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Commentarii Mathematici Helvetici

An example of an immersed complete genus one minimal surface in $\mathbb{R}^3$ with two convex ends

Barbara Nelli

Abstract. We prove the existence of a compact genus one immersed minimal surface M, whose boundary is the union of two immersed locally convex curves lying in parallel planes. M is a part of a complete minimal surface with two finite total curvature ends.

Mathematics Subject Classification (1991). 53A10, 53C42.

Keywords. Minimal surface, convex boundary, Weierstrass representation, elliptic functions.

1. Introduction

In 1978 Meeks conjectured that a connected minimal surface bounded by two convex curves in two parallel planes is topologically an annulus; hence it has genus zero. The conjecture has never been proved and the most general result, due to Schoen, is the following.

Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be any boundary consisting of two Jordan curves in parallel planes; assume that Γ is invariant by reflection through two planes P_1, P_2 orthogonal to the planes of the Γ_i and that both P_1 and P_2 divide Γ into pieces which are graphs with locally bounded slope over the dividing plane. Then any minimal surface spanning Γ is topologically an annulus and is an embedded surface meeting each parallel plane between the planes of the Γ_i in smooth Jordan curves.

In particular, if Γ_1 and Γ_2 are circles such that the line joining their centers is perpendicular to the planes in which they lie, then M is a catenoid (cf. [Sc]).

In 1991, Meeks and White studied the space of minimal annuli bounded by convex curves in parallel planes (cf. [MW]).

In this paper we prove the existence of a compact genus one immersed minimal surface M, whose boundary is the union of two immersed locally convex curves lying in parallel planes. In fact M is a part of a complete minimal surface with two finite total curvature ends.

The method we use to construct our surface is the following.

It is well known that a minimal surface of genus g and k ends can be described by its Weierstrass representation, that is a triple $\{\overline{R} \setminus [p_1, \dots, p_k], \omega = fdz, g\}$, where $\overline{R}$ is a compact Riemann surface of genus g, $p_1, \dots, p_k$ are points in $\overline{R}$, ω is a holomorphic differential on R and g is a meromorphic function on R.

In our setting $\overline{R}$ is a torus, so we can choose f and g to be elliptic functions. For references about the use of elliptic functions in the Weierstrass representation, see [A], [A1], [C], [C1], [R]).

I would like to thank Professor Harold Rosenberg for his continual encouragement and advice.

2. Statement of results

Consider the lattice $L(1, i)$ on $\mathbb{C}$ generated by 1 and i and let T^2 be the torus $\mathbb{C}/L(1, i)$. Let $\pi : \mathbb{C} \longrightarrow T^2$ be the standard projection to the quotient and set $p_o = \pi(0), p_1 = \pi(\frac{1}{2}), p_2 = \pi(\frac{1+i}{2})$ and $p_3 = \pi(\frac{i}{2})$. Finally, let $\wp$ be the Weierstrass function associated to the lattice $L(1, i)$ and $\wp'$ its derivative.

Theorem 2.1. *Let $f, g : T^2 \setminus \{p_o, p_2\} \longrightarrow \mathbb{C}$ be the two meromorphic functions defined by*

$$f = \wp^2 \quad g = \frac{\alpha \wp'}{\wp^3}$$

where α is a real constant depending only on $L(1, i)$ and $\wp$.

Then $\{T^2 \setminus [p_o, p_2], fdz, g\}$ is the Weierstrass representation of a complete genus one immersed minimal surface M with finite total curvature.

Remark 2.2. The ends of M cannot be embedded. In fact, if a complete finite total curvature minimal surface has two embedded ends, it is a catenoid (cf. [Sc]).

The functions f and g extend meromorphically to T^2 and we have $g(p_o) = 0$ and $g(p_2) = \infty$. Hence the limit normal vector at both ends of M is vertical. Then we have the following result.

Theorem 2.3. *There exists a positive constant $c \in \mathbb{R}$ such that $M \cap \{|x_3| \leq c\}$ is a compact genus one immersed minimal surface having the property that each of the boundary curves $M \cap \{x_3 = \pm c\}$ is a compact locally convex immersed curve.*

3. Proof of the theorems

We list some useful classical properties of the function $\wp$ (cf. [B], [WW]).

By abuse of notation, we often identify points of $\mathbb{C}$ with points of T^2. Let $'$ be the differentiation with respect to the variable $z \in \mathbb{C}$.

(i) $\wp$ is even and $\wp'$ is odd. We have $\wp(z),\ \wp'(z) \in \mathbb{R}$ when $z \in \mathbb{R}$, $\wp(p_1) = e_1 \in \mathbb{R}^*_+$, $\wp(p_2) = 0$ and $\wp(p_3) = -e_1$.

The following identities hold:
(ii) $(\wp')^2 = 4\wp(\wp^2 - e_1^2)$, $\wp'' = 2(3\wp^2 - e_1^2)$.
(iii) $\wp(z+p_1) = \dfrac{e_1(\wp(z)+e_1)}{\wp(z)-e_1}$, $\wp(z+p_3) = \dfrac{e_1(\wp(z)-e_1)}{\wp(z)+e_1}$, $\wp(z+p_2) = -\dfrac{e_1^2}{\wp(z)}$.
(iv) $\wp'(z+p_2) = e_1^2 \dfrac{\wp'(z)}{\wp(z)^2}$.
(v) $\wp(iz) = -\wp(z)$, $\wp'(iz) = i\wp'(z)$.
(vi) The local expansion of $\wp$ and $\wp'$ around p_o is

$$\wp(z) = \frac{1}{z^2} + \frac{e_1^2}{5} z^2 + O(z^6),$$

$$\wp'(z) = -\frac{2}{z^3} + \frac{2e_1^2}{5} z + O(z^5).$$

Proof of Theorem 2.1. It is sufficient to prove that the following conditions are satisfied.
(A) z is a pole of order m of $g \iff z$ is a zero of order $2m$ of f.
(B) $\int_\gamma (1+|g|^2)|f| = \infty$ for every divergent path γ in M.
(C) $\operatorname{Re} \int_\gamma fg = 0$ and $\int_\gamma fg^2 = \overline{\int_\gamma f}$ for every closed path in M.

Zeros and poles of f, g, fg, fg^2 in a fundamental region are as in figure 1.

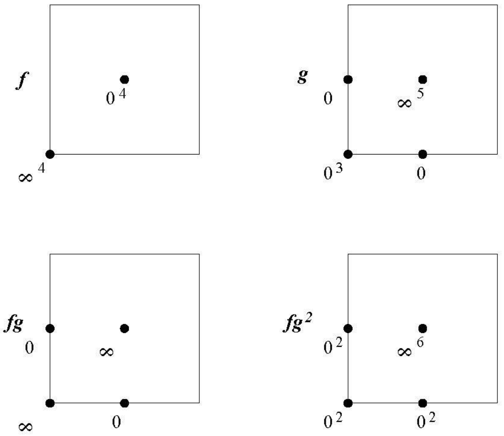

Figure 1.

The function g does not have poles in $T^2 \setminus \{p_o, p_2\}$, hence condition (A) is satisfied.

The expression of the metric on M in terms of $\wp$ is

$$ds = \left(1 + \alpha^2 \frac{|\wp'|^2}{|\wp|^6}\right) |\wp|^2$$

hence the metric is complete at the ends and condition (B) is satisfied.

We must verify (C) on paths that are not homologous to 0 in $T^2 \setminus \{p_o, p_2\}$, i.e. paths around p_o and p_2 and paths that generate the homology of T^2. Denote by $\alpha(p_o)$ and $\alpha(p_2)$ any closed path around p_o and p_2 respectively, and by γ_1 and γ_2 the following paths generating the homology of T^2 :

$$\gamma_1(t) = \frac{i}{4} + t \quad t \in [0,1]$$

$$\gamma_2(t) = \frac{1}{4} + it \quad t \in [0,1]$$

The functions f and fg^2 are even, so they have no residue at p_o, i.e.

$$\int_{\alpha(p_o)} fg^2 = \int_{\alpha(p_o)} f = 0$$

Furthermore

$$\mathrm{Re} \int_{\alpha(p_o)} fg = \mathrm{Re} \int_{\alpha(p_o)} \frac{\alpha \wp'}{\wp} = \mathrm{Re} \left[\mathrm{Res}_{p_o} (2\pi i \alpha \frac{\wp'}{\wp}) \right]$$

By the local expansion of $\wp$ and $\wp'$ around 0 we have that $\mathrm{Res}_{p_o}(2\pi i \alpha \frac{\wp'}{\wp}) = -4\pi i \alpha$, hence for $\alpha \in \mathbb{R}$ we have

$$\mathrm{Re} \int_{\alpha(p_o)} fg = 0$$

By (iii) and (iv) we have

$$f(z + p_2) = \frac{e_1^4}{\wp^2(z)},$$

$$fg^2(z + p_2) = \frac{\alpha^2}{e_1^4} (\wp'(z))^2.$$

Hence $f(z + p_2)$ and $fg^2(z + p_2)$ are even functions of z and this gives

$$\int_{\alpha(p_2)} fg^2 = \int_{\alpha(p_2)} f = 0.$$

By (iii) and (iv) we have

$$fg(z+p_2) = -\alpha\frac{\wp'(z)}{\wp(z)}.$$

Hence, by the computation above, for $\alpha \in \mathbb{R}$ we have

$$\mathrm{Re}\int_{\alpha(p_2)} fg = 0.$$

Now we verify (C) over γ_1 and γ_2. We have

$$\mathrm{Re}\int_{\gamma_i} fg = \mathrm{Re}\int_{\gamma_i} \alpha\frac{\wp'}{\wp} = \alpha[\ln|\wp|]_{\gamma_i(0)}^{\gamma_i(1)} = 0$$

by periodicity of $\wp$, as α is real.

Integral of f over γ_1 : by Cauchy theorem and periodicity we can move γ_1 up to the segment from p_3 to p_3+1, hence

$$\int_{\gamma_1} f = \int_0^1 f(p_3+t)dt = \int_0^1 e_1^2\frac{(\wp(t)-e_1)^2}{(\wp(t)+e_1)^2}dt$$

where the last equality is given by (iii).

Integral of f over γ_2 : we can move γ_2 to the vertical segment from p_1 to p_1+i, hence by (iii) and (iv)

$$\int_{\gamma_2} f = \int_0^1 f(p_1+t)idt = i\int_0^1 e_1^2\frac{(\wp(t)-e_1)^2}{(\wp(t)+e_1)^2}dt.$$

Integral of fg^2 over γ_1 : we can move γ_1 down to the real segment from p_o to p_o+1, hence

$$\int_{\gamma_1} fg^2 = \int_0^1 f(t)g^2(t)dt = \int_0^1 \alpha^2\frac{\wp'(t)^2}{\wp(t)^4}dt.$$

Integral of fg^2 over γ_2 : we can move γ_2 to the vertical segment from p_o to p_o+i, hence

$$\int_{\gamma_2} fg^2 = \int_0^1 f(it)g^2(it)idt = -i\int_0^1 \alpha^2\frac{\wp'(t)^2}{\wp(t)^4}dt.$$

Then α must satisfy

$$\alpha^2\int_0^1 \frac{\wp'(t)^2}{\wp(t)^4}dt = \int_0^1 e_1^2\frac{(\wp(t)-e_1)^2}{(\wp(t)+e_1)^2}dt.$$

If $t \in \mathbb{R}$ we have $\wp(t)$, $\wp'(t) \in \mathbb{R}$, hence the two integrals involved in the definition of α are positive real numbers. Furthermore they are convergent, so $\alpha \in \mathbb{R}$.

Since g and f extend meromorphically to T^2, M has finite total curvature. $\square$

Before proving Theorem 2.3 we need the following lemma.

Lemma 3.1. *Consider a minimal surface M with Weierstrass representation given by $\{f dz, g\}$ such that the vector corresponding to $g(0)$ is parallel to the x_3-axis. Then the planar curvature of the intersection curves of M with the horizontal planes is*

$$k = \frac{1}{|f^2 g|(1+|g|^2)} \mathrm{Re}\left(\overline{fg}\frac{g'}{g}\right).$$

Proof. Let $\theta = \arg g$ and s be the arc length of the curve $M \cap \{x_3 = c\}$; then $k(s) = \dfrac{d\theta}{ds}$. As $\arg g = \mathrm{Im}(\ln g)$, we have

$$k(s) = \frac{d\mathrm{Im}\ln g}{ds} = \mathrm{Im}(\frac{d\ln g}{dz}\frac{dz}{ds}) = \mathrm{Im}(\frac{g'}{g}\frac{dz}{ds}).$$

By the Weierstrass representation we have

$$x_3 = \mathrm{Re}\int fg.$$

Hence, on the curve $M \cap \{x_3 = c\}$, $\dfrac{dz}{ds}$ must satisfy

$$0 = \frac{d}{ds}\mathrm{Re}\int fg = \frac{1}{2}\mathrm{Re}(fg\frac{dz}{ds}).$$

By a straightforward computation we obtain

$$\frac{dz}{ds} = \frac{i}{(1+|g|^2)|f|}\frac{\overline{fg}}{|fg|}.$$

Then

$$k = \mathrm{Im}(\frac{i}{(1+|g|^2)|f|}\frac{\overline{fg}}{|fg|}\frac{g'}{g}) = \frac{1}{|f^2 g|(1+|g|^2)}\mathrm{Re}\left(\overline{fg}\frac{g'}{g}\right).$$

$\square$

Proof of Theorem 2.3. The third coordinate of M is given by

$$x_3 = \mathrm{Re}\int fg = \mathrm{Re}\int \alpha\frac{\wp'}{\wp} = \alpha\ln|\wp|,$$

since α is real. Then, any level curve is given by $|\wp| = c$ and next to the ends this is a compact immersed curve with only one component.

By a straightforward computation, we obtain

$$g'(z) = 2\alpha \left[\frac{5e_1^2 - 3\wp(z)^2}{\wp(z)^3} \right],$$

$$\frac{g'(z)}{g(z)} = \frac{2(5e_1^2 - 3\wp(z)^2)}{\wp'(z)},$$

$$\overline{f(z)g(z)} = \overline{\alpha}\frac{\overline{\wp'(z)}}{\overline{\wp(z)}}.$$

By using the expansion of $\wp$ and $\wp'$ at p_o we have

$$\overline{f(z)g(z)} \sim -2\frac{\overline{\alpha}}{\overline{z}},$$

$$\frac{g'(z)}{g(z)} \sim \frac{3}{z},$$

where $\sim$ denotes equality between the principal parts of the functions in a neighborhood of zero. Hence the sign of the curvature of the level curve next to the end p_o is the same as the sign of

$$\mathrm{Re}\left(\frac{-6\overline{\alpha}}{\overline{z}z}\right) = -\frac{6\alpha}{|z|^2},$$

α being real.

We use the equality

$$\overline{f(z+p_2)g(z+p_2)} = -\overline{f(z)g(z)}$$

and the fact that in a neighborhood of zero we have

$$\frac{g'(z+p_2)}{g(z+p_2)} = \frac{2(5\wp(z)^2 - 3e_1^2)}{\wp'(z)} \sim -\frac{5}{z},$$

to conclude that the sign of the curvature of the level curve next to the end p_2 is the same as the sign of

$$\mathrm{Re}\left(\frac{-10\overline{\alpha}}{\overline{z}z}\right) = -\frac{10\alpha}{|z|^2}$$

since α is real.

Thus, if we choose a negative α, the level curves are locally convex next to the two ends of M. ☐

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