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On foliated circle bundles over closed orientable 3-manifolds

Shigeaki Miyoshi

Abstract. We show that there exists a family of smooth orientable circle bundles over closed orientable 3-manifolds each of which has a codimension-one foliation transverse to the fibres of class C^0 but has none of class C^3. There arises a necessary condition induced from the Milnor-Wood inequality for the existence of a foliation transverse to the fibres of an orientable circle bundle over a closed orientable 3-manifold. We show that with some exceptions this necessary condition is also sufficient for the existence of a smooth transverse foliation if the base space is a closed Seifert fibred manifold.

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§ 1. Introduction and statements of results

Suppose $\xi = \{E \to \Sigma\}$ is an orientable circle bundle over a closed orientable surface Σ. In [M] and [W] the necessary and sufficient condition for the existence of a codimension-one foliation on E which is transverse to each fibre is obtained: Denote by $\chi(\xi)$ the Euler number of the circle bundle ξ and set $\chi_-(\Sigma) = \max\{0, -\chi(\Sigma)\}$, where $\chi(\Sigma)$ denotes the Euler characteristic of Σ. Then there exists a codimension-one foliation transverse to the fibres on the total space E if and only if $|\chi(\xi)| \leq \chi_-(\Sigma)$. Though a similar inequality is obtained in non-orientable case, we omit it for simplicity. We call this inequality *Milnor-Wood inequality*. In the case that the base is a 3-manifold, this Milnor-Wood inequality induces a necessary condition for the existence of a codimension-one foliation transverse to the fibres. That is, suppose $\xi = \{E \to M\}$ is an orientable circle bundle over a closed orientable 3-manifold M, and if there exists a codimension-one foliation transverse to the fibres on the total space E, then the following condition is satisfied:

(MW) : $|\langle e(\xi), z\rangle| \leq x(z)$ for any $z \in H_2(M; \mathbf{Z})$.

Here, $\langle\, ,\, \rangle$ denotes Kronecker product, $e(\xi) \in H^2(M;\mathbf{Z})$ is the Euler class of ξ and x is Thurston norm, that is, the pseudonorm on $H_2(M;\mathbf{Z})$ defined as follows: for any $z \in H_2(M;\mathbf{Z})$, $x(z)$ is defined to be the minimum $\chi_-(\Sigma)$ of all (singular or embedded) surfaces Σ in M each of which represents the given homology class z. Originally, Thurston norm was defined in [Th2] based on embedded surfaces representing the homology class and it was conjectured by Thurston that it coincides with the singular norm, the norm based on singular surfaces. D. Gabai showed in [Ga] that both of them coincide with the half of Gromov norm.

As another setting to study foliated circle bundles, bounded cohomology could be efficiently used. In fact, E. Ghys [Gh2] showed that the Gromov norm of the Euler class of a foliated circle bundle is less than or equal to 1/2. Also, he proved that a class in the second bounded cohomology $H_b^2(\Gamma;\mathbf{Z})$ of any discrete countable group Γ is realizable in $Homeo_+(S^1)$ if and only if it contains a cocycle taking only the values 0 and 1. Here, $Homeo_+(S^1)$ denotes the group of all orientation preserving homeomorphisms of the circle, and we say that a class $c \in H_b^2(\Gamma;\mathbf{Z})$ is *realizable* in $Homeo_+(S^1)$ if there is a homomorphism $\varphi : \Gamma \to Homeo_+(S^1)$ such that the class c is the pull-back of the bounded Euler class $eu \in H_b^2(Homeo_+(S^1);\mathbf{Z})$ by φ: $c = \varphi^*(eu)$.

In this paper, we consider the problem that asks if the condition (MW) is sufficient for the existence of a codimension-one foliation transverse to the fibres. In fact, we show the following:

Theorem 1. *There exists a family of smooth orientable circle bundles over closed orientable 3-manifolds each of whose total spaces has a codimension-one C^0 foliation transverse to the fibres and has none of class C^3.*

On the other hand, we have the following existence theorem:

Theorem 2. *Suppose $\xi = \{E \to M\}$ is a smooth orientable circle bundle over a closed orientable Seifert fibred manifold M. Assume that $H_1(M;\mathbf{Z})$ is torsion-free if the Euler number of the Seifert fibration is zero. Then, there exists a codimension-one C^∞ foliation transverse to the fibres on the total space E if and only if the Euler class $e(\xi)$ satisfies the condition (MW).*

For the definition of the Euler number of Seifert fibrations, see the proof of Theorem 2 in Section 4.

Remark 1. Concerning Theorem 2, note that the triviality of the Euler number does not necessarily imply the fact that the first homology group is torsion-free. That is, there is a Seifert fibred manifold M whose Euler number is zero such that $H_1(M;\mathbf{Z})$ has torsion.

Remark 2. Any torsion class in $H^2(M;\mathbf{Z})$ can be realized as the Euler class of a C^∞ foliated circle bundles over M. In fact, it can be proved that for any torsion class $e \in H^2(X;\mathbf{Z})$ of any dimensional closed manifold X there exists a C^∞ foliated circle bundle ξ over X such that $e(\xi) = e$.

Note. After this work was done and I talked the proof of Theorem 1 in a meeting at Atami, H. Minakawa informed me that he constructed a smooth orientable circle bundle over a closed orientable 3-manifold which admits C^0 transverse foliation to the fibres but none of class C^2.

§ 2. A rigidity theorem of Ghys

Let Σ be a closed orientable surface of genus greater than one. We denote by $Homeo_+(S^1)$ and $Diff^r_+(S^1)$ the group of all orientation preserving homeomorphisms of the circle and the group of all orientation preserving C^r diffeomorphisms of the circle ($0 \le r \le \infty$), respectively. Note that $Homeo_+(S^1) = Diff^0_+(S^1)$. Two homomorphisms $\psi_1, \psi_2 : \pi_1(\Sigma) \to Homeo_+(S^1)$ are said to be C^r *conjugate* if there is an orientation preserving C^r diffeomorphism of the circle $f (\in Diff^r_+(S^1))$ such that $\psi_1(\gamma) = f\psi_2(\gamma)f^{-1}$ for any $\gamma \in \pi_1(\Sigma)$.

A foliated circle bundle is completely determined by its *total holonomy homomorphism.* Precisely, the following holds (see [HH] for example):

Proposition. *The correspondence which assigns to a foliated circle bundle its total holonomy is a natural bijection between the set of all C^r (resp. C^0) isomorphism classes of orientable foliated circle bundles over a C^r manifold X and the set of all C^r (resp. topological) conjugacy classes of homomorphisms from $\pi_1(X)$ to $Diff^r_+(S^1)$ (resp. $Homeo_+(S^1)$).*

By this correspondence we consider that a homomorphism $\pi_1(X) \to Diff^r_+(S^1)$ is an equivalent of an orientable C^r foliated circle bundle over X.

Let $PSL(2,\mathbf{R})$ denote the projective special linear group of degree 2. It is well known that $PSL(2,\mathbf{R})$ is isomorphic to the group of all orientation preserving isometries of Poincaré disk of hyperbolic geometry. Moreover, $PSL(2,\mathbf{R})$ naturally acts the circle at infinity and therefore it may be considered as a subgroup of $Diff^\infty_+(S^1)$.

Suppose that $\psi_1, \psi_2 : \pi_1(\Sigma) \to PSL(2,\mathbf{R})$ are two injective homomorphisms and their images are cocompact, discrete subgroups of $PSL(2,\mathbf{R})$. Then it is known that ψ_1 and ψ_2 are topological conjugate. However, they are C^1 conjugate only if their images are conjugate in $PSL(2,\mathbf{R})$ (cf. [Gh1], [S]). See also [Gh2], [Ma1], [Ma2].

Consider now the minimum case in Milnor-Wood inequality, that is the case $\chi(\xi) = \chi(\Sigma)$. It is the case of the unit tangent circle bundle of a closed hyperbolic surface Σ. In the sense of smooth conjugacy, it is shown that this is the only foliated circle bundle in this case. That is, E. Ghys proved the following:

Theorem 2.1. ([Gh3]) *Suppose* $\psi : \pi_1(\Sigma) \to \mathit{Diff}^r_+(S^1)\,(3 \le r \le \infty)$ *is a homomorphism with* $\chi(\psi) = \chi(\Sigma)$. *Then, there exists an injective homomorphism* $\varphi : \pi_1(\Sigma) \to PSL(2, \mathbf{R})$ *whose image is a cocompact discrete subgroup such that* ψ *is* C^r *conjugate to* φ.

§ 3. Non-smoothable foliated circle bundles

In this section, we prove Theorem 1. First, we construct a circle bundle and then we show that it has the desired property.

Construction. Let Σ be a closed orientable surface of genus $g > 1$ and $M \to S^1$ an orientable Σ-bundle over the circle with the monodromy diffeomorphism $f : \Sigma \to \Sigma$. Assume f is not isotopic to a periodic diffeomorphism. This bundle $M \to S^1$ defines a simple foliation $\mathcal{F}$ whose leaves are the fibres of the bundle. Denote by $e(T\mathcal{F})$ the Euler class of the tangent bundle to $\mathcal{F}$. Let $\xi = \{E \to M\}$ be the orientable circle bundle over M whose Euler class $e(\xi)$ is equal to $e(T\mathcal{F})$. In other words, the bundle ξ is the unit tangent circle bundle to the simple foliation $\mathcal{F}$.

Verification. Now we prove that ξ has no C^3 transverse foliation. We consider $\Sigma \subset M$ as the fibre over the base point $0 \in \mathbf{R}/\mathbf{Z} = S^1$. First, note that by the definition of $e(\xi)$, ξ restricted to Σ is isomorphic to the unit tangent circle bundle over Σ as circle bundles:

$$\xi|\Sigma \cong \{T_1\Sigma \to \Sigma\} \tag{1}$$

Assume on the contrary to the assertion that there exists a homomorphism $\psi : \pi_1(M) \to \mathit{Diff}^r_+(S^1)\,(3 \le r \le \infty)$ such that $e(\psi) = e(\xi)$. Then, ξ is isomorphic to the foliated circle bundle $E_\psi \to M$ as circle bundles, where $E_\psi \to M$ is defined by the homomorphism ψ. Therefore, by Ghys' rigidity theorem and (1), $\psi|\Sigma$ is C^r conjugate to a representation of $\pi_1(\Sigma)$ into $PSL(2, \mathbf{R})$ with respect to a hyperbolic metric on Σ. That is, suppose $\pi_1(M)$ is presented as

$$\pi_1(M) = \left\langle a_i, b_i, t \;\middle|\; \prod_{i=1}^{g}[a_i, b_i], t a_i t^{-1} f_*(a_i)^{-1}, t b_i t^{-1} f_*(b_i)^{-1} \right\rangle,$$

where a_i and $b_i\,(i = 1, \ldots, g)$ are standard generators on Σ, then, up to C^r conjugacy, we can assume that $\psi(a_i), \psi(b_i) \in PSL(2, \mathbf{R})\,(i = 1, \ldots, g)$. Moreover, since $t a_i t^{-1} f_*(a_i)^{-1} = 1$, we have

$$\psi(t)\psi(a_i)\psi(t)^{-1} = \psi(f_*(a_i)). \tag{2}$$

Similarly, for $b_i \in \pi_1(M)$, we have

$$\psi(t)\psi(b_i)\psi(t)^{-1} = \psi(f_*(b_i)). \tag{3}$$

On the other hand, since the isometry group of a closed hyperbolic surface is finite, the non-periodic diffeomorphism f cannot be an isometry of Σ equipped with any hyperbolic metric. Therefore, there is a closed geodesic γ in Σ such that length(γ) $\neq$ length[$f(\gamma)$], where length[$f(\gamma)$] denotes the length of the unique closed geodesic which is freely homotopic to $f(\gamma)$. The length of a closed geodesic is related to the derivative of the corresponding element of $PSL(2, \mathbf{R})$ at its (expanding) fixed point: the length is equal to the logarithm of the derivative. Obviously this derivative, called the multiplier, is an invariant of smooth conjugacy. Therefore, the fact length(γ) $\neq$ length[$f(\gamma)$] contradicts (2) and (3).

Construction for C^0 case. We denote by $\widetilde{f}$ the extension of a lift of f to the circle at infinity of the Poincaré disk. It is known that $\widetilde{f}$ is homeomorphism of the circle but cannot be differentiable unless f is an isometry of Σ (cf. for example [CB] and [I]). Under the presentation of $\pi_1(M)$ as above, we define

$$\psi(\gamma) = \begin{cases} \varphi(a_i)\cdots & \text{if } \gamma = a_i \\ \varphi(b_i)\cdots & \text{if } \gamma = b_i \\ \widetilde{f}\cdots & \text{if } \gamma = t \end{cases}$$

on the generators, where φ is a faithful representation of $\pi_1(\Sigma)$ into $PSL(2, \mathbf{R})$ with respect to a hyperbolic metric on Σ. Then we have a homomorphism $\psi : \pi_1(M) \to Homeo_+(S^1)$ such that $e(\psi) = e(\xi)$. □

§ 4. Existence of foliated circle bundles

This section is devoted to the problem of existence of foliated circle bundles. The coefficient group of all homology and cohomology groups will be $\mathbf{Z}$ unless otherwise noted. Suppose that M is a closed orientable 3-manifold and a second cohomology class $e \in H^2(M)$ which satisfies the condition (MW) is given. That is, assume that for any $z \in H_2(M)$ the inequality $|\langle e, z\rangle| \leq x(z)$ holds. Then we consider the following question:

Question. Does there exist a C^r foliated circle bundle ξ over M such that $e(\xi) = e$?

As we proved Theorem 1 in the previous section, in the case $r \geq 3$ the answer is negative in general. However, we have an affirmative case. In fact, we will prove Theorem 2.

We will give a geometric proof here. An alternative proof based on the setting of group cohomology could be given and it would be shorter than the one presented here. However, the author believes that the geometric proof is also interesting in its own right.

Proof of Theorem 2. We only have to show that the condition (MW) is sufficient. Suppose that $M \to F$ is a Seifert fibration, where M is a closed orientable manifold, and that a class $e \in H^2(M)$ which satisfies the condition (MW) is given. The proof will be carried out in several steps.

First, we have the following:

Claim 1. *If F is non-orientable, then Thurston norm x is zero on $H_2(M)$.*

Proof. Suppose the genus of F is g. From the canonical presentation of $\pi_1(M)$ obtained from the fibration structure (cf. [H]), it can be seen that $H^1(M)$ is a free Abelian group of rank $g-1$. In fact, $H^1(M) = \mathrm{Hom}(\pi_1(M), \mathbf{Z})$ is generated by the homomorphisms $a_1^*, \ldots, a_{g-1}^*$ defined by

$$a_i^*(\gamma) = \begin{cases} 1 & \cdots \quad \text{if } \gamma = a_i \\ 0 & \cdots \quad \text{otherwise} \end{cases}$$

on the generators, where $a_1, \ldots, a_{g-1}$ are the generators coming from standard generators for a cross section over F minus singular points. Then the Poincaré dual of a_i^* is represented by a saturated torus in M which intersects a_i at exactly one point. Therefore $H_2(M)$ is generated by tori which implies that Thurston norm x is zero. □

Note that if the Thurston norm is zero, then the condition (MW) implies the class $e \in H^2(M)$ is torsion. As noted in Remark 2 in §1, any torsion class is realizable as the Euler class of a smooth foliated circle bundle. Therefore, from now on we assume the base surface F is orientable. Thus, suppose $\pi : M \to F$ is a Seifert fibration whose base surface F is orientable. Note that M is also assumed to be orientable. Let $S_1, \ldots, S_q$ be a non-empty collection of fibres in M, including all singular fibres. Denote by $N(S_i)$ a small saturated tubular neighbourhood of S_i in $M\,(1 \le i \le q)$ and set $M^* = M - \mathrm{int}(\cup_{i=1}^q N(S_i))$ and $F^* = \pi(M^*)$. Then the bundle $\pi : M^* \to F^*$ admits a cross section $s : F^* \to M^*$. Fix an orientation convention. Then the curve $s(F^*) \cap \partial N(S_i)$ represents a multiple of S_i in $\pi_1(N(S_i))$, say $-\beta_i[S_i]$. Also, suppose a regular fibre represents $\alpha_i[S_i]$ in $\pi_1(N(S_i))$. Then the *non-normalized Seifert invariant* is the collection of numbers

$$(g; (\alpha_1, \beta_1), \ldots, (\alpha_q, \beta_q))$$

which satisfy $g \ge 0$, $\alpha_i \ge 1$, $\gcd(\alpha_i, \beta_i) = 1$. Seifert invariant, up to a suitable equivalence, classifies such a Seifert fibration. Define $\chi(M \to F) = -\sum_i \frac{\beta_i}{\alpha_i}$. This number $\chi(M \to F)$ is independent of the choice of the expression of the Seifert invariant and is called the *Euler number* of the Seifert fibration $M \to F$. This Euler number is the characteristic number of Seifert fibrations. For more details, we refer to [NR] and [EHN].

Claim 2. *If $\chi(M \to F) \neq 0$, then x is zero on $H_2(M)$.*

Proof. By abelianizing the standard presentation of $\pi_1(M)$, we have $H_1(M) \cong \mathbf{Z}^{2g} \oplus \mathrm{Coker}(A)$, where $A : \mathbf{Z}^{q+1} \to \mathbf{Z}^{q+1}$ is a homomorphism with the matrix

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 \\ \alpha_1 & 0 & \cdots & 0 & \beta_1 \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & 0 & \vdots \\ & & & \alpha_q & \beta_q \end{pmatrix}.$$

Here missing entries are 0's and A acts on $\mathbf{Z}^{q+1}$ from the right. Then since

$$\begin{aligned} \det A &= (-1)^{q-1} \sum_{i=1}^{q} \beta_i \alpha_1 \cdots \widehat{\alpha_i} \cdots \alpha_q \\ &= (-1)^{q-1} \alpha_1 \cdots \alpha_q \sum_{i=1}^{q} \frac{\beta_i}{\alpha_i} \\ &= (-1)^{q} \alpha_1 \cdots \alpha_q \, \chi(M \to F) \\ &\neq 0, \end{aligned}$$

it follows that $H^1(M) = \mathrm{Hom}(H_1(M), \mathbf{Z})$ is isomorphic to $\mathbf{Z}^{2g}$ which is generated by the dual of the standard generators of $H_1(F)$ lifted on the cross section $s(F^*)$. As in the proof of Claim 1, it follows $H_2(M)$ is generated by tori, which implies the Thurston norm is zero. □

Now it remains the case $\chi(M \to F) = 0$. In this case, we can construct a transverse foliation $\mathcal{F}$ of $M \to F$ whose leaves are all compact (cf. Theorem 3.4 of [EHN]). The compact foliation $\mathcal{F}$ is just the fibre structure of an S^1 equivariant fibre bundle $M \to S^1$. We denote by $e(T\mathcal{F})$ the Euler class of the tangent bundle to $\mathcal{F}$. We have the following description of the class e:

Claim 3. *There exists a transverse foliation $\mathcal{F}$ of $M \to F$ whose leaves are all compact such that the class e is equal to a multiple of the Euler class $e(T\mathcal{F})$ in $H^2(M; \mathbf{Q})$.*

Proof. As is similar to the proof of Claim 1 and 2, it is easily seen that $H_2(M)$ is freely generated by vertical (i.e. saturated) tori and horizontal (i.e. transverse) closed surfaces. Since the class e satisfies the condition (MW), Kronecker products of e with tori vanish. On the other hand, a closed oriented horizontal surface S determines a transverse foliation $\mathcal{F}$ of $M \to F$ whose leaves are all compact and S is a leaf of $\mathcal{F}$. Kronecker products of $e(T\mathcal{F})$ with the vertical tori also vanish. Therefore, e and $e(T\mathcal{F})$ differ in a constant multiple as linear forms on $H_2(M;\mathbf{Q})$. □

By the hypothesis of the theorem, we are assuming that $H_1(M)$ is torsion-free. Therefore Claim 3 describes the given class completely. Suppose that $\mathcal{F}$ is a foliation on M as in Claim 3. Let $\widetilde{F}$ denote a leaf of the foliation $\mathcal{F}$. We denote the monodromy diffeomorphism by $f : \widetilde{F} \to \widetilde{F}$. Then it is clear that f is periodic and the quotient of $\widetilde{F}$ by f-action is F. Also the fibration $M \to F$ restricted to $\widetilde{F}$ is a branched covering $\widetilde{F} \to F$. Suppose the covering is m-fold. Then we can consider F as an orbifold with its uniformization $\widetilde{F} \to F$. Let $(g; (\alpha_1, \beta_1), \dots, (\alpha_q, \beta_q))$ be the Seifert invariant of $M \to F$. We set

$$\chi^{orb}(F) = \chi(F) - q + \sum_{i=1}^{q} \frac{1}{\alpha_i},$$

and call $\chi^{orb}(F)$ the *Euler characteristic* of the orbifold F. As the m-fold branched covering $\widetilde{F} \to F$ is exactly an m-fold orbifold covering, it follows that $\chi(\widetilde{F}) = m\chi^{orb}(F)$. We can assume that $\chi(\widetilde{F}) < 0$. Then by the assumption we have $\left|\left\langle e, [\widetilde{F}]\right\rangle\right| \leq \chi_-(\widetilde{F}) = -\chi(\widetilde{F}) = -m\chi^{orb}(F)$. Therefore, we have

$$-\chi(F) + q - \sum_{i=1}^{q} \frac{1}{\alpha_i} = -\chi^{orb}(F) \geq \frac{1}{m} \left|\left\langle e, [\widetilde{F}]\right\rangle\right|. \tag{4}$$

Denote by $\pi_1^{orb}(F)$ the fundamental group of the orbifold F (cf. [Th1]). Note that a Seifert fibration $M_1 \to F_1$ can be considered as an orbifold circle bundle (cf. [Th1]). As is similar to the case of circle bundles, it can be seen that a foliated Seifert fibration $M_1 \to F_1$ (that is, a Seifert fibration $M_1 \to F_1$ with a foliation transverse to the fibres on M_1) is determined by the associated homomorphism $\pi_1^{orb}(F_1) \to Homeo_+(S^1)$ and vice versa (cf. [EHN]). We also call such a homomorphism the *total holonomy homomorphism* of the foliated Seifert fibration. We will construct a homomorphism $\varphi : \pi_1^{orb}(F) \to PSL(2,\mathbf{R})$ such that the Euler number of the foliated Seifert fibration determined by φ is exactly equal to $\frac{1}{m}\left|\left\langle e, [\widetilde{F}]\right\rangle\right|$.

Assume first such a homomorphism $\varphi : \pi_1^{orb}(F) \to PSL(2,\mathbf{R})$ is constructed. Let $V \to F$ be the foliated Seifert fibration determined by φ. Then we pull it back over $\widetilde{F}$ so that we have a foliated circle bundle $\widetilde{V} \to \widetilde{F}$. By the naturality, the Euler number of $\widetilde{V} \to \widetilde{F}$ is $\left|\left\langle e, [\widetilde{F}]\right\rangle\right|$. Moreover, by the construction the action by f on $\widetilde{F}$ lifts on $\widetilde{V}$ and it preserves the foliation on $\widetilde{V}$. Let $\widetilde{V} \times \mathbf{R} \to \widetilde{F} \times \mathbf{R}$ be the product of the foliated circle bundle $\widetilde{V} \to \widetilde{F}$ with $\mathbf{R}$. Consider the equivalence relation on $\widetilde{F} \times \mathbf{R}$ and $\widetilde{V} \times \mathbf{R}$ generated by $(f(x), t) \sim (x, t+1)$. Then the quotient $\widetilde{F} \times \mathbf{R}/_\sim$ by the equivalence relation is just M and $\widetilde{V} \times \mathbf{R}/_\sim \to \widetilde{F} \times \mathbf{R}/_\sim$ defines a foliated circle bundle over M. Denote it by $\xi = \{E \to M\}$. The bundle ξ is exactly the pull-back of $V \to F$ by the projection of the Seifert fibration $M \to F$. It is now obvious by the construction that the Euler class $e(\xi)$ of ξ is equal to the given class e. Therefore, ξ is the desired foliated circle bundle.

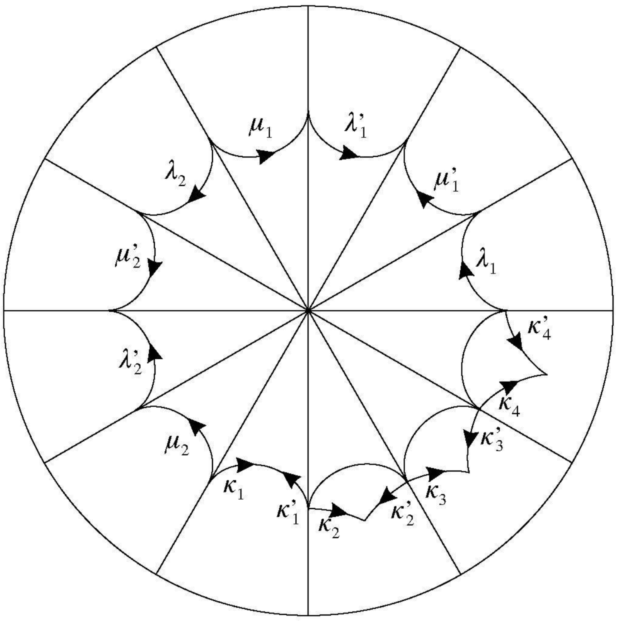

Figure 1

Finally, we construct the homomorphism $\varphi : \pi_1^{orb}(F) \to PSL(2, \mathbf{R})$. We mostly follow the proof of Poincaré's realization theorem of a Fuchsian group (cf. Theorem 4.3.2 of [K]). We use the unit disk model D of hyperbolic geometry. We can assume that $\alpha_i > 1$ for $i = 1, \ldots, q$. Let $P(t)$ be a regular hyperbolic polygon with $(4g+q)$ sides centered at the center of D each of whose vertex is situated at Euclidean distance $t\,(0 < t < 1)$ from the center of D. On the last q sides we add q external isosceles hyperbolic triangles such that the angles between the equal sides of the triangles are $2\pi/\alpha_1, \ldots, 2\pi/\alpha_q$. Note that if $\alpha_i = 2$, the corresponding triangle will degenerate. We denote by $Q(t)$ the union of $P(t)$ with these triangles. Label these sides $\lambda_1, \mu'_1, \lambda'_1, \mu_1, \ldots, \lambda_g, \mu'_g, \lambda'_g, \mu_g, \kappa_1, \kappa'_1, \ldots, \kappa_g, \kappa'_g$, and orient them as indicated in Figure 1. If t goes to 0, the hyperbolic area of $Q(t)$ tends to 0.

It can be seen that if t goes to 1, then the hyperbolic area of $Q(t)$ tends to $2\pi\{(2g-1)+\sum_{i=1}^{q}(1-\frac{1}{\alpha_i})\}$. Hence, by continuity and the inequality (4), there exists t_0 between 0 and 1 such that the hyperbolic area of $Q(t_0)$ is exactly equal to $\frac{2\pi}{m}\left|\left\langle e,[\widetilde{F}]\right\rangle\right|$. By the construction λ_i and λ_i' have the same hyperbolic length as do μ_i and μ_i', and κ_j and κ_j'. Then for each pair of geodesics there exists an orientation preserving isometry of D which maps one to the other. That is, there exist $A_i, B_i, C_j \in PSL(2,\mathbf{R})$ $(i=1,\dots,g; j=1,\dots,q)$ such that

$$A_i(\lambda_i') = \lambda_i,\ B_i(\mu_i') = \mu_i,\ C_j(\kappa_j') = \kappa_j.$$

Suppose $\pi_1^{orb}(F)$ is presented as

$$\left\langle a_i, b_i, c_j\,(i=1,\dots,g; j=1,\dots,q) \;\middle|\; \prod_{i=1}^{g}[a_i,b_i]\prod_{j=1}^{q}c_j,\ c_j^{\alpha_j}\right\rangle$$

Define a homomorphism $\varphi : \pi_1^{orb}(F) \to PSL(2,\mathbf{R})$ by

$$\varphi(\gamma) = \begin{cases} A_i & \cdots \quad \text{if } \gamma = a_i \\ B_i & \cdots \quad \text{if } \gamma = b_i \\ C_j & \cdots \quad \text{if } \gamma = c_j. \end{cases}$$

Then it can be easily seen that φ or the conjugacy of φ by the reflection of the circle is the desired homomorphism. □

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