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A theory of cobordism for non-spherical links

Vincent Blanlœil and Françoise Michel

Abstract. We define an equivalence relation, called algebraic cobordism, on the set of bilinear forms over the integers. When $n \geq 3$, we prove that two $2n - 1$ dimensional, simple fibered links are cobordant if and only if they have algebraically cobordant Seifert forms. As an algebraic link is a simple fibered link, our criterion for cobordism allows us to study isolated singularities of complex hypersurfaces up to cobordism.

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0. Introduction

In this work we present a cobordism theory for links which is motivated by the study of the topology of isolated singularities of complex hypersurfaces. Let us be more precise:

(0.1) Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$, be a holomorphic germ with an isolated singular point at the origin. We denote by D_δ^{2k} the compact ball of radius δ centred at 0 in \mathbb{C}^k , and by S_δ^{2k-1} its boundary. The orientation-preserving homeomorphism class of the pair $(D_\varepsilon^{2n+2}, f^{-1}(0) \cap D_\varepsilon^{2n+2})$ does not depend on the choice of a sufficiently small ε , by definition it is *the topological type* of f . The orientation preserving diffeomorphism class of the pair $(S_\varepsilon^{2n+1}, K(f))$, where $K(f) = (f^{-1}(0)) \cap S_\varepsilon^{2n+1}$ is the link of f . The Milnor's conic structure theorem (see [M3, 68]) shows that the link $K(f)$ determines the topological type of f . Moreover, J. Milnor has also proved that:

1. $f/|f| : S_\varepsilon^{2n+1} \setminus K(f) \rightarrow S^1$ is a differentiable fibration which is trivial on $U \setminus K(f)$, when U is a sufficiently "small" open tubular neighbourhood of $K(f)$.
2. The manifold $K(f)$ is $(n - 2)$ -connected.
3. The adherence F of a fiber of $f/|f|$ is a compact, oriented, $(n - 1)$ -connected

smooth submanifold of S_{ε}^{2n+1} having $K(f)$ as boundary. By definition F is the *Milnor fiber* of $K(f)$.

(0.2) More generally, we will say that a *link* is a $(n-2)$ -connected, oriented, smooth, closed, $(2n-1)$ dimensional submanifold of S^{2n+1} . A *knot* is a spherical link (i.e. a link abstractly homeomorphic to S^{2n-1}). It is well-known that, for any link K , there exists a smooth, compact, oriented $2n$ -submanifold F of S^{2n+1} , having K as boundary ; such a manifold F is called a *Seifert surface* for K .

(0.3) Following M. Kervaire [K1, 65], we say that two links K_0 and K_1 , abstractly diffeomorphic to the same manifold \mathcal{K} , are *cobordant* if there exists an embedding $\Phi, \Phi : \mathcal{K} \times [0, 1] \rightarrow S^{2n+1} \times [0, 1]$, such that:

$$\Phi(\mathcal{K} \times \{0\}) = K_0 \text{ and } \Phi(\mathcal{K} \times \{1\}) = -K_1,$$

where $-K_1$ is the link K_1 with the orientation reversed.

(0.4) Let F be a $2n$ dimensional oriented smooth manifold of S^{2n+1} , and let G be the quotient of $H_n(F, \mathbb{Z})$ by its \mathbb{Z} -torsion.

The *Seifert form* associated to F is the bilinear form $A : G \times G \rightarrow \mathbb{Z}$ defined as follows (see also [K2, 70] p.88 or [L2, 70], p.185): let (x, y) be in $G \times G$, then $A(x, y)$ is the linking number in S^{2n+1} of x and $i_+(y)$, where $i_+(y)$ is the cycle y "pushed" in $(S^{2n+1} \setminus F)$ by the positively oriented vector field normal to F in S^{2n+1} .

By definition a *Seifert form for a link K* is the Seifert form associated to a Seifert surface for K .

When $n \geq 2$, J. Levine ([L1, 69]) and M. Kervaire ([K2, 70]) gave a complete characterization of cobordism classes of knots in terms of Witt-equivalence classes of Seifert forms.

(0.5) A *simple link* is a link which has a $(n-1)$ -connected Seifert surface. A link K is a *simple fibered link* if there exists a differentiable fibration $\varphi : S^{2n+1} \setminus K \rightarrow S^1$, φ being trivial on $U \setminus K$, where U is a "small" open tubular neighbourhood of K , and having $(n-1)$ -connected fibers, the adherence of which are Seifert surfaces for K . In this paper we define in §1 (see (1.2)) an equivalence relation on integral bilinear forms which is much more sophisticated than "Witt-equivalence" and the theorems 2 and 3, stated in §1, imply:

Theorem A. *If $n \geq 3$, two simple fibered links are cobordant if and only if they have algebraically cobordant Seifert forms.*

(0.6) By definition an *algebraic link* is a link $K(f)$ associated, as described above, to a holomorphic germ f with an isolated singularity. Furthermore, Milnor's theory of singular complex hypersurfaces implies that algebraic links are simple fibered links. So theorem 2' and 3 stated in §1 imply:

Theorem B. *If $n \geq 3$, two algebraic links are cobordant if and only if the Seifert forms associated to their Milnor's fibers are algebraically cobordant.*

In [Lê, 72], D.T. Lê showed that two cobordant algebraic links of plane curves (i.e. when $n = 1$) are isotopic. In [DB-M, 93], P. du Bois and F. Michel found (using the classical cobordism theory for knots of M. Kervaire and J. Levine), for all $n \geq 3$, examples of non isotopic but cobordant algebraic knots. But in general algebraic links are not spherical links. So theorem B gives a cobordism theory for algebraic links.

Furthermore, having algebraically cobordant Seifert forms is also a necessary condition of cobordism for simple fibered links when n is 1 or 2. So we obtain in §5, without any restriction of dimension, a "Fox-Milnor" relation (see [F-M, 66]) for the Alexander polynomials of cobordant simple fibered links which implies:

(0.7) **Corollary.** *Let K_0 and K_1 be two algebraic links having respectively Δ_0 and Δ_1 as characteristic polynomials of monodromy. If K_0 and K_1 are cobordant then the product $\Delta_0 \cdot \Delta_1$ is a square in $\mathbb{Z}[X]$.*

(0.8) **Comments.** In [V1, 77] and [V2, 78] R. Vogt gave, when $n \geq 3$, a sufficient, but not necessary, condition of cobordism for simple links having torsion free homology groups. As shown in [DB-M, 93] the sufficient condition of cobordism for algebraic links given in [Sz, 89] by S. Szczepanski, cannot be true. So the problem of finding a criterion for cobordism of simple fibered links was largely open. Our definition of algebraic cobordism for Seifert forms solves the problem.

(0.9) In this paper we use the following **notations**: If X is a differentiable manifold we denote by ∂X its boundary, by $\overset{\circ}{X}$ its interior and by $H_k(X)$ the k^{th} -homology group of X with coefficients in \mathbb{Z} . If a is a k -cycle of X we denote by $[a]$ its homology class in $H_k(X)$. If G is an abelian group let $\text{rk}(G)$ be the rank of G , and $\text{Tors}(G)$ be the torsion subgroup of G .

1. Definitions and statement of results

Let \mathcal{A} be the set of bilinear forms defined on free \mathbb{Z} -modules G of finite rank.

Let ε be $+1$ or -1 .

(1.1) If A is in \mathcal{A} , let us denote by A^T the transpose of A , by S the ε -symmetric form $A + \varepsilon A^T$ associated to A , by $S^* : G \rightarrow G^*$ the adjoint of S (G^* being the dual $\text{Hom}_{\mathbb{Z}}(G; \mathbb{Z})$ of G), by $\overline{S} : \overline{G} \times \overline{G} \rightarrow \mathbb{Z}$ the ε -symmetric non degenerated form induced by S on $\overline{G} = G/\text{Ker } S^*$. A submodule M of G is pure if G/M is torsion free. If M is any submodule of G let us denote by M^\wedge the smallest pure submodule of G which contains M . In fact M^\wedge is equal to $(M \otimes \mathbb{Q}) \cap G$. For a submodule M of G we denote by \overline{M} the image of M in \overline{G} .

Definition. *Let $A : G \times G \rightarrow \mathbb{Z}$ be a bilinear form in \mathcal{A} . The form A is Witt associated to 0 if the rank m of G is even and if there exists a pure submodule M of rank $\frac{m}{2}$ in G such that A vanishes on M ; such a module M is called a*

metabolizer for A .

(1.2) **Definition.** Let $A_i : G_i \times G_i \rightarrow \mathbb{Z}$, $i=0,1$, be two bilinear forms in \mathcal{A} . Let G be $G_0 \oplus G_1$ and A be $(A_0 \oplus -A_1)$. The form A_0 is algebraically cobordant to A_1 if there exists a metabolizer M for A such that \bar{M} is pure in \bar{G} , an isomorphism φ from $\text{Ker } S_0^*$ to $\text{Ker } S_1^*$ and an isomorphism θ from $\text{Tors}(\text{Coker } S_0^*)$ to $\text{Tors}(\text{Coker } S_1^*)$ which satisfy the two following conditions:

- c.1: $M \cap \text{Ker } S^* = \{(x, \varphi(x)); x \in \text{Ker } S_0^*\}$,
- c.2: $d(S^*(M)^\wedge) = \{(x, \theta(x)); x \in \text{Tors}(\text{Coker } S_0^*)\}$, where d is the quotient map from G^* to $\text{Coker } S^*$.

In §2 (see (2.3)) we prove:

Theorem 1. Algebraic cobordism is an equivalence relation on the set \mathcal{A} .

(1.3) From now on, A_0 and A_1 will always be two Seifert forms associated to some $(n - 1)$ -connected Seifert surfaces F_0 and F_1 , of two simple links K_0 and K_1 . Let us justify the definition of algebraic cobordism. As a generalization of the Kervaire-Levine theory of knot cobordism we obtain in §3 (see (3.10)):

Proposition. If K_0 and K_1 are cobordant simple links, then $A = A_0 \oplus -A_1$ has a metabolizer.

Remark. Let ε be $(-1)^n$, then for $i=0,1$, $S_i = A_i + \varepsilon A_i^T$ is the intersection form on $H_n(F_i)$, $\text{Ker } S_i^*$ is the image of $H_n(K_i)$ in $H_n(F_i)$ and $\text{Coker } S_i^*$ is isomorphic to $\tilde{H}_{n-1}(K_i)$. So for spherical links, both $\text{Ker } S_i^*$ and $\text{Coker } S_i^*$ are zero, and conditions c.1 and c.2 in definition (1.2) vanish. Then, for spherical links, two Witt associated Seifert forms are algebraically cobordant, and we recover the Kervaire-Levine criterion for cobordism.

In the non-spherical case, the topology of the cobordism implies that the restriction of A_0 on $\text{Ker } S_0^*$ is isomorphic (on \mathbb{Z}) to the restriction of A_1 on $\text{Ker } S_1^*$ (it is easy to check it directly, and it is also implied by the more general proposition (3.10)). This necessary condition for cobordism is not implied by the fact that $A_0 \oplus -A_1$ is Witt associated to 0, but by condition c.1 in definition (1.2). The topology of the cobordism also implies that the linking forms on $\text{Tors}(H_{n-1}(K_i))$ are isomorphic. This necessary condition for cobordism is contained in point c.2 of definition (1.2).

(1.4) The major result of this work is theorem 2 proved in §3 (see (3.10) and (3.13)):

Theorem 2. Let K_0 and K_1 be two cobordant simple links. If K_0 and K_1 have $(n - 1)$ -connected Seifert surfaces F_0 and F_1 with unimodular Seifert forms A_0

and A_1 , then A_0 is algebraically cobordant to A_1 .

Remark. Let i be 0 or 1. Let us suppose that K_i is a simple fibered link and let F_i be a $(n-1)$ -connected fiber of a fibration $\varphi_i : S^{2n+1} \setminus K_i \rightarrow S^1$; then, the Seifert form A_i associated to F_i is unimodular. Conversely, if $n \geq 3$ and if A_i is unimodular then K_i is a simple fibered link (see [K-W, 77] chap. V, §3, p.118).

So, theorem 2 implies:

Theorem 2'. *Let K_0 and K_1 be two simple fibered links having F_0 and F_1 as $(n-1)$ -connected fibers of differentiable fibrations φ_0 and φ_1 . If K_0 is cobordant to K_1 , then the Seifert forms A_0 and A_1 , associated respectively to F_0 and F_1 , are algebraically cobordant.*

(1.5) Using classical methods of surgery, we prove in §4 (see (4.4) and (4.5)):

Theorem 3. *Let n be greater or equal to 3 and let K_0 and K_1 be two $2n-1$ dimensional simple links. If the Seifert forms A_0 and A_1 , associated to some $(n-1)$ -connected Seifert surfaces F_0 and F_1 of K_0 and K_1 , are algebraically cobordant then K_0 is cobordant to K_1 .*

(1.6) Proposition (3.10), which does not use (as remarked in (3.12)) any hypothesis on the Seifert forms, gives:

Theorem 4. *Let K_0 and K_1 be two cobordant simple links. If A_0 (resp. A_1) is a Seifert form associated to any $(n-1)$ -connected Seifert surface for K_0 (resp. K_1), then $A_0 \oplus -A_1$ has a metaboliser M such that $M \cap \text{Ker } S^* = \{(x, \varphi(x)); x \in \text{Ker } S_0^*\}$, where φ is an isomorphism between $\text{Ker } S_0^*$ and $\text{Ker } S_1^*$.*

2. Algebraic cobordism

(2.0) Let A_0 and A_1 be two algebraically cobordant forms, let A be the form $A_0 \oplus -A_1$ defined on $G = G_0 \oplus G_1$ and S be $A + \varepsilon A^T$. In this section we prove proposition (2.1) which shows that the algebraic cobordism between A_0 and A_1 allows us to describe S ; this characterization of S is fundamental to prove theorem 3 (see §4). Let M , φ and θ be as in (1.2), let m be $\text{rk}(G)$ and r be $\text{rk}(\text{Ker } S_0^*)$. Then definition (1.2) implies that $s = \text{rk}(S^*(M)) = \frac{1}{2} \text{rk}(S^*(G))$ and $\text{rk}(M) = r + s = \frac{m}{2}$.

We use the following notations: if E is any subset of G we denote by $\langle E \rangle$ the submodule of G , generated by E . If L is any submodule of G then:

$$L^\perp = \{x \in G \text{ s.t. } S(x, l) = 0 \forall l \in L\}$$

$$\text{Hom}_{\mathbb{Z}}(G|_L, \mathbb{Z}) = \{f \in G^* \text{ s.t. } f(l) = 0 \ \forall l \in L\}$$

Moreover if L_1 and L_2 are two submodules of G , orthogonal for S , we denote by $L_1 \oplus^\perp L_2$ their (orthogonal) direct sum.

Lemma. *We have: $S^*(G) \cap S^*(M)^\wedge = S^*(M^\perp)$.*

Proof. Let r be the rank of $\text{Ker } S_0^*$ and s be the rank of $S^*(M)$. As M is a metabolizer for S which fulfills condition c.1 in (1.2) we have:

$$\text{rk}(\text{Ker } S^*) = 2 \text{rk}(M \cap \text{Ker } S^*) = 2r, \text{rk}(S^*(G)) = 2s \text{ and } \text{rk}(M^\perp) = s + 2r. \text{ Hence } M^\perp = (M + \text{Ker } S^*)^\wedge \text{ and } S^*(M^\perp) \subset S^*(G) \cap S^*(M)^\wedge.$$

Moreover, $S^*(M)$ is of finite index in $\text{Hom}_{\mathbb{Z}}(G|_{M^\perp}; \mathbb{Z})$. As $\text{Hom}_{\mathbb{Z}}(G|_{M^\perp}; \mathbb{Z})$ is a pure submodule of G^* , we get $S^*(M)^\wedge = \text{Hom}_{\mathbb{Z}}(G|_{M^\perp}; \mathbb{Z})$. So if $S^*(x) \in S^*(M)^\wedge$, then $S^*(x, l) = 0$ for all l in M^\perp and x is in M^\perp . □

Since $S^*(M)$ is of finite index in $S^*(M)^\wedge$, one can write $(S^*(M)^\wedge)/S^*(M) \cong \bigoplus_{i=1}^s \mathbb{Z}/a_i \mathbb{Z}$ where $a_i \in \mathbb{N} \setminus \{0\}$ and a_i divides a_{i+1} (we do not exclude that there exists an integer l such that $a_i = 1$ for $i = 1, \dots, l$).

Proposition. *The submodule \overline{M} is pure in \overline{G} if and only if $S^*(M^\perp) = S^*(M)$.*

Proof. We suppose that \overline{M} is pure in \overline{G} . As $M \cap \text{Ker } S^* = \Delta(\varphi)$ has rank r , the rank of $M + \text{Ker } S^*$ is $s + 2r$. So $M + \text{Ker } S^*$ is of finite index in M^\perp . Let x be in M^\perp ; there exists a positive integer k such that $kx = y + m$, where y is in $\text{Ker } S^*$, m is in M ; so $\overline{m} = k\overline{x}$. Since \overline{M} is pure in \overline{G} then \overline{x} is in \overline{M} , so there exists y' in $\text{Ker } S^*$ such that $x + y'$ is in M . Finally $S^*(x) = S^*(x + y') \in S^*(M)$, and $S^*(M^\perp) \subset S^*(M)$. But $M \subset M^\perp$ so $S^*(M^\perp) = S^*(M)$.

We suppose that $S^*(M) = S^*(M^\perp)$. First we prove that $\overline{M^\perp}$ is pure in \overline{G} . Let z be in M^\perp with $\overline{z} = k\overline{x}$ where x is in G and k is a positive integer. So there exists y in $\text{Ker } S^*$ such that $kx = z + y$. For all m in M we have $S(kx, m) = S(z + y, m) = 0$, so $S(x, m) = 0$ and x is in M^\perp . Now we prove that $S^*(M^\perp) = S^*(M)$ implies $\overline{M} = \overline{M^\perp}$. Let z be in M^\perp . If $S^*(z) = f$ there exists m in M such that $S^*(m) = f$. So $z - m = y$ is in $\text{Ker } S^*$, and $\overline{z} = \overline{m}$ is in \overline{M} . Finally, since $\overline{M^\perp}$ is pure in \overline{G} and $\overline{M^\perp} \subset \overline{M}$ we get $\overline{M^\perp} = \overline{M}$ is pure in \overline{G} . □

By definition (1.2) \overline{M} is pure in \overline{G} , so lemma (2.0) and proposition (2.0), and, conditions c.1 and c.2 in definition (1.2) imply that $\text{Coker } S^*$ is isomorphic to

$$\mathbb{Z}^{2r} \oplus \left(\bigoplus_{i=1}^s \mathbb{Z}/a_i \mathbb{Z} \right)^2.$$

(2.1) **Proposition.** *There exists a basis $\mathcal{B} = \{m_i, m_i^*; i=1, \dots, s+r\}$ of G such that:*

1. $\{m_i; i=1, \dots, s+r\}$ is a basis of M ,
2. $\{m_i, m_i^*; i=s+1, \dots, s+r\}$ is a basis of $\text{Ker } S^*$ and $\{m_i^*; i=s+1, \dots, s+r\}$ is a basis of $\text{Ker } S_0^*$,
3. the submodules $\langle m_i, m_i^* \rangle, i=1, \dots, s+r$; are orthogonal for S , i.e.: $G = \bigoplus_{1 \leq i \leq s+r}^\perp \langle m_i, m_i^* \rangle$,
3. when $i=1, \dots, s, S(m_i, m_i^*) = a_i$.

Definition. *Such a basis is called a good basis of G associated to M .*

The form $S = A + \varepsilon A^T$ is always an even form. Moreover, when the a_i are odd we get the following corollary:

Corollary. *When the a_i are odd, the isomorphic class of S is given by $m = \text{rk}(G)$ and the isomorphic class of $\text{Coker } S^*$.*

Proof of proposition (2.1). In (2.0) we have seen that $S^*(M)^\wedge = \text{Hom}_{\mathbb{Z}}(G|_{M^\perp}; \mathbb{Z})$. Let M_0 be any direct summand complement of $(M \cap \text{Ker } S^*)$ in M . There exists a basis $\{m_i; i=1, \dots, s\}$ of M_0 and a basis $\{h_i; i=1, \dots, s\}$ of $\text{Hom}_{\mathbb{Z}}(G|_{M^\perp}; \mathbb{Z})$ such that $S^*(m_i) = a_i h_i$ where $a_i \in \mathbb{N} \setminus \{0\}$ and a_i divides a_{i+1} . Let m_1^* be any element in G such that $G = \text{Ker } h_1 \oplus \langle m_1^* \rangle$ and $h_1(m_1^*) = S(m_1, m_1^*).a_1^{-1} = 1$.

Claim. For all x in G , a_1 divides $S(x, m_1^*)$.

If $a_1 = 1$ it is obvious. If $a_1 > 1$, condition c.2 in (1.2) implies that $(S^*(G)^\wedge)/S^*(G)$ is isomorphic to $(S^*(M)^\wedge)/S^*(M)^2 \cong \left(\bigoplus_{i=1}^s \mathbb{Z}/a_i \mathbb{Z}\right)^2$ and the rank of $S^*(G)$ is $2s$.

So a_1 divides $S^*(x)$ for all x in G .

Now, we will construct an orthogonal complement $(M_1 \oplus R_1)$ for $\langle m_1, m_1^* \rangle$ in G such that:

- i) $M = \langle m_1 \rangle \oplus M_1$,
- ii) $\text{Ker } h_1 = M \oplus R_1$.

Let M_1 be the submodule of M generated by $m_i' = m_i - a_1^{-1} S(m_i, m_1^*).m_1$, $2 \leq i \leq s$, and $M \cap \text{Ker } S^*$. By construction M_1 is orthogonal to $\langle m_1, m_1^* \rangle$ and $M = \langle m_1 \rangle \oplus M_1$.

By construction $\text{Ker } h_1$ is orthogonal to m_1 and M is in $\text{Ker } h_1$.

If $\{x_i, i=2, \dots, s+r\}$ is a basis of any direct summand complement of M in $\text{Ker } h_1$, let R_1 be the submodule of $\text{Ker } h_1$ generated by x_i' where: $x_i' = x_i - a_1^{-1} S(x_i, m_1^*).m_1$. Then $\text{Ker } h_1 = \langle m_1 \rangle \oplus M_1 \oplus R_1$ and R_1 is orthogonal to m_1^* .

Now we have an orthogonal decomposition of G in $\langle m_1, m_1^* \rangle \oplus^\perp (M_1 \oplus R_1)$. By

induction on s we obtain an orthogonal decomposition:

$$G = (\oplus^\perp \langle m_i, m_i^* \rangle) \oplus^\perp (M_s \oplus R_s) \text{ where } \text{Ker } S^* = M_s \oplus R_s.$$

Let $\{m_{s+1}, \dots, m_{s+r}\}$ be any basis of $\text{Ker } S^* \cap M$. Thanks to condition c.1, $\text{Ker } S^* \cap M = \{(x, \varphi(x)); x \in \text{Ker } S_0^*\}$. So we can choose any basis $\{m_{s+1}^*, \dots, m_{s+r}^*\}$ of $\text{Ker } S_0^*$ to build up a basis of G which fulfills proposition (2.1). \square

(2.2) Now, we use the notations established in §1 and the following convention: if $f : R \rightarrow S$ is an isomorphism of \mathbb{Z} -modules, $\Delta(f)$ is the submodule $\{(x, f(x)); x \in R\}$ in $R \oplus S$. To prove theorem 1, we need the following proposition which gives an equivalent definition of algebraic cobordism.

Proposition. *Let A_0 and A_1 be in \mathcal{A} . Then A_0 is algebraically cobordant to A_1 if and only if there exists a pure submodule H of $G = G_0 \oplus G_1$ on which $A = A_0 \oplus -A_1$ vanishes, an isomorphism φ from $\text{Ker } S_0^*$ to $\text{Ker } S_1^*$ and an isomorphism θ from $\text{Tors}(\text{Coker } S_0^*)$ to $\text{Tors}(\text{Coker } S_1^*)$ such that:*

- c.11: $\Delta(\varphi) \subset H$,
- c.12: *the image \overline{H} of H in $\overline{G} = G/\text{Ker } S^*$ is a metabolizer for $\overline{S} = \overline{S_0} \oplus -\overline{S_1}$,*
- c.2: $d(S^*(H)^\wedge) = \Delta(\theta)$.

Proof. Let M, φ, θ be as in definition (1.2). Then M satisfies c.1 and c.2. The existence of φ shows that $\text{Ker } S_0^*$ and $\text{Ker } S_1^*$ have the same rank, r . So the rank of \overline{G} is $(m_0 + m_1 - 2r)$. By c.1 $M \cap \text{Ker } S^* = \Delta(\varphi)$ and $\text{rk}(M) = \frac{m_0 + m_1}{2}$ because M is a metabolizer for A . So $\text{rk}(\overline{M}) = \frac{m_0 + m_1}{2} - r$ and \overline{S} vanishes on \overline{M} . It implies that \overline{M} is a metabolizer for \overline{S} .

Conversely let H, φ and θ be as in the statement of proposition (2.1). As $\Delta(\varphi)$ is pure in H and in $\text{Ker } S^*$, there exists a direct sum decomposition $H \cap \text{Ker } S^* = \Delta(\varphi) \oplus M_0$. As $\text{Ker } S^*$ is pure in G , there exists also a direct sum decomposition $H = M_1 \oplus (H \cap \text{Ker } S^*)$. Let M be $M_1 \oplus \Delta(\varphi)$. By construction A vanishes on M , $M \cap \text{Ker } S^* = \Delta(\varphi)$ and $S^*(M) = S^*(H)$. So M, φ and θ satisfy c.1 and c.2 of definition (1.2). Furthermore, $\overline{H} = \overline{M_1} = \overline{M}$ and by c.12 the rank of \overline{H} is $\frac{m_0 + m_1}{2} - r$. But M_1 being isomorphic to $\overline{M_1}$, the rank of M is $\frac{m_0 + m_1}{2}$ and M is a metabolizer for A . \square

(2.3) *Proof of theorem 1.* The only non trivial property to check is the transitivity of the relation "algebraic cobordism".

(2.4) **Lemma.** *Let $B_i : G_i \times G_i \rightarrow \mathbb{Z}$ be in \mathcal{A} , $i = 0, 1, 2$. Let m_i be the rank of G_i . If there exists a metabolizer H_{01} (resp. H_{12}) for $B_0 \oplus -B_1$ (resp. $B_1 \oplus -B_2$) and if the B_i are non-degenerate, the form $B_0 \oplus -B_2$ vanishes on $H_{02} = \pi(L)$ and $\text{rk } H_{02} = \frac{1}{2} \text{rk}(G_0 \oplus G_2)$, where: $G = G_0 \oplus G_1 \oplus G_1 \oplus G_2$, $H = H_{01} \oplus H_{12}$,*

$\Delta = \{(y, y) \in G_1 \oplus G_1 ; y \in G_1\}$, $L = H \cap (G_0 \oplus \Delta \oplus G_2)$ and π is the projection of G on $G_0 \oplus G_2$.

Proof. As $B_0 \oplus -B_2$ vanishes on H_{02} by construction, it is sufficient to prove that the rank of H_{02} is $\frac{m_0+m_1}{2}$. The definition of H_{02} gives the following exact sequence:

$$0 \rightarrow L \cap \Delta \xrightarrow{i} L \xrightarrow{\pi} H_{02} \rightarrow 0.$$

So we get:

$$(*) \quad \text{rk}(L) = \text{rk}(L \cap \Delta) + \text{rk}(H_{02}).$$

If v is in H , there exists unique x in G_0 , y_1 and y_2 in G_1 and z in G_2 such that $v = (x, y_1, y_2, z)$. Let $\rho : H \rightarrow G_1 \oplus G_1$ be defined by $\rho(v) = (y_1 - y_2, 0)$. Let us denote by L_1 the image $\rho(H)$. By construction L is the kernel of ρ and we get the exact sequence: $0 \rightarrow L \xrightarrow{i} H \xrightarrow{\rho} L_1 \rightarrow 0$. Both this sequence and $(*)$ show:

$$(**) \quad \frac{m_0 + m_2 + 2m_1}{2} - \text{rk}(L_1) = \text{rk}(L \cap \Delta) + \text{rk}(H_{02}).$$

Claim. By $(B_1 \oplus -B_1)$, $\Delta \cap L$ is orthogonal to $L_1 \oplus \Delta$.

Indeed, Δ is self-orthogonal ; if (y, y) is in $\Delta \cap L$, then $(0, y)$ is in H_{01} and $(y, 0)$ is in H_{12} . On the other hand, an element of L_1 is of the form $(y_1, -y_2)$ where there exists (x, y_1) in H_{01} and (y_2, z) in H_{12} . So $B_1(y, y_1) = B_1(y_1, y) = 0$ and $-B_1(y, y_2) = -B_1(y_2, y) = 0$.

The rank of $L_1 \oplus \Delta$ is $m_1 + \text{rk}(L_1)$. The claim implies that the rank of the restriction of $B_1 \oplus -B_1$ to $(\Delta \cap L) \times (G_1 \oplus G_1)$ is smaller or equal to $m_1 - \text{rk}(L_1)$. But $B_1 \oplus -B_1$ is non-degenerate by hypothesis, so: $\text{rk}(\Delta \cap L) \leq m_1 - \text{rk}(L_1)$. By $(**)$ it implies: $\frac{m_0+m_2}{2} \leq \text{rk}(H_{02})$.

As B_0 and B_2 are non-degenerate by hypothesis and as $B_0 \oplus -B_2$ vanishes on H_{02} , $\text{rk}(H_{02}) \leq \frac{m_0+m_2}{2}$. It ends the proof of the lemma. \square

Let us go back to the proof of theorem 1. Let A_i be algebraically cobordant to A_{i+1} , $i = 0, 1$. Let $M_{i,i+1}$ be a metabolizer for $A_i \oplus -A_{i+1}$ with the isomorphisms φ_i and θ_i fulfilling conditions c.1 and c.2 in definition (1.2).

Let us take the following notations: $G = G_0 \oplus G_1 \oplus G_1 \oplus G_2$, $S_{02} = S_0 \oplus -S_2$, $G_{02} = G_0 \oplus G_2$, $S = S_0 \oplus -S_1 \oplus S_1 \oplus -S_2$, $\Delta = \{(x, x) ; x \in G_1\} \subset G_1 \oplus G_1$, d be the quotient map from G to $\text{Coker } S^*$ and d_{02} the quotient map from G_{02}^* to $\text{Coker } S_{02}^*$. Let π (resp. $\tilde{\pi}$) be the obvious projection from G (resp. $\text{Coker } S^*$) to $G_0 \oplus G_2$ (resp. $\text{Coker } S_{02}^*$). Since $\overline{M}_{i,i+1}$ is pure in $\overline{G}_i \oplus \overline{G}_{i+1}$ we have the following decompositions $M_{i,i+1}^\perp = \Delta(\varphi_i) \oplus \text{Ker } S_i^* \oplus R_{i,i+1}$ with $M_{i,i+1} = \Delta(\varphi_i) \oplus R_{i,i+1}$, and $\overline{R}_{i,i+1}$ is pure in $\overline{G}_i \oplus \overline{G}_{i+1}$. Let $Q_{i,i+1}$ be any direct summand complement of $M_{i,i+1}^\perp$ in $\overline{G}_i \oplus \overline{G}_{i+1}$. If $T_{i,i+1} = R_{i,i+1} \oplus Q_{i,i+1}$, then we have the following decomposition $G = \text{Ker } S_{01}^* \oplus \text{Ker } S_{12}^* \oplus T_{01} \oplus T_{12}$. Let us denote by T_0 (resp. T_1 , T_1' , T_2) the projection of T_{01} (resp. T_{01} , T_{12} , T_{12}) to G_0 (resp. G_1 , G_1 , G_2). We

modify R_{12} and Q_{12} by adding to them some elements of $\Delta(\varphi_1)$ in order to have $T_1 = T'_1$. Moreover, we have the following equalities: $G_i = \text{Ker } S_i^* \oplus T_i$ $i = 0, 1, 2$.

Let T_{02} be $T_{02} = \pi(T_{01} \oplus T_{12}) = T_0 \oplus T_2$. Let R_{02} be the smallest pure submodule of T_{02} which contains the projection of $(R_{01} \oplus R_{12}) \cap (G_0 \oplus \Delta \oplus G_2)$ on T_{02} : $R_{02} = (\pi((R_{01} \oplus R_{12}) \cap (G_0 \oplus \Delta \oplus G_2)))^\wedge$; and let A be $A_0 \oplus -A_2$, φ be $\varphi_1 \circ \varphi_0$ and θ be $-(\theta_1 \circ \theta_0)$.

By proposition (2.2), to prove that A_0 is algebraically cobordant to A_2 it is sufficient to prove that $H = \Delta(\varphi) \oplus R_{02}$ is a metabolizer for $A_0 \oplus -A_2$, and, H fulfill conditions c.11, c.12 and c.2 of (2.2). First we remark that H fulfills c.11 by definition.

(2.5) **Lemma.** *We have the equality $d_{02}(S_{02}^*(H)^\wedge) = \Delta(-\theta_1 \circ \theta_0)$.*

(2.6) **Lemma.** *The submodule H is a metabolizer for A , and \overline{H} is a metabolizer for $\overline{S_0} \oplus -\overline{S_2}$.*

Proof of lemma (2.5). By construction: $d(S^*(G)^\wedge) = \text{Tors}(\text{Coker } S^*)$ and $d_{02}(S_{02}^*(H)^\wedge) = \tilde{\pi}(d(S^*(L)^\wedge))$. But c.2 implies:

$d(S^*(L)^\wedge) = (\Delta(\theta_0) \oplus \Delta(\theta_1)) \cap d(S^*(G_0 \oplus \Delta \oplus G_2)^\wedge)$, so:

$d(S^*(L)^\wedge) = \{(x, \theta_0(x), y, \theta_1(y)); x \in \text{Tors}(\text{Coker } S_0^*), y = -\theta_0(x)\}$.

Finally: $d_{02}(S_{02}^*(H)^\wedge) = \{(x, -\theta_1 \circ \theta_0(x)); x \in \text{Tors}(\text{Coker } S_0^*)\} = \Delta(-\theta_1 \circ \theta_0)$. \square

Proof of lemma (2.6). The restriction $S_{i,i+1}|_{T_{i,i+1}}$ on $T_{i,i+1}$, of the ε -symmetric bilinear form $S_{i,i+1}$, is non-degenerate; and the submodule $R_{i,i+1}$ is a metabolizer for $S_{i,i+1}|_{T_{i,i+1}}$, $i = 0, 1$. By construction T_0 (resp. T_1, T_2) is the projection of T_{01} (resp. T_{01}, T_{12}) onto G_0 (resp. G_1, G_2). So we have $S_{i,i+1}|_{T_{i,i+1}} = S_i|_{T_i} \oplus -S_{i+1}|_{T_{i+1}}$. We use lemma (2.4) replacing B_i by $S_i|_{T_i}$, so $S_{02}|_{T_{02}}$ vanishes on R_{02} and $\text{rk } R_{02} = \frac{1}{2}\text{rk } T_{02}$. Since the pure submodule H of $G_{02} = \text{Ker } S_{02}^* \oplus T_{02}$ is defined by the equality $H = \Delta(\varphi) \oplus R_{02}$ then $\text{rk } H = \frac{1}{2}\text{rk } G_{02}$. Moreover for all h_1, h_2 in H there exist two integers a_1 and a_2 such that for $i = 1, 2$ we have: $a_i h_i = \pi(m_i)$ and $m_i = (x_i, \varphi_0(x_i), \varphi_0(x_i), \varphi(x_i)) + (m_{0,i}, m_{1,i}, m_{1,i}, m_{2,i})$ is in $M_{01} \oplus M_{12}$. So $A(h_1, h_2) = \frac{1}{a_1 a_2} (A_{01} \oplus -A_{12})(m_1, m_2) = 0$, so A vanishes on the pure submodule H of G_{02} . Finally H is a metabolizer for A . By construction $S_{02}|_{T_{02}}$ is isomorphic to $\overline{S_{02}}$, so as R_{02} is pure in T_{02} then $\overline{R_{02}}$ is a metabolizer for $\overline{S_{02}}$. \square

The above properties of H , and, lemmas (2.5) and (2.6) imply conditions c.12 and c.2 of proposition (2.2), and A_0 is algebraically cobordant to A_2 . This ends the proof of theorem 1. \square

3. The necessary condition to have a cobordism

Let K_0 and K_1 be two cobordant links. Let us denote by \mathcal{S} the product $S^{2n+1} \times [0, 1]$ and by Σ its oriented boundary. The definition of cobordism gives a submanifold $C = \Phi(\mathcal{K} \times [0, 1])$ of \mathcal{S} such that $\Sigma \cap C = K_0 \amalg (-K_1)$. Let N be $F_0 \cup C \cup (-F_1)$ where F_i is a Seifert surface for K_i . By construction N is a closed, compact, oriented, $2n$ -submanifold of \mathcal{S} .

(3.1) **Lemma.** *There exists a smooth oriented, compact, submanifold W of \mathcal{S} such that N is the boundary of W .*

Proof. This lemma is a consequence of classical obstruction theory. If $n \geq 3$ a proof is written in [L2, 70], p. 183. As the existence of W is fundamental to obtain theorem 2, we write a proof which works in any dimension.

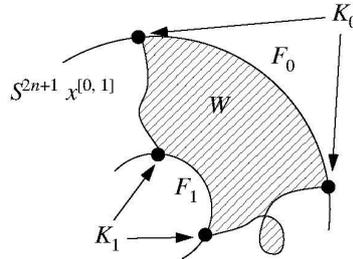
Let C_j for $j = 1, \dots, k$ be the k connected components of C . As C has a trivial normal bundle in \mathcal{S} , it is possible to choose disjoint, closed, tubular neighbourhoods U_j of C_j and a diffeomorphism $\Psi : C \times D^2 \rightarrow U = \coprod_{1 \leq j \leq k} U_j$.

Now we have meridians m_j on ∂U_j defined by: $m_j = \Psi(P_j \times S^1)$ where P_j is some point of C_j and m_j is oriented such that the linking number of m_j and C_j (in \mathcal{S}) is $+1$. Let X be $S \setminus \overset{\circ}{U}$, v be the diffeomorphism induced by the inclusion of ∂X in U , e be the excision isomorphism and ∂^i (resp. ∂_X^i) be the connectant homomorphism for the pair (\mathcal{S}, U) (resp. $(X, \partial X)$). Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
 \xrightarrow{\partial_X^0} & H^1(X, \partial X) & \xrightarrow{\rho} & H^1(X) & \xrightarrow{\sigma} & H^1(\partial X) & \xrightarrow{\partial_X^1} & H^2(X, \partial X) & \rightarrow \\
 & \cong \uparrow e & & \uparrow & & v \uparrow & & \cong \uparrow e & \\
 \xrightarrow{\partial^0} & H^1(\mathcal{S}, U) & \rightarrow & 0 = H^1(\mathcal{S}) & \rightarrow & H^1(U) & \xrightarrow{\cong \partial^1} & H^2(\mathcal{S}, U) & \rightarrow 0
 \end{array}$$

The commutativity of all the squares of the above diagram implies that the homomorphism ρ is zero so σ is injective and ∂_X^i is surjective for $0 \leq i \leq 2n-1$. We have the following direct sum decomposition: $H^1(\partial X) = \sigma(H^1(X)) \oplus v(H^1(U))$. Any element of $\sigma(H^1(X))$ is represented by a differentiable map from ∂X to S^1 , which is, up to homotopy, characterized by its degree on each meridian m_j , and which has a unique extension to X . Let $g : X \rightarrow S^1$ be the unique, up to homotopy, differentiable map which has degree $+1$ on each meridian. Thanks to the Thom-Pontriagin construction there exists a differentiable map $f : \Sigma \setminus (K_0 \amalg -K_1) \rightarrow S^1$ which has $\overset{\circ}{F}_0 \amalg (-\overset{\circ}{F}_1)$ as regular fiber and f has degree $+1$ on the meridians of the connected components of $K_0 \amalg (-K_1)$. So f and g have homotopic restrictions on $X \cap \Sigma$ and we can choose g such that its restriction on $X \cap \Sigma$ coincides with f .

Then g has a regular fiber \overline{W} such that $\overline{W} \cap \Sigma = (F_0 \amalg -F_1) \cap X$. The union of \overline{W} with a small collar in U is the manifold W such that $N = \partial W$. \square



(3.2) Let us take A_0 (resp. A_1) the Seifert form associated to a $(n - 1)$ -connected Seifert surface F_0 (resp. F_1) for K_0 (resp. K_1). Let $\tau : K_0 \rightarrow K_1$ be the diffeomorphism defined by: $\tau(P) = \Phi(\Phi^{-1}(P) \times \{1\})$ where P is any point of K_0 . The diffeomorphism τ induces isomorphisms $\theta_j : H_j(K_0) \rightarrow H_j(K_1)$ such that for any j -cycle x of K_0 , $(x, \theta_j(x))$ is a boundary in $C = \Phi(K \times [0, 1])$. Let $\chi_i : H_n(K_i) \rightarrow H_n(F_i)$ and $\lambda_i : H_n(F_i) \rightarrow H_n(N)$, $i = 0, 1$, be the homomorphisms induced by the inclusions $K_i \subset F_i \subset N$. The Mayer-Vietoris exact sequence associated to the decomposition of N in the union of $F_0 \cup C$ and $C \cup (-F_1)$ gives:

$$\rightarrow H_n(K_0) \xrightarrow{\chi} H_n(F_0) \oplus H_n(F_1) \xrightarrow{\lambda} H_n(N) \xrightarrow{\delta} H_{n-1}(K_0) \rightarrow$$

where $\chi = (\chi_0, \chi_1 \circ \theta_n)$ and $\lambda = (\lambda_0, \lambda_1)$

(3.3) **Remark.** Let m_i be $\text{rk}(H_n(F_i))$, m be $\text{rk}(H_n(N))$ and r be $\text{rk}(\chi(H_n(K_0)))$. By Poincaré duality $m = m_0 + m_1$, $r = \text{rk}(\delta(H_n(N)))$ and $r = \text{rk}(\text{Ker } S_i^*)$ where S_i^* is the adjoint of the intersection form S_i on $H_n(F_i)$.

(3.4) Construction of the isomorphisms $\varphi : \text{Ker } S_0^* \rightarrow \text{Ker } S_1^*$ and $\theta : \text{Tors}(\text{Coker } S_0^*) \rightarrow \text{Tors}(\text{Coker } S_1^*)$.

Let $S_{i*} : H_n(F_i) \rightarrow H_n(F_i, K_i)$ and $\partial : H_n(F_i, K_i) \rightarrow H_{n-1}(K_i)$ be the homomorphisms given by the long exact sequence for the pair (F_i, K_i) . Let $U : H^n(F_i) \rightarrow \text{Hom}_{\mathbb{Z}}(H_n(F_i); \mathbb{Z})$ be the universal coefficient isomorphism (F_i is $(n - 1)$ -connected) and let $P : H_n(F_i, K_i) \rightarrow H^n(F_i)$ be the Poincaré duality isomorphism. We have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & \chi_i(H_n(K_i)) & \rightarrow & H_n(F_i) & \xrightarrow{S_{i*}} & H_n(F_i, K_i) & \xrightarrow{\partial} & \partial(H_n(F_i, K_i)) & \rightarrow & 0 \\ & & \parallel & & \parallel & & \cong \downarrow U \circ P & & \downarrow \Delta_i & & \\ 0 & \rightarrow & \text{Ker } S_i^* & \rightarrow & H_n(F_i) & \xrightarrow{S_i^*} & \text{Hom}_{\mathbb{Z}}(H_n(F_i); \mathbb{Z}) & \xrightarrow{d} & \text{Coker } S_i^* & \rightarrow & 0 \end{array}$$

By definition $\Delta_i : \partial(\mathbb{H}_n(F_i, K_i)) \rightarrow \text{Coker } S_i^*$ is the quotient of the isomorphism $U \circ P$, so Δ_i is an isomorphism.

Let us consider again the isomorphism $\theta_j : \mathbb{H}_j(K_0) \rightarrow \mathbb{H}_j(K_1)$, which is defined in (3.2) thanks to the existence of the cobordism. Since F_i is $(n-1)$ -connected then $\partial(\mathbb{H}_n(F_i, K_i)) = \tilde{\mathbb{H}}_{n-1}(K_i)$ and $\theta_n(\text{Ker } \chi_0) = \text{Ker } \chi_1$, so $\theta_{n-1} \circ \partial(\mathbb{H}_n(F_0, K_0)) = \partial(\mathbb{H}_n(F_1, K_1))$.

Let θ be the restriction of the isomorphism $\Delta_1 \circ \theta_{n-1} \circ \Delta_0^{-1}$ on the \mathbb{Z} -torsion of $\text{Coker } S_0^*$.

Let φ be the restriction of θ_n on $\chi_0(\mathbb{H}_n(K_0))$. As $\chi_i(\mathbb{H}_n(K_i)) = \text{Ker } S_i^*$, so φ is defined on $\text{Ker } S_0^*$.

We denote by $\Delta(\varphi)$ the submodule $\{(x, \varphi(x)); x \in \text{Ker } S_0^*\}$ of $\mathbb{H}_n(F_0) \oplus \mathbb{H}_n(F_1)$.

(3.5) **Remark.** By construction φ fulfills: $\varphi \circ \chi_0 = \chi_1 \circ \theta_n$ and $\Delta(\varphi) = \chi(\mathbb{H}_n(K_0))$ where $\chi = (\chi_0, \chi_1 \circ \theta_n)$ as in (3.2).

(3.6) To prove theorem 2, we will construct a metabolizer M (in $\mathbb{H}_n(F_0 \amalg -F_1)$) for $A = A_0 \oplus -A_1$. This metabolizer M will fulfill conditions c.1 and c.2 in definition (1.2) of the algebraic cobordism, for the isomorphisms φ and θ defined in (3.4). To do that, we have to choose an oriented submanifold W of \mathcal{S} with $\partial(W) = N$ (thanks to (3.1) such a W exists). Let $j : \mathbb{H}_n(N) \rightarrow \mathbb{H}_n(W)$ be the homomorphism induced by the inclusion of N in W .

(3.7) **Lemma.** *The form $A = A_0 \oplus -A_1$ vanishes on $\lambda^{-1}(\text{Ker } j^\wedge)$.*

Proof. It is sufficient to prove that A vanishes on $\lambda^{-1}(\text{Ker } j)$. Let $a = [x]$ and $b = [y]$ be two homology classes in $\lambda^{-1}(\text{Ker } j)$. As λ is induced by the inclusion of $F_0 \amalg -F_1$ in N (see (3.2)), there exists two $(n+1)$ -chains α and β in W such that $\partial\alpha = x$ and $\partial\beta = y$. Let i_+ be the positively oriented normal vector field to W in \mathcal{S} . The intersection of α and $i_+(\beta)$ is zero. Hence the linking number in Σ of x and $i_+(y)$ is zero. But this linking number is, by definition, equal to $A(a, b)$, so $A(a, b) = 0$ and the lemma is proved. \square

(3.8) **Lemma.** *Let m be the rank of $\mathbb{H}_n(N)$. The rank of $\text{Ker } j$ is $\frac{m}{2}$.*

Proof. The long exact sequence for the pair (W, N) gives the exactness of:

$$0 \rightarrow \mathbb{H}_{2n+1}(W) \rightarrow \mathbb{H}_{2n+1}(W, N) \rightarrow \mathbb{H}_{2n}(N) \rightarrow \dots \rightarrow \mathbb{H}_{n+1}(W, N) \rightarrow \text{Ker } j \rightarrow 0$$

The alternating sum of the ranks in this exact sequence together with the Poincaré duality give:

$$\text{rk}(\text{Ker } j) = \frac{\text{rk}(\mathbb{H}_n(N))}{2} = \frac{m}{2}.$$

\square

(3.9) **Lemma.** *There exists a direct summand decomposition of $\lambda^{-1}(\text{Ker } j^\wedge)$ in*

$\Delta(\varphi) \oplus R_0 \oplus R$ where $\Delta(\varphi) = \{(x, \varphi(x)); x \in \text{Ker } S_0^*\}$, $R_0 = \lambda^{-1}(\text{Ker } j^\wedge) \cap \text{Ker } S_0^*$, and R is any direct summand complement of $\lambda^{-1}(\text{Ker } j^\wedge) \cap \text{Ker } S^*$ in $\lambda^{-1}(\text{Ker } j^\wedge)$.

Proof. As the considered submodules of $\lambda^{-1}(\text{Ker } j^\wedge)$ are pure, the lemma comes from the following equalities:

$$\begin{aligned} \chi(\text{H}_n(K_0)) &= \text{Ker } \lambda \subset \lambda^{-1}(\text{Ker } j^\wedge) \text{ (see (3.2)),} \\ \Delta(\varphi) &= \chi(\text{H}_n(K_0)) \text{ (see (3.5)),} \\ \text{Ker } S^* &= \chi(\text{H}_n(K_0)) \oplus \text{Ker } S_0^*. \end{aligned} \quad \square$$

(3.10) **Proposition.** *The submodule $M = \Delta(\varphi) \oplus R$ of $\lambda^{-1}(\text{Ker } j^\wedge)$ is a metabolizer for $A = A_0 \oplus -A_1$, which fulfills: $M \cap \text{Ker } S^* = \Delta(\varphi)$.*

Proof. By lemma (3.9), $M \cap \text{Ker } S^* = \Delta(\varphi)$. By (3.6), A vanishes on M . So we only have to show that M is of rank $\frac{m}{2}$. As remarked in (3.3), $r = \text{rk}(\delta(\text{H}_n(N)))$, so $\text{rk}(\delta(\text{Ker } j^\wedge)) \leq r$. Let us consider the following exact sequence induced by (3.2): $0 \rightarrow \Delta(\varphi) \xrightarrow{\lambda} \lambda^{-1}(\text{Ker } j^\wedge) \xrightarrow{\lambda} \text{Ker } j^\wedge \xrightarrow{\delta} \delta(\text{Ker } j^\wedge) \rightarrow 0$. This exact sequence together with the equalities: $\text{rk}(\text{Ker } j^\wedge) = \frac{m}{2}$ (see (3.8)), $\text{rk}(\Delta(\varphi)) = r$; give $\text{rk}(\lambda^{-1}(\text{Ker } j^\wedge)) = r + \frac{m}{2} - \text{rk}(\delta(\text{Ker } j^\wedge))$. So $\text{rk}(\lambda^{-1}(\text{Ker } j^\wedge)) \geq \frac{m}{2}$.

We can remark that if A is non degenerated (as supposed in theorem 2) then we have $\text{rk}(\lambda^{-1}(\text{Ker } j^\wedge)) \leq \frac{1}{2}\text{rk}(\text{H}_n(F_0) \oplus \text{H}_n(F_1)) = \frac{m}{2}$, because A vanishes on $\lambda^{-1}(\text{Ker } j^\wedge)$ (see (3.6)). So, if A is non degenerated, $\text{rk}(\lambda^{-1}(\text{Ker } j^\wedge)) = \frac{m}{2}$, $\text{rk}(\delta(\text{Ker } j^\wedge)) = r$, $\text{rk}(R_0) = 0$ and $M = \lambda^{-1}(\text{Ker } j^\wedge)$ is a metabolizer for A .

Come back to the general case. Let r_0 be the rank of R_0 . By construction: $\text{rk}(M) = \text{rk}(\lambda^{-1}(\text{Ker } j^\wedge)) - r_0 = r + \frac{m}{2} - \text{rk}(\delta(\text{Ker } j^\wedge)) - r_0$.

(3.11) **Lemma.** *The rank l of $\delta(\text{H}_n(N))/\delta(\text{Ker } j^\wedge)$ is greater or equal to r_0 .*

Proof. Let $\{e_j\}$, $j = 1, \dots, r_0$ be a basis of R_0 . Let $\{e_j^*\}$ be in $\text{H}_n(N) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $S_N(\lambda(e_j), e_j^*) = \delta_{ij}$ where S_N is the intersection form on $\text{H}_n(N) \otimes_{\mathbb{Z}} \mathbb{Q}$. The e_j^* exists because S_N is unimodular. Let R^* be the submodule of $\text{H}_n(N) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by $\{e_j^*\}$. Since $R_0 \cap \text{Ker } \lambda = \{0\}$, then $\text{rk}(\lambda(R_0)) = r_0$. As S vanishes on R_0 , then S_N vanishes on $\lambda(R_0)$. It implies that $\text{rk}(R^*) = \text{rk}(R_0) = r_0$, and $\text{Ker } j \cap R^* = \{0\}$. Since $R_0 \subset \text{Ker } S_0^*$, we have $S(x, y) = 0$ for all x in R_0 and all y in $\text{H}_n(F_0 \amalg -F_1)$. So $R^* \cap \lambda(\text{H}_n(F_0 \amalg -F_1)) = \{0\}$ and $\text{rk}(\delta(\text{H}_n(N))/\delta(\text{Ker } j^\wedge)) = l \geq \text{rk}(\delta(R^*)) = \text{rk}(R^*) = r_0$. □

In order to end the proof of (3.10), we only have to show that $\text{rk}(R) = \frac{m}{2} - r$. But $\text{rk}(\delta(\text{Ker } j^\wedge)) = r - l$; so we already have shown that $\text{rk}(R) = \text{rk}(M) - r = \frac{m}{2} - (r - l) - r_0$.

By lemma (3.11), we have $l - r_0 \geq 0$, so $\text{rk}(R) \geq \frac{m}{2} - r$. But $R \cap \text{Ker } S^* = \{0\}$ by construction, and the form \bar{S} induced by S on $\text{H}_n(F_0 \amalg -F_1)/\text{Ker } S^*$ is non-

degenerate of rank $m - 2r$. So $\text{rk}(R) \leq \frac{m}{2} - r$ because \overline{S} vanishes on $\overline{R} = R/(R \cap \text{Ker } S^*)$. \square

(3.12) **Remark.** We have found a metabolizer $M = \Delta(\varphi) \oplus R$ for A which fulfills condition c.1 of the algebraic cobordism without any condition on A . We already have got theorem 4 (see (1.6)). To prove condition c.2 and \overline{M} is pure in \overline{G} , we will have to choose $(n-1)$ -connected Seifert surfaces F_i for K_i on which the Seifert forms A_i are unimodular. So the following proposition (3.13) together with proposition (3.10) imply theorem 2 stated in (1.4).

Let θ_{n-1} be the isomorphism between $H_{n-1}(K_0)$ and $H_{n-1}(K_1)$ defined in (3.2), and let θ the isomorphism between $\text{Tors}(\text{Coker } S_0^*)$ and $\text{Tors}(\text{Coker } S_1^*)$ defined in (3.4). Using the notation of (2.2), let $\Delta(\theta_{n-1})$ (resp. $\Delta(\theta)$) be the group $\{(x, \theta_{n-1}(x)) ; x \in \text{Tors}(H_{n-1}(K_0))\}$ (resp. $\{(x, \theta(x)) ; x \in \text{Tors}(\text{Coker } S_0^*)\}$).

(3.13) **Proposition.** *If A_0 and A_1 are unimodular the metabolizer $M = \Delta(\varphi) \oplus R$ of $A = A_0 \oplus -A_1$, fulfills $d(S^*(M)^\wedge) = \Delta(\theta)$; and \overline{M} is pure in $H_n(F)/\text{Ker } S^*$.*

Proof. Let us denote $F_0 \amalg -F_1$ by F , $K_0 \amalg -K_1$ by K , and $S_0^* \oplus -S_1^*$ by S^* . We consider for F the following commutative diagram already constructed for F_i in (3.4):

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Ker } S_* & \hookrightarrow & H_n(F) & \xrightarrow{S_*} & H_n(F, K) & \xrightarrow{\partial} & \partial(H_n(F, K)) & \rightarrow & 0 \\ & & \parallel & & \parallel & & \cong \downarrow U \circ P & & \cong \downarrow \Delta_0 \oplus \Delta_1 & & \\ 0 & \rightarrow & \text{Ker } S^* & \hookrightarrow & H_n(F) & \xrightarrow{S^*} & \text{Hom}_{\mathbb{Z}}(H_n(F); \mathbb{Z}) & \xrightarrow{d} & \text{Coker } S^* & \rightarrow & 0 \end{array}$$

(3.14) **Lemma.** *The equality $d(S^*(M)^\wedge) = \Delta(\theta)$ is equivalent to the equality $\partial(S_*(M)^\wedge) = \Delta(\theta_{n-1})$.*

Proof. The lemma is a consequence of the two following statements:

The restriction of $\Delta_0 \oplus \Delta_1$ on $\Delta(\theta_{n-1})$ is an isomorphism to $\Delta(\theta)$ because $\theta \circ \Delta_0 = \Delta_1 \circ \theta_{n-1}$ by construction (see (3.4)).

The restriction of $\Delta_0 \oplus \Delta_1$ on $\partial(S_*(M)^\wedge)$ is an isomorphism to $d(S^*(M)^\wedge)$ because the commutativity of the above diagram gives $U \circ P(S_*(M)^\wedge) = S^*(M)^\wedge$. \square

Let $\kappa : H_n(N) \rightarrow H_n(N, C)$ be the homomorphism which is defined in the long exact sequence for the pair (N, C) and $\rho : H_n(N, C) \rightarrow N_n(F, K)$ be the inverse of the excision isomorphism induced by the inclusion of the pair $(F, K) \subset (N, C)$. Let $\xi = \rho \circ \kappa : H_n(N) \rightarrow H_n(F, K)$ and $\overline{\theta} = (\text{Id}, \theta_{n-1}) : H_{n-1}(K_0) \rightarrow H_{n-1}(K)$.

With the notations used in (3.2) we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 \rightarrow & H_n(K_0) & \xrightarrow{\chi} & H_n(F) & \xrightarrow{\lambda} & H_n(N) & \xrightarrow{\delta} & H_{n-1}(K_0) & \rightarrow \\
 (\star) & & & \parallel & \text{(I)} & \downarrow \xi & \text{(II)} & \downarrow \bar{\theta} & \\
 \rightarrow & H_n(K) & \xrightarrow{\chi_0 \oplus \chi_1} & H_n(F) & \xrightarrow{S_*} & H_n(F, K) & \xrightarrow{\partial} & H_{n-1}(K) & \rightarrow
 \end{array}$$

The square (I) is commutative by functoriality, and (II) is commutative by definition of ξ and $\bar{\theta}$.

(3.15) **Lemma.** *If A_0 and A_1 are unimodular, then we have $\delta(\text{Ker } j^\wedge) = \tilde{H}_{n-1}(K_0)$.*

We first show that lemma (3.15) implies proposition (3.13).

We show that \bar{M} is pure in $H_n(F)/\text{Ker } S^*$, which is equivalent to prove that the quotient $H_n(F)/(\text{Ker } S^* + M)$ is torsion free. Since $A = A_0 \oplus -A_1$ is non-degenerate $M = \lambda^{-1}(\text{Ker } j^\wedge)$. Furthermore by diagram (\star) we get $\lambda(\text{Ker } S^*) = \text{Ker } \xi$. Let pr be the projection of $H_n(N)$ on $H_n(N)/(\text{Ker } j^\wedge + \text{Ker } \xi)$, so $\text{Ker } (\text{pr} \circ \lambda) = M + \text{Ker } S^*$. The quotient of $\text{pr} \circ \lambda$ induces an injective map from $H_n(F)/(\text{Ker } S^* + M)$ into $H_n(N)/(\text{Ker } j^\wedge + \text{Ker } \xi)$.

Claim. The module $H_n(N)/(\text{Ker } j^\wedge + \text{Ker } \xi)$ is torsion free.

Proof of the claim. There exists $x_i, i = 1, \dots, r$, in $\text{Ker } j^\wedge$ such that $\tilde{H}_{n-1}(K_0) = \bigoplus_{i=1}^r \langle \delta(x_i) \rangle \oplus \text{Tors}(\tilde{H}_{n-1}(K_0))$. Let $(y_i)_{i=1, \dots, r}$ a basis of $\text{Ker } \xi$ such that $S_N(x_i, y_j) = \delta_{ij}$. By induction on r , we can construct these bases such that $H_n(N) = T \oplus^\perp T^\perp$ where $T = \bigoplus_{i=1}^r \langle x_i, y_i \rangle$. If we denote by D the module $D = T^\perp \cap \text{Ker } j^\wedge$ and by D^* any direct summand complement of D in T^\perp , then we get: $H_n(N)/(\text{Ker } \xi + \text{Ker } j^\wedge) \cong D^*$ which is torsion free. □

Finally $H_n(F)/(\text{Ker } S^* + M)$ is torsion free and \bar{M} is pure in $H_n(F)/(\text{Ker } S^*)$.

So if $n = 1$, the links K_0 and K_1 have torsion free homology groups (\mathcal{K} is a one dimensional compact manifold), so $\text{Tors}(\text{Coker } S^*) = \{0\}$ and we have already proved proposition (3.13).

Now let us take $n \geq 2$.

Thanks to lemma (3.14), the equality: $\Delta(\theta_{n-1}) = \partial(S_*(M)^\wedge)$ gives proposition (3.13). The above diagram (\star) and lemma (3.15) imply: $\bar{\theta}(H_{n-1}(K_0)) =$

$\Delta(\theta_{n-1}) \subset \partial(S_*(M)^\wedge)$. To show that the inclusion: $\Delta(\theta_{n-1}) \subset \partial(S_*(M)^\wedge)$ is an equality, it is sufficient to take any x in $(\partial(S_*(M)^\wedge) \cap \partial(H_n(F_0, K_0)))$, and to show that such a x is zero.

Let us denote by L (resp. L_i) the linking form on $\text{Tors}(H_{n-1}(K))$ (resp. $\text{Tors}(H_{n-1}(K_i))$). By definition (see remark (3.16)) such a form $L = L_0 \oplus -L_1$ is non degenerated and vanishes on $\partial(S_*(M)^\wedge)$ because $S_0 \oplus -S_1$ vanishes on M . Let $(y, \theta_{n-1}(y))$ be in $\Delta(\theta_{n-1})$. Then $L(x, (y, \theta_{n-1}(y))) = L_0(x, y) = 0$ for all $y \in \text{Tors}(H_{n-1}(K_0))$. The non degeneracy of L_0 implies $x = 0$. This ends the proof of proposition (3.13). \square

(3.16) **Remark.** The linking form L is defined as follows (see [L-L, 75] prop. 2.1): Let x, y be in $\text{Tors}(H_{n-1}(K))$ such that p and q are the smallest positive integers with $p.x = q.y = 0$. Let \bar{x} and \bar{y} be in $H_n(F)$ such that $\partial(S_*(\bar{x}) \otimes \frac{1}{p}) = x$ and $\partial(S_*(\bar{y}) \otimes \frac{1}{q}) = y$. Then: $L(x, y) \equiv \frac{1}{p.q} S(\bar{x}, \bar{y}) \pmod{\mathbb{Z}}$.

Proof of lemma (3.15). As shown in (3.10), if $A_0 \oplus -A_1$ is non degenerated, $M = \lambda^{-1}(\text{Ker } j^\wedge)$ has rank $\frac{m}{2}$ and is the chosen metabolizer. So λ induces a monomorphism $\bar{\lambda}$ on $H_n(F)/M$ to $H_n(N)/\text{Ker } j^\wedge$ and we get the following exact sequence:

$$0 \rightarrow H_n(F)/M \xrightarrow{\bar{\lambda}} H_n(N)/\text{Ker } j^\wedge \xrightarrow{\bar{\delta}} \tilde{H}_{n-1}(K_0)/\delta(\text{Ker } j^\wedge) \rightarrow 0.$$

As $\bar{\lambda}$ is injective and M is pure in $H_n(F)$ there exists two \mathbb{Z} -bases $\{\bar{e}_j; j=1, \dots, \frac{m}{2}\}$ of $H_n(F)/M$ and $\{\bar{k}_j; j=1, \dots, \frac{m}{2}\}$ of $H_n(N)/\text{Ker } j^\wedge$ such that $\bar{\lambda}(\bar{e}_j) = p_j \cdot \bar{k}_j$ with $p_j \in \mathbb{Z} \setminus \{0\}$. Let E (resp. H) be a direct summand complement of M (resp. $\text{Ker } j^\wedge$) in $H_n(F)$ (resp. $H_n(N)$). Let also $\{e_j; j=1, \dots, \frac{m}{2}\}$ (resp. $\{k_j; j=1, \dots, \frac{m}{2}\}$) be a \mathbb{Z} -basis of E (resp. H) such that $e_j \equiv \bar{e}_j \pmod{M}$ (resp. $k_j \equiv \bar{k}_j \pmod{\text{Ker } j^\wedge}$). By construction $\lambda(e_j) - p_j \cdot k_j = x \in \text{Ker } j^\wedge$. So there exists a $(n+1)$ -chain γ in W and a positive integer a such that: $\partial\gamma = a\lambda(e_j) - a p_j \cdot k_j$. Let ρ be a $(n+1)$ -chain of $S^{2n+1} \times [0, 1]$ with $\partial\rho = k_j$. So $a e_j$ is the boundary of $\gamma + a p_j \cdot \rho$ in $S^{2n+1} \times [0, 1]$.

Statement: for all m in M , p_j divides $A(e_j, m)$.

Let m be in $M = \lambda^{-1}(\text{Ker } j^\wedge)$ and Δ be a $(n+1)$ -chain in $S^{2n+1} \times [0, 1]$ such that $\partial\Delta = i_+(m)$. By definition $A(a e_j, m)$ is the intersection in $S^{2n+1} \times [0, 1]$ of $\gamma + a p_j \cdot \rho$ and Δ . But $\lambda(a m) \in \text{Ker } j$ so there exists a $(n+1)$ -chain μ in W such that $\partial\mu = a m$. We have $\partial(i_+(\mu)) = a i_+(m)$. Since $\partial(a \Delta) = a i_+(m)$, we get $\gamma \cap (a \Delta) = \gamma \cap (i_+(\mu)) = 0$. But $a > 0$, so $a(\gamma \cap \Delta) = 0$ implies $\gamma \cap \Delta = 0$. Finally $A(a e_j, m) = a p_j \cdot (\rho \cap \Delta)$ and p_j divides $A(e_j, m)$.

If A is unimodular the statement implies that $p_j = \pm 1$ for all $j = 1, \dots, \frac{m}{2}$. So $\bar{\lambda}$ is an isomorphism and his cokernel is zero. As asked we have got: $\delta(\text{Ker } j^\wedge) = \tilde{H}_{n-1}(K_0)$. This ends the proof of lemma (3.15). \square

(3.17) **Remark.** As above we can also prove that: for all m in M p_j divides $A(m, e_j)$.

4. The sufficient condition to have a cobordism

(4.1) Let K_0 and K_1 be two $2n - 1$ dimensional simple links, with $n \geq 3$. We suppose that there exists $(n - 1)$ -connected Seifert surfaces F_0 and F_1 , for K_0 and K_1 , such that the associated Seifert forms A_0 and A_1 are algebraically cobordant. We consider K_0 (resp. $-K_1$) as embedded in the sphere $S^{2n+1} \times \{0\}$ (resp. $S^{2n+1} \times \{1\}$) which are oriented as the boundary of $S^{2n+1} \times [0, 1]$.

Let x be in $S^{2n+1} \times \{0\}$ such that $(x \times [0, 1]) \cap (F_0 \amalg -F_1)$ is empty, and let U be a "small" open ball around x in $S^{2n+1} \times \{0\}$. The boundary S of the disk $D = (S^{2n+1} \times [0, 1]) \setminus (U \times [0, 1])$ contains $F_0 \amalg -F_1$. Let G be the closure of the connected sum, in S , of the interiors $\overset{\circ}{F}_0$ and $-\overset{\circ}{F}_1$. By construction $A = A_0 \oplus -A_1$ is the Seifert form of $K_0 \amalg -K_1$, associated to G .

(4.2) *Proof of theorem 3.* In order to prove theorem 3 we will do in D , an embed surgery on G , the result of which being a manifold \tilde{G} diffeomorphic to $\mathcal{K} \times [0, 1]$.

By proposition (2.1) we can choose a good basis $\mathcal{B} = \{(m_i, m_i^*); i=1, \dots, s+r\}$ of $H_n(G)$. Thanks to J. Milnor ([M1, 61] lemma 6 p. 50), any cycle of G can be represented by the image of an embedding of S^n . Furthermore:

(4.3) **Proposition.** *There exists $s+r$ disjoint embeddings $\psi_i : D^{n+1} \times D^n \rightarrow D$ such that for any $i \in \{1, \dots, s+r\}$ we have*

- 1- $[\psi_i(S^n \times \{0\})] = m_i,$
- 2- $(\psi_i(D^{n+1} \times D^n)) \cap G = \psi_i(D^{n+1} \times D^n) \cap S = \psi_i(S^n \times D^n).$

Proof. Let $\overline{\psi}_i : S^n \rightarrow G$ be an embedding of S^n which represents m_i . Let i, j with $i \neq j$, be in $\{1, \dots, s+r\}$, then m_i and m_j are in the metabolizer M and we have: $S(m_i, m_j) = A(m_i, m_j) + (-1)^n A(m_j, m_i) = 0$. Since $n \geq 3$, thanks to Whitney's procedure [Wh, 44] we can choose the $\overline{\psi}_i$ such that $\overline{\psi}_i(S^n) \cap \overline{\psi}_j(S^n) = \emptyset$. Since $n \geq 2$, the Whitney obstruction to extend $\overline{\psi}_i$ to disjoint embeddings ψ_i of D^{n+1} in the $(2n+2)$ -disk D , is the matrix $A(m_i, m_j)$ which is zero. Furthermore, $A(m_i, m_i) = 0$ is the classical obstruction to extend ψ_i to $\psi_i : D^{n+1} \times D^n \rightarrow D$. (see [Br, 72] and for details see [Bl, 94] proposition 5.1.2, p.58). We choose this extension ψ_i such that the restriction to $S^n \times D^n$ is a tubular neighbourhood of $\psi_i(S^n)$ in G . □

So thanks to proposition (4.3) we obtain a submanifold \tilde{G} of D as follows:

$$\tilde{G} = (G \setminus (\prod_{i=1}^{s+r} \psi_i(S^n \times D^n))) \cup (\prod_{i=1}^{s+r} \psi_i(D^{n+1} \times S^{n-1})).$$

(4.4) **Proposition.** *The inclusion k_0 (resp. k_1) of K_0 (resp. K_1) in \tilde{G} , induces isomorphisms $k_{0,j}$ (resp. $k_{1,j}$) from $H_j(K_0)$ (resp. $H_j(K_1)$) to $H_j(\tilde{G})$ for all j .*

(4.5) **Corollary.** *We have $H_*(\tilde{G}, K_0) = H_*(\tilde{G}, K_1) = 0$.*

This corollary (4.5) and the h-cobordism theorem imply that \tilde{G} is diffeomorphic to $K_0 \times [0, 1]$. More precisely $\dim \tilde{G} = 2n \geq 6$ and:

h-cobordism Theorem [M2, 65]. *Let \mathcal{M} be a k -dimensional differentiable compact manifold with $\partial\mathcal{M} = \mathcal{M}_0 \amalg \mathcal{M}_1$ such that \mathcal{M} , \mathcal{M}_0 and \mathcal{M}_1 are simply connected. If $H_*(\mathcal{M}, \mathcal{M}_0) = 0$ and $k \geq 6$ then \mathcal{M} is diffeomorphic to $\mathcal{M}_0 \times [0, 1]$.*

So to end the proof of theorem 3 we only have to prove proposition (4.4).

Proof of proposition (4.4). According to proposition (2.1), the intersection form on $H_n(F)$ splits in an orthogonal sum on the submodules $\langle m_i, m_i^* \rangle$, $i = 1, \dots, s+r$. So the proof of (4.4) when $s+r=1$ implies the general case.

Let us suppose that $\text{rk}(M) = 1$ and let m be a generator of M , then $H_n(G) = \langle m, m^* \rangle$. We denote by $\psi : D^{n+1} \times D^n \rightarrow D$ an embedding choosen as in proposition (4.3), by $\eta : S^n \rightarrow G$ an embedding such that $[\eta(S^n)] = m^*$, and by G_T the manifold $G_T = G \setminus \psi(S^n \times D^n)$.

(4.6) The Mayer-Vietoris sequence associated to the following decomposition of the manifold: $G = G_T \cup \psi(S^n \times D^n)$ gives:

$$\begin{aligned} 0 \rightarrow H_n(\psi(S^n \times S^{n-1})) \rightarrow H_n(G_T) \oplus H_n(\psi(S^n \times D^n)) \rightarrow H_n(G) \\ \xrightarrow{\delta} H_{n-1}(\psi(S^n \times S^{n-1})) \rightarrow H_{n-1}(G_T) \rightarrow 0. \end{aligned}$$

where δ is given by the intersection of cycles with m .

(4.7) The Mayer-Vietoris sequence associated to the following decomposition of the manifold: $\tilde{G} = G_T \cup \psi(D^{n+1} \times S^{n-1})$ gives:

$$\begin{aligned} 0 \rightarrow H_n(\psi(S^n \times S^{n-1})) \xrightarrow{\alpha} H_n(G_T) \rightarrow H_n(\tilde{G}) \xrightarrow{\gamma} H_{n-1}(\psi(S^n \times S^{n-1})) \\ \xrightarrow{\beta} H_{n-1}(\psi(D^{n+1} \times S^{n-1})) \oplus H_{n-1}(G_T) \rightarrow H_{n-1}(\tilde{G}) \rightarrow 0. \end{aligned}$$

Remark that the homomorphism β is injective into $H_{n-1}(\psi(D^{n+1} \times S^{n-1}))$, hence $\gamma = 0$ and the sequence (4.7) splits up into:

$$(4.8) \quad 0 \rightarrow H_n(\psi(S^n \times S^{n-1})) \xrightarrow{\alpha} H_n(G_T) \rightarrow H_n(\tilde{G}) \rightarrow 0,$$

$$(4.9) \quad 0 \rightarrow H_{n-1}(\psi(S^n \times S^{n-1})) \xrightarrow{\beta} H_{n-1}(\psi(D^{n+1} \times S^{n-1})) \oplus H_{n-1}(G_T) \rightarrow H_{n-1}(\tilde{G}) \rightarrow 0.$$

Since $\text{rk}(M) = 1 = s+r$ we have to consider the two following cases: $s=0, r=1$ and $s=1, r=0$.

★ 1^{st} case: $s=0$ and $r=1$, then $\text{Ker } S^* = \langle m, m^* \rangle$.

In sequence (4.6) we have $\text{Ker } \delta = \langle m, m^* \rangle$, then $H_n(G_T) = \langle [\psi(S^n \times \{1\})], [\eta(S^n)] \rangle$ and $H_{n-1}(G_T) = \langle [\psi(\{1\} \times S^{n-1})] \rangle$. In sequence (4.8) we have $\text{Im } \alpha =$

$\langle [\psi(S^n \times \{1\})] \rangle$, so $H_n(\tilde{G}) = \langle [\eta(S^n)] \rangle$. By construction of the good basis (2.1), $[\eta(S^n)]$ is a generator of $\text{Im}(H_n(K_0) \rightarrow H_n(G))$. So the inclusion of K_0 in \tilde{G} induces the isomorphism: $k_{0,n} : H_n(K_0) \xrightarrow{\cong} H_n(\tilde{G})$.

Since $H_{n-1}(G_T) = \langle [\psi(\{1\} \times S^{n-1})] \rangle$ in sequence (4.9), we have $H_{n-1}(\tilde{G}) = \langle [\psi(\{1\} \times S^{n-1})] \rangle$. Condition c.1 of the algebraic cobordism gives that there exists a in $\text{Ker } S_0^*$ such that $m = (a, \varphi(a))$. If we denote by $\gamma_0 : H_n(K_0) \rightarrow H_n(G)$ the homomorphism induced by the inclusion, then we can choose b in $H_{n-1}(K_0)$ such that $H_{n-1}(K_0) = \langle b \rangle$ and b is the dual of $\gamma_0^{-1}(a)$ for the intersection form of K_0 . There exists B in $H_n(G, K_0)$ such that $\partial B = b$ and the intersection between B and m is $+1$. The boundary of the n -chain $(B - (B \cap \psi(S^n \times \overset{\circ}{D}^n)))$ is homologous to the $(n-1)$ -cycle $b - (\psi(\{1\} \times S^{n-1}))$, hence b and $[\psi(\{1\} \times S^{n-1})]$ are homologous in $H_{n-1}(\tilde{G}) = \langle [\psi(\{1\} \times S^{n-1})] \rangle$. Thus the inclusion of K_0 in \tilde{G} induces the isomorphism: $k_{0,n-1} : H_{n-1}(K_0) \xrightarrow{\cong} H_{n-1}(\tilde{G})$.

★ 2^{nd} case: $s = 1$ and $r = 0$, then $\text{Ker } S^* = \{0\}$ and $H_n(K_0) = 0$. In sequence (4.6) we have $\text{Ker } \delta = \langle m \rangle$, then $H_n(G_T) = \langle [\psi(S^n \times \{1\})] \rangle$ and $H_{n-1}(G_T) = \langle [\psi(\{1\} \times S^{n-1})] \rangle$. In sequence (4.8) we have $\text{Im } \alpha = \langle [\psi(S^n \times \{1\})] \rangle$. Since $H_n(G_T) = \langle [\psi(S^n \times \{1\})] \rangle$ we have $H_n(\tilde{G}) = 0 = H_n(K_0)$.

- if $S_*(m)$ is indivisible (i.e. $H_{n-1}(K_0) = 0$), then δ in (4.6) is surjective. Thus $H_{n-1}(\tilde{G}) = 0 = H_{n-1}(K_0)$.

- If $a \neq 1$ is the greatest divisor of $S_*(m)$ (i.e. $H_{n-1}(K_0) \cong \mathbb{Z}/a\mathbb{Z}$) then condition c.2 of algebraic cobordism together with lemma (3.14) give that there exists c in $H_{n-1}(K_0)$ such that $\partial(\frac{1}{a} S_*(m)) = (c, \theta_{n-1}(c))$. Let b in $H_{n-1}(K_0)$ be the dual of c for the linking form of K_0 . There exists B in $H_n(G, K_0)$ such that $\partial B = b$ and the intersection between B and m is $+1$. As before the boundary of the n -chain $B - (B \cap \psi(S^n \times \overset{\circ}{D}^n))$ is the n -cycle $b - \psi(\{1\} \times S^{n-1})$, hence b and $[\psi(\{1\} \times S^{n-1})]$ are homologous in $H_{n-1}(G)$. Since $H_{n-1}(G_T) = \langle [\psi(\{1\} \times S^{n-1})] \rangle$ in sequence (4.9) we have $H_{n-1}(\tilde{G}) = \langle [\psi(\{1\} \times S^{n-1})] \rangle$. Thus b and $[\psi(\{1\} \times S^{n-1})]$ are homologous in $H_{n-1}(\tilde{G})$ and the inclusion of K_0 in \tilde{G} induces the isomorphism: $k_{0,n-1} : H_{n-1}(K_0) \xrightarrow{\cong} H_{n-1}(\tilde{G})$.

Since \tilde{G} is obtained by surgery on n -cycles, this surgery only modifies homology groups of dimensions n and $n - 1$. Hence for $k \neq n, n - 1$ we have $H_k(G) \cong H_k(K_0) \xrightarrow{k_{0,k}} H_k(\tilde{G})$. By symmetry we also have the same results with K_1 . Finally $k_{0,j}$ and $k_{1,j}$ are some isomorphisms for all j . This ends the proof of proposition (4.4), and the proof of theorem 3. \square

5. Appendix – Alexander polynomials of cobordant links.

Let K be a $2n - 1$ dimensional simple link, and $\varepsilon = (-1)^n$. One can associate a polynomial $\Delta \in \mathbb{Z}[X]$ to any Seifert surface F for the link K , defined by: $\Delta(X) =$

$\det(XA + \varepsilon A^T)$, where A is the Seifert form associated to F . Such a polynomial Δ is called a Alexander polynomial for the link K . Changing the Seifert surface to another multiplies Δ by $\pm X^m$ with m in \mathbb{Z} .

For a polynomial γ in $\mathbb{Z}[X]$ we define the polynomial γ^* by: $\gamma^*(X) = X^{\deg \gamma} \gamma(X^{-1})$.

(5.1) **Proposition.** *Let K_0 and K_1 be two cobordant simple $2n-1$ dimensional links. If Δ_0 and Δ_1 are Alexander polynomials for K_0 and K_1 , then there exists γ in $\mathbb{Z}[X]$ such that: $\gamma\gamma^* = \pm\Delta_0\Delta_1$.*

Remark. If F is the Milnor fiber of an algebraic link K , then the associated Alexander polynomial is the characteristic polynomial of the monodromy. Hence the above proposition and the monodromy theorem imply corollary (0.7).

Proof of proposition (5.1). We denote by F_0 and F_1 two $(n-1)$ -connected Seifert surfaces for K_0 and K_1 , and by A_0 and A_1 the associated Seifert forms. The links K_0 and K_1 are cobordant so proposition (3.10) implies that the form $A = A_0 \oplus -A_1$ has a metabolizer M . Therefore, there exists a basis for $H_n(F_0) \oplus H_n(F_1)$ such that in this basis the matrix for A is $\begin{pmatrix} 0 & B_1 \\ B_2 & B_3 \end{pmatrix}$ where B_i , $i=1,2,3$ are square matrices. We have $\Delta_0(X).\Delta_1(X) = \det(XA + \varepsilon A^T)$, hence $\Delta_0(X).\Delta_1(X) = \varepsilon.\det(XB_1 + \varepsilon B_2^T).\det(XB_2 + \varepsilon B_1^T)$. Let $\gamma(X)$ be $\det(XB_1 + \varepsilon B_2^T)$, then $\gamma^*(X) = \det(XB_2 + \varepsilon B_1^T)$. Finally we get $\gamma.\gamma^* = \pm\Delta_0.\Delta_1$. \square

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