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## Differential operators commuting with invariant functions

T. Levasseur and J. T. Stafford\*

**Abstract.** Let  $\mathfrak{g}$  be a reductive, complex Lie algebra, with adjoint group  $G$ , let  $G$  act on the ring of differential operators  $\mathcal{D}(\mathfrak{g})$  via the adjoint action and write  $\tau : \mathfrak{g} \rightarrow \mathcal{D}(\mathfrak{g})$  for the differential of this action. We prove that the commutant, in  $\mathcal{D}(\mathfrak{g})$ , of  $\mathcal{O}(\mathfrak{g})^G$  is the algebra generated by  $\mathcal{O}(\mathfrak{g})$  and  $\tau(\mathfrak{g})$ , thereby answering a question of Barlet.

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**Keywords.** Invariant differential operators, commuting differential operators, semi-simple Lie algebras, symmetric algebras.

### 1. Introduction

Fix a reductive, complex Lie algebra  $\mathfrak{g}$ , with adjoint group  $G$ , let  $G$  act on the ring of differential operators  $\mathcal{D}(\mathfrak{g})$  via the adjoint action and write  $\tau : \mathfrak{g} \rightarrow \mathcal{D}(\mathfrak{g})$  for the differential of this action. We identify  $\mathcal{O}(\mathfrak{g})$ , the ring of regular functions on  $\mathfrak{g}$ , with  $S(\mathfrak{g}^*)$  and let  $\mathcal{O}(\mathfrak{g})^G$  denote the subalgebra of  $G$ -invariant functions. The aim of this note is to prove:

**Theorem 1.1.** *The commutant  $\mathcal{C} = \mathcal{C}_{\mathcal{D}(\mathfrak{g})}(\mathcal{O}(\mathfrak{g})^G)$ , in  $\mathcal{D}(\mathfrak{g})$ , of  $\mathcal{O}(\mathfrak{g})^G$  is the algebra generated by  $\mathcal{O}(\mathfrak{g})$  and  $\tau(\mathfrak{g})$ .*

At the level of vector fields, this result follows from [5, Theorem 2.1], in the sense that Dixmier's result implies that  $\mathcal{C} \cap \text{Der } \mathcal{O}(\mathfrak{g}) = \mathcal{O}(\mathfrak{g})\tau(\mathfrak{g})$ . In [2], D. Barlet raised the question of whether Theorem 1.1 is true, since this would form a natural generalization of Dixmier's result. In the same paper, Barlet was able to prove the theorem in the case when  $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ . We would like to thank M. Raïs for bringing Barlet's question to our attention.

In the process of proving Theorem 1.1, we obtain a considerable amount of information about the structure of  $\mathcal{C}$ . Some particular properties are given in the next result. The unexplained definitions can be found in Section 3.

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**Proposition 1.2.**  *$\mathcal{C}$  is an Auslander-Gorenstein, CM domain and a maximal order in its quotient division ring.*

In fact, the main theorem of this paper is a result about commutative rings. To state this, let  $A$  denote the subalgebra of  $\mathcal{D}(\mathfrak{g})$  generated by  $\mathcal{O}(\mathfrak{g})$  and  $\tau(\mathfrak{g})$  and set  $E = \mathcal{O}(\mathfrak{g})\tau(\mathfrak{g}) \subset \text{Der } \mathcal{O}(\mathfrak{g})$ . If one filters  $\mathcal{D}(\mathfrak{g})$  and its subalgebras by degree of differential operators, then it is easy to see that the associated graded rings  $\text{gr } A$  and  $\text{gr } \mathcal{C}$  are domains with the same quotient field as the symmetric algebra  $\text{Sym}_{\mathcal{O}(\mathfrak{g})}(E)$ . Then, Theorem 1.1 and Proposition 1.2 follow easily from the following result.

**Theorem 1.3.** (i) *Let  $E = \mathcal{O}(\mathfrak{g})\tau(\mathfrak{g}) \subset \text{Der } \mathcal{O}(\mathfrak{g})$ . Then  $\text{Sym}_{\mathcal{O}(\mathfrak{g})}(E)$  is a factorial, complete intersection of Krull dimension  $2\dim \mathfrak{g} - \text{rk } \mathfrak{g}$ .*

(ii)  $\text{gr } A = \text{gr } \mathcal{C} = \text{Sym}_{\mathcal{O}(\mathfrak{g})}(E)$ .

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## 2. The symmetric algebra of the module generated by $\tau(\mathfrak{g})$

In this section we prove Theorem 1.3 from the introduction. We begin with some preliminary notation and results.

As before, we fix a complex, reductive Lie algebra  $\mathfrak{g}$  of dimension  $n$  and rank  $\ell$ . Write  $G$  for the adjoint group of  $\mathfrak{g}$ . Define the categorical quotient of  $\mathfrak{g}$  by  $\mathcal{O}(\mathfrak{g})//G = \text{Spec } \mathcal{O}(\mathfrak{g})^G$  and let  $u : \mathfrak{g} \rightarrow \mathfrak{g}//G$  denote the quotient morphism. We will write  $\mathcal{O}$  for  $\mathcal{O}(\mathfrak{g})$ . Define

$$\mathcal{X}_i = \{y \in \mathfrak{g} : \text{rk } d_y u \leq i\},$$

where  $d_y u : T_y \mathfrak{g} \rightarrow T_{u(y)} \mathfrak{g}//G$  denotes the differential of  $u$ . Observe that each  $\mathcal{X}_i$  is a closed  $G$ -subvariety of  $\mathfrak{g}$ . Recall that  $y \in \mathfrak{g}$  is called *regular* if its centralizer in  $\mathfrak{g}$  is of dimension  $\ell$ . Then [10, Theorem 10.1],  $\text{rk } d_y u = \ell$  if and only if  $y$  is regular.

We would like to thank D. Panyushev for the proof of the following proposition, which is considerably easier than our original proof.

**Proposition 2.1.** *One has:  $\text{codim } \mathcal{X}_i \geq \ell - i + 2$ , for  $0 \leq i \leq \ell - 1$ .*

*Proof.* Notice that  $u$  induces a surjective morphism  $\varpi : \mathcal{X}_i \rightarrow \mathcal{X}_i//G$  and that, for all  $x \in \mathcal{X}_i$ , the differential  $d_x \varpi$  is the restriction of  $d_x u$  to  $T_x \mathcal{X}_i$ . Set  $r = \max\{\text{rk } d_x \varpi : x \in \mathcal{X}_i\}$ . Then, by [7, Proposition III.10.6] and the definition of  $\mathcal{X}_i$ , we obtain that  $\dim \mathcal{X}_i//G \leq r \leq i$ .

Since  $\mathcal{X}_i$  is stable under the  $\mathbb{C}^*$ -action  $y \mapsto \lambda y$ ,  $\lambda \in \mathbb{C}^*$ , the point 0 belongs to each irreducible component of  $\mathcal{X}_i$ . Hence,  $\dim \mathcal{X}_i \leq \dim \mathcal{X}_i//G + \dim \varpi^{-1}(\varpi(0))$

(see [11, AI.3.3] or [7, Ex. II.3.22]). But  $\varpi^{-1}(\varpi(0)) = \mathcal{X}_i \cap \mathbf{N}$ , where  $\mathbf{N}$  denotes the nilpotent cone of  $\mathfrak{g}$ , and, since  $i \leq \ell - 1$ ,  $\mathcal{X}_i \cap \mathbf{N}$  is contained in the subvariety of non-regular nilpotent elements. Therefore  $\dim \varpi^{-1}(\varpi(0)) \leq n - \ell - 2$  and it follows that  $\dim \mathcal{X}_i \leq i + n - \ell - 2$ , as required.  $\square$

**Remark.** Proposition 2.1 generalizes the well-known fact that  $\mathcal{X}_{\ell-1}$  has codimension at least three (see, for example, [18, Theorem 4.12]). It is natural to conjecture that Proposition 2.1 can be improved to the statement that  $\text{codim } \mathcal{X}_i \geq 3(\ell - i)$  for  $0 \leq i \leq \ell - 1$ . D. Panyushev informs us that he has been able to prove this by a case by case analysis.

Fix a  $G$ -invariant, non-degenerate, symmetric bilinear form  $\kappa$  on  $\mathfrak{g}$  and let  $\tilde{\kappa} : \mathfrak{g}^* \rightarrow \mathfrak{g}$  be the induced isomorphism. Thus,  $\tilde{\kappa}$  induces an isomorphism between differential one-forms on  $\mathfrak{g}$  and vector fields on  $\mathfrak{g}$ . If  $f \in \mathcal{O}^G$ , then we define a  $G$ -invariant vector field  $\text{grad}(f) \in \mathcal{O} \otimes_{\mathbb{C}} \mathfrak{g}$  to be the image of  $df$  under  $\tilde{\kappa}$ . Equivalently, if we fix an orthonormal basis  $\{e_i\}$  of  $\mathfrak{g}$  and write  $x_i = e_i^* \in \mathfrak{g}^*$ , then

$$\text{grad}(f) = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \otimes e_j = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_j}. \quad (2.1)$$

By Chevalley's Theorem,  $\mathcal{O}^G$  is a polynomial ring, say  $\mathcal{O}^G = \mathbb{C}[u_1, \dots, u_\ell]$  for homogeneous, algebraically independent polynomials  $\{u_i\}_i$ . Set  $\nabla_i = \text{grad}(u_i)$ , for  $1 \leq i \leq \ell$ . If  $\tau : \mathfrak{g} \rightarrow \text{Der } \mathcal{O}$  is the differential of the adjoint action of  $G$  on  $\mathfrak{g}$ , then write  $E = \mathcal{O}\tau(\mathfrak{g})$ . We will also write  $\tau$  for the induced map:

$$\tau : \mathcal{O} \otimes_{\mathbb{C}} \mathfrak{g} \longrightarrow E \subseteq \text{Der } \mathcal{O}.$$

Notice that if  $\theta \in \mathcal{O} \otimes_{\mathbb{C}} \mathfrak{g}$ , the vector field  $\tau(\theta)$  is given by  $\tau(\theta)_y = [y, \theta_y]$  for all  $y \in \mathfrak{g}$ . It follows easily that if  $\theta$  is  $G$ -invariant, then  $\tau(\theta) = 0$ . In particular, one has  $\tau(\nabla_i) = 0$  for all  $i$ . In fact rather more is true:

**Lemma 2.2.** *There is a short exact sequence*

$$0 \longrightarrow \bigoplus_{i=1}^{\ell} \mathcal{O}\nabla_i \longrightarrow \mathcal{O} \otimes_{\mathbb{C}} \mathfrak{g} \xrightarrow{\tau} E \longrightarrow 0. \quad (2.2)$$

*Proof.* This is [16, Theorem 2.5.4]. Using the identification of  $\mathfrak{g}$  with  $\mathfrak{g}^*$  under  $\tilde{\kappa}$ , it also follows from [14, Theorem 1.9].  $\square$

**Corollary 2.3.** *If  $\text{Sym}_{\mathcal{O}}(E)$  denotes the symmetric algebra of the  $\mathcal{O}$ -module  $E$ , then,  $\text{Sym}_{\mathcal{O}}(E) \cong \text{Sym}_{\mathcal{O}}(\mathcal{O} \otimes_{\mathbb{C}} \mathfrak{g}) / (\nabla_1, \dots, \nabla_\ell)$ .*

*Proof.* This follows from the universal property of symmetric algebras.  $\square$

Set  $\text{Sym}(E) = \text{Sym}_{\mathcal{O}}(E)$ . The main aim of this section is to understand the structure of  $\text{Sym}(E)$ , for which we use the results from [1] and [8].

Let  $I_t(\mathbf{u})$  be the ideal generated by the  $t \times t$  minors of the matrix  $\mathbf{u} = [\frac{\partial u_i}{\partial x_j}]$  and consider the following condition for  $s \geq 0$ :

$$\text{ht } I_t(\mathbf{u}) \geq \ell - t + 1 + s, \quad \text{for } 1 \leq t \leq \ell. \quad (\mathcal{F}_s)$$

Observe that, if we regard the short exact sequence (2.2) as a sequence

$$0 \longrightarrow \mathcal{O}^\ell \xrightarrow{\beta} \mathcal{O}^n \longrightarrow E \longrightarrow 0,$$

then (2.1) implies that  $I_t(\mathbf{u})$  is the ideal generated by the  $t \times t$  minors of the map  $\beta$ . Thus, the ideals  $I_{n-t}(\mathbf{u})$  are nothing more than the Fitting ideals of  $E$  (see, for example, [17, 1.1]). In particular, they are independent of the presentation of  $E$  and our condition  $(\mathcal{F}_s)$  coincides with that of [8].

**Proposition 2.4.** (i) *The condition  $(\mathcal{F}_2)$  is satisfied by  $E$ .*

(ii)  *$\text{Sym}(E)$  is a factorial domain of Krull dimension  $2n - \ell$ . In particular,  $\text{Sym}(E)$  is a complete intersection and is Gorenstein.*

(iii) *If  $P$  is a prime ideal of  $\mathcal{O}$  with  $\text{ht } P \geq 2$ , then  $\text{ht } P \text{Sym}(E) \geq 2$ .*

*Proof.* Write  $\tilde{\mathcal{X}}_{i-1}$  for the zero set of  $I_i(\mathbf{u})$ ; thus

$$\tilde{\mathcal{X}}_{i-1} = \{x \in \mathfrak{g} : \text{rk}(\nabla_1(x), \dots, \nabla_\ell(x)) \leq i-1\}.$$

Since the  $\nabla_j$  are the images of the  $du_j$  under the isomorphism  $\tilde{\kappa}$ , clearly  $\tilde{\mathcal{X}}_{i-1} = \{x \in \mathfrak{g} : \text{rk}(d_x u_1, \dots, d_x u_\ell) \leq i-1\}$ . Since  $u_1, \dots, u_\ell$  define the quotient map  $u : \mathfrak{g} \rightarrow \mathfrak{g}/G$ , this implies that  $\tilde{\mathcal{X}}_i = \mathcal{X}_i$ . Hence, part (i) is a reformulation of Proposition 2.1.

By Lemma 2.2,  $E$  has projective dimension at most 1. Thus, part (ii) follows from part (i), combined with [1, Propositions 3 and 6]. By [8, Remarks, pp. 664-5], the condition of part (iii) is equivalent to the condition  $(\mathcal{F}_2)$ .  $\square$

We end this section by giving the geometric significance of Proposition 2.4. This should be compared with [9, § 2] which proves weaker results for much more general  $G$ -varieties.

The map  $\tau$  induces a homomorphism of algebras

$$\tilde{\tau} : \mathcal{O}(\mathfrak{g} \times \mathfrak{g}^*) = \text{Sym}_{\mathcal{O}}(\mathcal{O} \otimes \mathfrak{g}) \longrightarrow \mathcal{O}(T^*\mathfrak{g}) = \text{Sym}_{\mathcal{O}}(\text{Der } \mathcal{O}).$$

Clearly, the image of  $\tilde{\tau}$  is the subring  $\mathcal{O}[\tau(\mathfrak{g})]$  of  $\text{Sym}_{\mathcal{O}}(\text{Der } \mathcal{O})$  generated by  $\mathcal{O}$  and  $\tau(\mathfrak{g})$ . After identification of  $\mathfrak{g}^*$  with  $\mathfrak{g}$  through  $\tilde{\kappa}$ , the associated morphism to  $\tilde{\tau}$  is:

$$\nu : T^*\mathfrak{g} \cong \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \times \mathfrak{g}, \quad \nu(x, y) = (x, [y, x])$$

Let  $\tilde{T}\mathfrak{g}$  denote the closure of the image of  $\nu$ ; thus,  $\tilde{T}\mathfrak{g}$  is an irreducible affine subvariety of  $\mathfrak{g} \times \mathfrak{g}$  with coordinate ring  $\mathcal{O}(\tilde{T}\mathfrak{g}) \cong \mathcal{O}[\tau(\mathfrak{g})]$ .

**Corollary 2.5.** (i)  $\text{Sym}(E) = \mathcal{O}(\tilde{T}\mathfrak{g})$ .

(ii) *The variety  $\tilde{T}\mathfrak{g}$  is a factorial complete intersection in  $\mathfrak{g} \times \mathfrak{g}$ .*

*Proof.* By universality,  $\tilde{\tau}$  induces a surjective morphism  $\pi : \text{Sym}(E) \rightarrow \mathcal{O}[\tau(\mathfrak{g})]$ . If we prove that  $\dim \tilde{T}\mathfrak{g} \geq 2n - \ell$ , then the corollary will follow from Proposition 2.4(ii).

Let  $\rho : \tilde{T}\mathfrak{g} \rightarrow \mathfrak{g}$  denote the projection onto the first factor. By [11, AI.3.3] there exists a dense open subset  $U \subseteq \tilde{T}\mathfrak{g}$  such that  $\dim \tilde{T}\mathfrak{g} = \dim \mathfrak{g} + \dim \rho^{-1}(\rho(u))$  for all  $u \in U$ . Since  $\tilde{T}\mathfrak{g}$  is irreducible, we can pick  $u = (x, y) \in \rho^{-1}(\mathfrak{g}') \cap U$ , where  $\mathfrak{g}'$  denotes the set of generic elements in  $\mathfrak{g}$ . Now,

$$\rho^{-1}(\rho(u)) \supseteq \rho^{-1}(\rho(u)) \cap \text{Im } \nu = \{(x, [\mathfrak{g}, x])\}.$$

Since  $x$  is generic,  $\dim \rho^{-1}(\rho(u)) \geq \dim [\mathfrak{g}, x] = n - \ell$  and the result follows.  $\square$

### 3. The commutant of $\mathcal{O}(\mathfrak{g})^G$

As usual, we identify  $\mathcal{D}(\mathfrak{g})$ , as a vector space, with  $\mathcal{O} \otimes_{\mathbb{C}} S(\mathfrak{g})$ , where  $\mathcal{O} = \mathcal{O}(\mathfrak{g})$  and the symmetric algebra  $S(\mathfrak{g})$  is identified with the constant coefficient differential operators on  $\mathfrak{g}$ . We will always filter  $\mathcal{D}(\mathfrak{g})$  by degree of differential operators and so, as algebras,  $\text{gr } \mathcal{D}(\mathfrak{g}) = \text{Sym}_{\mathcal{O}}(\text{Der } \mathcal{O}) \cong \mathcal{O} \otimes_{\mathbb{C}} S(\mathfrak{g})$ . Write  $A$  for the subring of  $\mathcal{D}(\mathfrak{g})$  generated by  $\mathcal{O}$  and  $\tau(\mathfrak{g})$  and let  $\mathcal{C}$  denote the commutant of  $\mathcal{O}^G$ , as in the introduction. Obviously,  $A$  is contained in  $\mathcal{C}$ .

**Lemma 3.1.** *Let  $x \in \mathfrak{g}$  be a regular point and set  $R = \mathcal{O}_{\mathfrak{g}, x}$  for the local ring of  $\mathfrak{g}$  at  $x$ . Then, there exists a basis of derivations  $\{\partial_i : 1 \leq i \leq n\}$  of  $\text{Der } R$  such that  $\partial_i(u_j) = \delta_{ij}$  for all  $1 \leq i, j \leq \ell$  and  $R\tau(\mathfrak{g}) = \bigoplus_{i=\ell+1}^n R\partial_i$ .*

*Proof.* Let  $\mathbf{m}$  denote the maximal ideal of  $R$ . By [10, Theorem 0.1], the  $\{d_x u_i : 1 \leq i \leq \ell\}$  are linearly independent. The  $\{d_x u_i\}$  may also be regarded as elements of  $\mathbf{m}/\mathbf{m}^2$ , under the usual identification of  $T_x^* \mathfrak{g}$  with  $\mathbf{m}/\mathbf{m}^2$ . Thus, for some scalars  $\lambda_i$ , the set  $\{u_1 - \lambda_1, \dots, u_\ell - \lambda_\ell\}$  is part of a system of parameters, say  $\{z_1 = u_1 - \lambda_1, \dots, z_\ell = u_\ell - \lambda_\ell, z_{\ell+1}, \dots, z_n\}$  for  $\mathbf{m}$ . Let  $\partial_i \in \text{Der } R$  be defined by  $\partial_i(z_j) = \delta_{ij}$ .

If  $D \in \text{Der } R$ , then  $\tilde{D} = D - \sum_{i=1}^{\ell} D(u_i) \partial_i$  satisfies  $\tilde{D}(u_j) = 0$ , for  $1 \leq j \leq \ell$ . Thus,  $\tilde{D}(\mathcal{O}^G) = 0$  and so, by [5, Theorem 2.1] (or directly),  $\tilde{D} \in R\tau(\mathfrak{g})$ . Hence,  $\text{Der } R = R\tau(\mathfrak{g}) \oplus (\bigoplus_{i=1}^{\ell} R\partial_i)$ . Since

$$R\tau(\mathfrak{g}) \subseteq \{D \in \text{Der } R : D(u_j) = 0 \text{ for } 1 \leq j \leq \ell\} = \bigoplus_{i=\ell+1}^n R\partial_i,$$

the result follows.  $\square$

**Theorem 3.2.** *Let  $A$  and  $\mathcal{C}$  be given the filtrations induced from that on  $\mathcal{D}(\mathfrak{g})$ . Then  $\text{gr } A = \text{gr } \mathcal{C} \cong \text{Sym}(E)$ .*

*Proof.* Since  $\text{gr } \mathcal{C} \subset \text{gr } \mathcal{D}(\mathfrak{g}) \cong \mathcal{O}(T^*\mathfrak{g})$ , certainly  $\text{gr } A \subseteq \text{gr } \mathcal{C}$  are domains. Also, as  $\tau(\mathfrak{g})$  consists of derivations, we may regard  $\tau(\mathfrak{g}) \subseteq \text{Der } \mathcal{O} \subseteq \text{gr } \mathcal{D}(\mathfrak{g})$ . Hence the ring  $\mathcal{O}[\tau(\mathfrak{g})]$  is contained in  $\text{gr } A$  and, by Corollary 2.5(i), the natural map  $\pi : \text{Sym}(E) \rightarrow \mathcal{O}[\tau(\mathfrak{g})]$  is an isomorphism.

Let  $x \in \mathfrak{g}$  be a regular point and let  $\mathcal{S} = \{f \in \mathcal{O} : f(x) \neq 0\}$ . Given a ring  $C$  containing  $\mathcal{O}$ , we write  $C_x$  for the localization  $C_{\mathcal{S}}$  (given that it exists). Then, we claim that

$$(\text{gr } A)_x = (\text{gr } \mathcal{C})_x \cong \text{Sym}(E)_x, \quad (3.1)$$

where the isomorphism is induced by  $\pi^{-1}$ .

By mimicking the proof of Richardson's Lemma [11, II.3.4], one can show that this suffices to prove the theorem. In more detail, assume that (3.1) is true. Since  $\text{gr } \mathcal{C}$  and  $\text{Sym}(E)$  are domains, (3.1) certainly implies that

$$\text{Sym}(E) \xrightarrow{\pi} \text{gr } A \subseteq \text{gr } \mathcal{C}$$

and that  $\text{gr } \mathcal{C}$  and  $\text{Sym}(E)$  have the same field of fractions. Moreover,  $\{x \in \mathfrak{g} : (\text{gr } \mathcal{C})_x \neq \text{Sym}(E)_x\}$  is contained in the set of non-regular elements of  $\mathfrak{g}$ . By [10, Theorem 0.1], this is precisely the subspace  $\mathcal{X}_{\ell-1}$  and, by Proposition 2.1 or [18, Theorem 4.12],  $\text{codim } \mathcal{X}_{\ell-1} \geq 3$ . Thus, for any  $b \in \text{gr } \mathcal{C}$ , there exists an ideal  $I$  of  $\mathcal{O}$  of height at least 3 such that  $bI \subseteq \text{Sym}(E)$ . By Proposition 2.4(iii),  $\text{ht}_{\text{Sym}(E)} I \text{Sym}(E) \geq 2$ . Hence,  $b \in \text{Sym}(E)_{\mathbf{p}}$  for every height one prime  $\mathbf{p}$  of  $\text{Sym}(E)$ . Since  $\text{Sym}(E)$  is Cohen-Macaulay, it satisfies the  $(S_2)$ -condition [12, p. 125], and therefore  $b \in \text{Sym}(E)$ .

Thus, it remains to prove (3.1). Let  $R = \mathcal{O}_x = \mathcal{O}_{\mathfrak{g},x}$  and keep the notation of Lemma 3.1. It is immediate from that lemma that  $D \in \mathcal{D}(\mathfrak{g})_x$  satisfies  $[D, u_j] = 0$  if and only if  $D \in R\langle \partial_j : j \neq i \rangle$ . Consequently,  $\mathcal{C}_x = A_x = R\langle \partial_{\ell+1}, \dots, \partial_n \rangle$ .

Let  $\bar{\partial}_k$  denote the image of  $\partial_k$  in  $\text{gr } \mathcal{D}(\mathfrak{g})$ . Obviously, Lemma 3.1 also implies that  $R\tau(\mathfrak{g}) = \bigoplus_{k=\ell+1}^n R\bar{\partial}_k$ , where  $R\tau(\mathfrak{g})$  is now regarded as a subspace of  $\text{Der } R \subset \text{gr } \mathcal{D}(\mathfrak{g})_x \cong R \otimes_{\mathbb{C}} S(\mathfrak{g})$ . Thus,

$$\text{gr } \mathcal{C}_x = \text{gr } A_x = \text{gr } R\langle \partial_{\ell+1}, \dots, \partial_n \rangle = R[\bar{\partial}_{\ell+1}, \dots, \bar{\partial}_n] = R[\tau(\mathfrak{g})].$$

Since  $\pi : \text{Sym}(E)_x \rightarrow R[\tau(\mathfrak{g})]$  is an isomorphism, this completes the proof of (3.1) and hence of the theorem.  $\square$

The Gelfand-Kirillov dimension of a module  $M$  will be denoted  $\text{GKdim } M$ . If a Noetherian ring  $A$  has finite injective dimension, then  $A$  is called *Auslander-Gorenstein* if  $A$  satisfies the following condition: For any integers  $0 \leq i < j$  and finitely generated (right)  $A$ -module  $M$ , one has  $\text{Ext}_A^i(N, A) = 0$  for all (left)  $A$ -submodules  $N$  of  $\text{Ext}_A^j(M, A)$ . Set  $\text{j}(M) = \min\{j : \text{Ext}_A^j(M, A) \neq 0\}$ . The algebra  $A$  is *CM* if  $\text{j}(M) + \text{GKdim } M = \text{GKdim } A$  holds for all finitely generated, non-zero  $A$ -modules  $M$ .

**Corollary 3.3.** (i) *The commutant  $\mathcal{C}$  of  $\mathcal{O}^G$  in  $\mathcal{D}(\mathfrak{g})$  is the ring generated by  $\mathcal{O}$  and  $\tau(\mathfrak{g})$ . Moreover,  $\mathcal{C}$  is an Auslander-Gorenstein, CM, Noetherian domain and a maximal order.*

(ii) As a (left or right)  $\mathcal{O}$ -module,  $\mathcal{C} \cap \mathcal{D}(\mathfrak{g})_m$  is generated by the elements

$$\{\tau(\xi_1)\tau(\xi_2)\cdots\tau(\xi_k) : \xi_i \in \mathfrak{g} \text{ and } k \leq m\}.$$

(iii) The centre of  $\mathcal{C}$  is  $\mathcal{O}(\mathfrak{g})^G$ .

*Proof.* (i) By Theorem 3.2,  $\mathcal{C}$  is generated by  $\mathcal{O}$  and  $\tau(\mathfrak{g})$ . By that theorem and Proposition 2.4,  $\text{gr } \mathcal{C}$  satisfies the other conditions given in part (i). Let  $M = \bigcup_{n \in \mathbb{N}} M_n$  be a filtered right  $\mathcal{C}$ -module such that  $\text{gr } M$  is a finitely generated  $\text{gr } \mathcal{C}$ -module. By [3, Theorem 3.9],  $\mathcal{C}$  is Auslander–Gorenstein and  $j_{\mathcal{C}}(M) = j_{\text{gr } \mathcal{C}}(\text{gr } M)$ . However, [13, Corollary 1.4] implies that  $\text{GKdim } \text{gr } M = \text{GKdim } M$  and hence that  $\mathcal{C}$  is CM. Finally, [15] implies that  $\mathcal{C}$  is a maximal order.

(ii) This follows from the fact that, in  $\text{gr } \mathcal{C} = \text{Sym}(E)$ , a homogeneous element  $\bar{c}$  of degree  $m$  can be written  $\bar{c} = \sum f_{i_1, \dots, i_m} \xi_{i_1} \cdots \xi_{i_m}$ , for some  $f_{i_1, \dots, i_m} \in \mathcal{O}$  and  $\xi_{i_j} \in \tau(\mathfrak{g})$ .

(iii) Let  $Z$  denote the centre of  $\mathcal{C}$ . Clearly both  $\tau(\mathfrak{g})$  and  $\mathcal{O}$  commute with  $\mathcal{O}^G$  and so  $\mathcal{O}^G \subseteq Z$ . Conversely,  $Z$  is contained in the commutant, in  $\mathcal{D}(\mathfrak{g})$ , of  $\mathcal{O}$ . Hence,  $Z \subseteq \mathcal{O}$ . Since  $\mathcal{O}^G$  is the commutant, in  $\mathcal{O}$ , of  $\tau(\mathfrak{g})$ , the result follows.  $\square$

**Corollary 3.4.** *Both  $\mathcal{C}$  and  $\text{Sym}(E)$  are free (left or right) modules over  $\mathcal{O}(\mathfrak{g})^G$ .*

*Proof.* Set  $\mathcal{O} = \mathcal{O}(\mathfrak{g})$  and  $S = \text{Sym}(E) = \bigoplus_{m=0}^{\infty} \text{Sym}_m(E)$ . We first prove the result for  $\mathcal{C}$ , assuming that  $\text{Sym}_m(E)$  is a free  $\mathcal{O}^G$ -module for all  $m \in \mathbb{N}$ . Note that the isomorphism  $\text{gr } \mathcal{C} \cong S$  of Theorem 3.2 is a graded isomorphism of  $\mathcal{O}$ -algebras, for the natural graded structure of the two objects. In other words  $\mathcal{C}_m/\mathcal{C}_{m-1} \cong \text{Sym}_m(E)$ , for all  $m$ , where  $\mathcal{C}_m = \mathcal{C} \cap \mathcal{D}(\mathfrak{g})_m$ . Hence, each  $\mathcal{C}_m/\mathcal{C}_{m-1}$  is a free  $\mathcal{O}^G$ -module; it follows routinely that  $\mathcal{C}$  is also free over  $\mathcal{O}^G$ .

We now prove the result for  $\text{Sym}_m(E)$ . Note, first, that  $S$  is a quotient of the polynomial ring

$$T = \text{Sym}_{\mathcal{O}}(\mathcal{O} \otimes \mathfrak{g}) \cong \mathcal{O}[y_1, \dots, y_n] \cong \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n],$$

which we now grade by giving each generator  $x_i$  and  $y_j$  degree one. Since the  $u_i$  are homogeneous in  $\mathcal{O}$ , the  $\nabla_i = \text{grad}(u_i)$  are homogeneous in  $T$  and so, by Corollary 2.3,  $\text{Sym}_m(E)$  is a graded  $\mathcal{O}^G$ -module.

Set  $P = \sum_{i=1}^{\ell} u_i S$ . By [6, Proposition 2.16] and its proof (which depends upon a case by case analysis),  $S/P$  is a domain of dimension  $2n - 2\ell = \dim S - \ell$ . Hence, the  $u_j$  form a regular sequence in  $S$ , and therefore in each module  $\text{Sym}_m(E)$ . Thus, by [4, § 8, Proposition 8 and § 9, Corollaire 2],  $\text{Sym}_m(E)$  is a graded free  $\mathcal{O}^G$ -module.  $\square$

Corollary 3.3 and Corollary 3.4 should be compared with [9] which (as a very special case) shows that the commutant of  $\mathcal{D}(\mathfrak{g})^G$  is simply  $\mathbb{C}\langle\tau(\mathfrak{g})\rangle$  ( $\cong U(\mathfrak{g})$  when  $\mathfrak{g}$  is semisimple). Moreover, both rings are free modules over the centre of  $\mathcal{D}(\mathfrak{g})^G$  (which is also the centre of  $\mathbb{C}\langle\tau(\mathfrak{g})\rangle$ ).

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