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## The orbit space of the $p$ -subgroup complex is contractible

Peter Symonds

**Abstract.** We show that the quotient space of the  $p$ -subgroup complex of a finite group by the action of the group is contractible. This was conjectured by Webb.

**Mathematics Subject Classification (1991).** 20D30, 20J05, 57S17.

**Keywords.**  $p$ -subgroup, Brown-complex.

The  $p$ -subgroup complex (or Brown complex or Quillen complex) was introduced by K.S. Brown [B]. It is defined for a group  $G$  and a prime  $p$  and will be denoted by  $S_p$ . It is a simplicial complex in which the  $n$ -simplices are chains of non-trivial finite  $p$ -groups (with strict inclusions):

$$Q_0 < Q_1 < Q_2 < \cdots < Q_n,$$

with the face maps corresponding to inclusion of subchains. In other words,  $S_p$  is the geometric realisation of the poset of non-trivial  $p$ -subgroups of  $G$ .

This complex has played a prominent role in finite group theory since its introduction and the fundamental work of Quillen [Q]. For some more recent contributions see [ASe, ASm, KR, TW, W1, W2]. This paper consists of a proof of the following result.

**Theorem.** *Let  $G$  be a finite group and  $p$  a prime which divides  $|G|$ . Let  $S_p$  denote the  $p$ -subgroup complex for  $G$  (considered as a topological space). Then  $S_p/G$  is contractible.*

This was conjectured by Webb [W1, W2], who proved that  $S_p/G$  is mod- $p$  acyclic. When  $G$  is a group of Lie type in characteristic  $p$ , then  $S_p$  is equivariantly homotopy equivalent to the Tits building of  $G$ , for which the orbit space consists of just one simplex, so the conjecture was known to be true. Various cases were also considered by Thévenaz [T], who showed that the conjecture held when  $G$  was  $p$ -solvable, or when the Sylow  $p$ -subgroup was either abelian, generalized quaternion or TI.

Instead of  $S_p$  we shall consider a subcomplex  $\Delta$ , introduced by Robinson, in which the  $n$ -simplices are chains of  $p$ -groups (with strict inclusions), each one

normal in the others:

$$Q_0 \triangleleft Q_1 \triangleleft Q_2 \cdots \triangleleft Q_n, \quad Q_i \triangleleft Q_n, \quad 0 \leq i < n,$$

which we denote by  $(Q_0, \dots, Q_n)$ . This complex  $\Delta$  does not arise from a partially ordered set, but it is equivariantly homotopy equivalent to  $S_p$  (and to various other subgroup complexes too) [TW], and we actually prove that  $\Delta/G$  is contractible.

Now  $\Delta$  is a simplicial complex, but  $\Delta/G$  is naturally only a CW-complex. Each simplex of  $\Delta$  is naturally oriented, because it is a chain. This orientation is preserved by  $G$ , and so induces an orientation on  $\Delta/G$ .

*Proof.* We show that

- a)  $\pi_1(\Delta/G) = 1$   
and
- b)  $\tilde{H}_*(\Delta/G; \mathbb{Z}) = 0$ ,  
and invoke Whitehead's Theorem.

a) Let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Any class  $x \in \pi_1(\Delta/G, P)$  can be represented by a cellular loop  $s$ , i.e. a loop in the 1-skeleton which traverses each 1-cell at constant speed. This loop is determined by the sequence of directed 1-cells along which it travels.

Lift  $s$  to a cellular path  $\tilde{s}$  in  $\Delta$  starting at  $P$  and ending at some Sylow  $p$ -subgroup  $P'$ . Since  $\Delta$  is a simplicial complex,  $\tilde{s}$  is determined by the sequence of its vertices:

$$P \rightarrow Q_1 \rightarrow Q_2 \rightarrow \cdots \rightarrow Q_n \rightarrow P'.$$

There are two operations that we can perform on  $\tilde{s}$  which do not change its image in  $\pi_1(\Delta/G, P)$ .

- i) *Homotopy.* Change  $\tilde{s}$  by a homotopy in  $\Delta$  that fixes its endpoints.
- ii) *Change of Lift.* If  $g \in N_G(Q_j)$  then we can replace

$$P \rightarrow Q_1 \rightarrow Q_2 \rightarrow \cdots \rightarrow Q_j \rightarrow \cdots \rightarrow Q_n \rightarrow P'$$

by

$$P \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_{j-1} \rightarrow Q_j \rightarrow Q_{j+1}^g \rightarrow \cdots \rightarrow Q_n^g \rightarrow P'^g.$$

Define a height function  $h : \Delta \rightarrow \mathbb{R}$  by starting on the vertices with  $h(Q) = \log_p |Q|$  and then extending linearly on each simplex. Define the depth of a path  $\tilde{s}$  in  $\Delta$  to be  $d(\tilde{s}) = \min \{h(Q) \mid Q \text{ a vertex of } \tilde{s}\}$ .

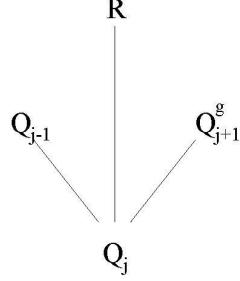
Now, for a given class  $c \in \pi_1(\Delta/G, P)$ , consider all the lifts starting at  $P$  of all the cellular paths representing  $c$ . Amongst these, restrict attention to those of maximal depth, and then choose one with the least possible number of vertices of minimal height. Call it  $\tilde{s}$ .

$$\tilde{s} : P \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_n \rightarrow P'.$$

Assume that  $c \neq 1$  so that there are at least three vertices. Let  $Q_j$  be a vertex of minimal height and let  $R$  be a Sylow  $p$ -subgroup of  $N_G(Q_j)$  containing  $Q_{j-1}$  (clearly  $Q_j \triangleleft Q_{j-1}$  since  $Q_j$  is of minimal height). Then for some  $g \in N_G(Q_j)$ ,  ${}^g R$  contains  $Q_{j+1}$ , and we can change the lift to obtain

$$s' : P \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_{j-1} \rightarrow Q_j \rightarrow Q_{j+1}^g \rightarrow \cdots \rightarrow Q_n^g \rightarrow P'^g.$$

We now have 1-simplices:



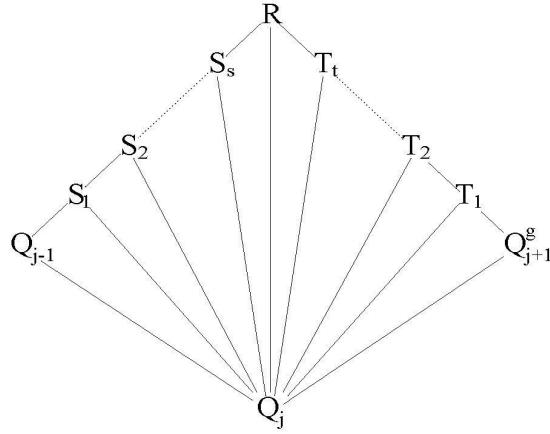
where  $Q_{j-1}, Q_{j+1}^g \leq R$  but they need not be normal. However there are sequences

$$Q_{j-1} \triangleleft S_1 \triangleleft \cdots \triangleleft S_s \triangleleft R$$

and

$$Q_{j+1}^g \triangleleft T_1 \triangleleft \cdots \triangleleft T_t \triangleleft R,$$

so we have 2-simplices:



We can now change the path  $s'$  by a homotopy to  $s''$ :

$$\begin{aligned} P &\rightarrow Q_1 \rightarrow \cdots \rightarrow Q_{j-1} \rightarrow S_1 \rightarrow \cdots \\ &\rightarrow S_s \rightarrow R \rightarrow T_t \rightarrow \cdots \rightarrow T_1 \rightarrow Q_{j+1}^g \rightarrow \cdots \rightarrow Q_n^g \rightarrow P'^g. \end{aligned}$$

But  $s''$  has fewer vertices of minimal height, a contradiction.

b) The case of homology is similar but a little more complicated. Clearly  $\tilde{H}_0(\Delta/G; \mathbb{Z}) = 0$ , i.e.  $\Delta/G$  is connected, because for every  $p$ -subgroup  $Q$  there is a sequence  $Q \triangleleft Q_1 \triangleleft \cdots \triangleleft Q_n$ , where  $Q_n$  is a Sylow  $p$ -subgroup of  $G$ . This yields a path from  $Q$  to  $Q_n$ , and all Sylow  $p$ -subgroups are conjugate. From now on we assume that  $n \geq 1$ .

Each  $n$ -cycle in the CW-homology of  $\Delta/G$  can be regarded as a linear combination  $s$  of oriented  $n$ -cells. This can be lifted to a linear combination  $\tilde{s}$  of  $n$ -simplices of  $\Delta$ ,  $\tilde{s} = \sum n_\sigma \sigma$ . We do not assume that this lifting is necessarily done in such a way that only one  $\sigma$  appears from each  $G$ -orbit.

There are two operations that we can perform on  $\tilde{s}$  which do not change its image in  $H_n(\Delta/G, \mathbb{Z})$ .

- i) *Homology.* Add a boundary (i.e. something homologous to zero).
- ii) *Change of Lift.* Any of the simplices can be replaced by another in the same  $G$ -orbit.

Define the height  $h(\sigma)$  of a simplex to be the height of its barycentre (i.e. the average height of its vertices) and its depth to be the minimum height of its codimension 1 faces. The depth of a chain is defined by  $d(\sum n_\sigma \sigma) = \min \{d(\sigma) | n_\sigma \neq 0\}$ .

Given a class  $c \in H_n(\Delta/G; \mathbb{Z})$  consider all the liftings  $\tilde{s}$  to  $\Delta$  of all cycles  $s$  representing  $c$ . Amongst these consider only those of maximal depth  $d$ , and write  $\tilde{s} = \sum n_\sigma \sigma$ . Now pick an  $\tilde{s}$  that minimizes the multiplicity,

$$m(\tilde{s}) = \sum_{d(\sigma)=d} |n_\sigma|.$$

Assume that  $c \neq 0$ , so there must be a simplex  $\rho_1$  with  $n_{\rho_1} \neq 0$  and  $d(\rho_1) = d$ . Now  $\rho_1$  has a face  $\mu = (Q_0 \triangleleft \cdots \triangleleft Q_{n-1})$  with  $h(\mu) = d$ . Let  $R_1$  be the vertex of  $\rho_1$  not in  $\mu$ . Then  $h(R_1) > h(Q_i)$  for any  $i$ , otherwise  $\rho_1$  would have a face of depth less than  $d$ , so  $\rho_1 = (\mu, R_1)$ .

Since the image of  $\tilde{s}$  in  $\Delta/G$  is a cycle, there must be another simplex  $\rho'$  with  $n_{\rho'} \neq 0$  such that some conjugate  $\rho_2 = h\rho'$  ( $h \in G$ ) also has a face  $\mu$ , and  $n_{\rho_1}$  and  $n_{\rho'}$  have opposite signs (but not necessarily the same absolute value). Again,  $\rho_2 = (\mu, R_2)$  by minimality and, by changing our attention to  $-c$  if necessary, we can assume that  $n_{\rho_1} > 0$  and  $n_{\rho'} < 0$ . Note that minimality under change of lift implies that the coefficient function  $n_\sigma$  can not take both positive and negative

values on the same orbit, so  $\rho' \neq \rho_1 \neq \rho_2$  and also  $n_{\rho_2} \leq 0$ . A change of lift alters  $\tilde{s}$  to

$$s' = \tilde{s} + \rho' - \rho_2 = \sum n_\sigma \sigma + \rho' - \rho_2 = \sum n'_\sigma \sigma,$$

where it is easy to check that  $d(s') = d(\tilde{s})$ ,  $m(s') = m(\tilde{s})$ ,  $n'_{\rho_1} > 0$  and  $n'_{\rho_2} < 0$ .

Now write

$$s' = \sum m_\sigma \sigma + \rho_1 - \rho_2 = t + \rho_1 - \rho_2,$$

so  $m_\sigma = n'_\sigma$  unless  $\sigma$  is  $\rho_1$  or  $\rho_2$ , and  $m_{\rho_1} = n'_{\rho_1} - 1 \geq 0$ ,  $m_{\rho_2} = n'_{\rho_2} + 1 \leq 0$ . Thus  $m(t) = m(\tilde{s}) - 2$ . Let  $R$  be a Sylow  $p$ -subgroup of  $\text{stab}_G(\mu)$  containing  $R_1$ . Then  $R_2 \leq R^g$  for some  $g \in \text{stab}_G(\mu)$ , so a change of lift alters  $s$  to  $s'' = t + \rho_1 - g\rho_2$ , where  $g\rho_2 = (\mu, {}^g R_2)$ .

Suppose, for the moment, that  $R_1 \neq R \neq R_2^g$ . Then we can find sequences of subgroups

$$R_1 \triangleleft S_1 \triangleleft \cdots \triangleleft S_s \triangleleft R$$

and

$${}^g R_2 \triangleleft T_1 \triangleleft \cdots \triangleleft T_t \triangleleft R.$$

Let

$$v_1 = (\mu, R_1, S_1) + (\mu, S_1, S_2) + \cdots + (\mu, S_s, R),$$

and

$$v_2 = (\mu, {}^g R_2, T_1) + (\mu, T_1, T_2) + \cdots + (\mu, T_t, R).$$

Then for  $i = 1, 2$ ,

$$(-1)^n \partial v_i = g^{i-1} \rho_i - (\mu, R) + X_i,$$

where  $X_i$  is a sum of cells which do not contain  $\mu$ , but their vertices which are not in  $\mu$  contain (as groups) all the vertices of  $\mu$ . It follows that  $X_i$  involves only cells of depth strictly greater than  $d$ , and therefore that

$$s'' = t + \rho_1 - g\rho_2 \equiv t + (-1)^n \partial(v_1 - v_2), \text{ modulo cells of depth greater than } d,$$

and a change by homology alters  $s'''$  to  $s''' = s'' - (-1)^n \partial(v_1 - v_2)$  and yields

$$s''' \equiv t, \text{ modulo cells of depth greater than } d.$$

But  $m(s'') = m(t) = m(\tilde{s}) - 2$ , a contradiction.

As for the remaining cases, if  $R_1 = R = {}^g R_2$  then  $s' = t$ . If  $R_1 \neq R = {}^g R_2$ , then  $(-1)^n \partial v_1 \equiv \rho_1 - g\rho_2$ , modulo cells of depth greater than  $d$ . The case  $R_1 = R \neq {}^g R_2$  is similar.

**Remark.** A relative version of this theorem also holds. Let  $Y$  be a set of subgroups of  $G$  that is closed under subgroups and conjugation. Let  $\Delta_Y$  be the subcomplex of  $\Delta$  in which we only allow chains of  $p$ -subgroups not in  $Y$ . Then if  $\Delta_Y$  is not empty,  $\Delta_Y/G$  is contractible.

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