

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 72 (1997)

Artikel: The zero-norm subspace of bounded cohomology
Autor: Soma, Teruhiko
DOI: <https://doi.org/10.5169/seals-54606>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 05.02.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

The zero-norm subspace of bounded cohomology

Teruhiko Soma

Abstract. Let Σ be a closed, orientable surface of genus > 1 . In this paper, non-trivial elements α of the third bounded cohomology $H_b^3(\Sigma; \mathbf{R})$ with $\|\alpha\| = 0$ are given constructively by using both a hyperbolic metric and a singular euclidean metric on $\Sigma \times \mathbf{R}$. Furthermore, it is shown that the dimension of the subspace $N^3(\Sigma)$ of $H_b^3(\Sigma; \mathbf{R})$ consisting of zero-norm elements is the cardinality of the continuum.

Mathematics Subject Classification (1991). 57M50, 55N99.

Keywords. Bounded cohomology, hyperbolic structures, measured foliations.

Introduction

Let X be a topological space and $C^k(X)$ the k -cochain group of real coefficient. The \mathbf{R} -subspace $C_b^k(X)$ of $C^k(X)$ consists of elements $c \in C^k(X)$ with

$$\|c\| = \sup\{|c(\sigma)|; \sigma: \Delta^k \longrightarrow X \text{ is a singular } k\text{-simplex}\} < \infty.$$

Consider the restriction $\delta_b^k = \delta^k|_{C_b^k(X)}: C_b^k(X) \longrightarrow C_b^{k+1}(X)$ of the coboundary operator $\delta^k: C^k(X) \longrightarrow C^{k+1}(X)$. Then, the cochain complex $(C_b^*(X), \delta_b^*)$ defines the *bounded cohomology*

$$H_b^*(X; \mathbf{R}) = Z_b^*(X)/B_b^*(X),$$

where $Z_b^k(X) = \text{Ker}(\delta_b^k)$, $B_b^k(X) = \text{Im}(\delta_b^{k-1})$. We refer to Gromov [7] for fundamental results on bounded cohomology. The *pseudonorm* $\|\alpha\|$ of $\alpha \in H_b^k(X; \mathbf{R})$ is defined by

$$\|\alpha\| = \inf\{\|c\|; c \in Z_b^k(X) \text{ with } [c] = \alpha\}.$$

We say that $N^k(X) = \{\alpha \in H_b^k(X; \mathbf{R}); \|\alpha\| = 0\}$ is the *zero-norm subspace* of $H_b^k(X; \mathbf{R})$. For any topological space X , Matsumoto–Morita [9] and Ivanov [8] proved independently that $N^k(X) = \{0\}$ whenever $k \leq 2$. At that moment, any examples of non-trivial $N^k(X)$ were not known for $k \geq 3$.

Here, we are mainly concerned with the case where the space X is a closed, connected, orientable surface Σ of genus > 1 . Then, the structure of the second bounded cohomology $H_b^2(\Sigma; \mathbf{R})$ was studied by Brooks–Series [2], Mitsumatsu [10], Barge–Ghys [1], Epstein–Fujiwara [4] and that of the third $H_b^3(\Sigma, \mathbf{R})$ by Yoshida [18], Soma [11], [12] and so on. We refer to Grigorchuk [6] for other useful references on bounded cohomology. Furthermore, the author showed in [13] that $N^3(\Sigma)$ is non-trivial by invoking Matsumoto–Morita [9, Theorem 2.3]. However, since the proof of their theorem relies on the Hahn–Banach theorem, we could not construct any non-trivial elements of $N^3(\Sigma)$ practically.

In this paper, non-trivial elements of $N^3(\Sigma)$ are given constructively by using both a hyperbolic metric and a singular euclidean metric on $\Sigma \times \mathbf{R}$, where the latter metric is defined by using a measured foliation associated to a pseudo-Anosov automorphism of Σ . A combination of these two metrics presents a continuous family $\{[c_{r,\varepsilon}]; 0 < r \leq 1\}$ of elements of $N^3(\Sigma \times \mathbf{R})$ which are linearly independent in $H_b^3(\Sigma \times \mathbf{R}; \mathbf{R}) \cong H_b^3(\Sigma; \mathbf{R})$, see Theorems 1 and 2 in §2 for details. In particular, it is shown that the dimension of the \mathbf{R} -vector subspace $N^3(\Sigma)$ of $H_b^3(\Sigma; \mathbf{R})$ is the cardinality of the continuum.

The key fact in our arguments is that the bounded 3-cocycle $c_{r,\varepsilon}$ given in §2 is the coboundary of a certain unbounded 2-cochain. For the proof, it is crucial that the 3-dimensional euclidean space \mathbf{E}^3 is the product metric space $\mathbf{E}^2 \times \mathbf{E}^1$. This is the main reason why we use a euclidean metric as well as a hyperbolic metric.

§1. Euclidean and hyperbolic structures on manifolds

Let Σ be a closed, connected and oriented surface of genus > 1 . A *measured foliation* \mathcal{F} on Σ is a topological foliation with finitely many prong singular points of degree ≥ 3 and equipped with the transverse measure. The set of singular points of \mathcal{F} is denoted by $S_{\mathcal{F}}$. An orientation-preserving homeomorphism $f: \Sigma \rightarrow \Sigma$ is called a *pseudo-Anosov automorphism* if there exists $\lambda = \lambda(f) > 1$ and a pair of mutually transverse, measured foliations $\mathcal{F}^s, \mathcal{F}^u$ with $S_{\mathcal{F}^s} = S_{\mathcal{F}^u} (= S(f))$ and $f(\mathcal{F}^s) = \lambda^{-1}\mathcal{F}^s, f(\mathcal{F}^u) = \lambda\mathcal{F}^u$. We refer to [3], [5] and [16] for the existence and fundamental properties of such automorphisms and for typical pictures of $\mathcal{F}^{s(u)}$ near $p \in S(f)$.

Note that the pair of these measured foliations $\mathcal{F}^u, \mathcal{F}^s$ determines an incomplete, euclidean structure, a smooth structure on $\Sigma^\circ = \Sigma - S(f)$. We will define a smooth structure on Σ extending that on Σ° . For any $n \in \mathbf{N}$ with $n \geq 3$, the euclidean 2-space $\mathbf{R}^2 = \mathbf{C}$; $(x, y) = x + \sqrt{-1}y$ is divided into the n sectors V_1, \dots, V_n such that

$$V_k = \left\{ r \exp(\sqrt{-1}\theta) \in \mathbf{C}; r \geq 0, \frac{2(k-1)\pi}{n} \leq \theta \leq \frac{2k\pi}{n} \right\}$$

for $k = 1, \dots, n$. The upper half plane $H = \{z \in \mathbf{C}; \operatorname{Im}(z) \geq 0\}$ admits the euclidean structure induced from that on $\mathbf{C} = \mathbf{R}^2$. Let $\chi_k: V_k \rightarrow H$ be the

homeomorphism defined by

$$\chi_k(r \exp(\sqrt{-1}\theta)) = r \exp\left(\sqrt{-1}\left(\frac{n\theta}{2} - (k-1)\pi\right)\right).$$

Note that the Jacobian of χ_k with respect to the standard euclidean coordinates on V_k and H is the constant $n/2$. Let $\mathcal{F}_H^s, \mathcal{F}_H^u$ be the measured foliation on H such that the set of leaves in \mathcal{F}_H^s (resp. \mathcal{F}_H^u) consists of straight lines parallel to (resp. straight rays orthogonal to) the x -axis ∂H and such that the transverse measures are induced from the euclidean metric on H . Then, the pair $\{\mathcal{F}_n^s, \mathcal{F}_n^u\}$ of measured foliations on \mathbf{R}^2 with the prong singular point $(0,0)$ of degree n is defined by

$$\mathcal{F}_n^s = \bigcup_{k=1}^n \chi_k^{-1}(\mathcal{F}_H^s), \quad \mathcal{F}_n^u = \bigcup_{k=1}^n \chi_k^{-1}(\mathcal{F}_H^u).$$

For a sufficiently small $\varepsilon > 0$, there exist mutually disjoint neighborhoods U_p of $p \in S(f)$ in Σ and homeomorphisms $\varphi_p: U_p \rightarrow D(\varepsilon) = \{z \in \mathcal{C}; |z| < \varepsilon\}$ such that $\varphi_p(\mathcal{F}_n^s|_{U_p}) = \mathcal{F}_n^s|_{D(\varepsilon)}, \varphi_p(\mathcal{F}_n^u|_{U_p}) = \mathcal{F}_n^u|_{D(\varepsilon)}$. For $V_k(\varepsilon) = \varphi_p^{-1}(D(\varepsilon) \cap V_k)$, the composition $\chi_k \circ \varphi_p|_{V_k(\varepsilon) - \{p\}}: V_k(\varepsilon) - \{p\} \rightarrow H - \{0\}$ is a locally isometric embedding if $V_k(\varepsilon) - \{p\}$ has the euclidean metric induced from that on Σ° . Regarding $\{(U_p, \varphi_p); p \in S(f)\}$ as a family of coordinate systems for Σ in $\cup_p U_p$, one can define the smooth structure on Σ extending that on Σ° . Then, $\Sigma \times I$ admits the product smooth structure, where I is the closed interval $[0, 1]$. From now on, we identify $\cup_p U_p \times I$ with $\cup_p D_p(\varepsilon) \times I$ via $\varphi_p \times \text{id}_I$'s, where $D_p(\varepsilon)$ are copies of $D(\varepsilon)$. Note that the homeomorphism $f: \Sigma \times \{0\} \rightarrow \Sigma \times \{1\}$ is not a diffeomorphism with respect to this smooth structure. So, we need another smooth structure on $\Sigma \times I$. For any t with $0 \leq t \leq 1$, consider the elliptic half-disk

$$E_t = \left\{ (x, y) \in \mathbf{R}^2; \lambda^{2+2t}x^2 + \lambda^{2-2t}y^2 = \varepsilon^2, y \geq 0 \right\}$$

in H . Set $W_{p,t} = \bigcup_{k=1}^n \chi_k^{-1}(E_t) \subset D_p(\varepsilon)$, and

$$X_p = \{(q, t); t \in I, q \in W_{p,t}\} \subset D_p(\varepsilon) \times I \subset \Sigma \times I.$$

For simplicity, we denote the product homeomorphism $\chi_k \times \text{id}_I: V_k \times I \rightarrow H \times I$ by $\widehat{\chi}_k$. The homeomorphism $\psi_p: X_p \rightarrow D_p(\varepsilon/\lambda) \times I$ is defined by

$$\psi_p(q, t) = \widehat{\chi}_k^{-1}(\lambda^t x, \lambda^{-t} y, t)$$

if $q \in \chi_k^{-1}(E_t)$ and $\chi_k(q) = (x, y)$. By taking $\{(X_p, \psi_p); p \in S(f)\}$ as a coordinate system for $\Sigma \times I$ instead of $\{(U_p \times I, \varphi_p \times \text{id}_I); p \in S(f)\}$, we have a new smooth structure on $\Sigma \times I$, and denote this smooth manifold by $\Sigma \times I^{\text{new}}$. Then, $f: \Sigma \times \{0\}^{\text{new}} \rightarrow \Sigma \times \{1\}^{\text{new}}$ is a diffeomorphism. In particular, the mapping torus $M = \Sigma \times I^{\text{new}} / \{(x, 0) \sim (f(x), 1)\}$ admits the induced smooth structure.

Let $\text{Vol}_{(1)}(B)$ (resp. $\text{Vol}_{(2)}(B)$) denote the volume of a compact 3-dimensional submanifold B in $X_p^\circ = X_p - \{p\} \times I$ (resp. in $D_p(\varepsilon/\lambda) \times I$) with respect to the incomplete euclidean metric on $X_p^\circ \subset \Sigma^\circ \times I$ (resp. the standard euclidean metric on $D_p(\varepsilon/\lambda) \times I$). Similarly, the areas of subsurfaces F in X_p° and $D_p(\varepsilon/\lambda) \times I$ are denoted by $\text{Area}_{(1)}(F)$ and $\text{Area}_{(2)}(F)$, respectively.

We denote the degree of \mathcal{F}^s (or \mathcal{F}^u) at $p \in S(f)$ by $n(p)$. Then, the following lemma holds.

Lemma 1. (i) For any compact 3-dimensional submanifold B of X_p° ,

$$\text{Vol}_{(1)}(B) = \frac{n(p)}{2} \text{Vol}_{(2)}(\psi_p(B)).$$

(ii) For any compact subsurface F of X_p° ,

$$\text{Area}_{(1)}(F) \leq \frac{n(p)\lambda}{2} \text{Area}_{(2)}(\psi_p(F)).$$

Proof. Since $X_k^\circ \subset D_p(\varepsilon)^\circ \times I = \bigcup_{k=1}^n V_k(\varepsilon)^\circ \times I$, if necessary dividing B and F into smaller pieces, we may assume that B and F are contained in $V_k(\varepsilon)^\circ \times I$ for some $k \in \{1, \dots, n(p)\}$, where $V_k(\varepsilon)^\circ = D_p(\varepsilon) \cap V_k - \{p\}$. Set $B' = \hat{\chi}_k(B)$ and $F' = \hat{\chi}_k(F)$. Recall that $\hat{\chi}_k|_{V_k(\varepsilon)^\circ \times I}: V_k(\varepsilon)^\circ \times I \rightarrow (H - \{0\}) \times I$ is a locally isometric embedding if $V_k(\varepsilon)^\circ$ has the incomplete euclidean metric induced from that on Σ° . For the diffeomorphism $\Psi: H \times I \rightarrow H \times I$ with $\Psi(x, y, t) = (\lambda^t x, \lambda^{-t} y, t)$, we have

$$\begin{aligned} \text{Vol}_{(1)}(B) &= \text{Vol}_{H \times I}(B') = \text{Vol}_{H \times I}(\Psi(B')) \quad \text{and} \\ \text{Area}_{(1)}(F) &= \text{Area}_{H \times I}(F') \leq \lambda \text{Area}_{H \times I}(\Psi(F')). \end{aligned}$$

Since $\hat{\chi}_k(\psi_p(B)) = \Psi(B')$ and $\hat{\chi}_k(\psi_p(F)) = \Psi(F')$ and since the Jacobian of $\chi_k: V_k \rightarrow H$ is $n(p)/2$, we have

$$\frac{n(p)}{2} \text{Vol}_{(2)}(\psi_p(B)) = \text{Vol}_{H \times I}(\Psi(B')) \quad \text{and} \quad \frac{n(p)}{2} \text{Area}_{(2)}(\psi_p(F)) \geq \text{Area}_{H \times I}(\Psi(F')).$$

This completes the proof. \square

Let $\rho: \widetilde{M} = \Sigma \times \mathbf{R} \rightarrow M$ be the infinite cyclic covering associated to $\pi_1(\Sigma) \subset \pi_1(M)$, and set $\widetilde{L} = S(f) \times \mathbf{R}$. Note that $\widetilde{M}^\circ = \Sigma^\circ \times \mathbf{R}$ has the product, incomplete euclidean metric induced from the euclidean metrics on Σ° and \mathbf{R} . For the euclidean area form η_{Σ° on Σ° , $\tilde{\eta} = \zeta^*(\eta_{\Sigma^\circ})$ is a 2-form on \widetilde{M}° , where $\zeta: \Sigma^\circ \times \mathbf{R} \rightarrow \Sigma^\circ$ is the orthogonal projection. The volume form $\widetilde{\Omega}_E$ on \widetilde{M}° is

given by $\tilde{\Omega}_E = \tilde{\eta} \wedge dt$. The diffeomorphism $\tilde{f}: \tilde{M} \rightarrow \tilde{M}$ with $\tilde{f}(x, t) = (f(x), t + 1)$ is the generator of the covering transformation group. Since $f|_{\Sigma^\circ}: \Sigma^\circ \rightarrow \Sigma^\circ$ is a euclidean-area-preserving diffeomorphism, $\tilde{f}|_{\tilde{M}^\circ}$ is a volume-preserving diffeomorphism, that is, $\tilde{f}^*(\tilde{\Omega}_E) = \tilde{f}^*(\tilde{\eta}) \wedge \tilde{f}^*(dt) = \tilde{\eta} \wedge dt = \tilde{\Omega}_E$. Thus, there exists the 3-form Ω_E in $M^\circ = M - L$ with $\rho^*(\Omega_E) = \tilde{\Omega}_E$, where $L = \rho(\tilde{L})$ is a link in M . Similarly, there exists a 2-form η_{M° on M° with $\rho^*(\eta_{M^\circ}) = \tilde{\eta}$. According to Thurston [17] (see also Sullivan [14]), the smooth manifold M admits a hyperbolic structure. For the hyperbolic volume form Ω_H on M , there exists a positive, smooth function $h: M^\circ \rightarrow \mathbf{R}$ with $\Omega_E = h\Omega_H$. We suppose that \tilde{M} admits the hyperbolic metric induced from that on M via ρ .

For the derivative $d\xi$ of the smooth embedding

$$\xi = \bigcup_{p \in S(f)} \psi_p^{-1}: \bigcup_{p \in S(f)} D_p(\varepsilon/\lambda) \times I \rightarrow \Sigma \times I^{\text{new}} \subset \tilde{M},$$

we set

$$\iota(\xi) = \inf \left\{ \|d\xi_x(v)\|_{\tilde{M}}; x \in \bigcup_{p \in S(f)} D_p(\varepsilon/\lambda) \times I, v \in TU_x \left(\bigcup_{p \in S(f)} D_p(\varepsilon/\lambda) \times I \right) \right\} > 0,$$

where $TU(\bigcup_{p \in S(f)} D_p(\varepsilon/\lambda) \times I)$ is the unit tangent bundle over the euclidean manifold $\bigcup_{p \in S(f)} D_p(\varepsilon/\lambda) \times I$. We note that the image $Y = \rho(\bigcup_{p \in S(f)} D_p(\varepsilon/\lambda) \times I)$ is a union of solid tori in M , and the complement $M - \text{int } Y$ of $\text{int } Y$ is a compact manifold.

Lemma 2. $K_1 = \sup\{h(s); s \in M^\circ\} < \infty$.

Proof. For any compact 3-dimensional submanifold B of $Y - L$, we have $\iota(\xi)^3 \text{Vol}_{(2)}(\tilde{B}) \leq \text{Vol}_M(B)$, where $\tilde{B} = \rho^{-1}(B) \cap (\Sigma \times I^{\text{new}})$. Then, by Lemma 1 (i), we have

$$\sup\{h(s); s \in M^\circ\} \leq \max \left\{ \max\{h(s); s \in M - \text{int } Y\}, \frac{n(f)}{2\iota(\xi)^3} \right\} < \infty,$$

where $n(f) = \max\{n(p); p \in S(f)\}$. This completes the proof. \square

Note that, in general, for a sequence $\{s_m\}$ in M° converging to a point in L , the limit $\lim_{m \rightarrow \infty} h(s_m)$ does not exist. Then, we can not extend h to a continuous map on M .

Let Q be a 2-dimensional subspace of $T_s(M^\circ)$ for $s \in M^\circ$. There exists a small, hyperbolic disk D centered at $x_0 \in \mathbf{H}^2$ and an embedding $i_Q: D \rightarrow M$ with $i_Q(x_0) = s$, $i_{Q*}(T_{x_0}(D)) = Q$ and such that i_Q is an isometry onto the image $i_Q(D)$ which is totally geodesic with respect to the hyperbolic metric on M . Let

$\varphi_Q: D^\circ \rightarrow \mathbf{R}$ be the smooth function with $i_Q^*(\eta_{M^\circ}) = \varphi_Q \cdot \eta_H$ on D° , where $D^\circ = D - i_Q^{-1}(L)$ and η_H is the hyperbolic area form on D . Then, we have

$$\int_{D^\circ} |i_Q^*(\eta_{M^\circ})| \leq \sup\{|\varphi_Q(x)|; x \in D^\circ\} \text{Area}_M(D^\circ), \quad (1.1)$$

where $\text{Area}_M(D^\circ)$ ($= \text{Area}_M(D)$) denotes the hyperbolic area of D° . Intuitively, $\varphi_Q(x_0)$ represents the ratio, in the cross section Q , of η_{M° to the hyperbolic metric at $s \in M$. It is easily seen that there exists the maximum

$$g(s) = \max\{|\varphi_Q(x_0)|; Q \text{ is a 2-dimensional subspace of } T_s(M^\circ)\},$$

and $g: M^\circ \rightarrow \mathbf{R}$ is a continuous, non-negative function. The following lemma is proved by the argument similar to that in Lemma 2.

Lemma 3. $K_2 = \sup\{g(s); s \in M^\circ\} < \infty$.

Proof. As in the proof of Lemma 2, for any compact subsurface F of $Y - L$, the inequality $\iota(\xi)^2 \text{Area}_{(2)}(\tilde{F}) \leq \text{Area}_M(F)$ holds, where $\tilde{F} = \rho^{-1}(F) \cap (\Sigma \times I^{\text{new}})$. If necessary dividing \tilde{F} into smaller pieces, we may assume that, for the inclusion $i: \tilde{F} \rightarrow \Sigma^\circ \times I$, the composition $\zeta \circ i$ is injective. Then, by the definition of $\tilde{\eta}$,

$$\int_{\tilde{F}} |i^*(\tilde{\eta})| = \text{Area}_{\Sigma^\circ}(\zeta(\tilde{F})) \leq \text{Area}_{(1)}(\tilde{F}).$$

By this inequality together with Lemma 1 (ii),

$$\int_F |i_F^*(\eta_{M^\circ})| \leq \text{Area}_{(1)}(\tilde{F}) \leq \frac{n(f)\lambda}{2\iota(\xi)^2} \text{Area}_M(F),$$

where $i_F: F \rightarrow Y - L \subset M^\circ$ is the inclusion. This shows that

$$\sup\{g(s); s \in M^\circ\} \leq \max\left\{\max\{g(s); s \in M - \text{int } Y\}, \frac{n(f)\lambda}{2\iota(\xi)^2}\right\} < \infty.$$

This completes the proof. \square

By Lemma 2, for any hyperbolically straight 3-simplex $\sigma: \Delta^3 \rightarrow \tilde{M}$,

$$\int_{\Delta_\sigma^{3^\circ}} |\sigma^*(\tilde{\Omega}_E)| = \int_{\Delta_\sigma^{3^\circ}} |(\rho \circ \sigma)^*(\Omega_E)| \leq K_1 \int_{\Delta_\sigma^{3^\circ}} |(\rho \circ \sigma)^*(\Omega_H)| = K_1 \text{Vol}(\Delta_\sigma^{3^\circ}),$$

where Δ_σ^3 denotes the 3-simplex Δ^3 with the hyperbolic metric induced from that on \tilde{M} via σ and $\Delta_\sigma^{3^\circ} = \Delta_\sigma^3 - \sigma^{-1}(\tilde{L})$. Since the hyperbolic volume $\text{Vol}(\Delta_\sigma^{3^\circ}) = \text{Vol}(\Delta_\sigma^3)$ is less than the volume \mathbf{v}_3 of a regular ideal simplex in \mathbf{H}^3 ,

$$\int_{\Delta_\sigma^{3^\circ}} |\sigma^*(\tilde{\Omega}_E)| < K_1 \mathbf{v}_3. \quad (1.2)$$

Similarly, by Lemma 3 together with the equation (1.1), for any straight 2-simplex $\tau: \Delta^2 \longrightarrow \widetilde{M}$,

$$\int_{\Delta_\tau^{2\circ}} |\tau^*(\widetilde{\eta})| = \int_{\Delta_\tau^{2\circ}} |(\rho \circ \tau)^*(\eta_{M^\circ})| \leq K_2 \text{Area}(\Delta_\tau^{2\circ}),$$

where Δ_τ^2 denotes the 2-simplex Δ^2 with the induced hyperbolic metric and $\Delta_\tau^{2\circ} = \Delta_\tau^2 - \tau^{-1}(\widetilde{L})$. Since $\text{Area}(\Delta_\tau^{2\circ}) = \text{Area}(\Delta_\tau^2) < \pi$,

$$\int_{\Delta_\tau^{2\circ}} |\tau^*(\widetilde{\eta})| < \pi K_2. \quad (1.3)$$

The inequalities (1.2) and (1.3) will be used in the next section.

§2. Zero-norm elements of bounded cohomology

For a topological space X , the *Gromov norm* of a singular k -chain $z = \sum_{i=1}^n a_i \sigma_i^k \in C_k(X)$ with real coefficients $a_i \in \mathbf{R}$ is defined by

$$\|z\| = \sum_{i=1}^n |a_i|.$$

Then, for any bounded k -cochain $c \in C_b^k(X)$, we have $|c(z)| \leq \|c\| \|z\|$.

For any $r \geq 0$, $\varepsilon > 0$, consider the continuous functions $\alpha_{r,\varepsilon}: \mathbf{R} \longrightarrow \mathbf{R}$ and $A_{r,\varepsilon}: \mathbf{R} \longrightarrow \mathbf{R}$ given by

$$\alpha_{r,\varepsilon}(t) = \min\{\varepsilon, |t|^{-r}\}, \quad A_{r,\varepsilon}(t) = \int_0^t \alpha_{r,\varepsilon}(u) du.$$

Note that $\lim_{t \rightarrow \infty} \alpha_{r,\varepsilon}(t) = 0$ if $r > 0$ and $\lim_{t \rightarrow \infty} A_{r,\varepsilon}(t) = \infty$ if $r \leq 1$. The compositions of the projection $\widetilde{M} = \Sigma \times \mathbf{R} \longrightarrow \mathbf{R}$ with $\alpha_{r,\varepsilon}$, $A_{r,\varepsilon}$ are also denoted by $\alpha_{r,\varepsilon}: \widetilde{M} \longrightarrow \mathbf{R}$ and $A_{r,\varepsilon}: \widetilde{M} \longrightarrow \mathbf{R}$, that is, $\alpha_{r,\varepsilon}(p, t) = \alpha_{r,\varepsilon}(t)$ and $A_{r,\varepsilon}(p, t) = A_{r,\varepsilon}(t)$. For a singular n -simplex $\tau: \Delta^n \longrightarrow \widetilde{M}$, $\text{straight}(\tau): \Delta^n \longrightarrow \widetilde{M}$ denotes the straight n -simplex obtained by straightening τ , see [15, Chapter 6] for details. Let $c_{r,\varepsilon} \in Z^3(\widetilde{M})$ be the 3-cycle defined by

$$c_{r,\varepsilon}(\sigma) = \int_{\Delta_{\text{straight}(\sigma)}^{3\circ}} \text{straight}(\sigma)^*(\alpha_{r,\varepsilon} \widetilde{\Omega}_E)$$

for any singular 3-simplex $\sigma: \Delta^3 \longrightarrow \widetilde{M}$. Intuitively, $c_{r,\varepsilon}(\sigma)$ represents the “euclidean” volume with weight $\alpha_{r,\varepsilon}$ of the “hyperbolically” straightened simplex. Since $\max\{|\alpha_{r,\varepsilon}(t)|; t \in \mathbf{R}\} = \varepsilon$, by (1.2),

$$|c_{r,\varepsilon}(\sigma)| \leq \varepsilon \int_{\Delta_{\text{straight}(\sigma)}^{3\circ}} |\text{straight}(\sigma)^*(\widetilde{\Omega}_E)| < \varepsilon K_1 \mathbf{v}_3.$$

This shows that $c_{r,\varepsilon} \in Z_b^3(\widetilde{M})$ and $\|c_{r,\varepsilon}\| \leq \varepsilon K_1 \mathbf{v}_3$.

In Theorem 1, we will show that the class $[c_{r,\varepsilon}] \in H_b^3(\widetilde{M}; \mathbf{R})$ is independent of ε if $r > 0$. However, Theorem 2 implies that $[c_{r,\varepsilon}]$ strictly depends on r if $0 \leq r \leq 1$.

Theorem 1. *If $0 \leq r \leq 1$, then $[c_{r,\varepsilon}] \neq 0$ in $H_b^3(\widetilde{M}; \mathbf{R})$. If $r > 0$, then for any $\varepsilon, \varepsilon' > 0$, $[c_{r,\varepsilon}] = [c_{r,\varepsilon'}]$ in $H_b^3(\widetilde{M}; \mathbf{R})$. In particular, if $0 < r \leq 1$, then $[c_{r,\varepsilon}]$ is a non-trivial element of $H_b^3(\widetilde{M}; \mathbf{R})$ with $\|[c_{r,\varepsilon}]\| = 0$.*

Proof. We set $\Sigma_n = \Sigma \times \{n\} \subset \widetilde{M}$ for $n \in \mathbf{Z}$. For a sufficiently small $\delta > 0$, let $\widehat{\Sigma}_0$ be an oriented surface piecewise smoothly embedded in $\Sigma \times [-\delta, \delta]$ each piece of which is a totally geodesic triangle with respect to the hyperbolic metric on \widetilde{M} and such that $\widehat{\Sigma}_0$ is isotopic to Σ_0 in $\Sigma \times [-\delta, \delta]$. Furthermore, we may take $\widehat{\Sigma}_0$ so that it satisfies (2.1).

$$\text{For any } p \in \Sigma, \widehat{\Sigma}_0 \text{ meets the line } \zeta^{-1}(p) \text{ in a single point.} \quad (2.1)$$

Let $z_0 \in Z_2(\widetilde{M})$ be a 2-cycle representing this hyperbolic triangulation of $\widehat{\Sigma}_0$. We set $\widehat{\Sigma}_n = \widehat{f}^n(\widehat{\Sigma}_0)$ and $z_n = \widehat{f}_*^n(z_0)$. Since $z_n - z_0$ is homologous to zero in \widetilde{M} , there exists a 3-chain $w_n \in C_3(\widetilde{M})$ consisting of straight 3-simplices and with $\partial w_n = z_n - z_0$. Note that $\widetilde{\eta} = \zeta^*(\eta_{\Sigma^\circ})$ and $\eta_{\Sigma^\circ} > 0$. Thus, we have

$$\int_{\widehat{\Sigma}_n^\circ} \widetilde{\eta} = \int_{\Sigma^\circ} \eta_{\Sigma^\circ} = \int_{\widehat{\Sigma}_0^\circ} |\widetilde{\eta}|,$$

where $\widehat{\Sigma}_n^\circ = \widehat{\Sigma}_n - \widehat{\Sigma}_n \cap \zeta^{-1}(S(f))$ and the second equality is derived from the property (2.1). We denote the value of these integrals by $K_3 > 0$.

If $[c_{r,\varepsilon}] = 0$ in $H_b^3(\widetilde{M}; \mathbf{R})$ for some $0 \leq r \leq 1$, then there would exist a bounded 2-cochain $a \in C_b^2(\widetilde{M})$ with $\delta_b^2(a) = c_{r,\varepsilon}$. This implies that, for any $n \in \mathbf{N}$,

$$|c_{r,\varepsilon}(w_n)| = |a(z_n - z_0)| \leq \|a\|(\|z_n\| + \|z_0\|) = 2\|a\| \|z_0\|.$$

Since $A_{r,\varepsilon}$ is an increasing function, $A_{r,\varepsilon}(n - \delta) \leq A_{r,\varepsilon} \leq A_{r,\varepsilon}(n + \delta)$ in $\Sigma_n \times [-\delta, \delta]$ and $A_{r,\varepsilon} \leq \varepsilon\delta$ in $\Sigma_0 \times [-\delta, \delta]$. Consider the 2-form $\theta_{r,\varepsilon} = A_{r,\varepsilon}\widetilde{\eta}$ on \widetilde{M}° . Since $d\theta_{r,\varepsilon} = \alpha_{r,\varepsilon}\widetilde{\Omega}_E$ and since $\text{straight}(w_n) = w_n$, the Stokes Theorem shows that

$$\begin{aligned} |c_{r,\varepsilon}(w_n)| &= \left| \int_{\widehat{\Sigma}_n^\circ} A_{r,\varepsilon}\widetilde{\eta} - \int_{\widehat{\Sigma}_0^\circ} A_{r,\varepsilon}\widetilde{\eta} \right| \\ &\geq A_{r,\varepsilon}(n - \delta) \int_{\widehat{\Sigma}_n^\circ} |\widetilde{\eta}| - \varepsilon\delta \int_{\widehat{\Sigma}_0^\circ} |\widetilde{\eta}| \\ &= (A_{r,\varepsilon}(n - \delta) - \varepsilon\delta)K_3. \end{aligned}$$

The condition $0 \leq r \leq 1$ implies that $\lim_{n \rightarrow \infty} A_{r,\varepsilon}(n - \delta) = \infty$ and hence $\lim_{n \rightarrow \infty} |c_{r,\varepsilon}(w_n)| = \infty$, a contradiction. It follows that $[c_{r,\varepsilon}]$ is a non-trivial element of $H_b^3(\widetilde{M}; \mathbf{R})$ for any $0 \leq r \leq 1$.

For any $\varepsilon, \varepsilon'$ with $\varepsilon > \varepsilon' > 0$, the 2-cochain $b \in C^2(\widetilde{M})$ is given by

$$b(\tau) = \int_{\Delta_{\text{straight}(\tau)}^{2\circ}} \text{straight}(\tau)^*(\theta_{r,\varepsilon} - \theta_{r,\varepsilon'})$$

for any singular 2-simplex $\tau: \Delta^2 \rightarrow \widetilde{M}$. Then, the coboundary of b is $\delta^2(b) = c_{r,\varepsilon} - c_{r,\varepsilon'}$. If $r > 0$, then

$$K_4 = \max\{|A_{r,\varepsilon}(t) - A_{r,\varepsilon'}(t)|; t \in \mathbf{R}\} = \int_0^{(\varepsilon')^{-1/r}} (\alpha_{r,\varepsilon}(u) - \alpha_{r,\varepsilon'}(u)) du < \infty.$$

By (1.3), we have

$$\begin{aligned} |b(\tau)| &= \left| \int_{\Delta_{\text{straight}(\tau)}^{2\circ}} \text{straight}(\tau)^*((A_{r,\varepsilon} - A_{r,\varepsilon'})\tilde{\eta}) \right| \\ &\leq K_4 \int_{\Delta_{\text{straight}(\tau)}^{2\circ}} |\text{straight}(\tau)^*(\tilde{\eta})| \\ &\leq \pi K_2 K_4. \end{aligned}$$

This shows that $b \in C_b^2(\widetilde{M})$ and hence $c_{r,\varepsilon} - c_{r,\varepsilon'} \in B_b^3(\widetilde{M})$ for $r > 0$. By the definition of the pseudonorm, for any $\varepsilon' > 0$, $\|[c_{r,\varepsilon}]\| = \|[c_{r,\varepsilon'}]\| \leq \varepsilon' K_1 \mathbf{v}_3$. Thus, we have $\|[c_{r,\varepsilon}]\| = 0$ whenever $r > 0$. \square

For two sequences $\{a_n\}, \{b_n\}$ with $a_n, b_n > 0$ ($n \in \mathbf{N}$), $a_n \sim b_n$ means that

$$0 < \liminf_{n \rightarrow \infty} \frac{a_n}{b_n} \leq \limsup_{n \rightarrow \infty} \frac{a_n}{b_n} < \infty.$$

The notation in the proof of Theorem 1 still works to prove Theorem 2.

Theorem 2. *For a fixed $\varepsilon > 0$, the elements $[c_{r,\varepsilon}]$ ($0 \leq r \leq 1$) are linearly independent in $H_b^3(\widetilde{M}; \mathbf{R})$.*

Proof. We suppose that

$$\gamma_1[c_{r_1,\varepsilon}] + \gamma_2[c_{r_2,\varepsilon}] + \cdots + \gamma_m[c_{r_m,\varepsilon}] = 0$$

for $0 \leq r_1 < r_2 < \cdots < r_m \leq 1$. Then, there exists a bounded 2-cochain $a \in C_b^2(\widetilde{M})$ with

$$\gamma_1 c_{r_1,\varepsilon} + \gamma_2 c_{r_2,\varepsilon} + \cdots + \gamma_m c_{r_m,\varepsilon} = \delta_b^2(a).$$

For the straight 3-chain $w_n \in C_3(\widetilde{M})$ given as above, we have

$$|\gamma_1 c_{r_1, \varepsilon}(w_n)| \leq \sum_{j=2}^m |\gamma_j c_{r_j, \varepsilon}(w_n)| + |\delta_b^2(a)(w_n)|.$$

The argument similar to that in the proof of Theorem 1 shows that

$$|\gamma_1| K_3(A_{r_1, \varepsilon}(n - \delta) - \varepsilon \delta) \leq \sum_{j=2}^m |\gamma_j| K_3(A_{r_j, \varepsilon}(n + \delta) + \varepsilon \delta) + 2\|a\| \|z_0\|,$$

and hence

$$|\gamma_1| \leq \frac{\sum_{j=2}^m |\gamma_j| (A_{r_j, \varepsilon}(n + \delta) + \varepsilon \delta) + 2K_3^{-1} \|a\| \|z_0\|}{A_{r_1, \varepsilon}(n - \delta) - \varepsilon \delta}. \quad (2.2)$$

Since $A_{r_1, \varepsilon}(n - \delta) \sim n^{1-r_1}$, $A_{r_j, \varepsilon}(n + \delta) \sim n^{1-r_j}$ if $r_j < 1$, and $A_{r_m, \varepsilon}(n + \delta) \sim \log n$ if $r_m = 1$, the right hand side of (2.2) converges to zero as $n \rightarrow \infty$. This shows that $\gamma_1 = 0$. Similarly, we have $\gamma_2 = \dots = \gamma_m = 0$. Thus, $[c_{r, \varepsilon}]$ ($0 \leq r \leq 1$) are linearly independent. \square

By Theorems 1 and 2, the continuous family $\{[c_{r, \varepsilon}]; 0 < r \leq 1\}$ consists of linearly independent elements in $N^3(\widetilde{M})$. Since the inclusion $i: \Sigma = \Sigma_0 \rightarrow \widetilde{M}$ is a homotopy equivalence, the induced homomorphism $i^*: (H_b^3(\widetilde{M}; \mathbf{R}), \|\cdot\|) \rightarrow (H_b^3(\Sigma; \mathbf{R}), \|\cdot\|)$ is isometrically isomorphic. Thus, we have the following corollary.

Corollary. *For any closed, connected, orientable surface Σ of genus > 1 , the dimension of the zero-norm subspace $N^3(\Sigma)$ of $H_b^3(\Sigma; \mathbf{R})$ is the cardinality of the continuum.* \square

References

- [1] J. Barge et E. Ghys, Surfaces et cohomologie bornée, *Invent. Math.* **92** (1988), 509–526.
- [2] R. Brooks and C. Series, Bounded cohomology for surface groups, *Topology* **23** (1984), 29–36.
- [3] A. Casson and S. Bleiler, Automorphisms of surfaces after Nielsen and Thurston, *London Math. Soc. Student Texts* 9, Cambridge Univ. Press, Cambridge 1988.
- [4] D. Epstein and K. Fujiwara, The second bounded cohomology of word-hyperbolic groups, *Topology* **36** (1997), 1275–1289.
- [5] A. Fathi, F. Laudenbach and V. Poénaru, Travaux de Thurston sur les surfaces, *Astérisque* no. 66-67, Soc. Math. France, 1979.
- [6] R. Grigorchuk, Some results on bounded cohomology, A. Duncan, N. Gilbert, J. Howie (Eds.), *Combinatorial and geometric group theory*, *London Math. Soc. Lecture Notes Series* **204**, Cambridge Univ. Press, Cambridge 1995, pp. 111–163.

- [7] M. Gromov, Volume and bounded cohomology, *Publ. Math. Inst. Hautes Etud. Sci.* **56** (1982), 5–100.
- [8] N. Ivanov, Second bounded cohomology group, *J. Soviet Math.* **52** (1990), 2822–2824.
- [9] S. Matsumoto and S. Morita, Bounded cohomology of certain groups of homeomorphisms, *Proc. Amer. Math. Soc.* **94** (1985), 539–544.
- [10] Y. Mitsumatsu, Bounded cohomology and l^1 -homology of surfaces, *Topology* **23** (1984), 465–471.
- [11] T. Soma, Bounded cohomology of closed surfaces, *Topology* **36** (1997), 1221–1246.
- [12] T. Soma, Bounded cohomology and topologically tame Kleinian groups, *Duke Math. J.* **88** (1997), 357–370.
- [13] T. Soma, Existence of non-Banach bounded cohomology, *Topology* (in press).
- [14] D. Sullivan, Travaux de Thurston sur les groupes quasi-fuchsien et les variétés hyperboliques de dimension 3 fibrés sur S^1 , *Séminaire Bourbaki 554, Lecture Notes in Math.*, vol. 842, Springer Verlag, Berlin and New York 1981, pp. 196–214.
- [15] W. Thurston, The geometry and topology of 3-manifolds, *Lecture Notes*, Princeton Univ., Princeton 1978.
- [16] W. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, *Bull. Amer. Math. Soc.* **19** (1988), 417–431.
- [17] W. Thurston, Hyperbolic structures on 3-manifolds II: Surface groups and 3-manifolds which fiber over the circle, preprint.
- [18] T. Yoshida, On 3-dimensional bounded cohomology of surfaces, *Homotopy Theory and Related Topics, Advanced Studies in Pure Math.*, vol. 9, Kinokuniya, Tokyo, 1986, pp. 173–176.

Teruhiko Soma
Department of Mathematical Sciences
College of Science and Engineering
Tokyo Denki University
Hatoyama-machi, Saitama-ken 350-03
Japan
e-mail: soma@r.dendai.ac.jp

(Received: November 26, 1996)