⌈**Commentarii Mathematici Helvetici**

Embeddable anticonformal autormorphisms of Riemann surfaces

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Abstract. Let S be a Riemann surface and f be an automorphism of finite order of S. We call f embeddable if there is a conformal embedding $e : S \to \mathbf{E}^3$ such that $e \circ f \circ e^{-1}$ is the restriction to $e(S)$ of a rigid motion. In this paper we show that an anticonformal automorphism of finite order is embeddable if and only if it belongs to one of the topological conjugation classes here described. For conformal automorphisms a similar result was known by R.A. Rüedy [R3].

Mathematics Subject Classification (1991). 30F99.

Keywords. Riemann surface, anticonformal automorphism, conformal embedding.

1. Introduction

Some of the best known examples of Riemann surfaces are given by smooth surfaces embedded in the euclidean space. The euclidean metric induces a conformal structure (the existence of such structure is given by the solutions of Beltrami equation). The Riemann surfaces constructed in such way are called *classic* surfaces and these classic surfaces are known from the work of both Riemann and Klein.

In fact Klein asked if every Riemann surface is conformally equivalent to a classic surface. The answer is positive and was given by A. Garsia [G] for compact surfaces and by R. A. Rüedy [R1] for the non-compact case.

Riemann surfaces with automorphisms play an important role in some aspects of this theory for instance in the study of Moduli. If a classic surface is invariant by the action of a rigid motion in the space, such motion, since it preserves the metric, induces an automorphism of the classic surface. We have in this way the first examples of Riemann surfaces with automorphisms. It is a natural question to ask which automorphisms are represented by these examples. In the case of conformal automorphisms Rüedy [R3] characterized the authomorphisms that can

be obtained in this way. In this paper we shall characterize the anticonformal ones. A related question is studied in [Z] and [C2]: when is the square of an anticonformal automorphism embeddable.

Let S be a Riemann surface and f be an automorphism of finite order of S. We shall call f embeddable if there is a conformal embedding $e : S \to \mathbf{E}^3$ such that $e \circ f \circ e^{-1}$ is the restriction to $e(S)$ of a rigid motion. Hence we shall study the automorphisms that are embeddable.

If f is an embeddable conformal automorphism then $e \circ f \circ e^{-1}$ must be a rotation and our definition of embeddable automorphism agrees with the definition given by Rüedy in [R3]. In order to give the characterization of embeddable automorphisms obtained in [R3] wee need first some definitions. Let $\pi_1 O(S/f)$ be the fundamental group of the orbifold S/f and $H_1 O(S/f)$ be its abelianization. Let $T : H_1 O(S/f) \to Z_n$ be the monodromy epimorphism of the covering $S \to S/F$, where n is the order of f. Let X_i be the elements of $H_1 O(S/f)$ representing by the boundaries of discs around the cone points of S/f with the orientation given by the orientation of S.

Theorem (Rüedy 1971 [R3]). *A conformal automorphism f of finite order n is embeddable if and only if:*

1. For each cone point of S/f, the corresponding element of $H_1 O(S/f)$, X_i, and $T(X_i)$ have order n.

2. The number of cone points of S/f is even. Let $2r$ be this number.

3. There is a partition C_1, C_2 of the set $\{1, \ldots, 2r\}$ $(= C_1 \cup C_2)$ such that

    a. $\#C_1 = \#C_2 = r$,

    b. $T(X_i) = -T(X_j)$ if $i \in C_1$, $j \in C_2$ and

    c. $T(X_i) = T(X_j)$ if $i, j \in C_s$, $s = 1, 2$.

Assume that f is an anticonformal automorphism of a Riemann surface. If f is embeddable and if f has order two then $e \circ f \circ e^{-1}$ is the restriction to $e(S)$ of a reflection in a plane or the reflection on a center. Thus for anticonformal involutions we shall prove the following result:

Proposition 1.1. *Let f be an anticonformal involution of a Riemann surface S then f is embeddable if and only if f satisfies the following condition:*

    Condition 1.1.: either S/f is orientable or S/f is non-orientable without boundary.

If f is embeddable and f has order greater than 2 then $e \circ f \circ e^{-1}$ is the restriction to $e(S)$ of the composition of a plane reflection with a rotation with axis orthogonal to the plane of reflection.

The topological classification of anticonformal automorphisms of order multiple of 4 is very different from the one for automorphisms of order $2q$ with q odd (see [C1] or [Y]). Thus also for our study we must distinguish the case of automorphisms

with order a multiple of 4.

Proposition 1.2. *Let f be an anticonformal automorphism of order $2q$ with q even. The automorphism f is embeddable if and only if it satisfies one of the two following conditions:*

either

Condition 2.1. f^2 has fixed points and it is embeddable as a conformal automorphism

or

Condition 2.2. f^2 is fixed point free and $h_1(f) = 0$, where $h_1(f)$ is the invariant defined in [C1] (if $Z \in H_1O(S/f)$ is the element of order two, of $H_1(S/f)$ then the condition $h_1(f) = 0$ is equivalent, by definition, to $T(Z) = 0$).

Assume that S/f is orientable. Let F_i be a boundary component of S/f we shall call E_i the element of $H_1O(S/f)$ representing by F_i with the orientation given by the orientation of S/f.

For anticonformal automorphisms of order $2q$ with q odd we have:

Proposition 1.3. *Let f be an anticonformal automorphism of a Riemann surface S of order $2q$ with q odd. Let t be the number of boundary components of S/f. The automorphism f is embeddable if and only if it satisfies one of the two following conditions:*

either

Condition 3.1. S/f is orientable with boundary (i.e. $t > 0$) and:

1. For each cone point of S/f, the corresponding element of $H_1O(S/f)$, X_i, has order q.

2. There exists a number $s \leq t$ such that:

    a. If r is the number of cone points then $r + s$ is even,

    b. There are partitions $C_1 \cup C_2 = \{1, \dots, r\}$ and $B_1 \cup B_2 = \{1, \dots, s\}$ such that

        $\#(C_1 \cup B_1) = \#(C_2 \cup B_2)$

        $T(\{X_i, E_j : i \in C_1, j \in B_1\}) = \{\alpha\}$ *and* $T(\{X_i, E_j : i \in C_2, j \in B_2\}) = \{-\alpha\}$, *where α is a generator of $\langle 2 \rangle \subset Z_{2q}$,*

        $T(E_j) = 0$ *for all $j > s$*

or

Condition 3.2. S/F is non-orientable and so without boundary ($t = 0$) and then:

1. For each cone point of S/f, the corresponding element of $H_1O(S/f)$, X_i, has order q.

2. Let r be the number of cone points, then $T(\{X_i : i = 1, \dots, r\}) = \{\pm\alpha\}$, where α is a generator of $\langle 2 \rangle \subset Z_{2q}$.

3. If h is the genus of S/f then $h + r$ is even.

Note that the conditions in the above propositions as in the theorem of Rüedy are of topological nature. Hence an automorphism is embeddable if and only if it belongs to some of the topological types as defined by the conditions given above in the statements of the three propositions and the theorem.

In Section 2 we shall prove that the embeddable anticonformal automorphisms satisfy the conditions 1.1, 2.1, 2.2, 3.1 and 3.2 in the propositions 1.1, 1.2 and 1.3. In Section 3, for each topological type of automorphisms described in the conditions of such propositions, we shall obtain smooth surfaces embedded in the euclidean space that are invariant for the action of a rigid motion and such that the automorphism given by the restriction of the rigid motion meets these conditions. In Section 4 we shall prove that we can deform the smooth surfaces constructed in Section 3 in order to obtain a conformal embedding of a Riemann surface with an anticonformal automorphism satisfying the conditions 1.1, 2.1, 2.2, 3.1 and 3.2 onto a classic surface such that the automorphism becomes the restriction of a rigid motion.

2. Topological restrictions

Let S be a Riemann surface and f be an anticonformal automorphism of finite order. Assume that f is embeddable. Let $e : S \to \mathbf{E}^3$ be the conformal embedding such that $e \circ f \circ e^{-1}$ is the restriction to $e(S)$ of a rigid motion. In this Section we shall obtain the topological restrictions on f that are imposed because it is embeddable.

Case 1. f has order 2.
There are two posssibilities:
Subcase 1.a. $e \circ f \circ e^{-1}$ is the restriction to $e(S)$ of a reflection on a plane π. Then S/f is homeomorphic to each one of the components of $e(S) - \pi$. Hence S/f is an orientable surface with boundary.
Subcase 1.b. $e \circ f \circ e^{-1}$ is the restriction to $e(S)$ of a reflection on a point. Hence S/f has no boundary and since f is anticonformal, S/f is non-orientable.

So we have:

Lemma 2.1. *Let f be an anticonformal involution of a Riemann surface S. If f is embeddable then either S/f is orientable or S/f is non-orientable without boundary.*

Case 2. f has order $2q$ with q even.
We have also two cases:
Subcase 2.a. f has order $2q$ with q even and f^2 has fixed points. Since $e \circ f \circ e^{-1}$ is the restriction to $e(S)$ of a rigid motion then $e \circ f^2 \circ e^{-1}$ is the restriction of a rotation. Hence f^2 is embeddable. Thus the next lemma is obvious:

Lemma 2.2.a. *Let f be an anticonformal automorphism of a Riemann surface S of order $2q$ with q even. If f^2 has fixed points and f is embeddable then f^2 is embeddable.*

Notice first that, by theorem 0.2 of [C1], if f is an anticonformal automorphism of a Riemann surface S of order $2q$ with q even and f^2 has fixed points then the topological type of f is given by the isotropy invariants and so this type is determined by the topological type of f^2. Hence the topological restrictions given by Rüedy for f^2 to be embeddable determine completely the possible topological types of anticonformal automorphisms in the conditions of Case 2.a.

Subcase 2.b. f has order $2q$ with q even and f^2 is fixed point free.

Lemma 2.2.b. *Let f be an anticonformal automorphism of a Riemann surface S of order $2q$ with q even. If f^2 is fixed point free and f is embeddable then $h_1(f) = 0$.*

Proof. Let $T : H_1O(S/f) \to Z_{2q}$ be the monodromy epimorphism of the covering $S \to S/f$. Let $[Z] \in H_1O(S/f)$ be the element of order two, notice that, since f has no fixed points, $H_1O(S/f)$ is the homology of a non-orientable surface. The invariant $h_1(f)$ in the theorem 0.2 of [C1] is given by $T[Z]$. Since f is anticonformal $e \circ f \circ e^{-1}$ is the restriction to $e(S)$ of an orientation reversing rigid motion of order $2q$ so must be the restriction of the composition of a reflection on a plane π composed with a rotation of finite order $2q$ and with axis ρ orthogonal to π. Let π_1 and π_2 be two semiplanes with common boundary ρ, making a dihedral angle $\frac{\pi}{q}$ and cutting $e(S)$ transversally.

The intersection $Z = \pi_1 \cap e(S)$ is a finite set of closed curves (there are no arcs because the axis ρ does not cut $e(S)$ since f^2 is fixed point free). Let R be the convex region of $\mathbf{E}^3$ determined by π_1 and π_2. The quotient surface $e(S)/(e \circ f \circ e^{-1})$ can be obtained from $e(S) \cap R$ by identifying the curves $\pi_1 \cap e(S)$ with the curves $\pi_2 \cap e(S)$ in the orientation reversing way. Let $e(S) \cap R/ \sim$ be the described model of $e(S)/(e \circ f \circ e^{-1})$. Let $[Z]$ be the element of $H_1O(S/f) = H_1O(e(S)/(e \circ f \circ e^{-1})) = H_1O(e(S) \cap R/ \sim)$ represented by $Z = \pi_1 \cap e(S)$ in $e(S) \cap R/ \sim$. This element is of order two because $[(e(S) \cap R)/ \sim] - Z = e(S) \cap R$ is orientable and $\partial(e(S) \cap R) = 2Z$. To find the topological type of f is necessary to compute $T[Z]$ (theorem 0.2 of [C1]). Since there are $2q$ half-planes which are images of π_1 by $e \circ f \circ e^{-1}$ then element $[Z]$ lifts to $2q$ copies in $e(S)$. Hence $T[Z] = 0$ in other words $h_1(f) = 0$.

Case 3. f has order $2q$ with q odd.

Subcase 3.a. f has order $2q$ with q odd and S/f is orientable. Then $e \circ f \circ e^{-1}$ is the restriction to $e(S)$ of the composition of a reflection in a plane π composed with a rotation of axis ρ orthogonal to π and with finite order q. Let P be one of the connected components of $\mathbf{E}^3 - \pi$. Then $e \circ f^2 \circ e^{-1}$ acts on $e(S) \cap P$

as a rotation of order q. The surface $e(S) \cap P$ is orientable with boundary, let $W = e(S) \cap P/e \circ f^2 \circ e^{-1} \simeq S/f$.

Let $X_1, \ldots, X_r$ be the elements of $H_1O(S/f) \simeq H_1O(W)$ represented by the boundaries of small discs around the cone points and $E_1, \ldots, E_k$ be the elements of $H_1O(S/f)$ represented by boundary components of S/f. The cone points in S/f appear as the projection of the points where the axis ρ meets $e(S) \cap P$.

Then all the X_i have order q and the $T(X_i)$ are all equal to $\pm\alpha$ where α is a generator of $\langle 2 \rangle \subset Z_{2q}$.

With respect to the E_i there are three types. Let us give ρ the orientation that together with the sense of rotation of $e \circ f^2 \circ e^{-1}$ produces the canonical orientation of the space.

1.- The E_i that lift to $e(S) \cap P$ to cycles that do not link with the axis ρ. For these $T(E_i) = 0$.

2.- The E_i that lift to boundaries of $e(S) \cap P$, $\bar{E}_i$, such taht $lk\ (\bar{E}_i, \text{axis } \rho) = 1$. Then $T(E_i) = \alpha$, and

3.- The E_i that lift to boundaries of $e(S) \cap P$, $\bar{E}_i$, such that $lk\ (\bar{E}_i, \text{axis } \rho) = -1$. Then $T(E_i) = -\alpha$.

Note that W is naturally embedded in $\mathbf{E}^3 \simeq \mathbf{E}^3/e \circ f^2 \circ e^{-1}$. Let W^* be the surface obtained by capping, with disjoint discs, the boundary components of W. Since $W*$ is an orientable closed surface in $\mathbf{E}^3$ the image of the axis ρ in $\mathbf{E}^3/e \circ f^2 \circ e^{-1}$ have intersection number zero with $W*$. This fact produces 2.b in the condition 3.1 of the proposition 1.3.

So we have:

Lemma 2.3.a. *Let f be an anticonformal automorphism of a Riemann surface S of order $2q$ with q odd and S/f orientable. If f is embeddable then f satisfies the condition 3.1 of proposition 1.3.*

Subcase 3.b. f has order $2q$ with q odd and S/f is non-orientable. Then $e \circ f \circ e^{-1}$ is the restriction to $e(S)$ of the composition of a reflection in a point Q composed with a rotation of finite order q and with axis ρ passing through Q. Hence S/f has no boundary. The cone points in S/f are given by the intersection of the axis of ρ with $e(S)$. If $X_i, \ldots, X_r$ are the elements of $H_1O(S/f)$ produced by these cone points then X_i has order q and $T(X_i) = \pm\alpha$ where α is a generator of Z_q, with $i = 1, \ldots, r$.

Let π_1 and π_2 be two semiplanes with common boundary the axis ρ, making a dihedral angle $\frac{\pi}{q}$ and cutting $e(S)$ transversally.

The intersection $\pi_i \cap e(S)$, $i = 1, 2$, is a finite set of closed curves and arcs with end points in the axis ρ. Remark that the fixed point, Q, of the rigid motion lies on ρ. Since Q determines two components in ρ then there are two types of arcs in $\pi_i \cap e(S)$, $i = 1, 2$: the arcs with end points in the same component of ρ determined by Q and the arcs with end points not in the same component. Let V denote the set of first type of arcs and W the set of the second type.

Let R be the convex region of $\mathbf{E}^3$ determined by π_1 and π_2. The quotient surface $e(S)/(e \circ f \circ e^{-1})$ can be obtained from $e(S) \cap R$ identifying the curves and arcs of $\pi_1 \cap e(S)$ with the curves and arcs of $\pi_2 \cap e(S)$ using as glueing map $e \circ f \circ e^{-1}$. Let $e(S) \cap R/ \sim$ denote this model of $e(S)/(e \circ f \circ e^{-1})$. Every arc in V produces in $e(S) \cap R/ \sim$ two cone points and each arc in W produces one cone point and a cross cap. Each closed curve in $\pi_i \cap e(S)$, $i = 1, 2$, produces two cross caps, let U the set of such closed curves. Note that $R \cap e(S)$ is topologically an orientable surface, let k be the genus of $R \cap e(S)$. Hence:

$$\text{number of cone points in } S/f + \text{ genus of } S/f =$$
$$(2\#V + \#W) + (\#W + 2\#U + 2k) = \text{even number.}$$

Thus f satisfies 3 of condition 3.2 in proposition 1.3.
Then:

Lemma 2.3.b. *Let f be an anticonformal automorphism of a Riemann surface S of order $2q$ with q odd and S/f non-orientable. If f is embeddable then f satisfies the condition 3.2 of proposition 1.3.*

Note that if S/f is non-orientable the elements $T(X_i)$ are topological invariants up sign, i.e. by an automorphism of $H_1 O(S/f)$ we can choose the sign of each $T(X_i)$ (cf. [C1]).

3. Topological realization

Let (S_1, f_1) and (S_2, f_2) be two pairs of surfaces and homeomorphisms, we shall say that they are topologically equivalent if there is a homeomorphism $h : S_1 \to S_2$ such that $h \circ f_1 = f_2 \circ h$. In this Section we shall prove that if f is an anticonformal automorphism of a Riemann surface S satisfying the topological conditions of the propositions 1.1, 1.2 and 1.3 then there is a smooth surface $F \subset \mathbf{E}^3$ and a rigid motion g in $\mathbf{E}^3$ such that $g(F) \subset F$ and (S, f) is topologically equivalent to (F, g). More precisely:

Lemma 3.1. *Let f be an anticonformal automorphism of a Riemann surface S. Assume that f is in one of the following cases:*
  – f is an involution and f satisfies condition 1.1 of proposition 1.1.
  – f has order $2q$ with q even and f satisfies either condition 2.1 or condition 2.2 of proposition 1.2.
  – f has order $2q$ with q odd and f satisfies either condition 3.1 or condition 3.2 of proposition 1.3.
  Then there is a smooth surface F in $\mathbf{E}^3$ and a rigid motion g such that $g(F) \subset F$ and (F, g^l), for some $l \in \{1, \ldots, 2q - 1\}$, is topologically equivalent to (S, f).

Case 1: f of order 2

Subcase a: S/f orientable of genus h with s boundary components
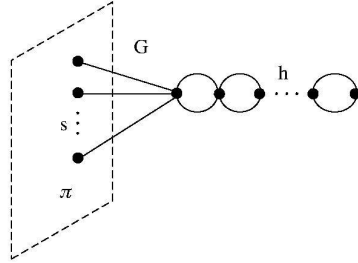
R = The semispace determined by π that contains G.
g = reflection on π

Subcase b: S/f nonorientable without boundary of genus h
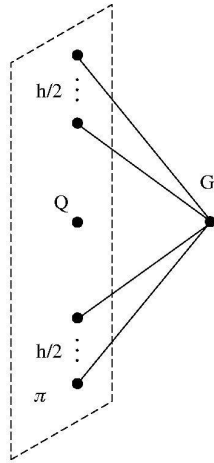
h even h odd
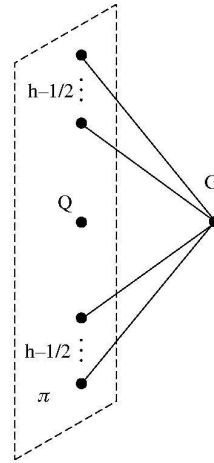
R = the semispace determined
by π containing G
g = reflection on Q
$N(G)$ is a neighborhood of G
such that $N(G) \cap \pi$ is g-equi-
variant

Figure 1.

Proof. The construction of (F, g) in each case is described in figures 1, 2 and 3. Each figure depicts a graph G in a region of the space R and a rigid motion g of finite order $2q$. If S/f is orientable then the surface F is $\cup_{i=0}^{2q-1} g^i(\partial N(G) \cap R)$, where $N(G)$ is a regular neighborhood of G such that $\partial N(G)$ is orthogonal to the planes π, π_1 and π_2 in the figures. If S/f is non-orientable the construction is similar but it is necessary to impose some conditions of g-equivariance on the neighborhood $N(G)$, such conditions are given on each Figure for each case.

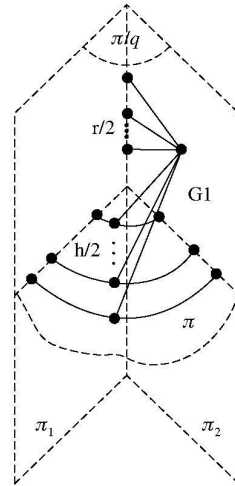Case 2: f or order $2q$ with q even (S/f nonorientable)
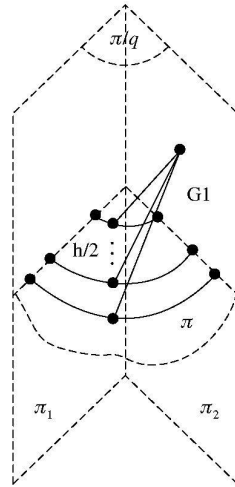
Subcase a: f^2 with $2r$ fixed points

r odd and S/f odd genus h r even and S/f even genus h
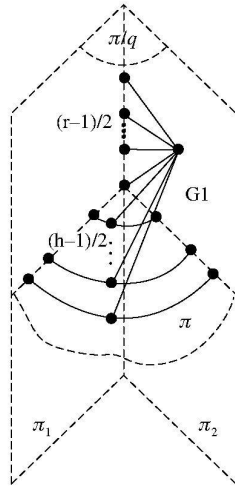
R = The convex region deter-
mined by π_1 and π_2
g = composition of rotation of
axis the intersection of π_1 and
π_2 and angle π/q with the re-
flection on the plane π.
$G = G_1 \cup g(G_1)$
$N(G)$ is a neighborhood of G
such that $g(N(G) \cap R) = N(G) \cap$
$g(R)$.

Subcase b: f^2 without fixed points

S/f of genus h (h must be even)

R = The convex region deter-
mined by π_1 and π_2
g = composition of rotation of
axis the intersection of π_1 and
π_2 and angle π/q with the re-
flection on the plane π.
$G = G_1 \cup g(G_1)$
$N(G)$ is a neighborhood of G
such that $g(N(G) \cap R) = N(G) \cap$
$g(R)$.

Figure 2.

Case 3: f of order $2q$ with q odd

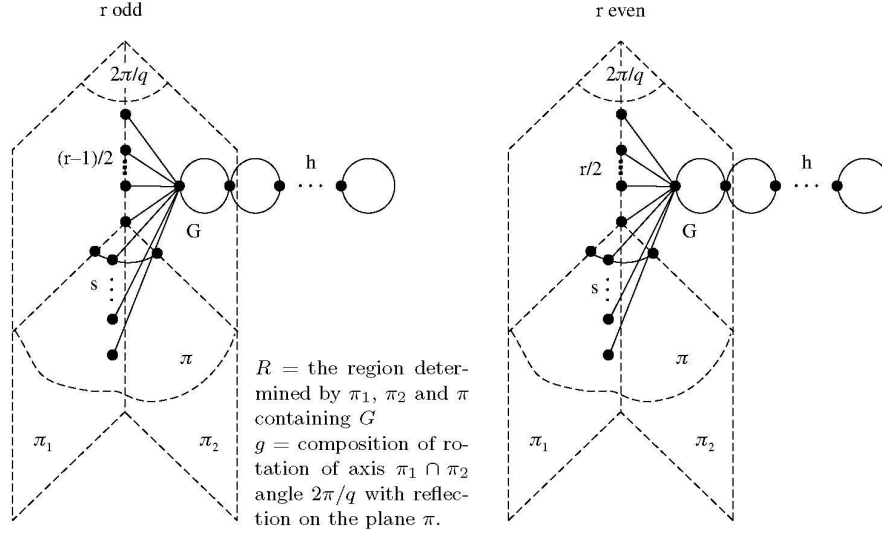Subcase a: S/f is orientable of genus h, with r cone points and s boundary components

Figure 3a

Note that in the case q even and f^2 with fixed points, the existence of the automorphism f ensures the condition $h+r$ even, where h is the genus of S/f and r is the number of cone points in S/f (cf. theorem 4 (condition iii) of [E]).

4. The conformal embedding

Let F be a classic surface and g be a rigid motion such that $g(F) \subset F$. Let N be a unit normal field on f and $\eta : F \to \mathbf{R}$ be a g-equivariant function i.e. $\eta \circ g = \eta$. We shall say that the surface $F + \eta N$ is a g-equivariant normal deformation of F. In this Section we shall prove the following result:

Lemma 4.1. *Let f be an anticonformal automorphism of a Riemann surface S, F be a classic surface and g be a finite order and orientation reversing rigid motion such that $g(F) \subset F$ and (S, f) is topologically equivalent to (F, g). Then there is a g-equivariant normal deformation F_η of the classic surface F such that there is a conformal embedding $e : S \to F_\eta \subset \mathbf{E}^3$ such that $g = e \circ f \circ e^{-1}$ on F_η.*

It is clear that the proposition 1.1 follows from the lemmae 2.1, 3.1 and 4.1, the proposition 1.2 follows from the lemmae 2.2a, 2.2b, 3.1 and 4.1 and the proposition

Case 3: f of order $2q$ with q odd

Subcase b: S/f is nonorientable of genus h, with r conic points and without boundary

R = The convex region determined by π_1 and π_2
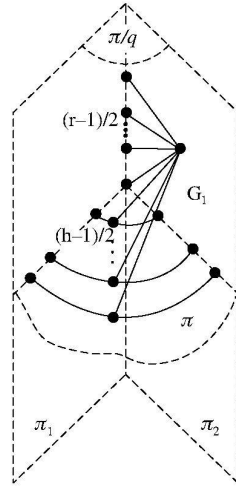
g = composition of rotation of axis the intersection of π_1 and π_2 and angle π/q with the reflection on the plane π.

$G = G_1 \cup g(G_1)$
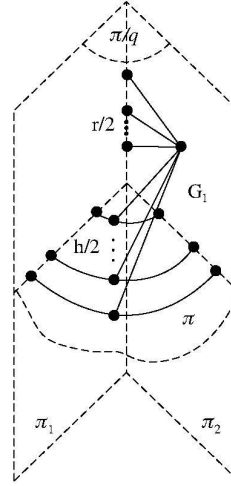
$N(G)$ such that $g(N(G) \cap R) = N(G) \cap g(R)$.

Figure 3b

1.3 follows from the lemmae 2.3.a, 2.3.b, 3.1 and 4.1.

Proof. Assume that S/f is uniformized by a crystallographic group with signature $\sigma = (g_1; \pm; [2, .\overset{r}{.}., 2], \{(-), .\overset{s}{.}., (-)\})$. Let T_σ be the Teichmüller space of crystallographic groups with signature σ (see [MS] and [BEGG]). The space T_σ is homeomorphic to a ball of real finite dimension. An element ρ of T_σ can be considered as a representation of an abstract NEC group with signature σ in $PSL(2, \mathbf{R})$ (modulo conjugation in $PSL(2, \mathbf{R})$). The automorphism f defines a canonical way to construct from an element ρ of T_σ, a subgroup $\Gamma_{\rho,f}$ of $im\rho$ such that $D/\Gamma_{\rho,f}$ is a Riemann surface with an anticonformal automorphism ϕ such that $D/\Gamma_{\rho,f}/\phi$ is a uniformized by $im\rho$ and $(D/\Gamma_{\rho,f}, \phi)$ is topologically equivalent to (S, f) (each topological type of anticonformal automorphisms defines an inclusion between abstract crystallographic groups). Then each element of T_σ determines a Teichmüller differential μ, that is defined for F and which is equivariant on the action of g.

Let ρ_0 be the element of T_σ such that $S/f = D/im\rho_0$ and ϵ be a positive real number. We shall follow [G], [R1] and [R2] to construct a continuous map

$$\Psi : \bar{B}_\epsilon(\rho_0) \subset T_\sigma \to T_\sigma$$

such that $d(\Psi(\rho), \rho) < \epsilon$ for all $\rho \in B_\epsilon(\rho_0)$, and such that $\Psi(\rho)$ is a g-equivariant normal deformation of F.

Assuming that we have such a Ψ then by Brouwer's theorem there is $\rho_0 = \Psi(\rho)$ for some $\rho \in B_\epsilon(\rho_0)$ and so we prove the existence of an g-equivariant normal deformation F_η.

Let μ be an g-equivariant Teichmüller differential on F corresponding to an element ρ of $\bar{B}_\epsilon(\rho_0)$. To define $\Psi(\rho)$ it is sufficient to define $\eta_\rho : S \to \mathbf{R}$, then $\Psi(\rho)$ is given by the surface $(e + \eta_\rho N \circ e)(S)$. In order to $\Psi(\rho)$ satisfy the condition $d(\Psi(\rho), \rho) < \epsilon$ we use the deformation lemma of Garsia ([G], [R1]).

Let ρ be in $\bar{B}_\epsilon(\rho_0)$, $\eta : F \to \mathbf{R}$ be a g-equivariant function and F_η be the g-equivariant normal deformation. Suppose that there is a K_0-quasiconformal mapping from $D/\Gamma_{\rho,f}$ onto F_η with dilatation $\leq 1 + \delta$ except on some portion P of F_η of real measure $\leq \gamma$. Suppose that (δ, γ) tends to $(0,0)$ and

$$\frac{1}{K(z)}\lambda(z)|dz + \mu(z)d\bar{z}| \leq |d((e + \eta_\rho N \circ e)(S))| \leq K(z)\lambda(z)|dz + \mu(z)d\bar{z}|, \quad (4.1)$$

where $\mu(z)$ is the g-equivariant Teichmüller differential corresponding to $D/\Gamma_{\rho,f}$, $1 \leq K(z) \leq K_0$ and $K(z) \leq 1 + \delta$ outside the set P. Thus $d_{T_g}(D/\Gamma_{\rho,f}, F_\eta)$ tends to zero, where T_g is the Teichmüller space of Riemann surfaces of genus g (= genus of S). The inclusion $T_\sigma \to T_g$ given by the inclusion between abstract NEC groups defined by f is an isometric embedding (see [MS]). Hence if ρ_1 is the point in T_σ corresponding to the NEC group uniformizing F_η/g then $d_{T_\sigma}(\rho, \rho_1)$ tends to zero.

In order to construct the continuous function $\Psi : \bar{B}_\epsilon(\rho) \to T_\sigma$ we must construct η_ρ for each ρ satisfying (4.1).

Let B be the boundary of a polygon that is a fundamental region of a group with signature σ and that uniformizes F/g.

Let $\bar{B}$ be the preimage of B in F by the natural projection. Then $F - \bar{B}$ is decomposed in $2q$ regions P_i, $i = 1, \ldots, 2q$. where $2q$ is the order of f. We construct the function η on P_1 following the construction given in [R1] (pages 425–426 and 433–437) and in the appendix of [R2]. Such construction is based on the fact that (4.1) can be replaced by

$$\frac{1}{K(z)}|\alpha_\mu^* dx + \beta_\mu^* dy| \leq \lambda^*(z)|\eta_x^* dx + \eta_y^* dy| \leq K(z)|\alpha_\mu^* dx + \beta_\mu^* dy|$$

where $\lambda^*(z)$, α_μ^*, β_μ^* are continuous real-valued functions. Choosing $\lambda^*(z)$ in a suitable way, we can force

$$\omega = \frac{1}{\lambda^*(z)}(\alpha_\mu^* dx + \beta_\mu^* dy)$$

to be exact. Then a solution of $d\eta^* = \omega$ give us, after some manipulations the function η satisfying (4.1) and vanishing on a small neighboorhood of the boundary of P_1.

Then we extend η equivariently by g, since μ is g-equivariant then η satisfies

(4.1) outside a neioghboorhood of $\bar{B}$ with measure converging to zero. Since η is g-equivariant we have the existence of the conformal embedding $e : S \to F_\eta$ such that $g = e \circ f \circ e^{-1}$.

Acknowledgement

The author would like to thank the referee for suggestions improving the exposition.

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