcal{L}_{\epsilon,\gamma}(x, \dot{x}) - \langle \frac{\bar{k}}{T}, \dot{x} \rangle - g_i(h_i) - \frac{\partial}{\partial t} S(x, t) - \frac{\partial}{\partial x} S(x, t) \dot{x}$$

then $\mathcal{L}^*(\Phi(x, t), t) \equiv 0$ and $\mathcal{L}^*$ increases quadratically with the distance $|(x, \dot{x}, t) - (\Phi(x, t), t)|$. Moreover S, which is found solving a Hamilton-Jacobi equation by the method of charachteristics, satisfies

$$S(x, t) - S(x, 0) = \int_0^t [\mathcal{L}(\Phi(x, t)) - \langle \frac{\bar{k}}{T}, \Phi_2(x, t) \rangle - g_i(h_i)] \mathrm{d}t$$

where Φ_2 denotes the second component of Φ. Since $\Phi(x, t)$ is asymptotic to Q^i_γ, by the last formula we get

$$\lim_{t \to \infty} \frac{1}{t} [S(x, t) - S(x, 0)] = 0 \quad \text{uniformly in } x. \tag{1.2}$$

Since μ is c-minimal and $\bar{G} \geq \alpha(\frac{\bar{k}}{T})$ we have that

$$0 \geq \int_{\mathbf{T}^m \times \mathbf{R}^m} [\mathcal{L}_{\epsilon,\gamma} - \frac{\bar{k}}{T} - \bar{G}] \mathrm{d}\mu$$

while by the Birkhoff ergodic theorem we can find a solution Q of the E-L equation such that

$$\int_{\mathbf{T}^m \times \mathbf{R}^m} [\mathcal{L}_{\epsilon,\gamma} - \frac{\bar{k}}{T} - \bar{G}] \mathrm{d}\mu = \lim_{T \to \infty} \frac{1}{T} \int_0^T [\mathcal{L}_{\epsilon,\gamma}(Q, \dot{Q}) - \langle \frac{\bar{k}}{T}, \dot{Q} \rangle - \bar{G}] \mathrm{d}t =$$

$$\lim_{T\to\infty}\frac{1}{T}\left\{\int_0^T \mathcal{L}^*(Q,\dot{Q},t)\mathrm{d}t + S(Q(T),T) - S(Q(0),0)\right\} + g_i(h_i) - \bar{G}.$$

By (1.2) and the above two formulas we get that

$$0 \geq \int_{\mathbf{T}^m\times\mathbf{R}^m}[\mathcal{L}_{\epsilon,\gamma} - \frac{\bar{k}}{\bar{T}} - \bar{G}]\mathrm{d}\mu = \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathcal{L}^*(Q,\dot{Q},t)\mathrm{d}t + g^i(h_i) - \bar{G} \geq g^i(h_i) - \bar{G}$$

where the last inequality is a consequence of $\mathcal{L}^* \geq 0$. Since $g_i(h_i) - \bar{G} > 0$ if $i \notin I$, the above formula implies that $i \in I$.

Let us now suppose by contradiction that $(Q(0), Q(0))$ does not stay on the local stable manifold; then $\mathcal{L}^*(Q(0),\dot{Q}(0),0) = \beta > 0$. By the ergodic theorem, $(Q,\dot{Q})$ enters frequently a neighbourhood of $(Q(0),\dot{Q}(0))$ where $\mathcal{L}^* \geq \frac{\beta}{2}$; since $\mathcal{L}^* \geq 0$, we have that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathcal{L}^*(Q,\dot{Q},t)\mathrm{d}t > 0.$$

By the last two formulas we have

$$0 \geq \int_{\mathbf{T}^m\times\mathbf{R}^m}[\mathcal{L}_{\epsilon,\gamma} - \frac{\bar{k}}{\bar{T}} - \bar{G}]\mathrm{d}\mu > g_i(h_i) - \bar{G} \geq 0$$

a contradiction. Thus Q stays on the local stable manifold of Q^i_γ; since it is recurrent, we get that Q must coincide with a translate of Q^i_γ. Thus μ is the pull-back of the Lebesgue measure by $(Q^i_\gamma, \dot{Q}^i_\gamma)$ and the lemma is proven. □

Lemma 1.2. *Let G1-5) hold, let $i \in (1,\dots,p)$, let $b(\dot{Q})$ and $\omega \in \mathbf{R}$ be such that $\epsilon\|b\|_{C^3} \leq \omega < \delta$ and $Q^i_\gamma(\mathbf{R}) \subset M^i_\omega$. Then there is $B > 0$ independent on ω such that, if Q satisfies $Q([0,T]) \subset M^i_\delta$ and $Q(0), Q(T) \in M^i_\omega$, we have*

$$\int_0^T [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle c,\dot{Q}\rangle - \bar{G}]\mathrm{d}t \geq -B(\omega + \gamma\bar{T}).$$

Proof. By G5), we have that $Q(i,h,\gamma)$ depends C^2 on (h,γ); this implies that, for some $s \in \mathbf{R}$,

$$\|Q(i,h,\gamma) - Q(i,h,0)(\cdot - s)\|_{C^1} \leq C\gamma.$$

Since $\mathcal{L}_{\epsilon,0}(Q(i,h,0)(t),\dot{Q}(i,h,0)(t))$ is constant, by the Lipschitz continuity of $\mathcal{L}_{\epsilon,\gamma}$ and the above formula we deduce that

$$\sup_{s,t\in\mathbf{R}}\left|\mathcal{L}_{\epsilon,\gamma}(Q^i_\gamma(t),\dot{Q}^i_\gamma(t)) - \langle\frac{\bar{k}}{\bar{T}},\dot{Q}^i_\gamma(t)\rangle - \mathcal{L}_{\epsilon,\gamma}(Q^i_\gamma(s),\dot{Q}^i_\gamma(s)) - \langle\frac{\bar{k}}{\bar{T}},\dot{Q}^i_\gamma(s)\rangle\right| \leq C'\gamma$$

which by the mean principle implies

$$\sup_{t\in\mathbf{R}}\Big|\mathcal{L}_{\epsilon,\gamma}(Q^i_\gamma(t),\dot{Q}^i_\gamma(t))-\langle\frac{\bar{k}}{\bar{T}},\dot{Q}^i_\gamma(t)\rangle-\bar{G}\Big|\le B'\gamma,\quad |\bar{G}-\frac{1}{2}|\le B'\gamma+\epsilon\|b\|. \quad (1.3)$$

By the Lipschitz continuity of $\mathcal{L}_{\epsilon,\gamma}$, the fact that $\epsilon\|b\|\le\omega$ and the above arguments we also deduce that

$$\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q})-\langle\frac{\bar{k}}{\bar{T}},\dot{Q}\rangle-\bar{G}\ge -B''(\omega+\gamma). \quad (1.4)$$

The last formula implies the thesis if $T\le 4$. Thus we can restrict ourselves to the case $T\ge 4$, where lemma A.1 and a standard calculation show that the function

$$F_T(Q_0,Q_1)=\min\{\int_0^T[\mathcal{L}_{\epsilon,\gamma}(u,\dot{u})-\langle\frac{\bar{k}}{\bar{T}},\dot{u}\rangle-\bar{G}]\mathrm{d}t\ :\ u(0)=Q_0,u(T)=Q_1\}$$

is Lipschitz of Lipschitz constant 2 for Q_0 in a 2δ-neighbourhood of $Q(0)$ and Q_1 in a 2δ-neighbourhood of $Q(T)$. This and the boundary conditions on Q imply that we can define a function $\bar{Q}$ such that $\bar{Q}(0),\bar{Q}(T)\in Q^i_\gamma(\mathbf{R})$, $\bar{Q}([0,T])\subset M^i_\delta$ and

$$\int_0^T[\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q})-\langle\frac{\bar{k}}{\bar{T}},\dot{Q}\rangle-\bar{G}]\mathrm{d}t\ge\int_0^T[\mathcal{L}_{\epsilon,\gamma}(\bar{Q},\dot{\bar{Q}})-\langle\frac{\bar{k}}{\bar{T}},\dot{\bar{Q}}\rangle-\bar{G}]\mathrm{d}t-4\omega. \quad (1.5)$$

We define a periodic orbit $\tilde{Q}$ in the following way: on $[0,T]$ $\tilde{Q}$ coincides with $\bar{Q}$, on $[T,T_1]$ $\tilde{Q}$ coincides with the segment of Q^i_γ connecting $Q(0)$ with $Q(T)$. By (1.3) we have

$$\Big|\int_T^{T_1}[\mathcal{L}_{\epsilon,\gamma}(\tilde{Q},\dot{\tilde{Q}})-\langle\frac{\bar{k}}{\bar{T}},\dot{\tilde{Q}}\rangle-\bar{G}]\mathrm{d}t\Big|\le B'\gamma(T-T_1)\le B'\gamma T^i_\gamma\le 2B'\gamma\bar{T} \quad (1.6)$$

where the last inequality comes from G5). By (1.5) and (1.6) we get that

$$\int_0^T[\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q})-\langle\frac{\bar{k}}{\bar{T}},\dot{Q}\rangle-\bar{G}]\mathrm{d}t\ge\int_0^{T_1}[\mathcal{L}_{\epsilon,\gamma}(\tilde{Q},\dot{\tilde{Q}})-\langle\frac{\bar{k}}{\bar{T}},\dot{\tilde{Q}}\rangle-\bar{G}]\mathrm{d}t-4\omega-2B'\gamma\bar{T}.$$

Defining $\mathcal{L}^*$ and S as in lemma 1.1 we get that

$$\int_0^{T_1}[\mathcal{L}_{\epsilon,\gamma}(\tilde{Q},\dot{\tilde{Q}})-\langle\frac{\bar{k}}{\bar{T}},\dot{\tilde{Q}}\rangle-\bar{G}]\mathrm{d}t=$$

$$\int_0^{T_1}\mathcal{L}^*(\tilde{Q},\dot{\tilde{Q}},t)\mathrm{d}t+S(\tilde{Q}(T_1),T_1)-S(\tilde{Q}(0),0)+g_i(h_i)-\bar{G}.$$

Putting the last two formulas together, recalling that $\mathcal{L}^* \geq 0$ and that $g_i(h_i) - \bar{G} \geq 0$ we get

$$\int_0^T [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{\bar{T}}, \dot{Q}\rangle - \bar{G}]dt \geq S(\tilde{Q}(T_1),T_1) - S(\tilde{Q}(0),0) - 4\omega - 2B'\gamma\bar{T} =$$

$$\int_0^{T_1} [\mathcal{L}_{\epsilon,\gamma}(\Phi(\tilde{Q}(0),t)) - \langle \frac{\bar{k}}{\bar{T}}, \Phi_2(\tilde{Q}(0),t)\rangle - \bar{G}]dt - 4\omega - 2B'\gamma\bar{T}.$$

But $\Phi(\tilde{Q}(0),t)$ is simply $(Q^i_\gamma(t+\tau), \dot{Q}^i_\gamma(t+\tau))$ because $\tilde{Q}(0) = Q^i_\gamma(\tau)$; if kT^i_γ is the biggest multiple of T^i_γ smaller than T_1, we get from the above formula and the definition of $\bar{G}$ that

$$\int_0^T [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{\bar{T}}, \dot{Q}\rangle - \bar{G}]dt \geq$$

$$\int_{kT^i_\gamma}^{T_1} [\mathcal{L}_{\epsilon,\gamma}(\Phi(\tilde{Q}(0),t)) - \langle \frac{\bar{k}}{\bar{T}}, \Phi_2(\tilde{Q}(0),t)\rangle - \bar{G}]dt - 4\omega - 2B'\gamma\bar{T}.$$

Since $T_1 - kT^i_\gamma \leq 2\bar{T}$, by (1.3) we get the thesis. □

We would like to show that the only $\frac{\bar{k}}{\bar{T}}$-minimal orbits are the ones supported by the Q^i_γ, $i \in I$. By lemma 1.1, it is sufficient to show that any $\frac{\bar{k}}{\bar{T}}$-minimal measure μ has support inside some $M^i_\delta \times \mathbf{R}^m$. Thus we need some condition which makes too costly for a $\frac{\bar{k}}{\bar{T}}$-minimal orbit to go outside M^i_δ frequently; essentially, this is condition (i) of G6) below. The two following hypotheses, G6) and G7), allow us to estimate the functional along orbits generic for μ and thus to get information on the support of μ; B is the same as in lemma 1.2 and B' is as in (1.3).

$G6)$

(i) $$\exists D \in (0,1) \quad \text{such that}$$

$$-\epsilon[V(Q,\dot{Q}) + b(\dot{Q})] - \gamma f(Q,\dot{Q}) \geq D\delta^2 \quad \text{if} \quad Q \in \mathbf{T}^m \setminus \bigcup_{i=1}^p M^i_{\frac{\delta}{2}}, \quad \dot{Q} \in \mathbf{R}^m$$

(ii) $$\exists \omega \in (0, \frac{\delta}{2}) \quad \text{such that} \quad \{Q^i_\gamma\}_{t\in\mathbf{R}} \subset M^i_\omega \forall i$$

and

$$\max_{t\in\mathbf{R}} [\mathcal{L}_{\epsilon,\gamma}(Q^i_\gamma(t), \dot{Q}^i_\gamma(t)) - \langle \frac{\bar{k}}{\bar{T}}, \dot{Q}^i_\gamma(t)\rangle] \leq$$

$$\inf\{\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle\frac{\bar{k}}{T},Q\rangle \ : \ (Q,\dot{Q}) \in \left(\mathbf{T}^m \setminus \bigcup_{i=1}^{p} M^i_\omega\right) \times \mathbf{R}^m\}$$

$$(iii) \qquad B(\omega+\gamma T) < \frac{D}{4}\delta^2, \qquad 2\epsilon\|b\|_{C^3} + B'\gamma \le \frac{D}{2}\delta^2, \qquad \epsilon\|b\|_{C^3} \le \omega.$$

G7) Any two couples of M^i_δ have empty intersection.

Theorem 1.3. *Let G1-7) hold. Then the ergodic $\frac{\bar{k}}{T}$-minimal measures are the pull-back of the Lebesgue measure by $(Q^i_\gamma, \dot{Q}^i_\gamma)$, for $i \in I$. In particular, $\alpha_\gamma(\frac{\bar{k}}{T}) = -\bar{G}$.*

Proof. We have seen in lemma 1.1 that if a $\frac{\bar{k}}{T}$-minimal measure has support in $M^i_\delta \times \mathbf{R}^m$, $i \in (1,\dots,p)$, then it must coincide with one the Q^i_γ. Thus the theorem is proven if we show that any ergodic $\frac{\bar{k}}{T}$-minimal measure has support in some $M^i_\delta \times \mathbf{R}^m$; we suppose by contradiction that there is μ, ergodic and $\frac{\bar{k}}{T}$-minimal, whose support is not contained in any $M^i_\delta \times \mathbf{R}^m$.
Let Q be an orbit generic for μ; we use Q to define three classes of intervals. Each interval P_l is maximal with respect to this property

$$\exists i \in (1,\dots,p) \ : \ Q(t) \in M^i_\delta \quad \forall t \in P_l, \quad Q(\partial P_l) \subset M^i_\omega.$$

The intervals R_l satisfy

$$\exists i \in (1,\dots,p) \ : \ Q(t) \in M^i_\delta \setminus (M^i_\omega)^\circ \quad \forall t \in R_l, \quad Q(\partial R_l) \subset \partial M^i_\omega \cup \partial M^i_{\frac{\delta}{2}}$$

where X° denotes the interior of X; the intervals S_l are the maximal ones such that

$$Q(t) \notin \bigcup_{i=0}^{p} (M^i_{\frac{\delta}{2}})^\circ \quad \forall t \in S_l, \quad \exists t \in S_l \ : \ Q(t) \notin M^i_\delta.$$

It is easy to see that $\mathbf{R}$ can be partitioned into these three families of intervals, and in such a way that a P_l is followed by a R_l which in turn is followed by a S_l which is again followed by a R_l; this R_l can be followed by a S_l or by a P_l and at this point the cycle begins again.
We are supposing that the support of μ is not contained in any $M^i_\delta \times \mathbf{R}^m$; by G7) it must intersect $(\mathbf{T}^m \setminus \bigcup_{i=1}^p M^i_\delta) \times \mathbf{R}^m$. Since Q is generic for μ, it enters $(\mathbf{T}^m \setminus \bigcup_{i=1}^p M^i_\delta) \times \mathbf{R}^m$ frequently; since by lemma A.1 of the appendix its speed is bounded, we get by G7) that

$$\liminf_{R\to\infty} \frac{m(\bigcup S_l \ : \ S_l \cap [0,R] \neq \emptyset)}{R} > 0 \tag{1.7}$$

where m denotes the Lebesgue measure. Moreover, the above formula implies that none of the P_l or of the R_l can be a half-line. We assert that the following inequalities hold

$$\forall l \quad \int_{S_l} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{\bar{T}}, \dot{Q}\rangle - \bar{G}]\mathrm{d}t \geq \frac{D}{4}\delta^2 \max(1, m(S_l)) \tag{1.8}$$

$$\forall l \quad \int_{R_l} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{\bar{T}}, \dot{Q}\rangle - \bar{G}]\mathrm{d}t \geq 0 \tag{1.9}$$

$$\forall l \quad \int_{P_l} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{\bar{T}}, \dot{Q}\rangle - \bar{G}]\mathrm{d}t \geq -B(\omega + \gamma\bar{T}). \tag{1.10}$$

We postpone the proof of (1.8)-(1.10) and see how they imply the thesis. If we number the S_l according to their order on $\mathbf{R}$ and set

$$t_0 = \min S_0 \qquad t_l = \max S_l$$

we get that

$$\frac{1}{t_l - t_0}\int_{t_0}^{t_l} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{\bar{T}}, \dot{Q}\rangle - \bar{G}]\mathrm{d}t =$$

$$\frac{1}{t_l - t_0}\Big\{ \sum_{s\,:\,P_s\cap[t_0,t_l]\neq\emptyset} \int_{P_s} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{\bar{T}}, \dot{Q}\rangle - \bar{G}]\mathrm{d}t +$$

$$\sum_{s=0}^{l} \int_{S_s} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{\bar{T}}, \dot{Q}\rangle - \bar{G}]\mathrm{d}t +$$

$$\sum_{s\,:\,R_s\cap[t_0,t_l]\neq\emptyset} \int_{R_s} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{\bar{T}}, \dot{Q}\rangle - \bar{G}]\mathrm{d}t\Big\} \geq$$

$$\frac{1}{t_l - t_0}\Big[\sum_{s\,:\,P_s\cap[t_0,t_l]\neq\emptyset} \int_{P_s} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{\bar{T}}, \dot{Q}\rangle - \bar{G}]\mathrm{d}t$$

$$+ \sum_{s=0}^{l} \int_{S_s} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{\bar{T}}, \dot{Q}\rangle - \bar{G}]\mathrm{d}t\Big] \tag{1.11}$$

where the last inequality is a consequence of (1.9). If we apply (1.8) and (1.10) and recall that between two S_s there is at most one P_s, we get that

$$\frac{1}{t_l - t_0}\int_{t_0}^{t_l} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{\bar{T}}, \dot{Q}\rangle - \bar{G}]\mathrm{d}t \geq \frac{1}{t_l - t_0}\sum_{s=0}^{l}[\frac{D}{4}\delta^2 \max(1, m(S_l)) - B(\omega + \gamma\bar{T})].$$

If we now apply (1.7) and (iii) of G6) we get that

$$\liminf_{l\to\infty}\frac{1}{t_l-t_0}\int_{t_0}^{t_l}[\mathcal{L}_{\epsilon,\gamma}(Q,\dot Q)-\langle\frac{\bar k}{T},\dot Q\rangle-\bar G]\mathrm{d}t>0.$$

Since $\bar G$ is the mean action of a periodic orbit, by the Birkhoff ergodic theorem the last formula implies a contradiction with the minimality of μ.

Thus the proof of the theorem reduces to the proof of (1.8)-(1.10). It is clear by (iii) of G6) that (1.10) is simply lemma 1.2; moreover, (1.9) is a direct consequence of (ii) of G6). Thus the only inequality we have to prove is (1.8). We distinguish two cases: $m(S_l)\ge 1$ and $m(S_l)\le 1$. In the first case we have by (i) of G6) that

$$\int_{S_l}[\mathcal{L}_{\epsilon,\gamma}(Q,\dot Q)-\langle\frac{\bar k}{T},\dot Q\rangle-\bar G]\mathrm{d}t\ge\int_{S_l}(-\frac{1}{2}-\epsilon\|b\|+D\delta^2-\bar G)\mathrm{d}t$$

which by (1.3) implies

$$\int_{S_l}[\mathcal{L}_{\epsilon,\gamma}(Q,\dot Q)-\langle\frac{\bar k}{T},\dot Q\rangle-\bar G]\mathrm{d}t\ge m(S_l)[D\delta^2-2\epsilon\|b\|-B'\gamma].$$

The last formula, by (iii) of G6), implies (1.8).

In the second case we use a method of [13]. On S_l, Q runs a distance at least δ in the direction orthogonal to $\frac{\bar k}{T}$; we have that, by (i) of G6),

$$\sqrt{D}\delta^2\le\sqrt{D}\delta\int_{S_l}|\dot Q-\frac{\bar k}{T}|\mathrm{d}t\le\int_{S_l}|\dot Q-\frac{\bar k}{T}|\cdot\sqrt{-\epsilon V(Q,\dot Q)-\epsilon b(\dot Q)-\gamma f(Q,\dot Q)}\mathrm{d}t\le$$

$$\left\{\int_{S_l}[-\epsilon V(Q,\dot Q)-\epsilon b(\dot Q)-\gamma f(Q,\dot Q)]\mathrm{d}t\cdot\int_{S_l}|\dot Q-\frac{\bar k}{T}|^2\mathrm{d}t\right\}^{\frac{1}{2}}\le$$

$$\int_{S_l}[\frac{1}{2}|\dot Q-\frac{\bar k}{T}|^2-\epsilon V(Q,\dot Q)-\epsilon b(\dot Q)-\gamma f(Q,\dot Q)]\mathrm{d}t.$$

From the last formula and G4) we get

$$\int_{S_l}[\mathcal{L}_{\epsilon,\gamma}(Q,\dot Q)-\langle\frac{\bar k}{T},\dot Q\rangle]\mathrm{d}t\ge-\frac{1}{2}+\sqrt{D}\delta^2\ge-\frac{1}{2}+D\delta^2$$

where the last inequality comes from the fact that $D\in(0,1)$. On the other side, by (1.3) and (iii) of G6), we get that

$$\bar G\le-\frac{1}{2}+B'\gamma+\epsilon\|b\|.$$

Always by (iii) of G6) the last two formulas imply (1.8) and thus the thesis. □

If Q is a path on the torus, let us denote by $[Q]$ one of its lifts to $\mathbf{R}^m$; $[Q]_\perp$ and $[Q]_\parallel$ will denote respectively the components of $[Q]$ orthogonal and parallel to $\frac{\bar{k}}{T}$. The next lemma gives an estimate on the action functional of an orbit in terms of its rotation number.

Lemma 1.4. *Let G1-7) hold. Then there is $F > 0$ such that, for any $T > 0$ and any orbit Q satisfying*

$$\exists t \in [0,T] \quad \textit{such that} \quad Q(t) \notin \cup_{i=0}^{p} M_\delta^i$$

we have

$$\int_0^T [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{T}, \dot{Q}\rangle + \alpha_\gamma(\frac{\bar{k}}{T})]\mathrm{d}t \geq F \cdot \Big| [Q(T)]_\perp - [Q(0)]_\perp \Big|.$$

Proof. We divide $[0,T]$ into intervals P_l, R_l and S_l exactly as we did in theorem 2.4.
From the arguments of [13] which implied (1.8) we get that

$$S_l = [a_l, b_l] \Rightarrow \int_{S_l} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{T}, \dot{Q}\rangle + \alpha_\gamma(\frac{\bar{k}}{T})]\mathrm{d}t \geq \frac{D}{4}\delta^2 \max(|b_l - a_l|, 1). \quad (1.12)$$

As in (1.11) we get that

$$\int_0^T [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{T}, \dot{Q}\rangle + \alpha_\gamma(\frac{\bar{k}}{T})]\mathrm{d}t \geq$$
$$\sum_{l\,:\,P_l\cap[0,T]\neq\emptyset} \int_{P_l} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{T}, \dot{Q}\rangle + \alpha_\gamma(\frac{\bar{k}}{T})]\mathrm{d}t$$
$$+ \sum_{l\,:\,S_l\cap[0,T]\neq\emptyset} \int_{S_l} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{T}, \dot{Q}\rangle + \alpha_\gamma(\frac{\bar{k}}{T})]\mathrm{d}t.$$

By the last formula, (1.12) and the fact that between two S_l there is at most one P_l we get that

$$\int_0^T [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{T}, \dot{Q}\rangle + \alpha_\gamma(\frac{\bar{k}}{T})]\mathrm{d}t \geq \sum_l \left[\frac{D}{4}\delta^2 \max(|b_l - a_l|, 1) - B(\omega + \gamma\bar{T})\right].$$

By G7) and lemma A.1 of the appendix it is clear that there is $C > 0$ such that

$$\sum_l |b_l - a_l| \geq C \Big| [Q(T)]_\perp - [Q(0)]_\perp \Big|.$$

The last two formulas imply the thesis. $\square$

Section 2

This section contains the results about the behaviour of α_γ and of the minimizing orbits. We begin with some definitions.

A solution u of the E-L equation of $\mathcal{L}_{\epsilon,\gamma}$ such that

$$d(u(t), Q^i_\gamma(t-a_{-\infty})) + |\dot{u}(t) - \dot{Q}^i_\gamma(t-a_{-\infty})| \to 0 \quad \text{for} \quad t \to -\infty$$

$$d(u(t), Q^j_\gamma(t-a_{\infty})) + |\dot{u}(t) - \dot{Q}^j_\gamma(t-a_{\infty})| \to 0 \quad \text{for} \quad t \to \infty$$

for two $a_{-\infty}, a_\infty \in \mathbf{R}$, is called a heteroclinic orbit connecting Q^i_γ with Q^j_γ if $i \neq j$, a homoclinic if $i = j$. Clearly, if there is a heteroclinic connection between Q^i_γ and Q^j_γ, the two orbits must have the same energy.

We recall some of the notations and results of [10]. If $\mu \in \mathcal{M}$, then there is a unique $\rho(\mu) \in \mathbf{R}^n$ (the "rotation number" of μ) such that

$$\forall c \in \mathbf{R}^n \qquad \langle \rho(\mu), c\rangle = \int_{\mathbf{T}^m \times \mathbf{R}^m} c \mathrm{d}\mu.$$

In the integral on the right c is seen as a function $c\colon \mathbf{T}^m \times \mathbf{R}^m \to \mathbf{R}$, $c\colon (a,b) \to \langle c, b\rangle$. If μ is c-minimal, then $\rho(\mu) \in \partial\alpha_\gamma(c)$, i. e. $\rho(\mu)$ is a subgradient of α_γ in c. We will denote by β_γ the polar of α_γ. An equivalent definition of β_γ is the following

$$\beta_\gamma(\rho) = \min\{\int_{\mathbf{T}^m \times \mathbf{R}^m} \mathcal{L}_{\epsilon,\gamma} \mathrm{d}\mu \ : \ \mu \in \mathcal{M}, \rho(\mu) = \rho\}.$$

Both α_γ and β_γ are convex and superlinear. In the following, we will denote by Λ_γ the minimal supporting domain of α_γ in $\frac{\bar{k}}{T}$. To prove theorem 2.3, we will need the following two lemmas.

Lemma 2.1. *Let G1-7) hold and let us define*

$$g\colon \mathbf{R} \to \mathbf{R} \qquad g\colon r \to \alpha_\gamma(r\frac{\bar{k}}{T}).$$

Then, if ϵ and γ are small enough, g is strictly convex in the point $r = 1$.

Proof. We set

$$\tilde{V}(Q,\dot{Q}) = \epsilon V(Q,\dot{Q}) + \epsilon b(\dot{Q}) + \gamma f(Q,\dot{Q}).$$

For $i \in I$ we consider

$$Q_\lambda(t) = Q^i_\gamma((1+\lambda)t)$$

a periodic orbit of period $T = \frac{T^i_\gamma}{1+\lambda}$ and see that

$$-\alpha_\gamma((1+\lambda)\frac{\bar{k}}{T}) \leq \frac{1}{T}\int_0^T [\mathcal{L}_{\epsilon,\gamma}(Q_\lambda, \dot{Q}_\lambda) - \langle (1+\lambda)\frac{\bar{k}}{T}, \dot{Q}_\lambda\rangle]\mathrm{d}t =$$

$$\frac{1}{T}\int_0^T [\frac{1}{2}|\dot{Q}_\lambda|^2 - \langle (1+\lambda)\frac{\bar{k}}{\bar{T}}, \dot{Q}_\lambda\rangle]\mathrm{d}t - \frac{1}{T}\int_0^T \tilde{V}(Q_\lambda, \dot{Q}_\lambda)\mathrm{d}t =$$

$$\frac{(1+\lambda)^2}{T^i_\gamma}\int_0^{T^i_\gamma} [\frac{1}{2}|\dot{Q}^i_\gamma|^2 - \langle \frac{\bar{k}}{\bar{T}}, \dot{Q}^i_\gamma\rangle]\mathrm{d}t - \frac{1}{T^i_\gamma}\int_0^{T^i_\gamma} \tilde{V}(Q^i_\gamma, (1+\lambda)\dot{Q}^i_\gamma)\mathrm{d}t =$$

$$-(1+\lambda)^2\alpha_\gamma(\frac{\bar{k}}{\bar{T}}) + \frac{(1+\lambda)^2}{T^i_\gamma}\int_0^{T^i_\gamma} \tilde{V}(Q^i_\gamma, \dot{Q}^i_\gamma)\mathrm{d}t -$$

$$\frac{1}{T^i_\gamma}\int_0^{T^i_\gamma} \tilde{V}(Q^i_\gamma, (1+\lambda)\dot{Q}^i_\gamma)\mathrm{d}t.$$

If we apply to the last integral the Taylor formula and G1), G5), we get that, for γ small enough,

$$\alpha_\gamma((1+\lambda)\frac{\bar{k}}{\bar{T}}) \geq (1+\lambda)^2\alpha_\gamma(\frac{\bar{k}}{\bar{T}}) - \frac{\lambda^2+2\lambda}{T^i_\gamma}\int_0^{T^i_\gamma} \tilde{V}(Q^i_\gamma, \dot{Q}^i_\gamma)\mathrm{d}t +$$

$$\frac{\lambda}{T^i_\gamma}\int_0^{T^i_\gamma} \frac{\partial}{\partial \dot{Q}}\tilde{V}(Q^i_\gamma, \dot{Q}^i_\gamma)\dot{Q}^i_\gamma\mathrm{d}t - 2\epsilon\lambda^2.$$

By G1) we have that

$$\left|\frac{1}{T^i_\gamma}\int_0^{T^i_\gamma} \tilde{V}(Q^i_\gamma, \dot{Q}^i_\gamma)\mathrm{d}t\right| \leq (2\epsilon+\gamma)$$

while by G1) and G5) we have that, for γ small enough,

$$\left|\frac{1}{T^i_\gamma}\int_0^{T^i_\gamma} \frac{\partial}{\partial \dot{Q}}\tilde{V}(Q^i_\gamma, \dot{Q}^i_\gamma)\dot{Q}^i_\gamma\mathrm{d}t\right| \leq 2(2\epsilon+\gamma).$$

From the last three formulas we get

$$\alpha_\gamma((1+\lambda)\frac{\bar{k}}{\bar{T}}) \geq (1+\lambda)^2\alpha_\gamma(\frac{\bar{k}}{\bar{T}}) - (\lambda^2+2\lambda)(2\epsilon+\gamma) - 2\lambda(2\epsilon+\gamma) - 2\epsilon\lambda^2$$

By (1.3) we have that, for ϵ and γ small, $\alpha_\gamma(\frac{\bar{k}}{\bar{T}}) \geq \frac{1}{2} - \epsilon\|b\| - B'\gamma \geq \frac{1}{4}$. Thus

$$\alpha_\gamma((1+\lambda)\frac{\bar{k}}{\bar{T}}) \geq \alpha_\gamma(\frac{\bar{k}}{\bar{T}}) + \frac{1}{8}\lambda^2 - C\lambda$$

which implies the thesis. □

From now on we will always suppose ϵ and γ so small that lemma 3.1 holds.

Lemma 2.2. *Let G1-7) hold, let $F > 0$ be the same as in lemma 1.4 and let us denote by π the affine hyperplane $\frac{\bar{k}}{T} + \left(\frac{\bar{k}}{T}\right)^{\perp}$; then if*

$$c \in \pi \cap B(\frac{\bar{k}}{T}, \frac{F}{2}) \tag{2.1}$$

we have $\alpha_\gamma(c) = \alpha_\gamma(\frac{\bar{k}}{T})$.

Proof. Since $(c - \frac{\bar{k}}{T}) \in (\frac{\bar{k}}{T})^{\perp}$ and Q^i_γ is homotopic to Q^i_0, we have that

$$\int_0^{T^i_\gamma} \langle c - \frac{\bar{k}}{T}, \dot{Q}^i_\gamma \rangle = 0. \tag{2.2}$$

Let us now consider $\mu \in \mathcal{M}$ c-minimizing; since any element in the ergodic decomposition of μ is c-minimizing, we can suppose μ ergodic. We begin to prove the lemma when $\rho(\mu) = r\frac{\bar{k}}{T}$ for some $r \in \mathbf{R}$. We have that

$$\int_{\mathbf{T}^m \times \mathbf{R}^m} (\mathcal{L}_{\epsilon,\gamma} - c)\mathrm{d}\mu = \int_{\mathbf{T}^m \times \mathbf{R}^m} (\mathcal{L}_{\epsilon,\gamma} - \frac{\bar{k}}{T})\mathrm{d}\mu + \int_{\mathbf{T}^m \times \mathbf{R}^m} (\frac{\bar{k}}{T} - c)\mathrm{d}\mu.$$

Since $\frac{\bar{k}}{T} - c \in \left(\frac{\bar{k}}{T}\right)^{\perp}$, we have by the above formula that μ is $\frac{\bar{k}}{T}$-minimal and thus one of the Q^i_γ by theorem 1.3. From (2.2) we now get

$$-\alpha_\gamma(c) = \frac{1}{T^i_\gamma} \int_0^{T^i_\gamma} [\mathcal{L}_{\epsilon,\gamma}(Q^i_\gamma, \dot{Q}^i_\gamma) - \langle c, \dot{Q}^i_\gamma \rangle]\mathrm{d}t = -\alpha_\gamma(\frac{\bar{k}}{T})$$

which is the thesis.
We now prove that, if μ is c-minimizing, then $\rho(\mu) = r\frac{\bar{k}}{T}$. Indeed, let us suppose by contradiction that

$$\rho(\mu) = r\frac{\bar{k}}{T} + v \quad r \in \mathbf{R} \quad v \in \left(\frac{\bar{k}}{T}\right)^{\perp} \setminus \{0\}.$$

By the Birkhoff ergodic theorem we have that there is an orbit Q and $T_k \to \infty$ such that

$$\frac{[Q(T_k)]_\perp - [Q(0)]_\perp}{T_k} \to v \tag{2.3}$$

$$\frac{1}{T_k} \int_0^{T_k} [\mathcal{L}_{\epsilon,\gamma}(Q, \dot{Q}) - \langle c, \dot{Q} \rangle]\mathrm{d}t \to -\alpha_\gamma(c). \tag{2.4}$$

By (2.4) we have that

$$-\alpha_\gamma(c) = \lim_{k\to\infty} \left[\frac{1}{T_k} \int_0^{T_k} [\mathcal{L}_{\epsilon,\gamma}(Q, \dot{Q}) - \langle \frac{\bar{k}}{T}, \dot{Q} \rangle]\mathrm{d}t + \frac{1}{T_k} \int_0^{T_k} \langle \frac{\bar{k}}{T} - c, \dot{Q} \rangle \mathrm{d}t \right].$$

Since by (2.3)

$$\frac{1}{T_k}\int_0^{T_k}\langle\frac{\bar{k}}{\bar{T}}-c,\dot{Q}\rangle \mathrm{d}t \to \langle\frac{\bar{k}}{\bar{T}}-c,v\rangle$$

we have that

$$-\alpha_\gamma(c)=\lim_{k\to\infty}\frac{1}{T_k}\int_0^{T_k}[\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q})-\langle\frac{\bar{k}}{\bar{T}},\dot{Q}\rangle]\mathrm{d}t+\langle\frac{\bar{k}}{\bar{T}}-c,v\rangle\geq$$

$$\lim_{k\to\infty}\frac{1}{T_k}\int_0^{T_k}[\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q})-\langle\frac{\bar{k}}{\bar{T}},\dot{Q}\rangle]\mathrm{d}t-\frac{F}{2}|v|$$

where the last inequality is a consequence of (2.1). We now apply lemma 1.4 and get

$$-\alpha_\gamma(c)\geq-\alpha_\gamma(\frac{\bar{k}}{\bar{T}})+F\cdot|v|-\frac{F}{2}|v|>-\alpha_\gamma(\frac{\bar{k}}{\bar{T}}).$$

On the other side we have that, for $i\in I$, (2.2) implies

$$-\alpha_\gamma(c)\leq\frac{1}{T^i_\gamma}\int_0^{T^i_\gamma}[\mathcal{L}_{\epsilon,\gamma}(Q^i_\gamma,\dot{Q}^i_\gamma)-\langle c,\dot{Q}^i_\gamma\rangle]\mathrm{d}t=$$

$$\frac{1}{T^i_\gamma}\int_0^{T^i_\gamma}[\mathcal{L}_{\epsilon,\gamma}(Q^i_\gamma,\dot{Q}^i_\gamma)-\langle\frac{\bar{k}}{\bar{T}},\dot{Q}^i_\gamma\rangle]\mathrm{d}t=-\alpha_\gamma(\frac{\bar{k}}{\bar{T}}).$$

The last two formulas are in contradiction. □

Theorem 2.3. *Let G1-7) hold and let π be the affine hyperplane $\frac{\bar{k}}{\bar{T}}+\left(\frac{\bar{k}}{\bar{T}}\right)^\perp$. Then*

(i) $\Lambda_\gamma\subset\pi$ and Λ_γ has nonempty interior relative to π; we denote this interior by $\Lambda_\gamma{}^\circ$. We also have that $\alpha_\gamma|_{\Lambda_\gamma}=\alpha_\gamma(\frac{\bar{k}}{\bar{T}})$.

(ii) If $c\in\Lambda_\gamma{}^\circ$, the only ergodic c-minimal measures are those supported by the Q^i_γ, $i\in I$.

(iii) If $c\in\Lambda_\gamma{}^\circ$ and if Q is a c-minimal orbit not coinciding with one of the Q^i_γ, $i\in I$, then Q is a heteroclinic but not a homoclinic connection.

Proof. We begin to prove point (i). Lemma 2.2 implies that $\pi\cap B(\frac{\bar{k}}{\bar{T}},\frac{F}{2})\subset\Lambda_\gamma$; since Λ_γ is convex, we have that $\Lambda_\gamma\cap\pi$ has nonempty interior relative to π and that $\frac{\bar{k}}{\bar{T}}$ belongs to the interior. By lemma 2.1 the intersection of Λ_γ with the ray $r\frac{\bar{k}}{\bar{T}}$ is a point; this and the previous observation imply that $\Lambda_\gamma\subset\pi$. By lemma 2.2 and the definition of minimal supporting domain, it now follows that $\alpha_\gamma|_{\Lambda_\gamma}=\alpha_\gamma(\frac{\bar{k}}{\bar{T}})$. To prove point (ii) we note that, by point (i), when $c\in\Lambda^\circ_\gamma$, the elements of $\partial\alpha_\gamma(c)$ are all collinear to $\frac{\bar{k}}{\bar{T}}$. Thus, if $c\in\Lambda_\gamma{}^\circ$ and μ is c-minimal we have that $\rho(\mu)=r\frac{\bar{k}}{\bar{T}}$

for some $r \in \mathbf{R}$. Since $c \in \pi$, we have that $c - \frac{\bar{k}}{T} \perp \rho(\mu)$ and thus

$$\int_{\mathbf{T}^m \times \mathbf{R}^m} (\mathcal{L}_{\epsilon,\gamma} - c) \mathrm{d}\mu = \int_{\mathbf{T}^m \times \mathbf{R}^m} (\mathcal{L}_{\epsilon,\gamma} - \frac{\bar{k}}{T}) \mathrm{d}\mu.$$

This means that μ is c-minimal iff it is $\frac{\bar{k}}{T}$-minimal; but we know from theorem 1.4 that the ergodic $\frac{\bar{k}}{T}$-minimal measures are the Q^i_γ, $i \in I$.

We now prove point (iii). First of all we can suppose that Q is not contained in any M^i_δ, $i \in (1, \ldots, p)$; otherwise it would be easy to show that Q is $\frac{\bar{k}}{T}$-minimal and then by the arguments of lemma 1.1 it would follow that Q coincides with some Q^i_γ, $i \in I$.

Thus let Q be c-minimal and such that

$$Q(0) \notin \bigcup_{i=1}^{p} M^i_\delta. \tag{2.5}$$

By [11], Q will accumulate, in the future and in the past, on some c-minimal measure which by point (ii) is one of the Q^i_γ, $i \in I$. Thus there is a sequence $\{(t_k, r_k)\}_{k \in \mathbf{Z}}$ and $i, j \in I$ such that

$$\lim_{k \to -\infty} t_k = -\infty, \qquad \lim_{k \to \infty} t_k = \infty$$

$$\begin{cases} d(Q(t_k), Q^i_\gamma(r_k)) + |\dot{Q}(t_k) - \dot{Q}^i_\gamma(r_k)| \to 0 & \text{for} \quad k \to -\infty \\ d(Q(t_k), Q^j_\gamma(r_k)) + |\dot{Q}(t_k) - \dot{Q}^j_\gamma(r_k)| \to 0 & \text{for} \quad k \to \infty. \end{cases} \tag{2.6}$$

We assert that in the above formula $i \neq j$. Indeed, let us suppose by contradiction that $i = j$. We begin to note that, by (2.6), if $|k|$ is big enough, then Q remains close to $(Q^i_\gamma, \dot{Q}^i_\gamma)$ during the intervals $[t_k, t_k + 2T^i_\gamma]$; thus, by the periodicity of Q^i_γ we can also suppose

$$d(Q(t_{-k}), Q(t_k)) \to 0 \quad \text{for} \quad k \to \infty. \tag{2.7}$$

We now distinguish two cases. In the first one there is a subsequence $k' \to \infty$ such that

$$\left| [Q(t_{-k'})]_\perp - [Q(t_{k'})]_\perp \right| \to 0.$$

Since $c \in \Lambda_\gamma$ the above formula implies

$$\left| \int_{t_{-k'}}^{t_{k'}} [\mathcal{L}_{\epsilon,\gamma}(Q, \dot{Q}) - \langle c, \dot{Q} \rangle + \alpha_\gamma(c)] \mathrm{d}t - \int_{t_{-k'}}^{t_{k'}} [\mathcal{L}_{\epsilon,\gamma}(Q, \dot{Q}) - \langle \frac{\bar{k}}{T}, \dot{Q} \rangle + \alpha_\gamma(\frac{\bar{k}}{T})] \mathrm{d}t \right| \to 0$$

$$\text{for} \quad k' \to +\infty$$

and thus Q is also $\frac{\bar{k}}{T}$-minimal. Thus Q is an orbit that accumulates, in the future and in the past, to the same Q^i_γ and moreover satisfies (2.5). It is now easy to see that we can apply the arguments of theorem 1.4 to show that

$$\int_{t_{-k'}}^{t_{k'}} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle \frac{\bar{k}}{T}, \dot{Q}\rangle + \alpha_\gamma(\frac{\bar{k}}{T})]\mathrm{d}t \geq \frac{D}{2}\delta^2.$$

But Q has boundary conditions close to the boundary conditions of Q^i_γ by (2.6) and (2.7); if we take into account the Lipschitz continuity of the action, the last formula contradicts the $\frac{\bar{k}}{T}$-minimality of Q.
By (2.7), the second alternative is that

$$\lim_{k\to\infty}\inf \Big|[Q(t_{-k})]_\perp - [Q(t_k)]_\perp\Big| \geq 2\pi. \tag{2.8}$$

In this case we consider an arbitrary sequence $\{c_k\} \subset {\Lambda_\gamma}^\circ$; by point (ii) the only periodic orbits realizing $\alpha_\gamma(c_k)$ are the Q^i_γ; thus

$$0 \leq \inf\{\int_0^T [\mathcal{L}_{\epsilon,\gamma}(u,\dot{u}) - \langle c_k, \dot{u}\rangle + \alpha_\gamma(c_k)]\mathrm{d}t \ : \ u(0) = u(T), T > 0\} \leq$$

$$\liminf_{k\to\infty} \int_{t_{-k}}^{t_k} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle c_k, \dot{Q}\rangle + \alpha_\gamma(c_k)]\mathrm{d}t =$$

$$\liminf_{k\to\infty} \left[\int_{t_{-k}}^{t_k} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle c, \dot{Q}\rangle + \alpha_\gamma(c)]\mathrm{d}t + \int_{t_{-k}}^{t_k} \langle c - c_k, \dot{Q}\rangle \mathrm{d}t\right] \tag{2.9}$$

where the second inequality is a consequence of (2.7) and of the Lipschitz continuity of the functional. If we specialize c_k by

$$c - c_k = -\eta \frac{[Q(t_k)]_\perp - [Q(t_{-k})]_\perp}{|[Q(t_k)]_\perp - [Q(t_{-k})]_\perp|}$$

we see that, since $c \in {\Lambda_\gamma}^\circ$, if $\eta > 0$ is small enough, then $c_k \in {\Lambda_\gamma}^\circ \forall k$. Thus by (2.8) and (2.9) we get

$$0 \leq \liminf_{k\to\infty} \int_{t_{-k}}^{t_k} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle c, \dot{Q}\rangle + \alpha_\gamma(c)]\mathrm{d}t - 2\pi\eta$$

which we can re-write as

$$\liminf_{k\to\infty} \int_{t_{-k}}^{t_k} [\mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle c, \dot{Q}\rangle + \alpha_\gamma(c)]\mathrm{d}t$$

$$\geq \int_0^{T_\gamma^i} [\mathcal{L}_{\epsilon,\gamma}(Q_\gamma^i, \dot{Q}_\gamma^i) - \langle c, \dot{Q}_\gamma^i\rangle + \alpha_\gamma(c)]\mathrm{d}t + 2\pi\eta$$

since the second integral is 0 by point (ii). However, Q satisfies (2.7) and $Q(t_{-k})$, $Q(t_k)$ tend to the same point of Q_γ^i. By the Lipschitz continuity of the action functional, the above formula contradicts the c-minimality of Q. We have thus proven that Q cannot accumulate, in the future and in the past, on the same Q_γ^i. In an analogous way it can be shown that, if Q accumulates on Q_γ^j, say in the future, then for t big enough it will always stay inside M_δ^j; using the arguments of lemma 1.1 one can show that this implies that Q stays on the stable manifold of Q_γ^j. Analogously, one shows that Q stays on the unstable manifold of Q_γ^i and point (iii) is proven. □

We want to compare the sets Λ_γ for different values of γ; to do this we translate the point $(\frac{\bar{k}}{T}, \alpha_\gamma(\frac{\bar{k}}{T}))$ to the origin, setting

$$\tilde{\mathcal{L}}_{\epsilon,\gamma}(Q,\dot{Q}) = \mathcal{L}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle\frac{\bar{k}}{T}, \dot{Q}\rangle + \alpha_\gamma(\frac{\bar{k}}{T})$$

and defining

$$-\tilde{\alpha}_\gamma(c) = \min\{\int_{\mathbf{T}^m\times\mathbf{R}^m} (\tilde{\mathcal{L}}_{\epsilon,\gamma} - c)\mathrm{d}\mu \ : \ \mu \in \mathcal{M}\}.$$

Clearly, $\tilde{\alpha}_\gamma(0) = 0$; moreover, $(\frac{\bar{k}}{T} + c)$-minimal orbits and measures of $\mathcal{L}_{\epsilon,\gamma}$ are c-minimal orbits and measures of $\tilde{\mathcal{L}}_{\epsilon,\gamma}$, and vice-versa. If we denote by $\tilde{\Lambda}_\gamma$ the minimal supporting domain of $\tilde{\alpha}_\gamma$ containing 0, then $\tilde{\Lambda}_\gamma \subset \tilde{\pi} = \left(\frac{\bar{k}}{T}\right)^\perp$ and $\tilde{\Lambda}_\gamma$ has nonempty interior relative to $\tilde{\pi}$. Moreover, by point (i) of theorem 2.3, we have $\tilde{\Lambda}_\gamma = \{\tilde{\alpha}_\gamma = 0\}$.

Theorem 2.4. *Let G1-7) hold. Then , given $\eta > 0$, there is $\gamma_0 > 0$ such that, if $|\gamma| \leq \gamma_0$, we have that $\tilde{\Lambda}_\gamma$ is contained in a η-neighbourhood of $\tilde{\Lambda}_0$, and $\tilde{\Lambda}_0$ is contained in a η-neighbourhood of $\tilde{\Lambda}_\gamma$.*

Proof. We begin to show that

$$|\tilde{\alpha}_\gamma - \tilde{\alpha}_0| \leq |\gamma|. \tag{2.10}$$

Indeed, it is one of the results of [11] that

$$\tilde{\alpha}_\gamma(c) = \liminf_{n\to\infty} \min\{\frac{1}{n}\int_0^n [\tilde{\mathcal{L}}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle c, \dot{Q}\rangle]\mathrm{d}t \ : \ Q \in \text{A. C.}([0,n], \mathbf{T}^m),$$
$$Q(0) = Q(n)\}.$$

Since by G1)

$$\left|\frac{1}{n}\int_0^n [\tilde{\mathcal{L}}_{\epsilon,\gamma}(Q,\dot{Q}) - \langle c,\dot{Q}\rangle]\mathrm{d}t - \frac{1}{n}\int_0^n [\tilde{\mathcal{L}}_{\epsilon,0}(Q,\dot{Q}) - \langle c,\dot{Q}\rangle]\mathrm{d}t\right| \leq \gamma$$

we have (2.9).
We now prove that, for γ small enough, $\tilde{\Lambda}_\gamma$ is contained in a η-neighbourhood of $\tilde{\Lambda}_0$. Let us suppose by contradiction that this is not true. Then there are $c_k \in \mathbf{R}^m$ and $\gamma_k \to 0$ such that

$$d(c_k, \tilde{\Lambda}_0) \geq \eta \quad \text{and} \quad c_k \in \tilde{\Lambda}_{\gamma_k}, \quad \text{i.e} \quad \tilde{\alpha}_{\gamma_k}(c_k) = 0. \tag{2.11}$$

Since by [10] α_γ is superlinear we have that $\tilde{\Lambda}_0$ is bounded; we can now suppose that $\{c_k\}$ is bounded and thus that $c_k \to c$, with $d(c, \tilde{\Lambda}_0) \geq \eta$. By (2.10) and (2.11) we have that

$$0 = \tilde{\alpha}_{\gamma_k}(c_k) \to \tilde{\alpha}_0(c)$$

which implies $c \in \tilde{\Lambda}_0$, a contradiction.
We now prove that, for γ small enough, $\tilde{\Lambda}_0$ is contained in a η-neighbourhood of $\tilde{\Lambda}_\gamma$. Actually, we will prove a stronger assertion: if $c \in \tilde{\Lambda}_0^\circ$, then, for γ small enough, the only c-minimal ergodic measures for $\tilde{\mathcal{L}}_{\epsilon,\gamma}$ are those supported by the Q^i_γ. Let us suppose by contradiction that this is not true. Then there is $c \in \tilde{\Lambda}_0^\circ$, $\gamma_k \to 0$ and a sequence μ_k of ergodic measures, each c-minimal for $\mathcal{L}_{\epsilon,\gamma_k}$, such that μ_k does not coincide with any of the $Q^i_{\gamma_k}$. We note that μ_k cannot be supported in $(\cup_{i=1}^p M^i_\delta) \times \mathbf{R}^m$ since in this case we could show easily that μ_k is $\frac{k}{T}$-minimal and then, by lemma 1.1, that μ_k coincides with one of the $Q^i_{\gamma_k}$, $i \in I$. Let us consider an orbit Q_k, generic for μ_k; we have that Q_k is c-minimal for $\mathcal{L}_{\epsilon,\gamma_k}$ and that Q_k stays frequently outside $\cup_{i=1}^p M^i_\delta$. Thus after a translation in time we have that $Q_k(0) \notin \cup_{i=1}^p M^i_\delta$. By a diagonalization argument, it is easy to see that Q_k converges in $C^1_{\mathrm{loc}}(\mathbf{R}, \mathbf{T}^m)$ to Q^1, a c-minimal orbit for $\mathcal{L}_{\epsilon,0}$ such that $Q^1(0) \notin \cup_{i=1}^p M^i_\delta$; by theorem 2.3, this is a heteroclinic connection, say between Q^i_0 and Q^j_0. Actually, it is possible to show that there are

$$t^1_k \to -\infty \qquad t^2_k \to \infty$$

$$Q_k|_{[t^1_k, t^2_k]} \to Q^1 \quad \text{in} \quad C^1$$

$$\liminf_{k\to\infty} \int_{t^1_k}^{t^2_k} [\tilde{\mathcal{L}}_{\epsilon,\gamma_k}(Q_k,\dot{Q}_k) - \langle c,\dot{Q}_k\rangle + \tilde{\alpha}_{\gamma_k}(c)]\mathrm{d}t$$

$$\geq \lim_{n\to\infty} \int_{-n}^n [\tilde{\mathcal{L}}_{\epsilon,0}(Q^1,\dot{Q}^1) - \langle c,\dot{Q}^1\rangle + \tilde{\alpha}_0(c)]\mathrm{d}t. \tag{2.12}$$

To prove the latter fact we fix $\nu > 0$ and choose T and k so big that

$$d(Q^1(-T), Q^i_{\gamma_k}(\mathbf{R})) + d(Q^1(T), Q^j_{\gamma_k}(\mathbf{R})) \leq \nu.$$

Moreover, for k big enough we have that

$$d(Q_k(t_k^1), Q^i_{\gamma_k}(\mathbf{R})) + d(Q_k(t_k^2), Q^j_{\gamma_k}(\mathbf{R})) \leq \nu.$$

By (1.3) we get

$$\int_{t_k^1}^{-T} [\tilde{\mathcal{L}}_{\epsilon,\gamma}(Q^i_{\gamma_k}, \dot{Q}^i_{\gamma_k}) - \langle c, \dot{Q}^i_{\gamma_k}\rangle + \tilde{\alpha}_{\gamma_k}(c)]\mathrm{d}t$$

$$+ \int_{T}^{t_k^2} [\tilde{\mathcal{L}}_{\epsilon,\gamma}(Q^i_{\gamma_k}, \dot{Q}^i_{\gamma_k}) - \langle c, \dot{Q}^i_{\gamma_k}\rangle + \tilde{\alpha}_{\gamma_k}(c)]\mathrm{d}t \geq -2B'\gamma_k \bar{T}.$$

From the last three formulas, the minimality of Q and the Lipschitz continuity of the action functional we get that

$$\int_{t_k^1}^{t_k^2} [\tilde{\mathcal{L}}_{\epsilon,\gamma_k}(Q_k, \dot{Q}_k) - \langle c, \dot{Q}_k\rangle + \tilde{\alpha}_{\gamma_k}(c)]\mathrm{d}t \geq$$

$$-8\nu - 2B'\gamma_k\bar{T} + \int_{-T}^{T} [\tilde{\mathcal{L}}_{\epsilon,\gamma_k}(Q_k, \dot{Q}_k) - \langle c, \dot{Q}_k\rangle + \tilde{\alpha}_{\gamma_k}(c)]\mathrm{d}t.$$

Since $Q_k \to Q^1$ in C^1_{loc}, we have that

$$\int_{-T}^{T} [\tilde{\mathcal{L}}_{\epsilon,\gamma_k}(Q_k, \dot{Q}_k) - \langle c, \dot{Q}_k\rangle + \tilde{\alpha}_{\gamma_k}(c)]\mathrm{d}t \to \int_{-T}^{T} [\tilde{\mathcal{L}}_{\epsilon,0}(Q^1, \dot{Q}^1) - \langle c, \dot{Q}^1\rangle + \tilde{\alpha}_0(c)]\mathrm{d}t$$

and, from the last two formulas, we deduce (2.12).
Since Q_k passes frequently outside $\cup_{i=1}^p M^i_\delta$, we can find $T_k^2 > t_k^2$ and t_k^3 such that $Q_k(T_k^2) \notin \cup_{i=1}^p M^i_\delta$ and

$$t_k^3 - T_k^2 \to \infty$$

$$Q_k(\cdot - T_k^2)|_{[t_k^2 - T_k^2, t_k^3 - T_k^2]} \to Q^2$$

$$\liminf_{k\to\infty} \int_{t_k^2}^{t_k^3} [\tilde{\mathcal{L}}_{\epsilon,\gamma_k}(Q_k, \dot{Q}_k) - \langle c, \dot{Q}_k\rangle + \alpha_{\gamma_k}(c)]\mathrm{d}t$$

$$\geq \lim_{n\to\infty} \int_{-n}^{n} [\tilde{\mathcal{L}}_{\epsilon,0}(Q^2, \dot{Q}^2) - \langle c, \dot{Q}^2\rangle + \alpha_0(c)]\mathrm{d}t$$

and Q^2 is a heteroclinic connection for $\mathcal{L}_{\epsilon,0}$, connecting Q_0^j and Q_0^l. Thus we build a chain of heteroclinic connections, $\{Q^i\}_{i=1}^j$ for $\mathcal{L}_{\epsilon,0}$ such that the ω-limit of Q^i

coincides with the α-limit of Q^{i+1} and is one Q_0^l; moreover, since the number of different Q_0^l is smaller than p, for some $s \leq p$ we must have that the ω-limit of Q^s coincides with the α-limit of Q^1. If we apply to this heteroclinic chain the arguments of point *(iii)* of theorem 2.3, we see that

$$\sum_{i=1}^{s} \lim_{n \to \infty} \int_{-n}^{n} [\tilde{\mathcal{L}}_{\epsilon,0}(Q^i, \dot{Q}^i) - \langle c, \dot{Q}^i \rangle + \tilde{\alpha}_0(c)] \mathrm{d}t = \nu > 0.$$

But this implies that, for k big enough,

$$\int_{t_k^1}^{t_k^{s+1}} [\tilde{\mathcal{L}}_{\epsilon,\gamma_k}(Q_k, \dot{Q}_k) - \langle c, \dot{Q}_k \rangle + \tilde{\alpha}_{\gamma_k}(c)] \mathrm{d}t$$

$$= \sum_{i=1}^{s} \int_{t_k^i}^{t_k^{i+1}} [\tilde{\mathcal{L}}_{\epsilon,\gamma_k}(Q_k, \dot{Q}_k) - \langle c, \dot{Q}_k \rangle + \tilde{\alpha}_{\gamma_k}(c)] \mathrm{d}t \geq \frac{\nu}{2}.$$

But $Q_k(t_k^1)$ and $Q_k(t_k^{j+1})$ are close to the same point on $Q_{\gamma_k}^i$ and thus, by the Lipschitz continuity of the functional, the above formula contradicts the c-minimality of Q_k. □

We now specialize to $\mathbf{T}^2$, where we can get sharper results. Since quasi-integrable hamiltonian systems on $\mathbf{T}^2$ are twist maps (see for instance [2]) what follows is just a re-formulation of well-known results for twist maps. In particular, the following lemma can also be read as a consequence of the fact that, in two degrees of freedom, α is differentiable.

Lemma 2.5. *Let $m = 2$ and let G1-7) hold. Let c belong to the boundary of $\tilde{\Lambda}_\gamma$ relative to $\tilde{\pi}$. Then the only ergodic c-minimal measures are those supported by the Q_γ^i for $i \in I$. Moreover, if Q is a c-minimal orbit not coinciding with one of the Q_γ^i then Q is a homoclinic or heteroclinic connection.*

Proof. It is a well-known fact (see for instance [7], [1]) that, for $d \in \mathbf{R}^2$, two d-minimal orbits can intersect only once. Now let c be as in the hypotheses. Clearly the measures supported on the Q_γ^i, $i \in I$, are c-minimal, since the mean action is continuous in c and they are $\bar{c}$-minimal for $\bar{c}$ in the interior of $\tilde{\Lambda}_\gamma$. Let now μ be c-minimal and ergodic and Q generic for μ; by proposition 5 of [11] Q is c-minimal. Since Q and Q_γ^i intersect only once, and since we are on the two-torus, we conclude that

$$\rho(\mu) = \lim_{t \to \infty} \frac{[Q](T) - [Q](0)}{T} = r \frac{\bar{k}}{\bar{T}}.$$

We have already seen in the proof of theorem 2.3 that this implies that μ is $\frac{\bar{k}}{\bar{T}}$-minimal and thus one of the Q_γ^i.

We now prove the second assertion. It is easy to show by a comparison argument that Λ_γ has diameter smaller than $C\sqrt{\epsilon}$; indeed, if $|c-\frac{\bar{k}}{T}|\geq C\sqrt{\epsilon}$ for a suitable $C>0$, we see that the orbit $Q(t)=ct$ has mean action smaller than $-\tilde{\alpha}_\gamma(\frac{\bar{k}}{T})$. By lemma A.1 in the appendix this implies that, for c as in the hypotheses, an orbit Q which is c-minimal satisfies $\dot{Q}_\|(t)>0\forall t$. Since the Q^i_γ are the only c-minimal measures, [11] implies that there is a sequence $t_k\to\infty$ and $i\in I$ such that $(Q(t_k),\dot{Q}(t_k))$ converges to a point of $(Q^i_\gamma,\dot{Q}^i_\gamma)$. Thus we can suppose that, for k big enough, $Q(t)$ remains close to $Q^i_\gamma(\mathbf{R})$ on all the intervals $[t_k,t_k+2\bar{T}]$. Let us suppose that infinitely many of the segments $[t_k,t_k+2\bar{T}]$ lie on the left of $Q^i_\gamma(\mathbf{R})$. Then Q cannot go away from $Q^i_\gamma(\mathbf{R})$ on the left (since $\dot{Q}_\|>0$ it would intersect itself more that twice) nor on the right, since in this case, after intersecting $Q^i_\gamma(\mathbf{R})$, it should intersect itself infinitely many times to return close to $Q^i_\gamma(\mathbf{R})$. Thus for T big enough $Q([T,\infty))\subset M^i_\delta$; the arguments of lemma 1.1 now imply that Q is on the stable manifold of Q^i_γ. Analogously, one shows that Q is on the stable manifold of Q^j_γ for some $j\in I$ and the lemma is proven. □

Proposition 2.6. *Let $m=2$ and G1-7) hold. Then there is a neighbourhood U of $\tilde{\Lambda}_\gamma$ such that, if $c\in(U\setminus\tilde{\Lambda}_\gamma)\cap\tilde{\pi}$, the c-minimal orbits are approximated by bi-infinite sequences of heteroclinic or homoclinic connections.*

Proof. Let c be as in the hypotheses and let Q be a c-minimal orbit. We begin to note that Q cannot be contained in any M^i_δ, otherwise by the technique of lemma 1.1 Q would coincide with a Q^i_γ, $i\in I$. And since by [11]

$$-\tilde{\alpha}_\gamma(c)=\liminf_{T\to\infty}\frac{1}{T}\int_0^T[\tilde{\mathcal{L}}_{\epsilon,\gamma}(Q,\dot{Q})-\langle c,\dot{Q}\rangle]\mathrm{d}t$$

we would have that $\tilde{\alpha}_\gamma(c)=0$, contradicting the fact that $c\notin\tilde{\Lambda}_\gamma$.
Let us now suppose by contradiction that there is a sequence $\{c_n\}\subset\mathbf{R}^2$ such that

$$\forall n\in\mathbf{N}\quad c_n\in(U\setminus\tilde{\Lambda}_\gamma)\cap\tilde{\pi},\quad d(c_n,\tilde{\Lambda}_\gamma)\to 0 \tag{2.14}$$

and that, for each c_n, there is a c_n-minimal orbit Q_n such that

$$Q_n(0)\notin\bigcup_{i\in I}M^i_\delta \tag{2.15}$$

and $(Q_n(0),\dot{Q}_n(0))$ differs more than $\eta>0$ from the initial condition of any homoclinic or heteroclinic connection. It is easy to see by a diagonalization argument that $\{Q_n\}$ converges, up to a subsequence, to a c-minimal orbit Q, in the $C^1_{\mathrm{loc}}(\mathbf{R},\mathbf{T}^2)$ topology. By (2.14) $c\in\partial\tilde{\Lambda}_\gamma$ and thus, by lemma 2.5, Q is either one of the Q^i_γ or one of the heteroclinic or homoclinic connections between them But by (2.15) Q cannot be one of the Q^i_γ; it must thus be a homoclinic or heteroclinic connection; but this contradicts the fact that $(Q_n(0),\dot{Q}_n(0))$ differs more than $\eta>0$ from the initial condition of any homoclinic or heteroclinic connection. □

Appendix

We begin defining the norm of the perturbation. By $\mathbf{C}$ we will denote the complex field. If $P \subset \mathbf{R}^m$, we define a complex neighbourhood of $\mathbf{T}^m \times P$ by

$$U(P,R,s) = W_s(\mathbf{T}^n) \times V_R(P) \subset \mathbf{C}^m \times \mathbf{C}^m$$

where

$$W_s(\mathbf{T}^n) = \{\theta \in \mathbf{C}^m \ : \ \max_i |\mathrm{Im}\theta_i| < s\}$$

$$V_R(P) = \{I \in \mathbf{C}^m \ : \ \inf_{x\in P} |I - x| < R\}.$$

In other words, W_s is the complex strip around the torus $\mathbf{T}^n$. If f is analytic in $U(P,R,s)$ with Fourier development

$$f(Q,\dot{Q}) = \sum_{k\in\mathbf{Z}^n} a_k(\dot{Q}) e^{i\langle k,Q\rangle}$$

we define its norm by

$$|f|(P,R,s) = \sup_{\dot{Q}\in V_R(P)} \sum_{k\in\mathbf{Z}^n} |a_k(\dot{Q})| e^{|k|s}.$$

These norms are equivalent to the sup-norm in a complex strip around $\mathbf{T}^m \times P$ and in particular they bound higher order derivatives (see [12] for the precise estimates.)
We will consider lagrangians of the following form

$$L_\epsilon) \qquad \mathcal{L}(Q,\dot{Q}) = A(\dot{Q}) - h(Q,\dot{Q}) \qquad (Q,\dot{Q}) \in \mathbf{T}^m \times \mathbf{R}^m$$

with A real analytic, $0 < M \leq \frac{\partial^2}{\partial \dot{Q}^2} A(\dot{Q}) \leq M'$ $\forall \dot{Q} \in \mathbf{R}^m$ and h real analytic satisfying

$$|h|(\mathbf{R}^m, R, s) \leq \epsilon \tag{A.1}$$

for some $R, s, \epsilon > 0$.

In the following, we will consider M, M', R and s as fixed and we will take ϵ as small as we need. By C_i we will always denote a positive constant independent on ϵ. Since by $L_\epsilon)$ we have that A is convex, we have that ∇A is an invertible map; we will denote its inverse by $(\nabla A)^{-1}$.

Lemma A.1. *Let $\mathcal{L}$ satisfy $L_\epsilon)$. Then there are $\epsilon_0 > 0$ and $C_0 > 0$ such that, if $\epsilon \in [0, \epsilon_0]$, $c \in B(0,2)$ and Q is c-minimal, we have*

$$\forall t \qquad |\dot{Q}(t) - (\nabla A)^{-1}(c)| \leq C_0 \sqrt{\epsilon}.$$

Proof. The proof consists in one of the arguments of [3]. By (A.1) we have that there is $C_1 > 0$ such that

$$|h(Q,\dot{Q})| + |\frac{\partial}{\partial Q} h(Q,\dot{Q})| + |\frac{\partial}{\partial \dot{Q}} h(Q,\dot{Q})| \le C_1 \epsilon \qquad \forall (Q,\dot{Q}) \in \mathbf{T}^m \times \mathbf{R}^m \quad (A.2)$$

where we used $|\cdot|$ to denote both the absolute value of a number and the Euclidean norm in $\mathbf{R}^m$. It is a subproduct of proposition 4 of [10] that there is $\Gamma > 0$ such that, for $c \in B(0,2)$, c-minimal orbits have speed bounded by Γ. By the E-L equation, this fact and the above formula imply that, for ϵ small enough,

$$\forall t \qquad |\ddot{Q}(t)| \le C_2 \epsilon. \qquad (A.3)$$

We assert that

$$b - a > 2\pi \sqrt{\frac{M'}{C_1 \epsilon}} \Longrightarrow \left| \frac{[Q(b)] - [Q(a)]}{b-a} - (A')^{-1}(c) \right| < \frac{\sqrt{8 C_1 \epsilon}}{\sqrt{M}}. \qquad (A.4)$$

Clearly, the last formula together with (A.3) gives us the thesis. We prove (A.4): let us suppose by contradiction that, for some $b - a > 2\pi\sqrt{\frac{M'}{C_1\epsilon}}$, we have

$$\left| \frac{[Q(b)] - [Q(a)]}{b-a} - (\nabla A)^{-1}(c) \right| \ge \frac{\sqrt{8 C_1 \epsilon}}{\sqrt{M}}. \qquad (A.5)$$

Let us define, for $l \in 2\pi \mathbf{Z}^m$

$$Q_1 \colon \mathbf{R} \to \mathbf{R}^m$$

$$Q_1(t) = \begin{cases} [Q(t)] & t \le a \\ [Q(a)] + \dfrac{[Q(b)] - [Q(a)] + l}{b-a}(t-a) & a \le t \le b \\ [Q(t)] + l & b \le t. \end{cases}$$

Since $b - a > 2\pi\sqrt{\frac{M'}{C_1\epsilon}}$ we can choose $l \in 2\pi\mathbf{Z}^m$ in such a way that

$$\left| \frac{[Q(b)] - [Q(a)] + l}{b-a} - (\nabla A)^{-1}(c) \right| \le \sqrt{\frac{C_1 \epsilon}{M'}}. \qquad (A.6)$$

If we project Q_1 on $\mathbf{T}^m$ we obtain an orbit which coincides with Q for $t \notin (a,b)$; c-minimality of Q yields

$$\int_a^b [A(\dot{Q}) - \langle c, \dot{Q} \rangle - h(Q,\dot{Q})] \mathrm{d}t \le \int_a^b [A(\dot{Q}_1) - \langle c, \dot{Q}_1 \rangle - h(Q_1, \dot{Q}_1)] \mathrm{d}t. \qquad (A.7)$$

By (A.2) we get that

$$\int_a^b [A(\dot{Q}_1) - \langle c, \dot{Q}_1\rangle - h(Q_1, \dot{Q}_1)]\mathrm{d}t \leq$$

$$(b-a)\left[A\left(\frac{[Q(b)]-[Q(a)]+l}{b-a}\right) - \langle c, \frac{[Q(b)]-[Q(a)]+l}{b-a}\rangle + C_1\epsilon\right]. \qquad (A.8)$$

We also have that

$$\int_a^b [A(\dot{Q}) - \langle c, \dot{Q}\rangle - h(Q, \dot{Q})]\mathrm{d}t$$

$$\geq (b-a)\left[A\left(\frac{[Q(b)]-[Q(a)]}{b-a}\right) - \langle c, \frac{[Q(b)]-[Q(a)]}{b-a}\rangle - C_1\epsilon\right]. \qquad (A.9)$$

By (A.8) and (A.9) we get

$$\frac{1}{b-a}\left[\int_a^b [A(\dot{Q}_1) - \langle c, \dot{Q}_1\rangle - h(Q_1, \dot{Q}_1)]\mathrm{d}t - \int_a^b [A(\dot{Q}) - \langle c, \dot{Q}\rangle - h(Q, \dot{Q})]\mathrm{d}t\right] \leq$$

$$A\left(\frac{[Q(b)]-[Q(a)]+l}{b-a}\right) - \langle c, \frac{[Q(b)]-[Q(a)]+l}{b-a}\rangle$$

$$-A\left(\frac{[Q(b)]-[Q(a)]}{b-a}\right) + \langle c, \frac{[Q(b)]-[Q(a)]}{b-a}\rangle + 2C_1\epsilon.$$

From (A.5), (A.6) and the convexity hypothesis on A we get

$$\frac{1}{b-a}\left[\int_a^b [A(\dot{Q}_1) - \langle c, \dot{Q}_1\rangle - h(Q_1, \dot{Q}_1)]\mathrm{d}t - \int_a^b [A(\dot{Q}) - \langle c, \dot{Q}\rangle - h(Q, \dot{Q})]\mathrm{d}t\right] \leq$$

$$\frac{1}{2}M'\left(\frac{C_1\epsilon}{M'}\right) - \frac{1}{2}M\left(\frac{8C_1\epsilon}{M}\right) + 2C_1\epsilon < 0.$$

The last formula contradicts (A.7) and thus (A.4) holds. □

We now re-formulate a lemma of [12] in the Lagrangian framework.

Lemma A.2. *There are $C_9, C_{10}, C_{11} > 0$ such that the following holds. Let $\mathcal{L}$ satisfy L_ϵ), let $(\bar{k}, \bar{T}) \in \mathbf{Z}^m \times \mathbf{R}^+$ satisfy $|\frac{\bar{k}}{\bar{T}}| \leq 2$ and let $r \in (0, \frac{R}{2})$ be such that*

$$\epsilon \leq C_9 r^2. \qquad (A.10)$$

Then there is a real analytic, symplectic change of coordinates, $\Phi\colon (q, \dot{q}) \to (Q, \dot{Q})$ defined in

$$U((\nabla A)^{-1}\frac{\bar{k}}{\bar{T}}, \frac{r}{4}, \frac{s}{6})$$

such that in the new coordinates $(q, \dot{q})$ *we have*

$$\mathcal{L}(Q(q,\dot{q}), \dot{Q}(q,\dot{q})) = \tilde{\mathcal{L}}(q,\dot{q})$$

$$\tilde{\mathcal{L}}(q,\dot{q}) = \tilde{A}(\dot{q}) - V(q,\dot{q}) - f(q,\dot{q})$$

where

$$\frac{M}{2} \le \nabla^2 \tilde{A}(p) \le 2M' \qquad \forall p \in \mathbf{R}^m$$

$$V(q,\dot{q}) = \sum_{k \perp \bar{k}} b_k(\dot{q}) e^{i\langle k,q\rangle} \tag{A.11}$$

$$|f|((A')^{-1}\frac{\bar{k}}{\bar{T}}, \frac{r}{4}, \frac{s}{6}) \le \exp(-\frac{C_{10}}{\bar{T}r}) \tag{A.12}$$

If

$$h(Q,\dot{Q}) = \sum_{k\in\mathbf{Z}^m, k\neq 0} a_k(\dot{Q}) e^{i\langle k,Q\rangle}$$

then we have

$$|V - \sum_{k\neq 0, k\perp\bar{k}} a_k(\dot{Q}) e^{i\langle k,Q\rangle}|((A')^{-1}\frac{\bar{k}}{\bar{T}}, \frac{r}{4}, \frac{s}{6}) \le C_{11}\frac{\epsilon^2}{r^3}. \tag{A.13}$$

Proof. The Legendre transform brings $\mathcal{L}$ into the hamiltonian

$$H(Q,P) = B(P) + g(Q,P)$$

where B is the polar of A and

$$g = \sum_{k\in\mathbf{Z}^m} c_k(P) e^{i\langle k,Q\rangle}, \quad |g|(B(0,2), \frac{R}{2}, s) \le C_{12}\epsilon.$$

The above estimate follows easily from the formula of the Legendre transform and the Cauchy inequalities of [12].
A part of the proof of theorem 4 of [12] consists in showing that the domain $V_r(\frac{\bar{k}}{\bar{T}}) \cap \mathbf{R}^m$ is α, K-nonresonant modulo $\mathbf{Z}^m \cap (\frac{\bar{k}}{\bar{T}})^\perp$, with

$$\alpha = \frac{3}{2}M'Kr, \qquad K = \frac{\pi}{M'\bar{T}r}.$$

This simply means that, if $P \in V_r(\frac{\bar{k}}{\bar{T}}) \cap \mathbf{R}^m$, then

$$|\langle B'(P), k\rangle| \ge \alpha \qquad \forall k \in \mathbf{Z}^m \cap (\frac{\bar{k}}{\bar{T}})^\perp \cap B(0,K).$$

By (A.10) we can now apply the "normal form lemma" of [12] with the above constants and find a symplectic analytic change of coordinates

$$\Psi\colon (q,p) \to (Q,P)$$

with the following properties: Ψ is defined in $V(\frac{\bar{k}}{\bar{T}}, \frac{r}{2}, \frac{s}{6})$ and has an analytic generating function $\tilde{S}(q,P)$, where $q \in \mathbf{R}^m$, the universal cover of $\mathbf{T}^m$. In other words, $\mathrm{d}\tilde{S} = \tilde{c}\mathrm{d}q + \mathrm{d}S(q,P)$, with $\tilde{c} \in \mathbf{R}^m$ and S defined on $\mathbf{T}^m \times \mathbf{R}^m$. Moreover, in the new coordinates we have

$$H(Q(q,p),P(q,p)) = \tilde{H}(q,p) = \tilde{B}(p) + \tilde{g}(q,p) + \tilde{f}(q,p)$$

$$p\mathrm{d}q = P\mathrm{d}Q + \tilde{c}\mathrm{d}q + \mathrm{d}S \qquad (A.14)$$

where

$$\tilde{g}(q,p) = \sum_{k \perp \bar{k}} \tilde{c}_k(p) e^{i\langle k,q\rangle}$$

$$|\tilde{f}|(\frac{\bar{k}}{\bar{T}}, \frac{r}{2}, \frac{s}{6}) \le \exp(-\frac{C_{14}}{\bar{T}r})$$

$$|g - \sum_{k \perp \bar{k}} c_k e^{i\langle k,Q\rangle}|(\frac{\bar{k}}{\bar{T}}, \frac{r}{2}, \frac{s}{6}) \le C_{15}\frac{\epsilon^2}{r^2}$$

$$|\tilde{B} - B|(\frac{\bar{k}}{\bar{T}}, \frac{r}{2}, \frac{s}{6}) \le C_{15}\frac{\epsilon}{r}.$$

If we now apply again the Legendre transform to $\tilde{H}$, and compare it with the Legendre transform of H, which is $\mathcal{L}$, it is easy to see that we get a lagrangian $\tilde{\mathcal{L}}$ satisfying (A.11)-(A.13). □

We now explain how we are going to use the above lemma in the spirit of [8]. First of all, we restrict ourselves to diffusion far away from 0 and ∞; we will fix once for all the set $B(0,2) \setminus B(0,\frac{1}{2}) \subset \mathbf{R}^m$ as the one to which c will belong. Moreover, to simplify calculations, we will suppose that $A(\dot{Q}) = \frac{1}{2}|\dot{Q}|^2$. By the Dirichlet approximation theorem, we have that there is $C_{16} > 0$ such that

$$\forall \epsilon, Q > 0 \quad \forall c \in B(0,2) \setminus B(0,\frac{1}{2}) \quad \exists \bar{T} \in [1,Q], \exists \bar{k} \in \mathbf{Z}^n \quad \text{such that}$$

$$\left| c - \frac{\bar{k}}{\bar{T}} \right| \le C_{16} \cdot \frac{1}{\bar{T}Q^{\frac{1}{m-1}}}. \qquad (A.15)$$

In lemma A.2 we now take $r = D\frac{\epsilon^{\frac{1}{2m}}}{\bar{T}}$; we want that any $c \in B(0,2) \setminus B(0,\frac{1}{2})$ stays in $B(\frac{\bar{k}}{\bar{T}}, \frac{r}{8})$ for some $\frac{\bar{k}}{\bar{T}}$. By (A.15) this is possible and we can choose $\bar{T}$ not bigger than Q, with

$$Q = \left(\frac{C_{17}}{D\epsilon^{\frac{1}{2m}}} \right)^{m-1}.$$

However, we also want (A.10) to hold; since $\bar{T} \in [1, Q]$ this means that

$$\epsilon \le C_9 r^2 = C_9 \left(\frac{D\epsilon^{\frac{1}{2m}}}{\bar{T}} \right)^2$$

must hold for $0 \le \bar{T} \le Q$; from the above formula we thus get

$$\epsilon \le C_{18} D^{2m} \epsilon$$

which is always true for D big enough. With this choice of D, any $c \in B(0,2) \setminus B(0, \frac{1}{2})$ belongs to some $B(\frac{\bar{k}}{\bar{T}}, \frac{r}{8})$ defined as above. Moreover, by our definition of r and $\bar{T}$ we have that

$$r \ge C_{19} D^m \sqrt{\epsilon}$$

so that, taking D big enough, we can be sure by lemma A.1 that, for $c \in B(\frac{\bar{k}}{\bar{T}}, \frac{r}{8})$, the c-minimal orbits lie in $U(\frac{\bar{k}}{\bar{T}}, \frac{r}{4}, \frac{s}{6})$, where the normal form is defined. In particular, the support of the c-minimal measures lies inside $U(\frac{\bar{k}}{\bar{T}}, \frac{r}{4}, \frac{s}{6})$; thus, if μ is c-minimal, we get by (A.14) that

$$\int_{\mathbf{T}^m \times \mathbf{R}^m} (\mathcal{L} - c) \mathrm{d}\mu = \int_{\mathbf{T}^m \times \mathbf{R}^m} (\tilde{\mathcal{L}} - \Phi_*(c) + \tilde{c} + \mathrm{d}S) \mathrm{d}\tilde{\mu}$$

where $\tilde{\mu} = \Phi_* \mu$. Exactly with the same proof than in [10], section 2, it can now be shown that

$$\int_{\mathbf{T}^m \times \mathbf{R}^m} \mathrm{d}S \mathrm{d}\tilde{\mu} = 0$$

and thus we get

$$\int_{\mathbf{T}^m \times \mathbf{R}^m} (\mathcal{L} - c) \mathrm{d}\mu = \int_{\mathbf{T}^m \times \mathbf{R}^m} (\tilde{\mathcal{L}} - \Phi_*(c) + \tilde{c}) \mathrm{d}\tilde{\mu}$$

which shows that μ is c-minimal if and only if $\tilde{\mu}$ is $(\Phi_*(c) - \tilde{c})$-minimal. As a consequence of the above formula, we also have that $\alpha_{\mathcal{L}}(c) = \alpha_{\tilde{\mathcal{L}}}(\Phi_*(c) + \tilde{c})$, where $-\alpha_{\mathcal{L}}(c)$ is the minimum of $\int_{\mathbf{T}^m \times \mathbf{R}^m} (\mathcal{L} - c) \mathrm{d}\mu$ on the measures invariant for $\mathcal{L}$. Let now Q be c-minimal, and let $(q, \dot{q})$ be its image. We have by (A.14) that

$$\forall a < b \in \mathbf{R} \quad \int_a^b [\mathcal{L}(Q, \dot{Q}) - \langle c, \dot{Q} \rangle] \mathrm{d}t$$

$$= \int_a^b [\tilde{\mathcal{L}}(q, \dot{q}) - \langle \Phi_*(c) - \tilde{c}, \dot{q} \rangle] \mathrm{d}t + S(q(b), \dot{q}(b)) - S(q(a), \dot{q}(a)).$$

Thus the fact that Q is c-minimal iff q is $(\Phi_*(c) - \tilde{c})$-minimal would follow if we could restrict ourselves to variations Q_1 such that $(Q_1(a), \dot{Q}_1(a)) = (Q(d), \dot{Q}(d))$

and$(Q_1(b), \dot{Q}_1(b)) = (Q(e), \dot{Q}(e))$. It is easy to see that this apparently weaker definition of c-minimal orbit is equivalent to the one we gave in the introduction. Thus every c- minimal orbit for $c \in B(\frac{\bar{k}}{T}, \frac{r}{8})$, with r defined as above, lives in the domain of the normal form and is a $(\Phi_*(c) - \tilde{c})$-minimal orbit in the new coordinates; analogously, the $(\Phi_*(c) - \tilde{c})$-minimal orbits in the new coordinates are c-minimal orbits in the old ones.

We conclude with one last remark: in section 1, we consider lagrangians $\mathcal{L}(Q, \dot{Q})$ defined for $\dot{Q} \in \mathbf{R}^m$. This is possible because, given the normal form $\mathcal{L}$ of lemma A.2, we can extend it outside $B(\frac{\bar{k}}{T}, \frac{r}{4})$ in a C^3 way. Since by lemma A.1 c-minimal orbits for $c \in B(\frac{\bar{k}}{T}, \frac{r}{8})$ do not exit $B(\frac{\bar{k}}{T}, \frac{r}{4})$, this extension has no influence on our results.

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