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The orbit space of the p -subgroup complex is contractible

Peter Symonds

Abstract. We show that the quotient space of the p -subgroup complex of a finite group by the action of the group is contractible. This was conjectured by Webb.

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The p -subgroup complex (or Brown complex or Quillen complex) was introduced by K.S. Brown [B]. It is defined for a group G and a prime p and will be denoted by S_p . It is a simplicial complex in which the n -simplices are chains of non-trivial finite p -groups (with strict inclusions):

$$Q_0 < Q_1 < Q_2 < \cdots < Q_n,$$

with the face maps corresponding to inclusion of subchains. In other words, S_p is the geometric realisation of the poset of non-trivial p -subgroups of G .

This complex has played a prominent role in finite group theory since its introduction and the fundamental work of Quillen [Q]. For some more recent contributions see [ASe, ASm, KR, TW, W1, W2]. This paper consists of a proof of the following result.

Theorem. *Let G be a finite group and p a prime which divides $|G|$. Let S_p denote the p -subgroup complex for G (considered as a topological space). Then S_p/G is contractible.*

This was conjectured by Webb [W1, W2], who proved that S_p/G is mod- p acyclic. When G is a group of Lie type in characteristic p , then S_p is equivariantly homotopy equivalent to the Tits building of G , for which the orbit space consists of just one simplex, so the conjecture was known to be true. Various cases were also considered by Thévenaz [T], who showed that the conjecture held when G was p -solvable, or when the Sylow p -subgroup was either abelian, generalized quaternion or TI.

Instead of S_p we shall consider a subcomplex Δ , introduced by Robinson, in which the n -simplices are chains of p -groups (with strict inclusions), each one

normal in the others:

$$Q_0 \triangleleft Q_1 \triangleleft Q_2 \cdots \triangleleft Q_n, \quad Q_i \triangleleft Q_n, \quad 0 \leq i < n,$$

which we denote by (Q_0, \dots, Q_n) . This complex Δ does not arise from a partially ordered set, but it is equivariantly homotopy equivalent to S_p (and to various other subgroup complexes too) [TW], and we actually prove that Δ/G is contractible.

Now Δ is a simplicial complex, but Δ/G is naturally only a CW-complex. Each simplex of Δ is naturally oriented, because it is a chain. This orientation is preserved by G , and so induces an orientation on Δ/G .

Proof. We show that

a) $\pi_1(\Delta/G) = 1$

and

b) $\tilde{H}_*(\Delta/G; \mathbb{Z}) = 0$,

and invoke Whitehead's Theorem.

a) Let P be a Sylow p -subgroup of G . Any class $x \in \pi_1(\Delta/G, P)$ can be represented by a cellular loop s , i.e. a loop in the 1-skeleton which traverses each 1-cell at constant speed. This loop is determined by the sequence of directed 1-cells along which it travels.

Lift s to a cellular path \tilde{s} in Δ starting at P and ending at some Sylow p -subgroup P' . Since Δ is a simplicial complex, \tilde{s} is determined by the sequence of its vertices:

$$P \rightarrow Q_1 \rightarrow Q_2 \rightarrow \cdots \rightarrow Q_n \rightarrow P'.$$

There are two operations that we can perform on \tilde{s} which do not change its image in $\pi_1(\Delta/G, P)$.

- i) *Homotopy.* Change \tilde{s} by a homotopy in Δ that fixes its endpoints.
- ii) *Change of Lift.* If $g \in N_G(Q_j)$ then we can replace

$$P \rightarrow Q_1 \rightarrow Q_2 \rightarrow \cdots \rightarrow Q_j \rightarrow \cdots \rightarrow Q_n \rightarrow P'$$

by

$$P \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_{j-1} \rightarrow Q_j \rightarrow Q_{j+1}^g \rightarrow \cdots \rightarrow Q_n^g \rightarrow P'^g.$$

Define a height function $h : \Delta \rightarrow \mathbb{R}$ by starting on the vertices with $h(Q) = \log_p |Q|$ and then extending linearly on each simplex. Define the depth of a path \tilde{s} in Δ to be $d(\tilde{s}) = \min \{h(Q) \mid Q \text{ a vertex of } \tilde{s}\}$.

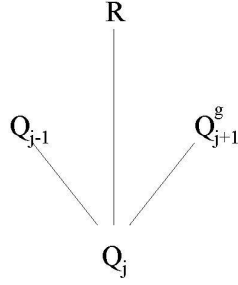
Now, for a given class $c \in \pi_1(\Delta/G, P)$, consider all the lifts starting at P of all the cellular paths representing c . Amongst these, restrict attention to those of maximal depth, and then choose one with the least possible number of vertices of minimal height. Call it \tilde{s} .

$$\tilde{s} : P \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_n \rightarrow P'.$$

Assume that $c \neq 1$ so that there are at least three vertices. Let Q_j be a vertex of minimal height and let R be a Sylow p -subgroup of $N_G(Q_j)$ containing Q_{j-1} (clearly $Q_j \triangleleft Q_{j-1}$ since Q_j is of minimal height). Then for some $g \in N_G(Q_j)$, gR contains Q_{j+1} , and we can change the lift to obtain

$$s' : P \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_{j-1} \rightarrow Q_j \rightarrow Q_{j+1}^g \rightarrow \cdots \rightarrow Q_n^g \rightarrow P'^g.$$

We now have 1-simplices:



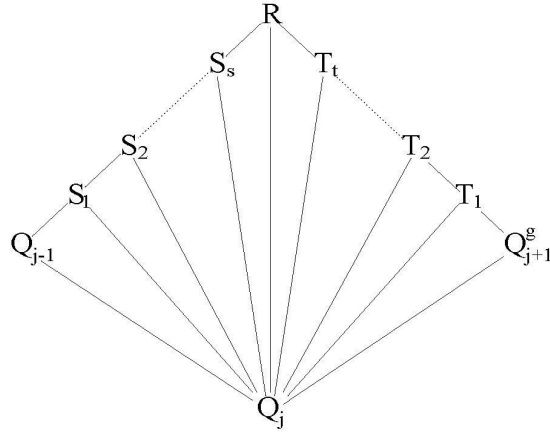
where $Q_{j-1}, Q_{j+1}^g \leq R$ but they need not be normal. However there are sequences

$$Q_{j-1} \triangleleft S_1 \triangleleft \cdots \triangleleft S_s \triangleleft R$$

and

$$Q_{j+1}^g \triangleleft T_1 \triangleleft \cdots \triangleleft T_t \triangleleft R,$$

so we have 2-simplices:



We can now change the path s' by a homotopy to s'' :

$$\begin{aligned} P &\rightarrow Q_1 \rightarrow \cdots \rightarrow Q_{j-1} \rightarrow S_1 \rightarrow \cdots \\ &\rightarrow S_s \rightarrow R \rightarrow T_t \rightarrow \cdots \rightarrow T_1 \rightarrow Q_{j+1}^g \rightarrow \cdots \rightarrow Q_n^g \rightarrow P'^g. \end{aligned}$$

But s'' has fewer vertices of minimal height, a contradiction.

b) The case of homology is similar but a little more complicated. Clearly $\tilde{H}_0(\Delta/G; \mathbb{Z}) = 0$, i.e. Δ/G is connected, because for every p -subgroup Q there is a sequence $Q \triangleleft Q_1 \triangleleft \cdots \triangleleft Q_n$, where Q_n is a Sylow p -subgroup of G . This yields a path from Q to Q_n , and all Sylow p -subgroups are conjugate. From now on we assume that $n \geq 1$.

Each n -cycle in the CW-homology of Δ/G can be regarded as a linear combination s of oriented n -cells. This can be lifted to a linear combination \tilde{s} of n -simplices of Δ , $\tilde{s} = \sum n_\sigma \sigma$. We do not assume that this lifting is necessarily done in such a way that only one σ appears from each G -orbit.

There are two operations that we can perform on \tilde{s} which do not change its image in $H_n(\Delta/G, \mathbb{Z})$.

- i) *Homology*. Add a boundary (i.e. something homologous to zero).
- ii) *Change of Lift*. Any of the simplices can be replaced by another in the same G -orbit.

Define the height $h(\sigma)$ of a simplex to be the height of its barycentre (i.e. the average height of its vertices) and its depth to be the minimum height of its codimension 1 faces. The depth of a chain is defined by $d(\sum n_\sigma \sigma) = \min \{d(\sigma) | n_\sigma \neq 0\}$.

Given a class $c \in H_n(\Delta/G; \mathbb{Z})$ consider all the liftings \tilde{s} to Δ of all cycles s representing c . Amongst these consider only those of maximal depth d , and write $\tilde{s} = \sum n_\sigma \sigma$. Now pick an \tilde{s} that minimizes the multiplicity,

$$m(\tilde{s}) = \sum_{d(\sigma)=d} |n_\sigma|.$$

Assume that $c \neq 0$, so there must be a simplex ρ_1 with $n_{\rho_1} \neq 0$ and $d(\rho_1) = d$. Now ρ_1 has a face $\mu = (Q_0 \triangleleft \cdots \triangleleft Q_{n-1})$ with $h(\mu) = d$. Let R_1 be the vertex of ρ_1 not in μ . Then $h(R_1) > h(Q_i)$ for any i , otherwise ρ_1 would have a face of depth less than d , so $\rho_1 = (\mu, R_1)$.

Since the image of \tilde{s} in Δ/G is a cycle, there must be another simplex ρ'_1 with $n_{\rho'_1} \neq 0$ such that some conjugate $\rho_2 = h\rho'_1$ ($h \in G$) also has a face μ , and n_{ρ_1} and $n_{\rho'_1}$ have opposite signs (but not necessarily the same absolute value). Again, $\rho_2 = (\mu, R_2)$ by minimality and, by changing our attention to $-c$ if necessary, we can assume that $n_{\rho_1} > 0$ and $n_{\rho'_1} < 0$. Note that minimality under change of lift implies that the coefficient function n_σ can not take both positive and negative

values on the same orbit, so $\rho' \neq \rho_1 \neq \rho_2$ and also $n_{\rho_2} \leq 0$. A change of lift alters \tilde{s} to

$$s' = \tilde{s} + \rho' - \rho_2 = \sum n_{\sigma} \sigma + \rho' - \rho_2 = \sum n'_{\sigma} \sigma,$$

where it is easy to check that $d(s') = d(\tilde{s})$, $m(s') = m(\tilde{s})$, $n'_{\rho_1} > 0$ and $n'_{\rho_2} < 0$.

Now write

$$s' = \sum m_{\sigma} \sigma + \rho_1 - \rho_2 = t + \rho_1 - \rho_2,$$

so $m_{\sigma} = n'_{\sigma}$ unless σ is ρ_1 or ρ_2 , and $m_{\rho_1} = n'_{\rho_1} - 1 \geq 0$, $m_{\rho_2} = n'_{\rho_2} + 1 \leq 0$. Thus $m(t) = m(\tilde{s}) - 2$. Let R be a Sylow p -subgroup of $\text{stab}_G(\mu)$ containing R_1 . Then $R_2 \leq R^g$ for some $g \in \text{stab}_G(\mu)$, so a change of lift alters s to $s'' = t + \rho_1 - g\rho_2$, where $g\rho_2 = (\mu, {}^gR_2)$.

Suppose, for the moment, that $R_1 \neq R \neq R_2^g$. Then we can find sequences of subgroups

$$R_1 \triangleleft S_1 \triangleleft \cdots \triangleleft S_s \triangleleft R$$

and

$${}^gR_2 \triangleleft T_1 \triangleleft \cdots \triangleleft T_t \triangleleft R.$$

Let

$$v_1 = (\mu, R_1, S_1) + (\mu, S_1, S_2) + \cdots + (\mu, S_s, R),$$

and

$$v_2 = (\mu, {}^gR_2, T_1) + (\mu, T_1, T_2) + \cdots + (\mu, T_t, R).$$

Then for $i = 1, 2$,

$$(-1)^n \partial v_i = g^{i-1} \rho_i - (\mu, R) + X_i,$$

where X_i is a sum of cells which do not contain μ , but their vertices which are not in μ contain (as groups) all the vertices of μ . It follows that X_i involves only cells of depth strictly greater than d , and therefore that

$$s'' = t + \rho_1 - g\rho_2 \equiv t + (-1)^n \partial(v_1 - v_2), \text{ modulo cells of depth greater than } d,$$

and a change by homology alters s''' to $s''' = s'' - (-1)^n \partial(v_1 - v_2)$ and yields

$$s''' \equiv t, \text{ modulo cells of depth greater than } d.$$

But $m(s''') = m(t) = m(\tilde{s}) - 2$, a contradiction.

As for the remaining cases, if $R_1 = R = {}^gR_2$ then $s' = t$. If $R_1 \neq R = {}^gR_2$, then $(-1)^n \partial v_1 \equiv \rho_1 - g\rho_2$, modulo cells of depth greater than d . The case $R_1 = R \neq {}^gR_2$ is similar.

Remark. A relative version of this theorem also holds. Let Y be a set of subgroups of G that is closed under subgroups and conjugation. Let Δ_Y be the subcomplex of Δ in which we only allow chains of p -subgroups not in Y . Then if Δ_Y is not empty, Δ_Y/G is contractible.

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