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An example of an immersed complete genus one minimal surface in \mathbb{R}^3 with two convex ends

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Abstract. We prove the existence of a compact genus one immersed minimal surface M , whose boundary is the union of two immersed locally convex curves lying in parallel planes. M is a part of a complete minimal surface with two finite total curvature ends.

Mathematics Subject Classification (1991). 53A10, 53C42.

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1. Introduction

In 1978 Meeks conjectured that a connected minimal surface bounded by two convex curves in two parallel planes is topologically an annulus; hence it has genus zero. The conjecture has never been proved and the most general result, due to Schoen, is the following.

Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be any boundary consisting of two Jordan curves in parallel planes; assume that Γ is invariant by reflection through two planes P_1, P_2 orthogonal to the planes of the Γ_i and that both P_1 and P_2 divide Γ into pieces which are graphs with locally bounded slope over the dividing plane. Then any minimal surface spanning Γ is topologically an annulus and is an embedded surface meeting each parallel plane between the planes of the Γ_i in smooth Jordan curves.

In particular, if Γ_1 and Γ_2 are circles such that the line joining their centers is perpendicular to the planes in which they lie, then M is a catenoid (cf. [Sc]).

In 1991, Meeks and White studied the space of minimal annuli bounded by convex curves in parallel planes (cf. [MW]).

In this paper we prove the existence of a compact genus one immersed minimal surface M , whose boundary is the union of two immersed locally convex curves lying in parallel planes. In fact M is a part of a complete minimal surface with two finite total curvature ends.

The method we use to construct our surface is the following.

It is well known that a minimal surface of genus g and k ends can be described

by its Weierstrass representation, that is a triple $\{\overline{R} \setminus [p_1, \dots, p_k], \omega = f dz, g\}$, where \overline{R} is a compact Riemann surface of genus g , p_1, \dots, p_k are points in \overline{R} , ω is a holomorphic differential on R and g is a meromorphic function on R .

In our setting \overline{R} is a torus, so we can choose f and g to be elliptic functions. For references about the use of elliptic functions in the Weierstrass representation, see [A], [A1], [C], [C1], [R]).

I would like to thank Professor Harold Rosenberg for his continual encouragement and advice.

2. Statement of results

Consider the lattice $L(1, i)$ on \mathbb{C} generated by 1 and i and let T^2 be the torus $\mathbb{C}/L(1, i)$. Let $\pi : \mathbb{C} \rightarrow T^2$ be the standard projection to the quotient and set $p_o = \pi(0)$, $p_1 = \pi(\frac{1}{2})$, $p_2 = \pi(\frac{1+i}{2})$ and $p_3 = \pi(\frac{i}{2})$. Finally, let \wp be the Weierstrass function associated to the lattice $L(1, i)$ and \wp' its derivative.

Theorem 2.1. *Let $f, g : T^2 \setminus \{p_o, p_2\} \rightarrow \mathbb{C}$ be the two meromorphic functions defined by*

$$f = \wp^2 \quad g = \frac{\alpha \wp'}{\wp^3}$$

where α is a real constant depending only on $L(1, i)$ and \wp .

Then $\{T^2 \setminus [p_o, p_2], f dz, g\}$ is the Weierstrass representation of a complete genus one immersed minimal surface M with finite total curvature.

Remark 2.2. The ends of M cannot be embedded. In fact, if a complete finite total curvature minimal surface has two embedded ends, it is a catenoid (cf. [Sc]).

The functions f and g extend meromorphically to T^2 and we have $g(p_o) = 0$ and $g(p_2) = \infty$. Hence the limit normal vector at both ends of M is vertical. Then we have the following result.

Theorem 2.3. *There exists a positive constant $c \in \mathbb{R}$ such that $M \cap \{|x_3| \leq c\}$ is a compact genus one immersed minimal surface having the property that each of the boundary curves $M \cap \{x_3 = \pm c\}$ is a compact locally convex immersed curve.*

3. Proof of the theorems

We list some useful classical properties of the function \wp (cf. [B], [WW]).

By abuse of notation, we often identify points of \mathbb{C} with points of T^2 . Let $'$ be the differentiation with respect to the variable $z \in \mathbb{C}$.

(i) \wp is even and \wp' is odd. We have $\wp(z), \wp'(z) \in \mathbb{R}$ when $z \in \mathbb{R}$, $\wp(p_1) = e_1 \in \mathbb{R}_+^*$, $\wp(p_2) = 0$ and $\wp(p_3) = -e_1$.

The following identities hold:

(ii) $(\wp')^2 = 4\wp(\wp^2 - e_1^2)$, $\wp'' = 2(3\wp^2 - e_1^2)$.

(iii) $\wp(z + p_1) = \frac{e_1(\wp(z) + e_1)}{\wp(z) - e_1}$, $\wp(z + p_3) = \frac{e_1(\wp(z) - e_1)}{\wp(z) + e_1}$, $\wp(z + p_2) = -\frac{e_1^2}{\wp(z)}$.

(iv) $\wp'(z + p_2) = e_1^2 \frac{\wp'(z)}{\wp(z)^2}$.

(v) $\wp(iz) = -\wp(z)$, $\wp'(iz) = i\wp'(z)$.

(vi) The local expansion of \wp and \wp' around p_o is

$$\wp(z) = \frac{1}{z^2} + \frac{e_1^2}{5}z^2 + O(z^6),$$

$$\wp'(z) = -\frac{2}{z^3} + \frac{2e_1^2}{5}z + O(z^5).$$

Proof of Theorem 2.1. It is sufficient to prove that the following conditions are satisfied.

(A) z is a pole of order m of $g \iff z$ is a zero of order $2m$ of f .

(B) $\int_\gamma (1 + |g|^2)|f| = \infty$ for every divergent path γ in M .

(C) $\operatorname{Re} \int_\gamma fg = 0$ and $\int_\gamma fg^2 = \overline{\int_\gamma f}$ for every closed path in M .

Zeros and poles of f, g, fg, fg^2 in a fundamental region are as in figure 1.

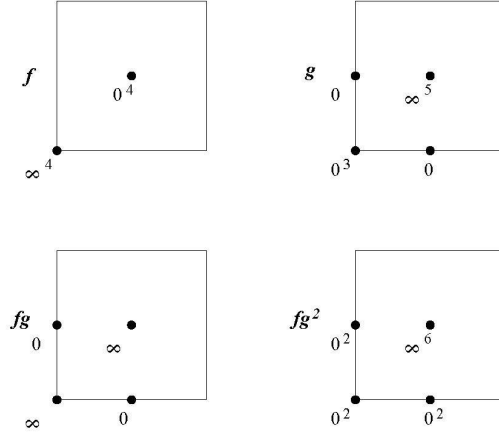


Figure 1.

The function g does not have poles in $T^2 \setminus \{p_o, p_2\}$, hence condition (A) is satisfied.

The expression of the metric on M in terms of \wp is

$$ds = \left(1 + \alpha^2 \frac{|\wp'|^2}{|\wp|^6}\right) |\wp|^2$$

hence the metric is complete at the ends and condition (B) is satisfied.

We must verify (C) on paths that are not homologous to 0 in $T^2 \setminus \{p_o, p_2\}$, i.e. paths around p_o and p_2 and paths that generate the homology of T^2 . Denote by $\alpha(p_o)$ and $\alpha(p_2)$ any closed path around p_o and p_2 respectively, and by γ_1 and γ_2 the following paths generating the homology of T^2 :

$$\gamma_1(t) = \frac{i}{4} + t \quad t \in [0, 1]$$

$$\gamma_2(t) = \frac{1}{4} + it \quad t \in [0, 1]$$

The functions f and fg^2 are even, so they have no residue at p_o , i.e.

$$\int_{\alpha(p_o)} fg^2 = \int_{\alpha(p_o)} f = 0$$

Furthermore

$$\operatorname{Re} \int_{\alpha(p_o)} fg = \operatorname{Re} \int_{\alpha(p_o)} \frac{\alpha \wp'}{\wp} = \operatorname{Re} \left[\operatorname{Res}_{p_o} \left(2\pi i \alpha \frac{\wp'}{\wp} \right) \right]$$

By the local expansion of \wp and \wp' around 0 we have that $\operatorname{Res}_{p_o} (2\pi i \alpha \frac{\wp'}{\wp}) = -4\pi i \alpha$, hence for $\alpha \in \mathbb{R}$ we have

$$\operatorname{Re} \int_{\alpha(p_o)} fg = 0$$

By (iii) and (iv) we have

$$f(z + p_2) = \frac{e_1^4}{\wp^2(z)},$$

$$fg^2(z + p_2) = \frac{\alpha^2}{e_1^4} (\wp'(z))^2.$$

Hence $f(z + p_2)$ and $fg^2(z + p_2)$ are even functions of z and this gives

$$\int_{\alpha(p_2)} fg^2 = \int_{\alpha(p_2)} f = 0.$$

By (iii) and (iv) we have

$$fg(z + p_2) = -\alpha \frac{\wp'(z)}{\wp(z)}.$$

Hence, by the computation above, for $\alpha \in \mathbb{R}$ we have

$$\operatorname{Re} \int_{\alpha(p_2)} fg = 0.$$

Now we verify (C) over γ_1 and γ_2 . We have

$$\operatorname{Re} \int_{\gamma_i} fg = \operatorname{Re} \int_{\gamma_i} \alpha \frac{\wp'}{\wp} = \alpha [\ln |\wp|]_{\gamma_i(0)}^{\gamma_i(1)} = 0$$

by periodicity of \wp , as α is real.

Integral of f over γ_1 : by Cauchy theorem and periodicity we can move γ_1 up to the segment from p_3 to $p_3 + 1$, hence

$$\int_{\gamma_1} f = \int_0^1 f(p_3 + t) dt = \int_0^1 e_1^2 \frac{(\wp(t) - e_1)^2}{(\wp(t) + e_1)^2} dt$$

where the last equality is given by (iii).

Integral of f over γ_2 : we can move γ_2 to the vertical segment from p_1 to $p_1 + i$, hence by (iii) and (iv)

$$\int_{\gamma_2} f = \int_0^1 f(p_1 + t) i dt = i \int_0^1 e_1^2 \frac{(\wp(t) - e_1)^2}{(\wp(t) + e_1)^2} dt.$$

Integral of fg^2 over γ_1 : we can move γ_1 down to the real segment from p_o to $p_o + 1$, hence

$$\int_{\gamma_1} fg^2 = \int_0^1 f(t) g^2(t) dt = \int_0^1 \alpha^2 \frac{\wp'(t)^2}{\wp(t)^4} dt.$$

Integral of fg^2 over γ_2 : we can move γ_2 to the vertical segment from p_o to $p_o + i$, hence

$$\int_{\gamma_2} fg^2 = \int_0^1 f(it) g^2(it) i dt = -i \int_0^1 \alpha^2 \frac{\wp'(t)^2}{\wp(t)^4} dt.$$

Then α must satisfy

$$\alpha^2 \int_0^1 \frac{\wp'(t)^2}{\wp(t)^4} dt = \int_0^1 e_1^2 \frac{(\wp(t) - e_1)^2}{(\wp(t) + e_1)^2} dt.$$

If $t \in \mathbb{R}$ we have $\wp(t), \wp'(t) \in \mathbb{R}$, hence the two integrals involved in the definition of α are positive real numbers. Furthermore they are convergent, so $\alpha \in \mathbb{R}$.

Since g and f extend meromorphically to T^2 , M has finite total curvature. \square

Before proving Theorem 2.3 we need the following lemma.

Lemma 3.1. *Consider a minimal surface M with Weierstrass representation given by $\{f dz, g\}$ such that the vector corresponding to $g(0)$ is parallel to the x_3 -axis. Then the planar curvature of the intersection curves of M with the horizontal planes is*

$$k = \frac{1}{|f^2 g|(1 + |g|^2)} \operatorname{Re} \left(\overline{f g} \frac{g'}{g} \right).$$

Proof. Let $\theta = \arg g$ and s be the arc length of the curve $M \cap \{x_3 = c\}$; then $k(s) = \frac{d\theta}{ds}$. As $\arg g = \operatorname{Im}(\ln g)$, we have

$$k(s) = \frac{d \operatorname{Im} \ln g}{ds} = \operatorname{Im} \left(\frac{d \ln g}{dz} \frac{dz}{ds} \right) = \operatorname{Im} \left(\frac{g'}{g} \frac{dz}{ds} \right).$$

By the Weierstrass representation we have

$$x_3 = \operatorname{Re} \int f g.$$

Hence, on the curve $M \cap \{x_3 = c\}$, $\frac{dz}{ds}$ must satisfy

$$0 = \frac{d}{ds} \operatorname{Re} \int f g = \frac{1}{2} \operatorname{Re} \left(f g \frac{dz}{ds} \right).$$

By a straightforward computation we obtain

$$\frac{dz}{ds} = \frac{i}{(1 + |g|^2)|f|} \frac{\overline{f g}}{|f g|}.$$

Then

$$k = \operatorname{Im} \left(\frac{i}{(1 + |g|^2)|f|} \frac{\overline{f g}}{|f g|} \frac{g'}{g} \right) = \frac{1}{|f^2 g|(1 + |g|^2)} \operatorname{Re} \left(\overline{f g} \frac{g'}{g} \right).$$

\square

Proof of Theorem 2.3. The third coordinate of M is given by

$$x_3 = \operatorname{Re} \int f g = \operatorname{Re} \int \alpha \frac{\wp'}{\wp} = \alpha \ln |\wp|,$$

since α is real. Then, any level curve is given by $|\wp| = c$ and next to the ends this is a compact immersed curve with only one component.

By a straightforward computation, we obtain

$$g'(z) = 2\alpha \left[\frac{5e_1^2 - 3\wp(z)^2}{\wp(z)^3} \right],$$

$$\frac{g'(z)}{g(z)} = \frac{2(5e_1^2 - 3\wp(z)^2)}{\wp'(z)},$$

$$\overline{f(z)g(z)} = \overline{\alpha} \frac{\overline{\wp'(z)}}{\wp(z)}.$$

By using the expansion of \wp and \wp' at p_o we have

$$\overline{f(z)g(z)} \sim -2\frac{\overline{\alpha}}{\overline{z}},$$

$$\frac{g'(z)}{g(z)} \sim \frac{3}{z},$$

where \sim denotes equality between the principal parts of the functions in a neighborhood of zero. Hence the sign of the curvature of the level curve next to the end p_o is the same as the sign of

$$\operatorname{Re}\left(\frac{-6\overline{\alpha}}{\overline{z}z}\right) = -\frac{6\alpha}{|z|^2},$$

α being real.

We use the equality

$$\overline{f(z+p_2)g(z+p_2)} = -\overline{f(z)g(z)}$$

and the fact that in a neighborhood of zero we have

$$\frac{g'(z+p_2)}{g(z+p_2)} = \frac{2(5\wp(z)^2 - 3e_1^2)}{\wp'(z)} \sim -\frac{5}{z},$$

to conclude that the sign of the curvature of the level curve next to the end p_2 is the same as the sign of

$$\operatorname{Re}\left(\frac{-10\overline{\alpha}}{\overline{z}z}\right) = -\frac{10\alpha}{|z|^2}$$

since α is real.

Thus, if we choose a negative α , the level curves are locally convex next to the two ends of M . \square

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